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ON PROPER 2-LABELLINGS DISTINGUISHING BY SUMS, MULTISETS OR PRODUCTS

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Abstract

Given a graph G, a k-labelling ℓ of G is an assignment $\ell : E(G) \to \{1, \ldots, k\}$ of labels from $\{1, \ldots, k\}$ to the edges. We say that ℓ is s-proper, m-proper or p-proper, if no two adjacent vertices of G are incident to the same sum, multiset or product, respectively, of labels.

Proper labellings are part of the field of distinguishing labellings, and have been receiving quite some attention over the last decades, in particular in the context of the well-known 1-2-3 Conjecture. In recent years, quite some progress was made towards the main questions of the field, with, notably, the analogues of the 1-2-3 Conjecture for m-proper and p-proper labellings being solved. This followed mainly from a better global understanding of these types of labellings.

In this note, we focus on a question raised by Paramaguru and Sampathkumar, who asked whether graphs with m-proper 2-labellings always admit s-proper 2-labellings. A negative answer to this question was recently given by Luiz, who provided infinite families of counterexamples. We give a more general result, showing that recognising graphs with m-proper 2-labellings but no s-proper 2-labellings is an NP-hard problem. We also prove a similar result for m-proper 2-labellings and p-proper 2-labellings, and raise a few directions for further work on the topic.

Keywords: proper labelling, sum of labels, multiset of labels, product of labels.

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1. Introduction

Let G be a graph. By a *labelling* ℓ of G, we mean an assignment of *labels* (numbers) from a given set to the edges of G. For a set S of labels, ℓ is called an S-*labelling* if it assigns labels from S, while, if $S = \{1, \ldots, k\}$ for some $k \geq 1$, then we call ℓ a *k-labelling*. In so-called **distinguishing labellings**, we are interested in designing labellings that permit to distinguish certain pairs of vertices accordingly to some function inferred by the assigned labels. As attested by the survey [12] by Gallian, this general definition is very flexible, and it gave birth, throughout the years, to a tremendous number of such distinguishing labelling notions.

In this work, we are interested in a subset of distinguishing labellings, called **proper labellings**. In proper labellings, the pairs of vertices that are required to be distinguished are the pairs of adjacent vertices. Regarding the distiguishing function, there are again many possibilities. We are here interested in three such functions, being the *sums*, *multisets* and *products* of labels incident to the vertices. Formally, let us consider a graph G, together with a labelling ℓ. For any vertex v of G, we denote by $\sigma(v)$ the sum of labels assigned by ℓ to the edges incident to v. Similarly, we denote by $\mu(v)$ and $\rho(v)$ the multiset and product, respectively, of these incident labels. Now, we say that ℓ is *s-proper* if no two adjacent vertices u and v of G are incident to the same sum of labels, i.e., if $\sigma(u) \neq \sigma(v)$ for every edge $uv \in E(G)$. Similarly, we say that ℓ is *m-proper* and *p-proper* if every two adjacent vertices are distinguished through the functions μ and ρ, respectively.

In proper labellings, the goal is generally to design labellings that are not only proper, but also k-labellings for k as small as possible. This leads to the definition of three parameters, denoted $\chi_{\mathrm{S}}(G)$, $\chi_{\mathrm{M}}(G)$ and $\chi_{\mathrm{P}}(G)$ for a given graph G, which denote the smallest $k \geq 1$ such that s-proper, m-proper and p-proper, respectively, k-labellings of G exist. It is worth pointing out now that these three parameters are not defined for all graphs G, as it can easily be observed that K_2, the complete graph on 2 vertices, admits no proper labellings at all. However, greedy arguments can be employed to show that this is the only pathological connected graph. Consequently, the parameters χ_{S}, χ_{M} and χ_{P} are more precisely studied in the context of *nice graphs*, which are those graphs which do not have K_2 as a connected component.

The notion of s-proper labellings emerged in the literature as a local version of the *irregularity strength* of graphs, which was initially introduced, back in 1988, by Chartrand *et al.* [9]. Since then, s-proper labellings have been studied for their own interest, and they have actually been attracting a lot of attention due to the intriguing **1-2-3 Conjecture**.

1-2-3 Conjecture (Karoński, Łuczak, Thomason [15]). *If G is a nice graph, then $\chi_{\mathrm{S}}(G) \leq 3$.*

For details on the 1-2-3 Conjecture, we refer the interested reader to [19]. Let us just mention, for now, that this conjecture has been proven to be quite challenging. To date, the best result towards the conjecture is that $\chi_{\mathrm{S}}(G) \leq 5$ for every nice graph G (see [14]). The conjecture was proven to hold for 3-colourable graphs [15], and simple graph classes such as complete graphs [8], for which, already, all of the labels $1, 2, 3$ are required.

The general hardness behind the 1-2-3 Conjecture is one of the main reasons that led people to consider m-proper labellings instead, as it can be observed that an s-proper labelling is always m-proper. This observation led the authors of [1] to consider a multiset version of the 1-2-3 Conjecture (asking, naturally, whether $\chi_{\mathrm{M}}(G) \leq 3$ for every nice graph G). Right away, this presumption that distinguishing vertices via multisets is easier than distinguishing via sums was proven correct, as the authors showed that $\chi_{\mathrm{M}}(G) \leq 4$ for all nice graphs G. More recently, this was definitely confirmed, with Vučković proving, in [22], the multiset version of the 1-2-3 Conjecture.

