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A σ_3 CONDITION FOR ARBITRARILY PARTITIONABLE GRAPHS

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Abstract

A graph G of order n is arbitrarily partitionable (AP for short) if, for every partition $(\lambda_1, \ldots, \lambda_p)$ of n, there is a partition $(V_1, \ldots, V_p)$ of $V(G)$ such that $G[V_i]$ is a connected graph of order λ_i for every $i \in \{1, \ldots, p\}$. Several aspects of AP graphs have been investigated to date, including their connection to Hamiltonian graphs and traceable graphs. Every traceable graph (and, thus, Hamiltonian graph) is indeed known to be AP, and a line of research on AP graphs is thus about weakening, to APness, known sufficient conditions for graphs to be Hamiltonian or traceable.

In this work, we provide a sufficient condition for APness involving the parameter $\overline{\sigma_3}$, where, for a given graph G, the parameter $\overline{\sigma_3}(G)$ is defined as the minimum value of $d(u)+d(v)+d(w)-|N(u)\cap N(v)\cap N(w)|$ for a set $\{u, v, w\}$ of three pairwise independent vertices u, v, and w of G. Flandrin, Jung, and Li proved that any graph G of order n is Hamitonian provided G is 2-connected and $\overline{\sigma_3}(G) \geq n$, and traceable provided $\overline{\sigma_3}(G) \geq n-1$. Unfortunately, we exhibit examples showing that having $\overline{\sigma_3}(G) \geq n-2$ is not a guarantee for G to be AP. However, we prove that G is AP provided G is 2-connected, $\overline{\sigma_3}(G) \geq n-2$, and G has a perfect matching or quasi-perfect matching.

Keywords: arbitrarily partitionable graph, partition into connected subgraphs, σ_3 condition, Hamiltonian graph, traceable graph.

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1. Introduction

This paper deals with so-called *arbitrarily partitionable graphs*, which are defined formally as follows. Let G be an n-graph, *i.e.*, a graph of order n. Let also $\pi = (\lambda_1, \ldots, \lambda_p)$ be an *n-partition*, i.e., a partition of n (that is, $\lambda_1 + \cdots + \lambda_p = n$). We say that π is *realisable* in G if $V(G)$ can be partitioned into p parts $V_1, \ldots, V_p$ such that $G[V_i]$ is a connected graph of order λ_i for every $i \in \{1, \ldots, p\}$, and we call $(V_1, \ldots, V_p)$ a *realisation* (of π in G). Finally, G is called *arbitrarily partitionable* (AP for short) if all n-partitions are realisable in G. In other words, G is AP if we can partition G into arbitrarily many connected subgraphs, regardless of their requested orders.

AP graphs were introduced independently by Barth, Baudon, and Puech and Horňák and Woźniak [1, 10] in early 2000s, but the problem of partitioning graphs into connected subgraphs has been attracting attention since at least the 1970s, recall for instance the influential Győri-Lovász Theorem [8, 12]. Since then, quite some aspects of AP graphs have been investigated in the literature, including structural aspects, algorithmic questions, numerous variants, and others. For a recent survey on the topic, we refer the interested reader to, e.g., [2].

AP graphs are objects of interest for numerous reasons. A notable one is the fact that they sort of lie in-between two other important types of graph notions, being perfect matchings and Hamitonian cycles. Note indeed that, if n is even, then any realisation of the n-partition $(2, \ldots, 2)$ in an n-graph forms a perfect matching, while, if n is odd, then any realisation of the n-partition $(2, \ldots, 2, 1)$ forms a quasi-perfect matching. Remark also that adding edges to an AP graph cannot make it loose its APness, and, from this simple observation, we get that any graph spanned by an AP graph is AP itself. Since paths are obviously AP, this implies that Hamiltonian graphs and even traceable graphs (graphs having a Hamiltonian path) are AP. Thus, having a perfect matching or a quasi-perfect matching is a necessary condition for a graph to be AP, while being traceable is a sufficient condition for a graph to be AP.

These simple thoughts lead to one of the most interesting lines of research regarding AP graphs, which is on the weakening, to APness, of sufficient conditions for Hamiltonicity and traceability. The general idea is that one can consider any of the numerous sufficient conditions for a graph to be Hamiltonian or traceable, and investigate whether it can be weakened to a sufficient condition for APness. This line of research was initiated by Marczyk in [13], in which he focused on the parameter σ_2, being defined as

$$\sigma_2(G) = \min \{d(u) + d(v) : u \text{ and } v \text{ are independent vertices of } G\}$$

for any graph G. By a famous result of Ore [16], recall indeed that any connected n-graph G is Hamiltonian whenever $\sigma_2(G) \geq n$, while G is traceable whenever $\sigma_2(G) \geq n - 1$. In [13], Marczyk proved that G is AP provided $\sigma_2(G) \geq n - 2$ and $\alpha(G) \leq \lceil n/2 \rceil$ (that is, provided G has a perfect matching or a quasi-perfect matching). Later on, in [9, 14], Marczyk, together with Horňák, Schiermeyer, and Woźniak, improved this sufficient condition to G satisfying only $\sigma_2(G) \geq n-5$ and additional conditions (such as the previous condition on $\alpha(G)$, and also conditions on n).

Sufficient conditions for Hamiltonicity and traceability being surely one of the most investigated and rich areas of graph theory, the work of Marczyk opened the way to many promising investigations in that line. For instance, in [11], Kalinowski, Pilśniak, Schiermeyer, and Woźniak, motivated by similar conditions for Hamiltonicity and traceability, exhibited sufficient conditions in terms of size (number of edges) guaranteeing a graph is AP. In [3], Bensmail and Li considered several sufficient conditions for Hamiltonicity and traceability (covering squares of graphs and forbidden structures) and proved that some of these weaken to APness, while some do not.

The current work takes place in the line of the previous investigations, and, more particularly, relates to the initiating work of Marczyk on the topic. Precisely, we deal with the parameter σ_3 being defined as

$$\sigma_3(G) = \min\{d(u) + d(v) + d(w) : u,\ v, \text{ and } w \text{ are independent vertices of } G\}$$

for a given graph G. As reported in several surveys on the topic (such as, e.g., [7, 18]), this parameter σ_3 has indeed be employed, together with other graph properties (such as connectivity and claw-freeness), to express sufficient conditions for graphs to be Hamiltonian, and sometimes more or less than that. As far as we are aware, this was also considered in the context of AP graphs. Indeed, at the occasion of the 18th Workshop "Colourings, Independence and Domination" (CID 2013) led in 2013, Brandt was invited to give a lecture, entitled "Finding Vertex Decompositions in Dense Graphs", during which he announced the following result.

Theorem 1 (Brandt). *If G is a connected n-graph with $\sigma_3(G) \geq n$, then G is* AP *if and only if G admits a perfect matching or a quasi-perfect matching.*

Unfortunately, it seems that Brandt has never published a proof of Theorem 1, and the only remains of his investigations to date are the title of his talk (mentioned in the preface [4]), as well as the corresponding abstract, which can be found online and which we report word for word in concluding Section 4 for the sake of keeping track of it. On the positive side, this abstract provides hints regarding the main lines of Brandt's proof of Theorem 1.