In parallel with these results, a product version of the 1-2-3 Conjecture, stating that $\chi_{\mathrm{P}}(G) \leq 3$ for every nice graph G, was introduced by Skowronek-Kaziów in [20]. An interesting fact is that, due to 2 and 3 being coprime, m-proper 3-labellings and p-proper 3-labellings are actually not so distant objects (see [5] for more details), which sort of gave the impression that the product version of the 1-2-3 Conjecture might be of intermediate difficulty, between the sum and multiset versions, yet closer to the latter one. Again, this was confirmed recently in [6], by Bensmail, Hocquard, Lajou and Sopena proving the product version of the 1-2-3 Conjecture in full.

We have thus reached a point of time where quite some progress towards the 1-2-3 Conjecture has been recently made, through the upper bound, 5, being very close to what is conjectured exactly, and two related conjectures, which for quite some time seemed of close hardness, being proved. What made this recent progress possible, is definitely a better understanding over the inherent behaviour of s-proper, m-proper and p-proper 3-labellings, which on themselves, gave birth to interesting side investigations (see [3] and the references there for examples). As a matter of fact, even if the 1-2-3 Conjecture turned out to be proven soon, there would still remain lots of interesting questions to answer towards fully understanding proper labellings.

One of these questions, which is precisely the one we investigate throughout this note, is about the real difference between s-proper, m-proper and p-proper labellings. As partly mentioned earlier, s-properness and p-properness both imply m-properness, that is $\chi_{\mathrm{M}}(G) \leq \min\{\chi_{\mathrm{S}}(G), \chi_{\mathrm{P}}(G)\}$ for every nice graph G. This led, in particular, to the following question:

Question 1 (Paramaguru, Sampathkumar [18]). *Does every nice graph G verify $\chi_{\mathrm{M}}(G) = \chi_{\mathrm{S}}(G)$?*

Note that $\chi_{\mathrm{S}}(G) = 1$ for a graph G if and only if G is *locally irregular*, i.e., does not have adjacent vertices with the same degree [2], in which context also $\chi_{\mathrm{M}}(G) = 1$. Now, since the multiset version of the 1-2-3 Conjecture holds [22], we have $\chi_{\mathrm{M}}(G) \leq 3$ for every nice graph G. So the natural question to ask in order to advance towards answering Question 1, is whether there exist graphs G

such that $2 \leq \chi_{\mathrm{M}}(G) < \chi_{\mathrm{S}}(G)$. Now, if we assume that the 1-2-3 Conjecture holds as well, then this question reduces to the following.

Question 2. *Are there graphs G with $\chi_{\mathrm{M}}(G) = 2 < 3 = \chi_{\mathrm{S}}(G)$?*

Infinitely many graphs having the properties described in Question 2 have actually been exhibited recently by Luiz in [16]. For instance, any split graph obtained from a complete graph on at least six vertices by attaching a pendant degree-1 vertex to any vertex has the desired properties. This leads to the question of whether all graphs with the properties described in Question 2 are always that easy to describe. In this work, we focus on that very question, and prove that determining whether $\chi_{\mathrm{S}}(G) = 3$ for a given graph G with $\chi_{\mathrm{M}}(G) = 2$ is NP-hard. Thus, recognising graphs G with $\chi_{\mathrm{M}}(G) = 2 < 3 = \chi_{\mathrm{S}}(G)$ cannot be done in polynomial time, unless P=NP. We also prove that such a result holds for m-proper and p-proper labellings.

This note is organised as follows. We start, in Section 2, by recalling a few facts on s-proper, m-proper and p-proper labellings. In Section 3, we give a first general result on m-proper and p-proper labellings, from which we provide a first insight into our proof arguments. From these, we then give our main result in Section 4, from which we deduce that many more diverse graphs with the properties described in Question 2 exist. We conclude in Section 5 with a few more remarks and questions related to our investigations.

2. General Tools and Previous Results

We start by recalling a few facts on proper labellings, which were already observed in previous works (such as [8]). We begin with the following, which is evident.

Observation 3. *Let G be a nice graph, and ℓ be a k-labelling of G. If $uv \in E(G)$ is an edge with $d(u) = 1$ (and $d(v) > 1$ since G is nice), then $\sigma(u) \neq \sigma(v)$.*

Another obvious observation, is the fact that p-proper labellings cannot assign label 0. This is because if we have $\ell(uv) = 0$ for an edge uv of a graph G by a labelling ℓ, then $\rho(u) = \rho(v) = 0$. Due to this property, whenever dealing with p-proper S-labellings throughout this note (in particular in upcoming Observation 5 and Corollary 6), we implicitly assume that $0 \notin S$.

Observation 4. *p-proper labellings cannot assign label 0.*

The next result is another elementary one of the field. Yet, it has several interesting consequences. Particularly, it gives contexts in which s-proper, m-proper and p-proper 2-labellings stand as equivalent objects.