Our investigations in the current work were inspired by Brandt's result, and our original intent was to provide a result that would be sort of reminiscent of Theorem 1. As a result, we deal with a parameter that is very close to the parameter σ_3, as it relies on a slightly different notion of degree sum for triples of pairwise independent vertices. That is, for a graph G and any three pairwise distinct independent vertices u, v, and w of G, set

$$d^*(u, v, w) = d(u) + d(v) + d(w) - |N(u) \cap N(v) \cap N(w)|.$$

Now, define

$$\overline{\sigma_3}(G) = \min\{d^*(u,v,w) : u,\ v, \text{ and } w \text{ are independent vertices of } G\}.$$

Note that $\overline{\sigma_3}(G) \leq \sigma_3(G)$ for every graph G. It turns out that this parameter $\overline{\sigma_3}$ has also been used to express sufficient conditions for Hamiltonicity and traceability. In particular, Flandrin, Jung, and Li proved in [6] that any 2-connected n-graph G with $\overline{\sigma_3}(G) \geq n$ is Hamiltonian, while any connected n-graph G with $\overline{\sigma_3}(G) \geq n-1$ is traceable. [18] reports that the latter bound for traceability was proved to also hold when $\overline{\sigma_3}(G) \geq n-2$ provided G fulfils additional strong conditions (involving 2-connectivity and claw-freeness).

The current work is dedicated to proving the next result, which provides a new sufficient condition, involving the parameter $\overline{\sigma_3}$, for a graph to be AP.

Theorem 2. *If G is a 2-connected n-graph with $\overline{\sigma_3}(G) \geq n-2$, then G is* AP *if and only if G admits a perfect matching or a quasi-perfect matching.*

We searched the literature for a while, and, as far as we can tell, it seems that Theorem 2 does not follow immediately from existing results on the parameter $\overline{\sigma_3}$ (such as conditions implying traceability). We have to add, however, that the literature on the topic is quite vast, rich, and that several old works mentioned, e.g., in surveys seem to be impossible to access easily nowadays. Thus, it is hard to be fully certain we have not missed something. However, even if a previous result implying Theorem 2 was to exist, we believe the proof we give would remain of interest, as it relies on understanding how APness and a parameter such as $\overline{\sigma_3}$ relate in general.

We start by introducing useful material in Section 2, before focusing on proving Theorem 2 in Section 3. We discuss our result in Section 4, in which we also discuss Brandt's Theorem 1.

2. Preliminaries

In our proof of Theorem 2, we will often deal with particular situations in which the following notations and terminology will be useful. Let G be a graph, and let $P = v_1 \cdots v_p$ be a path of G. For a vertex $v_i \in V(P)$, we sometimes denote by v_i^- and v_i^+ the vertices v_{i-1} and v_{i+1} (assuming they exist). For a set $S \subseteq V(P)$ of vertices of P, we denote by S^- and S^+ the sets $\{v_i^- : v_i \in S\}$ and $\{v_i^+ : v_i \in S\}$, respectively. These notions will be particularly useful when dealing with vertices of G having all their neighbours on P. Namely, for a $u \in V(G)$ (we might have $u \in V(P)$) such that $N(u) \subseteq V(P)$, we will have to deal with the sets $N(u)^-$ and $N(u)^+$ in many occasions.

In the proof of Theorem 2, we will sometimes obtain a realisation in G of some partition by first "splitting" (sometimes only part of) P into parts of certain size containing consecutive vertices. Note that this is indeed a legitimate way to proceed, as any set of consecutive vertices of P induces a connected graph. In such occasions, we will process the vertices of P one by one, from one end-vertex to the other (either as going from v_1 to v_p, or conversely), and *pick* parts as going along. Assuming we want to split P into connected parts of size $\lambda_1, \dots, \lambda_p$, where $\lambda_1 + \cdots + \lambda_p \leq |V(P)|$, the first λ_1 vertices of P (following the considered ordering) will form one part, the next λ_2 vertices will form another part, and so on. When building one of these parts, we will sometimes have to *grab* a vertex u of G from some v_i on the way, meaning that, at the very moment where v_i is added to some part, assuming $v_i u$ is an edge and the current part still misses vertices to reach the desired size, then we add u to the current part before resuming the picking process with the vertex succeeding v_i in the considered ordering. As will be apparent later on, we will use this picking process in a rather flexible way (sometimes different from what we described above), and, in every such occasion, we will make sure to describe the exact process properly to avoid any ambiguity.

Still about P, note that the way we denote its consecutive vertices $v_1, \dots, v_p$ above yields a virtual orientation of P from the first vertex (v_1) to the last (v_p), where v_i is considered to be a *predecessor* of v_j if $i < j$, while v_j is considered to be a *successor* of v_i in such situations. Note that these notions also make sense for the parts we pick during the picking process above. One has to be careful, however, that the notions of preceding and succeeding parts are not with respect to the virtual orientation of P, but rather with the ordering in which the consecutive vertices of P as considered through the process. In particular, when picking parts as going from v_p to v_1, note that these notions are reversed compared to those we get as going from v_1 to v_p.

We finish off with a useful result on partitioning graphs with long paths into connected subgraphs. Assume G is an n-graph, and let π be an n-partition. We denote by $\mathrm{sp}(\pi)$ the *spectrum* of π, being the set of distinct element values that appear in π. An important and crucial fact is that if P is long enough, then all n-partitions with sufficiently many distinct element values are realisable in G. This is captured in the following result of Ravaux.

Theorem 3 (Ravaux [17])**.** *If G is a connected n-graph with a path of length $n - \alpha$, then every n-partition π with $|\mathrm{sp}(\pi)| \geq \alpha$ is realisable in G.*

Before we proceed to the proof of Theorem 2 in the next section, we first recall some easy properties of longest paths in connected graphs.

Lemma 4. *Let $P = v_1 \cdots v_p$ be a longest path in a connected graph G, and set $R = V(G) \setminus V(P)$. Then the following items hold for every vertex $u \in R$.*

(1) *$\{u, v_1, v_p\}$ is a stable set.*

(2) *$d(v_1) = |N(v_1)^-|$ and $d(v_p) = |N(v_p)^+|$.*

(3) *$N_P(u) \subseteq V(P) \setminus (N(v_1)^- \cup N(v_p)^+)$.*

(4) $N_P(u) \cap N_P(u)^- = N_P(u) \cap N_P(u)^+ = \emptyset$.

(5) *If $G[R]$ has at least one edge, then*

(5a) $N(v_1)^- \cap N(v_p)^+ = \emptyset$;

(5b) $d_P(u) + d(v_1) + d(v_p) \leq p$.

Proof. (1) follows from the fact that having an edge joining two vertices in $\{u, v_1, v_p\}$ would lead, because G is connected, to deducing a path of G longer than P, a contradiction.

(2) follows directly from the fact that v_1 and v_p, because P is a longest path of G, have all their neighbours on P.

(3) is because if u had a neighbour v_i with $v_i \in N(v_1)^-$ (or $v_i \in N(v_p)^+$), then $uv_i \cdots v_1 v_{i+1} \cdots v_p$ (or $uv_i \cdots v_p v_{i-1} \cdots v_1$) would be a path of G longer than P, a contradiction.

(4) is because if u had two neighbours v_i and v_{i+1} that are consecutive on P, then $v_1 \cdots v_i u v_{i+1} \cdots v_p$ would be a path of G longer than P, another contradiction.

Assume now $G[R]$ has at least one edge. First, note that if there was some $v_i \in N(v_1)^- \cap N(v_p)^+$, then $v_1 \cdots v_{i-1} v_p \cdots v_{i+1} v_1$ would be a cycle of length $p - 1$, and, from the facts that G is connected and that $G[R]$ contains edges, we would come up with a path of G longer than P, a contradiction. Thus, such a v_i cannot exist, which proves (5a). Now, since (3) and (5a) hold, note that we have $d_P(u) \leq p - d(v_1) - d(v_p)$. Thus $d_P(u) + d(v_1) + d(v_p) \leq p$, and (5b) also holds. ■

3. Proof of Theorem 2

We start by establishing facts on the structure of 2-connected n-graphs G with $\overline{\sigma_3}(G) \geq n - 2$.

Lemma 5. *Let $P = v_1 \cdots v_p$ be a longest path in a 2-connected n-graph G with $\overline{\sigma_3}(G) \geq n - 2$, and set $R = V(G) \setminus V(P)$. If G is not traceable, i.e., $p < n$, then the following items hold.*

(1) *R is a stable set.*

(2) $|R| \leq 2$.

(3) *If $R = \{u_1, u_2\}$, then*

(3a) $|N(u_1) \cap N(u_2)| \geq 2$;

(3b) *denoting by v_{z_1} and v_{z_2} the first and last vertices of $N(u_1) \cup N(u_2)$ on P, respectively, and setting $Z = \{v_{z_1}, \ldots, v_{z_2}\}$, we have*

$$|N_Z(v_1) \cup N(u_1) \cup N(u_2)| > 1 + \frac{z_2 - z_1}{2}.$$

Proof. Set $r = |R|$. Regarding (1), assume, towards a contradiction, that $G[R]$ has at least one edge. Let $u \in R$ be a vertex incident to an edge of $G[R]$. Then

$$d(u) + d(v_1) + d(v_p) - |N(u) \cap N(v_1) \cap N(v_p)| \geq \overline{\sigma_3}(G) \geq n - 2 = p + r - 2.$$

However, by Lemma 4(5b), we have

$$d_P(u) + d(v_1) + d(v_p) \leq p.$$

Since $d(u) = d_P(u) + d_R(u)$, it follows that

$$d_R(u) \geq r - 2 + |N(u) \cap N(v_1) \cap N(v_p)|.$$

Now suppose $d_R(u) = r-2$. Then $N(u) \cap N(v_1) \cap N(v_p) = \emptyset$ and $d_P(u) + d(v_1) + d(v_p) = p$. Thus, by Lemma 4(2) and (5a), we have

$$d_P(u) = p - \left(\left|N(v_1)^-\right| + \left|N(v_p)^+\right|\right) = p - \left|N(v_1)^- \cup N(v_p)^+\right|.$$

From Lemma 4(3), it thus follows that every vertex of P that is not a neighbour of u lies in $N(v_1)^- \cup N(v_p)^+$. Now, because G is connected, we can further assume that there is a v_i such that uv_i is an edge. Note that $v_i \notin \{v_1, v_2, v_{p-1}, v_p\}$, as otherwise, because u is incident to edges of $G[R]$, we could find a path of G longer than P. Thus, v_{i-2} and v_{i+2} exist. Also, uv_{i-1} and uv_{i+1} cannot be edges by Lemma 4(4). So, by earlier arguments, we have that v_{i-1} and v_{i+1} belong both to $N(v_1)^- \cup N(v_p)^+$.

• If v_{i-1} and v_{i+1} belong to distinct of $N(v_1)^-$ and $N(v_p)^+$, then either $v_{i-1} \in N(v_1)^-$ and $v_{i+1} \in N(v_p)^+$, or $v_{i-1} \in N(v_p)^+$ and $v_{i+1} \in N(v_1)^-$. In the first case, note that $v_i \in N(v_1) \cap N(v_p) \cap N(u)$, which is not allowed. In the second case, we have $v_{i-2} \in N(v_p)$ and $v_{i+2} \in N(v_1)$, in which case we can find a path of G longer than P, a contradiction. One such path first follows $v_1 \cdots v_{i-2} v_p \cdots v_i$, then goes to u, and then traverses edges of $G[R]$.

• If v_{i-1} and v_{i+1} belong both to $N(v_1)^-$ or $N(v_p)^+$, then, assuming, without loss of generality, they belong both to $N(v_1)^-$, we have $v_{i-1}, v_{i+1} \in N(v_1)^-$, and thus $v_i, v_{i+2} \in N(v_1)$. Again, a path longer than P can be deduced: start, e.g., by following $v_p \cdots v_{i+2} v_1 \cdots v_i$, then go to u, and lastly traverse edges of $G[R]$.

In both cases we thus come up with a path of G longer than P, which is a contradiction. So R must be a stable set, which proves (1).

We now prove (2). Set $R = \{u_1, \dots, u_r\}$. Since R is stable due to (1), every vertex $u \in R$ satisfies $N(u) \subset \{v_1, \dots, v_p\}$. Recall also that, by Lemma 4(1), vertex u can be adjacent to neither v_1 nor v_p. Thus the vertices in R have all their neighbours in $\{v_2, \dots, v_{p-1}\}$.

Note that if u_i and u_j are two distinct vertices of R, then $|N(u_j) \cap N(u_i)^+| \leq 1$. Indeed, assume there are two distinct vertices $v_{i_1+1}, v_{i_2+1} \in N(u_j) \cap N(u_i)^+$, where $i_1 < i_2$. Then $v_{i_1}, v_{i_2} \in N(u_i)$, and $v_1 \cdots v_{i_1} u_i v_{i_2} \cdots v_{i_1+1} u_j v_{i_2+1} \cdots v_p$ is a path of G of order $p+2$, a contradiction to the maximality of P. Similarly, $|N(u_j) \cap N(u_i)^-| \leq 1$.

Towards a contradiction to (2), assume $r \geq 3$. So, u_1, u_2, and u_3 exist. We define the following metric ρ for the vertices in $\{v_1, \ldots, v_{p-1}\}$. We set $\rho(1) = -1$. Then, for every $i \in \{2, \ldots, p-1\}$ such that $\rho(i-1)$ is defined, we define $\rho(i)$ as follows:

- if v_i is adjacent to no vertex in $\{u_1, u_2, u_3\}$, then $\rho(i) = \rho(i-1) - 1$;
- if v_i is adjacent to exactly one vertex in $\{u_1, u_2, u_3\}$, then $\rho(i) = \rho(i-1)$;
- otherwise, if v_i is adjacent to two or three vertices in $\{u_1, u_2, u_3\}$, then $\rho(i) = \rho(i-1) + 1$.

In other words, $\rho(i)$ refers to the difference $X - Y$, where X denotes the number of edges incident to $v_1, \ldots, v_i$ going to vertices in $\{u_1, u_2, u_3\}$ (with the subtlety that for a common neighbour of u_1, u_2, and u_3 we count only two edges) while Y denotes the number of vertices in $\{v_1, \ldots, v_i\}$, which is precisely i. Note that, indeed, we have $\rho(1) = -1$ since v_1 is incident to none of u_1, u_2, and u_3.

Note that, for every $i \geq 3$, we cannot have $\rho(i) = \rho(i-1) + 1 = \rho(i-2) + 2$ because two consecutive v_i's cannot both be adjacent to the same u_i by the maximality of P (recall Lemma 4(4)). Thus, for the $\rho(i)$'s to grow by 2, some v_i must be adjacent to two or three vertices in $\{u_1, u_2, u_3\}$, some next v_i's, say $v_{i+1}, \ldots, v_{j-1}$, must then be adjacent to exactly one vertex in $\{u_1, u_2, u_3\}$ each, before the next vertex, v_j, is adjacent to two vertices in $\{u_1, u_2, u_3\}$. Actually, the first v_i in that sequence cannot be adjacent to the three vertices in $\{u_1, u_2, u_3\}$, as, by an earlier remark, this would make it impossible for v_{i+1} to have neighbours in $\{u_1, u_2, u_3\}$. Thus, this first v_i must be adjacent to exactly two vertices in $\{u_1, u_2, u_3\}$. More generally, note that if some v_i neighbours all three vertices in $\{u_1, u_2, u_3\}$, then both v_{i-1} and v_{i+1} cannot have neighbours in $\{u_1, u_2, u_3\}$.