Observation 5. *Let G be a nice graph, and ℓ be an $\{a,b\}$-labelling of G for any two distinct real numbers a,b. For $\sigma(u) \neq \sigma(v)$ to hold for some $uv \in E(G)$ with $d(u)=d(v)$, the number of edges labelled a incident to u must be different from the number of edges labelled a incident to v (and similarly for the numbers of edges incident to u and v labelled b). Consequently, if $\sigma(u) \neq \sigma(v)$, then also both $\mu(u) \neq \mu(v)$ and $\rho(u) \neq \rho(v)$.*

Proof. If we denote by $n_x(u)$ and $n_x(v)$ the number of edges incident to u and v, respectively, assigned some label x by ℓ, then the facts that $d(u)=d(v)$ and $n_a(u)=n_a(v)$ imply that also $n_b(u)=n_b(v)$, and from that we deduce that $\sigma(u)=\sigma(v)$, $\mu(u)=\mu(v)$ and $\rho(u)=\rho(v)$ (assuming $0 \notin \{a,b\}$ in this last case, recall Observation 4). From these observations, we deduce that if uv is an edge of G with $d(u)=d(v)$, then, so that $\sigma(u) \neq \sigma(v)$ by an $\{a,b\}$-labelling of G, it must be that $n_a(u) \neq n_a(v)$ and thus that $n_b(u) \neq n_b(v)$. Under those assumptions, note that we also have $\mu(u) \neq \mu(v)$ and $\rho(u) \neq \rho(v)$, as claimed. ■

Corollary 6. *In regular graphs, finding an s-proper, m-proper or p-proper $\{a,b\}$-labelling for some distinct real numbers a,b, is equivalent to finding an s-proper, m-proper or p-proper $\{a',b'\}$-labelling for any distinct a',b'.*

In the next sections, the main results we prove relate to the following decision problems.

S-proper 2-Labelling
Input: A graph G.
Question: Do we have $\chi_{\mathrm{S}}(G) \leq 2$, i.e., does G admit s-proper 2-labellings?

P-proper 2-Labelling
Input: A graph G.
Question: Do we have $\chi_{\mathrm{P}}(G) \leq 2$, i.e., does G admit p-proper 2-labellings?

Note that these problems are clearly in NP. They were also showed to be NP-hard in general, first by Dudek and Wajc in [11] for general graphs. Later on, Ahadi, Dehghan and Sadeghi proved in [10] that these problems remain NP-hard when restricted to cubic graphs. From Corollary 6, this implies the following.

Theorem 7 (Ahadi, Dehghan, Sadeghi [10])**.** *For any two distinct a,b, deciding whether a (cubic) graph admits an s-proper, m-proper or p-proper $\{a,b\}$-labelling is NP-hard.*

3. Multisets Versus Products

The main result to be established in this section, relies on the following simple fact, already established e.g. in [17].

Observation 8. *Let G be a nice graph with a vertex v, and let H be obtained from G by attaching a pendant path (v, a, b, c, d) of length 4 at v. If ℓ is a p-proper 2-labelling of H, then the restriction of ℓ to G is also p-proper. Conversely, any p-proper 2-labelling of G can be extended to one of H.*

Proof. Assume ℓ is a p-proper 2-labelling of H. So that $\rho(c) \neq \rho(d)$, note that we must have $\ell(bc) = 2$. From this, so that $\rho(a) \neq \rho(b)$, we must have $\ell(va) = 1$, meaning that $\ell(va)$ does not contribute to $\rho(v)$. We thus deduce that, for ℓ to be p-proper in H, the restriction of ℓ to the edges of G must also be p-proper in G.

Regarding the last part of the statement, consider a p-proper 2-labelling ℓ of G. We extend this labelling to a p-proper 2-labelling of H, by first setting $\ell(va) = 1$ and $\ell(bc) = 2$, and then setting either $\ell(ab) = 1$ and $\ell(cd) = 2$ (if $\rho(v) \neq 1$) or $\ell(ab) = 2$ and $\ell(cd) = 1$ (otherwise). It can be checked that this raises no conflict in both cases. ■

We now prove that P-proper 2-Labelling is NP-hard even when restricted to graphs G with $\chi_{\mathrm{M}}(G) = 2$. So, assuming that P≠NP, not only does our result imply that there exist infinitely many graphs G with $\chi_{\mathrm{M}}(G) = 2 < 3 = \chi_{\mathrm{P}}(G)$, but also that recognising the graphs G that verify $\chi_{\mathrm{M}}(G) = \chi_{\mathrm{P}}(G) = 2$ cannot be done in polynomial time.

Theorem 9. P-proper 2-Labelling *is NP-hard for graphs G with $\chi_{\mathrm{M}}(G) = 2$.*

Proof. The proof is by reduction from the P-proper 2-Labelling problem in cubic graphs, which is NP-hard (recall Theorem 7). Let G be a cubic graph, an instance of P-proper 2-Labelling. We can assume that G is connected. We can also assume that G is not K_4 (since $\chi_{\mathrm{P}}(K_4) = 3$, see [8]), thus that its chromatic number is at most 3 by Brooks' Theorem [7]. In other words, we can assume we also have a proper vertex-colouring $\phi : V(G) \to \{0, 1, 2\}$ of G.

We construct a graph H verifying $\chi_{\mathrm{M}}(H) = 2$, and such that $\chi_{\mathrm{P}}(H) = 2$ if and only if $\chi_{\mathrm{P}}(G) = 2$. For that, we start from G, consider every vertex $v \in V(G)$ in turn, and attach at v exactly $\phi(v)$ new disjoint pendant paths of length 4. Note that the construction of H is clearly achieved in polynomial time.