We claim that $\rho(i) \leq 1$ for every $i \in \{1, \ldots, p-1\}$. Assume this is wrong, that is we have $\rho(i) = 2$ for some $i \in \{2, \ldots, p-1\}$. Because $\rho(1) = -1$, by the remarks above there must be an x such that $\rho(x-1) = -1$ and $\rho(x) = 0$, a $y > x+1$ such that $\rho(x) = \rho(x+1) = \cdots = \rho(y-1) = 0$ and $\rho(y) = 1$, and a $z > y+1$ such that $\rho(y) = \rho(y+1) = \cdots = \rho(z-1) = 1$ and $\rho(z) = 2$. Still by the remarks above, we have that v_x, v_y, and v_z must each be adjacent to exactly two vertices in $\{u_1, u_2, u_3\}$, while all of $v_{x+1}, \ldots, v_{y-1}, v_{y+1}, \ldots, v_{z-1}$ must be adjacent to exactly one vertex in $\{u_1, u_2, u_3\}$ each. Without loss of generality, assume v_x is adjacent to u_1 and u_2. Then v_{x+1} must be adjacent to u_3. Also, there cannot be another i such that $v_i \in N(u_1) \cup N(u_2)$ and $v_{i+1} \in N(u_3)$, by an earlier remark. This implies that v_y cannot be adjacent to u_3, and v_y must thus be adjacent to u_1 and u_2, while v_{y-1} must be adjacent to u_3. This implies there cannot be another i such that $v_i \in N(u_3)$ and $v_{i+1} \in N(u_1) \cup N(u_2)$. Now we get to a contradiction whatever two vertices in $\{u_1, u_2, u_3\}$ are adjacent to v_z, regardless of what other vertex in $\{u_1, u_2, u_3\}$ is adjacent to v_{z-1}.

Thus, we must have $\rho(p-1) \leq 1$, which means that the number of edges incident to $v_1, \ldots, v_p$ going to vertices in $\{u_1, u_2, u_3\}$ (counting only two edges for the v_i's being common neighbours of u_1, u_2, and u_3) is at most p. In other words, $d^*(u_1, u_2, u_3) \leq p$. Since $n = p + r$ and $r \geq 3$, we have $p \leq n-3$, and thus $\overline{\sigma_3}(G) \leq d^*(u_1, u_2, u_3) < n-2$, a contradiction to the fact that $\overline{\sigma_3}(G) \geq n-2$. Thus, $r \leq 2$, and (2) holds.

We now prove (3a), given that (1) and (2) hold. Set $A = N(u_1) \setminus N(u_2)$, $B = N(u_2) \setminus N(u_1)$, and $C = N(u_1) \cap N(u_2)$. Then, $N(u_1) \cup N(u_2) = A \cup B \cup C$. Set also $|A| = a$, $|B| = b$, and $|C| = c$. Recall that v_1 cannot have neighbours in $\{u_1, u_2\}$, A^+, B^+, or C^+ (by Lemma 4(1) and (3)). Therefore, we have

$$d(v_1) \leq p - 1 - a - b - c.$$

Since $d(u_1) + d(u_2) = a + b + 2c$, it follows that

$$d^*(v_1, u_1, u_2) \leq p - 1 + c - |N(v_1) \cap C|.$$

However, $d^*(v_1, u_1, u_2) \geq n - 2 = p$, and hence

$$|N(v_1) \cap C| \leq c - 1.$$

Thus, $c \geq 1$. Now suppose $c = 1$. Then $|N(v_1) \cap C| = 0$. Thus, assuming v_x is the vertex in C, we have $v_x \notin N(v_1)$. However, v_{x-1} lies neither in $N(u_1)$ nor in $N(u_2)$ by Lemma 4(4). Thus, $v_x \notin A^+ \cup B^+ \cup C^+$, and thus

$$d(v_1) \leq p - 2 - a - b - c = p - 3 - a - b,$$

which implies that

$$d^*(v_1, u_1, u_2) \leq (p - 3 - a - b) + (a + b + 2) = p - 1 = n - 3,$$

contradicting that $\overline{\sigma_3}(G) \geq n-2$. Thus, (3a) holds.

Let us now focus on proving (3b). Set $X = \{v_1, \ldots, v_{z_1-1}\}$ and $Y = \{v_{z_2+1}, \ldots, v_p\}$. In the calculation of $d^*(v_1, u_1, u_2)$, every vertex in $N(v_1) \cup N(u_1) \cup N(u_2)$ is counted at most twice, and hence

$$d^*(v_1, u_1, u_2) \leq |N_X(v_1)| + 2|N_Z(v_1) \cup N(u_1) \cup N(u_2)| + |N_Y(v_1)|.$$

Now, if $|N_Z(v_1) \cup N(u_1) \cup N(u_2)| \leq 1 + \frac{z_2 - z_1}{2}$, then

$$d^*(v_1, u_1, u_2) \leq (z_1 - 2) + (2 + z_2 - z_1) + (p - 1 - z_2) = p - 1 = n - 3,$$

which contradicts that $\overline{\sigma_3}(G) \geq n-2$. Thus, (3b) holds. ■

We are now ready to prove Theorem 2.

***Proof of Theorem* 2.** As mentioned in Section 1, admitting a perfect matching or a quasi-perfect matching is a necessary condition for a graph to be AP. We can thus focus on proving the other direction of the equivalence. Let G be a 2-connected n-graph with $\overline{\sigma_3}(G) \geq n-2$ such that G admits a perfect matching or a quasi-perfect matching. Our goal is to prove that G is AP, that is, that every n-partition π is realisable in G.

Let P be a longest path of G, and set $R = V(G) \setminus V(P)$. Then, by Lemma 5, R is a stable set with cardinality $r \leq 2$. Since P is a path of length $n-(r+1) \geq n-3$, it follows from Theorem 3 that π admits a realisation in G if $|\mathrm{sp}(\pi)| \geq r+1$. Thus, we may assume that $|\mathrm{sp}(\pi)| \leq r$.

We consider various scenarios depending on r, π, and neighbours of vertices in R. In each possible case, we describe a picking process yielding a realisation of π in G, or arrive at contradictions to the fact that $\overline{\sigma_3}(G) \geq n-2$.

Case 1. $r = 2$ and $|\mathrm{sp}(\pi)| = 2$. Set $R = \{u_1, u_2\}$. By Lemma 5(3a), u_1 and u_2 have at least two common neighbours on P, which we denote by v_x and v_y, where v_x can be assumed to be their common neighbour with the lowest index, while v_y can be assumed to be that with the largest index. Then, $x < y$.

Set $\pi = (\alpha, \dots, \alpha, \beta, \dots, \beta)$, where $\alpha < \beta$. We start by applying the picking process, described in Section 2, along P, as going from v_1 to v_p, picking parts of size α first (as many such parts as indicated by π), and, once no more parts of size α must be picked, then picking parts of size β (because $|V(P)| = n-2$, the process, if led entirely, would actually end up with a part of size $\beta - 2$). Once v_x is added to a (possibly partial) part, we pause the process, and analyse the situation we have reached. Several scenarios can occur, where throughout what follows, an *α-part* (or *β-part*) refers to a part that is intended to have cardinality α (or β) through the picking process.