The fact that $\chi_{\mathrm{P}}(G) = 2$ if and only if $\chi_{\mathrm{P}}(H) = 2$ follows mainly from Observation 8. That is, a p-proper 2-labelling of H directly infers one of G, due to the fact that the paths of length 4 do not contribute to the products of the vertices in $V(H) \cap V(G)$. Regarding the other direction, a p-proper 2-labelling of G can be extended to the pendant paths we have added to form H, by repeated applications of the arguments in the proof of Observation 8. Thus, we have the desired equivalence.

To see now that we always have $\chi_{\mathrm{M}}(H) = 2$ (regardless of G), it suffices to note that, for any two vertices u and v, by any labelling of a graph, we have $\mu(u) \neq \mu(v)$ whenever $d(u) \neq d(v)$. Now, for any vertex $v \in V(H) \cap V(G)$, note that $d_H(v) = d_G(v) + \phi(v)$. In particular, because ϕ is a proper $\{0, 1, 2\}$-vertex-colouring of G, for every two adjacent vertices $u, v \in V(H) \cap V(G)$, we have $d_H(u) \neq d_H(v)$, for $d_H(u), d_H(v) \in \{3, 4, 5\}$, and thus $\mu(u) \neq \mu(v)$ by any labelling of H. Similarly, if a vertex $v \in V(H) \cap V(G)$ is incident to a pendant path (added when constructing H from G), and is thus adjacent to a degree-2 vertex u, then we have $d_H(v) \geq 4 > 2 = d_H(u)$, and again $\mu(u) \neq \mu(v)$ by any labelling. Thus, designing an m-proper 2-labelling of H falls down to 2-labelling the pendant paths in an m-proper way. From this, we deduce that assigning label 1 to all edges of $E(H) \cap E(G)$, and, then, for every pendant path (v, a, b, c, d), where $v \in V(H) \cap V(G)$ (and thus $d_H(d) = 1$), assigning label 2 to dc and cb and label 1 to ba and av, results in an m-proper 2-labelling of H. ∎

4. Multisets Versus Sums

Before proving our main result, Theorem 12, we first need to introduce a few gadgets and constructions, and to point out some of their properties.

The gadget D is depicted in Figure 1(a). Throughout this section, we deal with the vertices and edges of D following the notation given in Figure 1. We call s and t the two *ends* of D, while we say that edges sv_1 and tv_4 are its two *outputs*. This gadget D was already used in [10, 13], where it was noticed it has the following labelling properties.

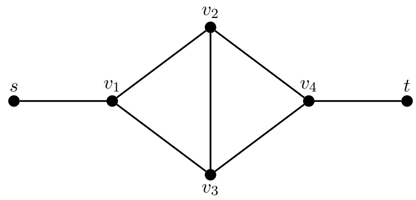
Theorem 10. *D fulfils the following properties.*

1. *D admits s-proper 2-labellings ℓ where $\ell(sv_1) = 1$;*
2. *D admits s-proper 2-labellings ℓ where $\ell(sv_1) = 2$;*
3. *If ℓ is any s-proper 2-labelling of D, then*
   (a) *$\ell(sv_1) = \ell(tv_4)$;*
   (b) *$\sigma(v_1) = \sigma(v_4) = 4$ if $\ell(sv_1) = 1$, and $\sigma(v_1) = \sigma(v_4) = 5$ otherwise.*

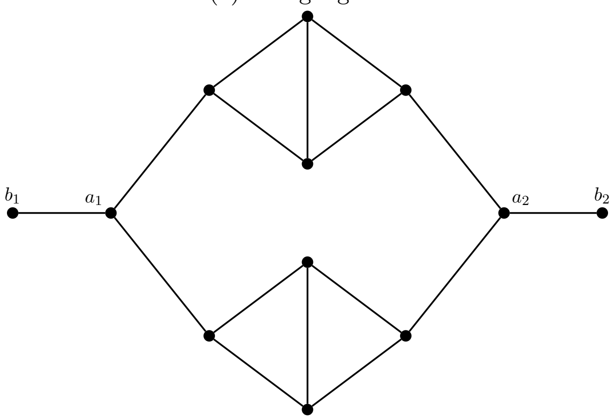
Proof. The vertices of D being of degree 1 and 3 only, it can be observed (through Observations 3 and 5) that if we have an s-proper 2-labelling of D and reverse all labels (1's become 2's, and *vice versa*), then what results is an s-proper 2-labelling of D. From this, we get that if Item 1 holds then Item 2 holds. Similarly, if the first part of Item 3(b) holds, then the second part holds.

Let ℓ be an s-proper 2-labelling of D. By the previous arguments, we can assume that $\ell(v_2v_3) = 1$. Assume first that $\ell(v_2v_1) = \ell(v_2v_4) = 1$. Then $\sigma(v_2) = 3$. So that $\sigma(v_2) \neq \sigma(v_3)$, at least one of $\ell(v_3v_1)$ and $\ell(v_3v_4)$ must be 2. There are two cases.

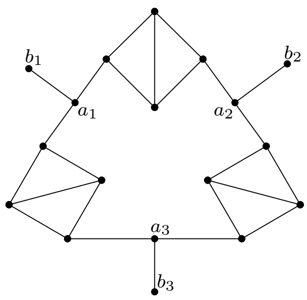
- Assume first that $\ell(v_3v_1) \neq \ell(v_3v_4)$, say $\ell(v_3v_1) = 1$ and $\ell(v_3v_4) = 2$. Thus, $\sigma(v_3) = 4$. Now observe that, regardless of $\ell(v_1s)$, we must have $\sigma(v_1) \in \{3, 4\} = \{\sigma(v_2), \sigma(v_3)\}$, resulting in a conflict, thus a contradiction.