Case 1.1. v_x is the ith vertex of an α-part with $i \leq \alpha - 2$, or of a β-part with $i \leq \beta - 2$. In this case, we grab u_1 and u_2 from v_x before resuming the process (taking into account that the current part has two vertices more). Once the whole process finishes, we end up with a realisation of π in G. Note in particular that there is a part containing v_x, u_1, and u_2, which induces a connected graph due to the edges $v_x u_1$ and $v_x u_2$.

Case 1.2. v_x is the $(\alpha - 1)$th vertex of an α-part. Here, we grab u_1 and u_2 from v_x, to turn the current part into a part of size $\alpha + 1 \leq \beta$. We then resume the process along $v_{x+1} \cdots v_p$ so that this part of size $\alpha + 1$ is complemented to a part of size β, before picking the remaining parts arbitrarily. Eventually, a realisation of π in G results.

Case 1.3. v_x is the αth vertex of an α-part.

Case 1.3.1. $\beta \geq \alpha + 2$. Here, we grab u_1 and u_2 from v_x to turn the current part into a part of size $\alpha+2$, and then resume the picking process along $v_{x+1} \cdots v_p$, taking into account that this part must be complemented to a part of size β, and might thus miss vertices (when $\beta > \alpha + 2$). Eventually, the process ends up with a realisation of π in G.

Case 1.3.2. $\beta = \alpha + 1$. Start by grabbing u_1 from v_x to turn the current part into a part of size β. Note that, if we resume the picking process, then there remains at least one part of size α to pick. If there also remain parts of size β to be picked, then Theorem 3 implies we can pick the remaining parts along $v_{x+1} \cdots v_p$ in such a way that u_2 can be grabbed from v_y. From here, by then going on arbitrarily along $v_{y+1} \cdots v_p$, we can obtain a realisation of π in G. Otherwise, it means that $\pi = (\alpha, \ldots, \alpha, \alpha + 1)$ for some α, that is $\beta = \alpha + 1$ and only one part of size β is requested. If $\alpha = 1$, then obviously a realisation of π in G exists. If $\alpha = 2$, then note that a realisation of $\pi = (2, \ldots, 2, 3)$ exists if and only if $(2, \ldots, 2, 1)$ is realisable in G, that is, if and only if G admits a quasi-perfect matching, an assumption that was made on G. So, we can further assume $\alpha \geq 3$.

Let z_1, z_2, and Z be as defined in Lemma 5(3b). Note that the number of integers in $\{z_1, \ldots, z_2\}$ that are congruent to 0 modulo α is at most

$$1 + \frac{z_2 - z_1}{\alpha} < 1 + \frac{z_2 - z_1}{2}$$

since $\alpha \geq 3$. But, by Lemma 5(3b),

$$|N_Z(v_1) \cup N(u_1) \cup N(u_2)| \geq 1 + \frac{z_2 - z_1}{2},$$

and hence there is a vertex $v_i \in N_Z(v_1) \cup N(u_1) \cup N(u_2)$ such that $i \not\equiv 0 \bmod \alpha$.

- If v_i is a neighbour of u_1 or u_2, then consider running the picking process along P again, as going from v_1 to v_p, first picking parts of size α as much as possible, before eventually picking the part of size $\alpha+1$ (if led entirely without grabbing u_1 or u_2, then note that the last part would actually be of size $\alpha-1$). When reaching v_i, because $i \not\equiv 0 \bmod \alpha$, then u_1 or u_2 can be grabbed from v_i. Then run the picking process till the end to have a partition of G into connected subgraphs of order α missing only one of u_1 and u_2, and eventually add the missing vertex to any of the parts it is joined to, to turn it into a part of size $\beta = \alpha + 1$.

- If v_i is a neighbour of v_1, then run the picking process as in the previous case but omitting v_1, that is, pick parts of size α as going from v_2 to v_p. Additionally, during the process, grab a vertex in $\{u_1, u_2\}$ from v_{z_1}, and grab v_1 from v_i (this is possible, by all hypotheses; in particular, note that v_{z_1} and v_i necessarily belong to different parts). Once the process finishes, this partitions all vertices of G but one in $\{u_1, u_2\}$ into connected subgraphs of order α. Then add the missing vertex to any adjacent part.

In both situations, we thus end up with a realisation of π in G, which deals with *Case* 1.3.2 and thus with the whole of *Case* 1.3.

We now consider cases where v_x, through the picking process picking parts of size α first and then parts of size β, gets added to a β-part. Since *Case* 1.1 does not apply, we can assume v_x is the $(\beta - 1)$th or βth vertex of that part. Note that, at this point of the process, after completing the part of size β containing v_x (by adding v_{x+1} if that part misses one vertex), we can assume there remain at least two parts of size β to be picked (as otherwise the rest of the vertices of G would be β vertices only, inducing a connected graph due to v_y, where $v_y \neq v_{x+1}$, and from this a realisation of π would be obtained). Recall also that we have picked all required parts of size α earlier in the process.

Case 1.4. v_x is the βth vertex of a β-part.

Case 1.4.1. $\beta \geq \alpha + 2$. By "replacing" one of the previous parts of size α constructed earlier through the picking process with a part of size β (or, in other words, by running the picking process so that we pick a first part of size β, then all parts of size α, and eventually all remaining parts of size β), we essentially "shift" the succeeding parts towards v_p by exactly $\beta - \alpha$ vertices, including the part that contains v_x, which makes it now possible, after removing the last $\beta - \alpha$ vertices of the part, to grab u_1 and u_2 from v_x (note indeed that shifting parts this way cannot make v_x "change" part, as this would require $\beta - \alpha \geq \beta$, which is not possible since $1 \leq \alpha < \beta$). Then by resuming the picking process, we eventually get a realisation of π in G, once the process achieves.

Case 1.4.2. $\beta = \alpha + 1$. Note that if $\alpha = 1$, then $\beta = 2$, and a realization of $\pi = (1, \dots, 1, 2, \dots, 2)$ in G exists since G admits a perfect matching or a quasi-perfect matching. Thus, we can assume $\alpha \geq 2$. Recall that, in the present case, the part containing v_x misses no vertex. Then, just as in *Case* 1.4.1, replace a previous part of size α with a part of size $\alpha + 1$ to shift parts towards v_p by one vertex, so that, after removing the last vertex of the part containing v_x, we can now grab u_1 from v_x. It remains the vertices of $\{v_{x+1}, \dots, v_p\} \cup \{u_2\}$ to partition, which induce a connected graph due to v_y. Also, the remaining part sizes form a partition of the form $(\alpha, \alpha + 1, \dots, \alpha + 1)$. Theorem 3 tells it is possible to pick parts so that, eventually, a realisation of π in G results.

Case 1.5. v_x is the $(\beta - 1)$th vertex of a β-part.

Case 1.5.1. $\beta \geq \alpha + 2$. If $\alpha \neq 1$, then we are done by proceeding as in previous *Case* 1.4. That is, by replacing a preceding part of size α with a part of size β (to shift parts towards v_p), having v_x changing part requires $\beta - \alpha \geq \beta - 1$, which holds only if $\alpha = 1$. Thus, this cannot occur here. By removing the last

$\beta - \alpha + 1$ vertices of v_x's part, we can then grab u_1 and u_2 from v_x, and then resume the picking process to eventually obtain a realisation of π in G.

Now, if $\alpha = 1$, then we can assume that $\pi = (1, \beta, \ldots, \beta)$, that is, π contains only one 1. Indeed, if π had at least two 1's, then a realisation of π in G could be obtained by considering $\{u_1\}$ and $\{u_2\}$ as parts of size 1, and then applying the picking process along P as going from v_1 to v_p, picking parts of size β. So assume $\pi = (1, \beta, \ldots, \beta)$. We can further assume that $\beta \geq 3$, since G was assumed to admit a quasi-perfect matching. The same arguments as in *Case* 1.3.2 can now be employed to prove that, for $Z = \{v_{z_1}, \ldots, v_{z_2}\}$ as defined in Lemma 5(3b), there is some $v_i \in N_Z(v_1) \cup N(u_1) \cup N(u_2)$ with $i \not\equiv 0 \bmod \beta$ and $z_1 < i < z_2$.