(a) The gadget D.



(b) The 2-necklace D_2.



(c) The 3-necklace D_3.

Figure 1. Gadgets used to prove Theorem 12.

- Assume now that $\ell(v_3v_1) = \ell(v_3v_4) = 2$. Then $\sigma(v_3) = 5$. So that $\sigma(v_1), \sigma(v_4) \notin \{3, 5\} = \{\sigma(v_2), \sigma(v_3)\}$, note that we must have $\ell(v_1s) = \ell(v_4t) = 1$, which indeed yields $\sigma(v_1) = \sigma(v_4) = 4$, as claimed.

If $\ell(v_2v_1) = \ell(v_2v_4) = 2$, then, so that $\sigma(v_2) \neq \sigma(v_3)$, either $\ell(v_3v_1) = \ell(v_3v_4) = 1$, in which case we fall into the previous case (up to renaming the vertices) or $\ell(v_3v_1) \neq \ell(v_3v_4)$. In that last case, if we have, say, $\ell(v_3v_1) = 1$ and $\ell(v_3v_4) = 2$, then $\{\sigma(v_2), \sigma(v_3)\} = \{4, 5\}$; note that, again, regardless of $\ell(v_1s)$, we must have $\sigma(v_1) \in \{4, 5\} = \{\sigma(v_2), \sigma(v_3)\}$, thus a conflict.

Lastly, if $\ell(v_2v_1) \neq \ell(v_2v_4)$, then, so that $\sigma(v_2) \neq \sigma(v_3)$, we must have

$\ell(v_3v_1) = \ell(v_3v_4)$. Up to renaming the vertices of D, this is a case we have covered earlier. ∎

Using copies of the gadget D, we can construct bigger gadgets. For any $k \geq 1$, the *k-necklace* D_k is obtained in the following way (see Figures 1(b) and (c) for examples).

- Start from k disjoint copies $G_0, \ldots, G_{k-1}$ of the gadget D. For every $i \in \{0, \ldots, k-1\}$, we denote by x_i and y_i the ends of G_i.
- For every $i \in \{0, \ldots, k-1\}$, identify y_i and $x_{(i+1) \bmod k}$ to connect the G_i's in a cyclic fashion. Note that all vertices have degree 3, except for exactly k of them, which we denote by $a_1, \ldots, a_k$, which result from the identifications.
- For every $i \in \{1, \ldots, k\}$, attach a pendant vertex b_i at a_i.

We call the b_i's the *ends* of D_k, while we call the a_ib_i's its *outputs*. Labelling properties of the gadget D actually infer labelling properties for necklaces; in particular we have the following.

Theorem 11. *For any $k \geq 1$, the k-necklace D_k fulfils the following properties.*

1. *By any s-proper 2-labelling of D_k, all outputs must be assigned the same label;*
2. *D_k admits s-proper 2-labellings assigning label 1 to the outputs;*
3. *D_k admits s-proper 2-labellings assigning label 2 to the outputs;*
4. *If ℓ is any s-proper 2-labelling of D_k, then $\sigma(a_1) = \cdots = \sigma(a_k) = 3$ if $\ell(a_1b_1) = 1$, and $\sigma(a_1) = \cdots = \sigma(a_k) = 6$ otherwise.*

Proof. D_k having vertices of degree 1 and 3 only, by Observations 3 and 5 we get that, upon reversing labels by s-proper 2-labellings, Item 3 holds as soon as Items 1 and 2 hold.

Let ℓ be an s-proper 2-labelling of D_k. First off, we claim that, for each of the k copies $G_0, \ldots, G_{k-1}$ of the gadget D forming D_k, its two outputs must be assigned the same label by ℓ. In other words, we claim that the $2k$ outputs of $G_0, \ldots, G_{k-1}$ must be assigned the same label. Indeed, assume this is wrong. Recall that, by Item 3(a) of Theorem 10, for any copy of D in D_k, its two outputs must be assigned the same label by ℓ. Now, by our hypothesis, there must be a vertex a_i in D_k resulting from the identification of two ends from two copies G_{i-1} and G_i of D such that, say, the outputs of G_{i-1} are assigned label 1 by ℓ while the outputs of G_i are assigned label 2. By Item 3(b) of Theorem 10, this means that a_i is adjacent to a vertex v of G_{i-1} that has sum 4 by ℓ, and to a vertex u of G_i that has sum 5. Then, there is a conflict either between a_i and v (if $\ell(a_ib_i) = 1$) or between a_i and u (if $\ell(a_ib_i) = 2$).

Thus, the $2k$ outputs of the copies of D in D_k must be assigned the same label by ℓ. Assume this label 1, and focus on any vertex a_i resulting from the identification of two ends of G_{i-1} and G_i, being copies of D. Then the two neighbours of a_i in G_{i-1} and G_i, by Item 3(b) of Theorem 10, have sum 4 by ℓ. So that there is no conflict between a_i and these vertices, we must have $\ell(a_ib_i) = 1$ so that $\sigma(a_i) = 3$. Similar arguments show that we must have $\ell(a_ib_i) = 2$ in case the other two edges incident to a_i are assigned label 2, in which case $\sigma(a_i) = 6$. ■

We are now ready to prove our main result. Again, this result implies that there exist infinitely many graphs G verifying $\chi_{\mathrm{M}}(G) = 2 < 3 = \chi_{\mathrm{S}}(G)$. Even worse, recognising such graphs cannot be done in polynomial time, unless P=NP.