- If v_i neighbours u_1 or u_2, then a realisation of π in G can be obtained by picking parts of size β along $v_1, \ldots, v_p$ so that u_1 or u_2 is grabbed from v_i during the process (which is possible since $i \not\equiv 0 \bmod \beta$), and eventually having the remaining of u_1 and u_2 forming the desired part of size $\alpha = 1$.

- If v_i neighbours v_1, then a realisation of π in G can be obtained by picking parts of size β along $v_2, \ldots, v_p$ so that, during the process, u_1 is grabbed from v_x and v_1 is grabbed from v_i (which is possible since $i \not\equiv 0 \bmod \beta$), and eventually defining $\{u_2\}$ as the desired part of size $\alpha = 1$.

Case 1.5.2. $\beta = \alpha + 1$. For similar reasons as in *Case* 1.5.1, we can assume $\alpha \geq 2$. Now, since the part containing v_x misses only one vertex, then, as in previous *Case* 1.4.2, we are done through replacing one of the previous parts of size α with a part of size $\beta = \alpha + 1$, as it shifts parts toward v_p by one vertex and makes it possible now, because $\beta \geq 3$, after removing the last vertex of the part containing v_x, to grab both u_1 and u_2 from v_x.

Case 2. $r = 2$ and $|\mathrm{sp}(\pi)| = 1$. We again set $R = \{u_1, u_2\}$, and denote by v_x and v_y two common neighbours (on P) of u_1 and u_2, where x and y satisfy the same conditions as in *Case* 1. Set $\pi = (\lambda, \ldots, \lambda)$. We can suppose that $\lambda \geq 3$ since G admits a perfect matching. Now apply the picking process along P as going from v_1 to v_p, picking parts of size λ as long as possible (in particular, if the process was achieved completely, then the part containing v_p would be of size $\lambda - 2$). We consider a few cases.

Case 2.1. v_x is the ith vertex of a λ-part with $i \leq \lambda - 2$. We can here grab u_1 and u_2 from v_x and resume the process to obtain a realisation of π in G.

Case 2.2. v_x is the λth vertex of a λ-part. Repeat the picking process, but as going from v_p to v_1 instead. It can be checked that because $n \equiv 0 \bmod \lambda$ and $p = n - 2$, this time when treating v_x the part misses at least one vertex. If the part misses at least two vertices, then we fall back into previous *Case* 2.1. The remaining case is when the part misses exactly one vertex, which is precisely next *Case* 2.3.

Case 2.3. v_x is the $(\lambda-1)$th vertex of a λ-part. At this point of the picking process, note that if we grab u_1 from v_x and resume the process, then, later on, v_y can be assumed to be the λth vertex added to its part, as otherwise we could just grab u_2 from v_y, and, upon leading the picking process to its end, get a realisation of π in G.

To make things clear, let us run the picking process from v_1 to v_p again, picking parts of size λ, with the exception that we make sure v_x is the last vertex of its part X (thus of size $\lambda-1$), and the very last part (containing v_p) is of size $\lambda-1$. Regarding the resulting parts preceding X, note that u_1 and u_2 can be assumed to only neighbour the λth vertex as otherwise, if they neighboured another of these vertices, say, v_z, then a realisation of π would be obtained through running the picking process and grabbing u_1 and u_2 from v_z and v_y (as, indeed, grabbing one of u_1 and u_2 from v_z, because $\lambda \geq 3$, would have the effect to shift succeeding parts one vertex towards v_1, and thus make it possible to grab the second vertex from v_y). Now, obviously, in all parts succeeding X, vertices u_1 and u_2 can be assumed to only neighbour the λth vertex (because v_x neighbours both u_1 and u_2, and its part X has size $\lambda-1$). Regarding v_1, we recall it cannot neighbour a vertex in $N(u_1)^+ \cup N(u_2)^+$ by Lemma 4(3). So if u_1 and/or u_2 neighbour the λth vertex of some part, then v_1 cannot neighbour the first vertex of the next part.

Similar assumptions can be made regarding the neighbourhood of v_1, u_1, and u_2 in X. By definition, the $(\lambda-1)$th vertex of X, which is v_x, is actually the last vertex of X, and it neighbours both u_1 and u_2. By an earlier remark, this means v_1 cannot neighbour the first vertex of the part succeeding X. Now let v_z be any vertex of X different from v_x. By our choice of v_x, it cannot be that v_z neighbours both u_1 and u_2. If v_z neighbours v_1 and, say, u_1, then v_{z+1} cannot neighbour any of v_1 and u_1 (by Lemma 4(3) and (4)), and it can be assumed that v_{z+1} does not neighbour u_2. Indeed, because v_{z+1} does not neighbour u_1, we have $v_{z+1} \neq v_x$ since v_x neighbours both u_1 and u_2. This means v_{z+1} would be, at worst, the $(\lambda-2)$th vertex of X, and thus v_z would be, at worst, the $(\lambda-3)$th vertex of X. Here, a realisation of π in G could be obtained by running the picking process as earlier, grabbing u_1 from v_z and u_2 from v_{z+1} during the process, thereby forming a part of size λ, and resuming the picking process. Thus, we can assume each v_z of the first $\lambda-2$ vertices of X either neighbours at most one vertex in $\{v_1, u_1, u_2\}$, or it neighbours two vertices in $\{v_1, u_1, u_2\}$ in which case the next vertex has no neighbour in that set (in particular, this implies $z \neq x-1$).

Under all those assumptions, let us now determine an upper bound for $d^*(v_1, u_1, u_2)$. To that aim, we consider the parts we have constructed above one by one, as going from v_2 to v_p. As considering every v_i one by one this way, we determine a value $\rho(i)$ deduced from $\rho(i-1)$, the exact same way we did to prove Lemma 5(2). Essentially, for every $i \in \{2, \ldots, p\}$, the value $\rho(i)$ is the number of edges incident to the vertices in $\{v_2, \ldots, v_i\}$ going to $\{v_1, u_1, u_2\}$ (counting only two incident edges when the three edges exist) minus $|\{v_2, \ldots, v_i\}|$. We consider two main cases.

• Assume the first part is X. That is, v_1 and v_x belong to the same part, X, which, recall, is of size $\lambda - 1$. As a base case, note that $\rho(2) = 0$ if v_2 neighbours v_1 only, and that $\rho(2) = 1$ if v_2 also neighbours one of u_1 and u_2. In the latter case, by a remark above, we know that v_3 neighbours no vertex in $\{v_1, u_1, u_2\}$, and thus $\rho(3) = 0$. In the former case, v_3 is either adjacent to no vertex in $\{v_1, u_1, u_2\}$, in which case $\rho(3) = \rho(2) - 1 = -1$, to only one vertex in $\{v_1, u_1, u_2\}$, in which case $\rho(3) = \rho(2) = 0$, or to v_1 and exactly one vertex in $\{u_1, u_2\}$, in which case $\rho(3) = \rho(2) + 1 = 1$ and we additionally know that $\rho(4) = 0$. By repeating these arguments to all vertices of X one by one, for every $i \in \{2, \ldots, x - 1\}$ it can be determined that $\rho(i)$ is at most 1, and for $\rho(i)$ to be exactly 1 it must be that v_i neighbours v_1 and exactly one vertex in $\{u_1, u_2\}$, in which case $\rho(i + 1)$ is sure to be 0. Since v_x is adjacent to both u_1 and u_2, we get that $\rho(x - 1)$ must be at most 0, and thus $\rho(x)$ is at most 1.

We now get to the part succeeding X. For the first vertex, v_{x+1}, we actually have $\rho(x + 1) = \rho(x) - 1 \leq 0$ since v_x is a common neighbour of u_1 and u_2. From here, recall that, for the next $\lambda - 1$ vertices, only the last one of the part, i.e., $v_{x+\lambda}$, can neighbour u_1 and/or u_2. Thus, for every $i \in \{2, \ldots, \lambda\}$, we have $\rho(x + i) \leq \rho(x + 1) \leq 0$. Now, regarding $v_{x+\lambda}$, either it neighbours no vertex in $\{v_1, u_1, u_2\}$ and we have $\rho(x+\lambda) = \rho(x+\lambda-1)-1 \leq -1$, it neighbours only one vertex in $\{v_1, u_1, u_2\}$ and we have $\rho(x + \lambda) = \rho(x + \lambda - 1) \leq 0$, or it neighbours two vertices in $\{v_1, u_1, u_2\}$ and we have $\rho(x + \lambda) = \rho(x + \lambda - 1) + 1 \leq 1$. In the latter case, we also know that $v_{x+\lambda+1}$ neighbours no vertex in $\{v_1, u_1, u_2\}$ and $\rho(x + \lambda + 1) \leq 0$.

These arguments then repeat from the vertices of a part to the vertices of the succeeding part. In particular, recall that the last part, that containing v_p, is of size $\lambda - 1 \geq 2$. This means that v_{p-1} cannot be the λth vertex of a part, and, in particular, $\rho(p-1) \leq 0$. Now, since v_p can neighbour no vertex in $\{v_1, u_1, u_2\}$, we have $\rho(p) \leq -1$. Since $|\{v_2, \ldots, v_p\}| = p-1$, by the definition of ρ we deduce that $d^*(v_1, u_1, u_2) \leq p - 2 = n - 4$. This is a contradiction to the fact that $\overline{\sigma_3}(G) \geq n - 2$.

• Similar arguments can be employed when the first part is not X. Recall that this first part is of size λ and that only its λth vertex can neighbour u_1 and/or u_2. So we have $\rho(2) = 0$, while $\rho(3), \ldots, \rho(\lambda - 1) \leq 0$. Now, if v_λ neighbours at most one vertex in $\{v_1, u_1, u_2\}$ then we have $\rho(\lambda) \leq \rho(\lambda-1) \leq 0$, while if v_λ neighbours at least two vertices in $\{v_1, u_1, u_2\}$ then we have $\rho(\lambda) = \rho(\lambda - 1) + 1 \leq 1$. In the latter case, note that, by definition of v_x, the at least two neighbours of v_λ cannot include u_1 and u_2, so they must be v_1 and exactly one of u_1 and u_2, implying that $v_{\lambda+1}$ cannot be adjacent to v_1. So, if the next part is not X, then

$\rho(\lambda+1)=\rho(\lambda)-1\leq 0$, since $v_{\lambda+1}$ cannot neighbour u_1 and/or u_2 in this case. From these arguments, we deduce that the same calculations hold for all parts preceding X.

Assume now the first vertex of X is $v_{q\lambda+1}$ and that $\rho(q\lambda)$ was computed. By the arguments above, we have $\rho(q\lambda)\leq 1$. Also, if $\rho(q\lambda)=1$, then $v_{q\lambda}$ is adjacent to v_1 and one of u_1 and u_2, implying that $v_{q\lambda}$ cannot be adjacent to two vertices in $\{v_1,u_1,u_2\}$. Thus, we deduce that $\rho(q\lambda+1)\leq 1$. If we now employ the exact same arguments as when we dealt with X in the previous case, we here deduce that for v_x, the last vertex of X, we have $\rho(x)\leq 2$. Also, since v_x is adjacent to both u_1 and u_2, we have that v_{x+1} has no neighbour in $\{v_1,u_1,u_2\}$, and thus $\rho(x+1)\leq 1$.

By now repeating the same arguments as earlier to the consecutive parts succeeding X, we here deduce that $\rho(p-1)\leq 1$, and thus $\rho(p)\leq 0$. Since $|\{v_2,\dots,v_p\}|=p-1$, then $d^*(v_1,u_1,u_2)\leq p-1=n-3$, which is another contradiction to $\overline{\sigma_3}(G)\geq n-2$.

Case 3. $r=1$. Set $R=\{u\}$ and $\pi=(\lambda,\dots,\lambda)$. Since G is assumed to admit a perfect matching or a quasi-perfect matching, we may assume $\lambda\geq 3$.

Set $I=\{v_i : i\equiv 0 \bmod \lambda\}$. Note that if u is adjacent to a vertex $v_i\notin I$, then a realisation of π in G can be obtained by running the picking process from v_1 to v_p picking parts of size λ, and grabbing u from v_i. So we can assume that all neighbours of u lie in I.

Let us denote by $i_1,\dots,i_d$ the indexes, where $i_1<\cdots<i_d$, such that $v_{i_1},\dots,v_{i_d}$ are the $d=d(u)$ neighbours of u. Note that $d\geq 2$ since G is 2-connected. Also, $v_{i_1},\dots,v_{i_d}\in I$. Regarding the upcoming arguments, we partition the vertices of P into three sets V_1, V_2, and V_3 as follows: V_1 is $\{v_i : i<i_1\}$, V_2 is $\{v_i : i\in\{i_1,\dots,i_d\}\}$, and V_3 is $\{v_i : i>i_d\}$. Note that $v_1\in V_1$, $v_p\in V_3$, and $v_{i_1},v_{i_d}\in V_2$. So, none of the three sets is empty. Since $d\geq 2$ and $\lambda\geq 3$, also $|V_2|\geq 4$.

We now analyse the possible neighbours of v_1 (and, similarly, of v_p) through the next claim.

Claim 6. *A realisation of π in G can be constructed in the following contexts:*

(1) *if v_1 has a neighbour v_i that does not lie in $V_1\cup I$;*

(2) *if $v_i\in V_1\cap I$ is a vertex such that v_pv_i is an edge, and there is some $v_j\in V_1\setminus I$ with $j>i$ such that v_1v_j is an edge.*

Proof. Regarding (1), assume v_1 has a neighbour v_i that does not lie in $V_1\cup I$. Thus, $i>i_1$. A realisation of π in G can then be obtained as follows. Start running the picking process along P as going from v_p to v_2, picking parts of size λ. When reaching v_i, then, because $v_i\notin I$, at least one more vertex must be added to the part. Then grab v_1 from v_i, before resuming the process. Later on, when reaching v_{i_1}, note that the part must be missing vertices. So, here, just grab u from v_{i_1} before resuming the process. Once it achieves, this results in a realisation of π.