Theorem 12. S-proper 2-Labelling *is NP-hard for graphs G with $\chi_{\mathrm{M}}(G) = 2$.*

Proof. The proof starts similarly as that of Theorem 9, and is done by a reduction from the S-proper 2-Labelling problem in cubic graphs. Let G be a connected cubic graph given together with a proper vertex-colouring $\phi : V(G) \to \{0, 2, 4\}$, where G is an instance of S-proper 2-Labelling. We construct, in polynomial time, a graph H with $\chi_{\mathrm{M}}(H) = 2$, and such that $\chi_{\mathrm{S}}(G) = 2$ if and only if $\chi_{\mathrm{S}}(H) = 2$.

The construction of H is mainly achieved by using an α-necklace with sufficiently many outputs, connected to the rest of the graph in some fashion to guarantee that 1) any s-proper 2-labelling must propagate in a very particular way, and that 2) some adjacent vertices have distinct degrees so that they cannot be an obstacle to m-properness. We voluntarily avoid specifying the value of α for now, and just assume it is sufficiently large so that we always have unused outputs in hands whenever we need some. Let us just mention that α will be a linear function of $|V(G)|$.

Before describing the heart of the reduction, let us make some preparations. We start from A, an α-necklace with α outputs. Recall that by any s-proper 2-labelling, all outputs of A must be assigned the same label, and that this label can be 1 or 2 (recall Theorem 11). We first modify A, to force this label on the outputs to be 1. For that, we first need to point out the following easy observation.

Claim 13. *Let $u_1v_1, \dots, u_kv_k$ be $k \geq 1$ pairwise distinct outputs of A, where the v_i's are the degree-1 vertices. For every $i \in \{1, \dots, k\}$, attach a pendant path (v_i, x_i, y_i) of length 2 at v, resulting in a new graph A'. Then*

1. *every s-proper 2-labelling of A can be extended to one of A';*
2. *if ℓ is any s-proper 2-labelling of A', then, for every $i \in \{1, \dots, k\}$*
   (a) *$\ell(u_iv_i) \neq \ell(x_iy_i)$;*
   (b) *$\sigma(x_i) = 3$ if $\ell(u_iv_i) = 1$, and $\sigma(x_i)$ can be anything in $\{2, 3\}$ otherwise.*

Proof. Item 2(a) is necessary, as it is the only reason that guarantees $\sigma(v_i) \neq \sigma(x_i)$. To see now that Item 1 holds, consider an s-proper 2-labelling ℓ of A, and consider extending it to the edges v_ix_i and x_iy_i of A', for some $i \in \{1, \dots, k\}$. If $\ell(u_iv_i) = 1$, then recall that $\sigma(u_i) = 3$ by Theorem 11. In that case, we must set $\ell(v_ix_i) = 1$ and $\ell(x_iy_i) = 2$, which yields $\sigma(v_i) = 2$, $\sigma(x_i) = 3$ and $\sigma(y_i) = 2$, thus no conflicts. Now, if $\ell(u_iv_i) = 2$, then, by Theorem 11, recall that $\sigma(u_i) = 6$. In that case, we can set e.g. $\ell(v_ix_i) = 1$ (or $\ell(v_ix_i) = 2$) and $\ell(x_iy_i) = 1$, which yields $\sigma(v_i) = 3$ ($\sigma(v_i) = 4$, respectively), $\sigma(x_i) = 2$ ($\sigma(x_i) = 3$, respectively) and $\sigma(y_i) = 1$, raising no conflicts. From these arguments, ℓ can thus be extended to A', and Items 1 and 2(b) hold. □

In what is to come, we will need to "extend" some outputs of A as described in Claim 13. That is, given an output uv of A where v is the degree-1 vertex, by *extending* uv we mean attaching a pendant path (v, x, y) of length 2 at v. We regard the pendant edge xy as another output of the resulting graph, its end being y. Conversely, uv is no longer regarded as an output of the resulting graph. So, from A, by extending some outputs we get another graph A' with α outputs which can now be of two possible types: an output of A' either is an *extended output*, resulting precisely from an output extension, or is one of the initial outputs of A that was not extended, which we call *regular outputs* from now on to avoid any confusion. Due to Claim 13, the main difference, in brief, between regular and extended outputs, is that regular outputs should all be assigned the same label by an s-proper 2-labelling, while all extended outputs must be assigned the other label.

We are now ready to modify A to force the label of its outputs by any s-proper 2-labelling. For that, we proceed as follows. Take one output e with end u of A, and two more outputs f_1, f_2. Start by extending f_1, f_2, resulting in two new extended outputs f_1', f_2'. Calling v_1 and v_2, respectively, their ends, we modify A by first identifying v_1 and v_2 to a new vertex v, and adding an edge between u and v. This results in a new graph A', in which we no longer regard e, f_1', f_2' as outputs, and which thus has $\alpha - 3$ outputs (all of which are regular). A' also has the following properties.

Claim 14. *A' fulfils the following properties.*

1. *By any s-proper 2-labellings of A', all (regular) outputs must be assigned label 1;*
2. *A' admits s-proper 2-labellings.*

Proof. We start by proving the first item. Assume Item 1 is wrong, and that A' admits s-proper 2-labellings ℓ where all outputs are assigned label 2. By Theorem 11, following the terminology above, by the construction of A' from A, the edge e must be assigned label 2 by ℓ, while f_1', f_2', being extensions of f_1, f_2, must be assigned label 1 (by Claim 13). This implies that we must have $\sigma(u) = \sigma(v)$ regardless of $\ell(uv)$, a contradiction.

Assume now that ℓ is an s-proper 2-labelling of A where all outputs are assigned label 1 (which exists by Theorem 11). We claim it can be extended to A', thereby proving the second item of the claim. By Claim 13, ℓ extends from f_1, f_2 to f'_1, f'_2, and it must be that $\ell(f'_1) = \ell(f'_2) = 2$, while the ends of f'_1, f'_2 other than v have sum 3. By setting $\ell(uv) = 1$, we obtain $\sigma(u) = 2$ and $\sigma(v) = 5$, thus raising no conflicts. In particular, recall that the neighbour of u different from v must have sum 3 by Theorem 11. $\square$

At this point, we know that A' admits s-proper 2-labellings, all of which must assign label 1 to all of the $\alpha - 3$ (regular) outputs. We now describe how to construct H from G and A'. Start from G and A'. Consider then every vertex v of G in turn, and pick $x = \phi(v)$ new regular outputs $e_1, \dots, e_x$ of A', as well as $y = \frac{4-\phi(v)}{2}$ other new regular outputs $f_1, \dots, f_y$. Start by extending each of $f_1, \dots, f_y$, resulting in y extended outputs $f'_1, \dots, f'_y$. Note that $x + y \in \{2, 3, 4\}$. Now, just identify v and each of the $x + y$ ends of $e_1, \dots, e_x, f'_1, \dots, f'_y$. Once all vertices v of G have been treated that way, the resulting graph is our H.

Before proceeding with proving the equivalence between G and H, let us first comment on α, the number of outputs that A must have for the whole construction to be achieved. For a vertex $v \in V(H) \cap V(G)$ with $\phi(v) = 0$, we have attached two extended outputs at v (and no regular outputs). If $\phi(v) = 2$, then we have attached two regular outputs and one extended output at v. Last, if $\phi(v) = 4$, then we have attached four regular outputs (and no extended outputs). Thus, the number of attached outputs (regardless of their type) of A' is at most $4|V(G)|$, and remember that three additional outputs were necessary to go from A to A'. Thus it is sufficient to have α linear in $|V(G)|$, and the construction of H from G and A' is clearly achieved in polynomial time.

We now prove that we have the desired equivalence between G and H.

• Assume first that H admits an s-proper 2-labelling ℓ. Recall that all regular outputs of A' must be assigned label 1 by ℓ due to Claim 14, and that all extended outputs must be assigned label 2 by Claim 13. By how H was constructed, in particular with respect to ϕ, note that, for every vertex $v \in V(H) \cap V(G)$, the edges of $E(A')$ incident to v have their assigned labels summing up to exactly 4. In particular, for every edge uv in $E(H) \cap E(G)$, we have $\sigma(u) \neq \sigma(v)$, and, thus, $\sigma(u) - 4 \neq \sigma(v) - 4$. This implies that the restriction of ℓ to the edges of G is also s-proper.

• Conversely, assume G admits an s-proper 2-labelling ℓ. We claim that ℓ can be extended to an s-proper 2-labelling of H. To see this is true, start from ℓ being a partial labelling of H (following the labelling ℓ of G), and simply extend this ℓ to the edges of A' by considering an s-proper 2-labelling of A' (such a labelling exists, by Claim 14). Let us denote by ℓ' the resulting 2-labelling of H. We claim ℓ' is s-proper. First off, for every vertex $v \in V(H) \cap V(G)$, as mentioned earlier by ℓ' we have $\sigma(v) = x + 4$, where x is the value of $\sigma(v)$ by ℓ. Since ℓ is s-proper in G, this implies that, for every $uv \in E(H) \cap E(G)$, by ℓ' we have $\sigma(u) \neq \sigma(v)$. By how ℓ' was obtained, also $\sigma(u) \neq \sigma(v)$ by ℓ' for every $uv \in E(H) \cap E(A')$ such that $u, v \notin V(A') \cap V(G)$. It remains to prove that we also have $\sigma(u) \neq \sigma(v)$ by ℓ' for every two adjacent vertices u, v with $uv \in E(H) \cap E(A)'$, $u \in V(H) \cap V(G)$ and $v \notin V(H) \cap V(G)$. Due to the outputs of A' attached at u, which, by ℓ', bring exactly 4 to $\sigma(u)$, and $d_G(u) = 3$, note that $\sigma(u) \geq 7$ by ℓ'. On the other hand, v is either of degree 3 (case of a regular output uv attached at u) or 2 (otherwise, case of an extended output uv), meaning that $\sigma(v) \leq 6$ by ℓ'. Thus, we always have $\sigma(u) > \sigma(v)$, and ℓ' is thus s-proper.

We conclude the proof by showing that $\chi_{\mathrm{M}}(H) = 2$, regardless of G. This is by the following arguments. For every vertex $v \in V(H) \cap V(G)$, note that we have $d_H(v) = d_G(v) + \phi(v) + \frac{4-\phi(v)}{2}$. Since G is cubic, we thus have $d_H(v) = 7$ if $\phi(v) = 4$, $d_H(v) = 6$ if $\phi(v) = 2$, and $d_H(v) = 5$ otherwise, i.e., if $\phi(v) = 0$. Since ϕ is proper, this means that, for every $uv \in E(H) \cap E(G)$, we have $d_H(u) \neq d_H(v)$, implying that $\mu(u) \neq \mu(v)$ by any 2-labelling of H. Similarly, for every vertex $v \in (V(H) \cap V(A')) \setminus V(G)$, we have $d_H(v) \leq 3$, implying that $d_H(u) \neq d_H(v)$ for every $u \in V(H) \cap V(G)$, thus $\mu(u) \neq \mu(v)$ again by any 2-labelling of H. All these arguments imply that if we just start from an s-proper 2-labelling of A' (which exists by Claim 14), and extend it to the whole of H by assigning arbitrary labels to the edges of $E(H) \cap E(G)$, then what results is an m-proper 2-labelling of H. Thus, $\chi_{\mathrm{M}}(H) = 2$. ■

5. Discussion

Through Theorem 12, we have provided a positive answer to Question 2, thus a negative answer to Question 1, going beyond the results of Luiz from [16]. A general way to pursue the investigations on this topic, could be by wondering about classes of graphs for which Question 1 can be answered (positively or negatively), and similarly for its counterpart for p-proper labellings.

• In the case of nice trees T, it has been known for long that $\chi_{\mathrm{S}}(T) \leq 2$ always holds (see [8]), which yields a positive answer to Question 1 for trees. This is a neat difference with p-proper labellings, as Szabo Lyngsie proved in [17] that there exist infinitely many trees T with $\chi_{\mathrm{P}}(T) = 3$, which, fortunately, can be recognised in polynomial time. Thus, for trees T, we sometimes have $\chi_{\mathrm{M}}(T) = 2 < 3 = \chi_{\mathrm{P}}(T)$, but such situations can be recognised easily.

• Regarding bipartite graphs, it was proved by Thomassen, Wu and Zhang [21] that bipartite graphs G with $\chi_{\mathrm{S}}(G) = 3$ form exactly the class of the so-called *odd multi-cacti*, which can be recognised in polynomial time. Looking at the structure of odd multi-cacti, it is not too complicated to prove that each graph

G that belongs to this class, also verifies $\chi_{\mathrm{M}}(G) = 3$ (refer e.g. to [4] for more insight in the structure of these graphs). In other words, the answer to Question 1 is also yes for bipartite graphs. Regarding p-proper labellings, we have already mentioned earlier that trees form a context in which we sometimes do not have equality between the parameters χ_{M} and χ_{P}. Regarding the complexity of deciding whether this is the case or not for a given bipartite graph, it is still unclear, as we still do not know whether bipartite graphs G with $\chi_{\mathrm{P}}(G) \leq 2$ can be recognised in polynomial time (see [17] for partial results).

• Regarding graphs with bounded maximum degree, we note that our reduction in the proof of Theorem 12 builds graphs of maximum degree at most 7. A question could thus be whether Theorem 12 also holds for graphs with maximum degree less than 6.

Note that graphs G with maximum degree 2 are paths and cycles, for which it is not complicated to prove, through Observations 3 and 5, that we always have $\chi_{\mathrm{M}}(G) = \chi_{\mathrm{S}}(G) = 2$. We note also that for any $\Delta \in \{3, 4, 5, 6\}$, we can construct graphs with maximum degree Δ that are counterexamples to Question 1. Let us give an example for $\Delta = 3$, which generalises easily for any $\Delta \in \{4, 5, 6\}$. Reusing the terminology from the proof of Theorem 12, start from a 6-necklace D_6, with outputs $e_1, \ldots, e_6$. Now extend e_4, e_5, e_6 to new extended outputs e'_4, e'_5, e'_6. Denote by $v_1, \ldots, v_6$ the ends of $e_1, e_2, e_3, e'_4, e'_5, e'_6$, respectively. Now identify v_4 and v_5 to a new vertex x, and similarly identify v_2 and v_3 to a new vertex y, and just add the edges v_1x and v_6y, to obtain a graph G. It is not too hard to check that $\Delta(G) = 3$, and that $\chi_{\mathrm{M}}(G) = 2 < 3 = \chi_{\mathrm{S}}(G)$, by similar arguments as in the proof of Theorem 12. In particular, note that, just as in the proof with A', the edge v_1x forces all outputs of D_6 to be assigned label 1 by an s-proper 2-labelling of G, while the edge v_6y, by similar arguments, forces all outputs of D_6 to be assigned label 2. It follows that $\chi_{\mathrm{S}}(G) > 2$. On the other hand, due to the number of outputs of D_6 we have identified, note that v_1 and x, and similarly v_6 and y, have different degrees and thus cannot be in conflict when considering multisets.

Regarding the same concerns for p-proper labellings, note that the reduction in the proof of Theorem 9 provides graphs with maximum degree 5. Thus, a similar question as above concerns the complexity of the same problem for graphs with maximum degree 3 or 4, given, again, that some of these graphs G sometimes verify $\chi_{\mathrm{M}}(G) = 2 < 3 = \chi_{\mathrm{P}}(G)$ (which can be proved reusing the same ideas as above, but with the labelling ideas from the proof of Theorem 9).

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