Now consider (2). Let $v_i \in V_1 \cap I$ be a vertex such that $v_p v_i$ is an edge, and assume there is a $v_j \in V_1 \setminus I$ with $j > i$ such that $v_1 v_j$ is an edge. A realisation of π in G can here be obtained as follows. Let α be the smallest index with $\alpha > j$ such that $v_\alpha \in I$. We run the picking process along the path $v_{\alpha-1} \cdots v_2$ as going from $v_{\alpha-1}$ to v_2, picking parts so that the first part (containing $v_{\alpha-1}$ and v_j) has size $\lambda - 1$, the part containing v_i has size $\lambda - 1$, and the other parts have size λ. Note that the hypothesis on α and j implies this is possible (in particular, the choice of α implies that $\alpha - 1 - i > \lambda - 1$, so v_j and v_i cannot belong to the same part). Then, grab v_1 from v_j to form a part of size λ, and grab v_p from v_i to form another part of size λ. Now resume the process, but along $v_{p-1} \cdots v_\alpha$ as going from v_{p-1} to v_α, picking parts of size α. In particular, when considering v_{i_d}, more vertices must be added to that part, and we can freely grab u from v_{i_d} before resuming the process. Once the process achieves, this results in a realisation of π in G. □

Back to the proof of *Case* 3, assume now that Claim 6 cannot be applied to deduce a realisation of π in G. That is, we can assume all neighbours of v_1 lie in $V_1 \cup I$, and that if $v_i \in V_1 \cap I$ is a vertex such that $v_p v_i$ is an edge, then there does not exist any $v_j \in V_1 \setminus I$ with $j > i$ such that $v_1 v_j$ is an edge. Likewise, we can assume similar things for v_p, that is, all its neighbours lie in $V_3 \cup I$, and if $v_i \in V_3 \cap I$ is a vertex such that $v_1 v_i$ is an edge, then there cannot be any $v_j \in V_3 \setminus I$ with $j < i$ such that $v_p v_j$ is an edge.

Let us now analyse the possible neighbours in $\{v_1, v_p, u\}$ for the other vertices, in $\{v_2, \ldots, v_{p-1}\}$, with respect to all observations we made so far.

- Any $v_i \in I$ can be a neighbour of any number of vertices in $\{v_1, v_p, u\}$.
- Any $v_i \in V_2 \setminus I$ cannot have any neighbour in $\{v_1, v_p, u\}$.
- Any $v_i \in V_1 \setminus (I \cup \{v_1\})$ has v_1 as its only potential neighbour in $\{v_1, v_p, u\}$, but this can only occur if there is no $j < i$ such that $v_j \in I \cap N(v_p)$.
- Any $v_i \in V_3 \setminus (I \cup \{v_p\})$ has v_p as its only potential neighbour in $\{v_1, v_p, u\}$, but this can only occur if there is no $j > i$ such that $v_j \in I \cap N(v_1)$.

We now count edges incident to v_1, v_p, and u, omitting one edge for each of their common neighbours. This is exactly the quantity $d^*(v_1, v_p, u)$. In order to make the calculations easier, we split the analysis into three main cases.

Case 3.1. v_1 has no neighbour in $I \cap V_3$, and v_p has no neighbour in $I \cap V_1$. In that case, all neighbours of v_1 lie in $V_1 \cup (I \cap V_2)$, while all neighbours of v_p lie in $V_3 \cup (I \cap V_2)$. Recall also that all neighbours of u lie in $I \cap V_2$. Then $d^*(v_1, v_p, u)$

is at most

$$(|V_1| - 1) + (|V_3| - 1) + (2|I \cap V_2|).$$

Now, since $d \geq 2$ and $\lambda \geq 3$, from this we deduce that

$$d^*(v_1, v_p, u) \leq (i_1 - 2) + (p - 1 - i_d) + 2\left(1 + \frac{i_d - i_1}{\lambda}\right) < p - 1 = n - 2,$$

which is a contradiction to the fact that $\overline{\sigma_3}(G) \geq n - 2$.

Case 3.2. v_p has a neighbour in $I \cap V_1$, and v_1 has no neighbour in $I \cap V_3$. In this case, let v_x be the neighbour of v_p in $I \cap V_1$ with the lowest index. Set $V_1' = \{v_1, \ldots, v_{x-1}\}$ and $V_1'' = \{v_x, \ldots, v_{i_1-1}\}$. Note that $V_1 = V_1' \cup V_1''$. Also, $|V_1''|$ is a multiple of λ.

Note that, by Claim 6, all neighbours of v_1 lie in $V_1' \cup (I \cap V_1'') \cup (I \cap V_2)$, while all neighbours of v_p lie in $(I \cap V_1'') \cup (I \cap V_2) \cup V_3$. Also, we still have that all neighbours of u lie in $I \cap V_2$. Thus, $d^*(v_1, v_p, u)$ is at most

$$(|V_1'| - 1) + (2|I \cap V_1''|) + (|V_3| - 1) + (2|I \cap V_2|).$$

Since $d \geq 2$ and $\lambda \geq 3$, we thus have

$$d^*(v_1, v_p, u) \leq (x - 2) + 2\left(\frac{i_1 - x}{\lambda}\right) + (p - 1 - i_d) + 2\left(1 + \frac{i_d - i_1}{\lambda}\right) < p - 1 = n - 2,$$

another contradiction to $\overline{\sigma_3}(G)$ being at least $n - 2$.

Case 3.3. v_p has a neighbour in $I \cap V_1$, and v_1 has a neighbour in $I \cap V_3$. Here, let v_x be the neighbour of v_p in $I \cap V_1$ with the lowest index, and v_y be the neighbour of v_1 in $I \cap V_3$ with the largest index. Set $V_1' = \{v_1, \ldots, v_{x-1}\}$ and $V_1'' = \{v_x, \ldots, v_{i_1-1}\}$, and $V_3' = \{v_{i_d+1}, \ldots, v_y\}$ and $V_3'' = \{v_{y+1}, \ldots, v_p\}$. Note that $V_1 = V_1' \cup V_1''$ and that $V_3 = V_3' \cup V_3''$. Also, $|V_1''|$ and $|V_3'|$ are multiples of λ.

In the current case, all neighbours of v_1 lie in $V_1' \cup (I \cap V_1'') \cup (I \cap V_2) \cup (I \cap V_3')$, while all neighbours of v_p lie in $(I \cap V_1'') \cup (I \cap V_2) \cup (I \cap V_3') \cup V_3''$. Again, all neighbours of u lie in $I \cap V_2$. Thus, $d^*(v_1, v_p, u)$ is at most

$$(|V_1'| - 1) + (2|I \cap V_1''|) + (2|I \cap V_3'|) + (|V_3''| - 1) + (2|I \cap V_2|).$$

We thus have

$$d^*(v_1, v_p, u) \leq (x - 2) + 2\left(\frac{i_1 - x}{\lambda}\right) + 2\left(\frac{y - i_d}{\lambda}\right) + (p - y - 1) + 2\left(1 + \frac{i_d - i_1}{\lambda}\right),$$

which, because $d \geq 2$ and $\lambda \geq 3$, is strictly less than $p - 1 = n - 2$. This, again, contradicts that $\overline{\sigma_3}(G) \geq n - 2$.

In all cases we thus get a contradiction to $\overline{\sigma_3}(G) \geq n - 2$; this concludes the proof. ∎

4. Conclusion

Regarding Theorem 2, it is worth reporting that there are examples of graphs showing that some of the requirements in the statement are mandatory. In particular, both the condition on $\alpha(G)$ and on the connectivity of G cannot be dropped out.

• First, consider, as G, the complete bipartite graph $K_{k,k+2}$ for any $k \geq 2$. Note that G has order $2k+2$, which is even, and that G is 2-connected because $k \geq 2$. Furthermore, every three pairwise independent vertices u, v, and w of G must belong to the same partition class, from which we deduce that $d^*(u,v,w) \geq 2k = n-2$, and thus $\overline{\sigma_3}(G) \geq n-2$. Also, the longest paths of G go through $n-1$ vertices, and thus, by Theorem 3, all n-partitions π with $|\mathrm{sp}(\pi)| \geq 2$ are realisable in G. Thus, if G is not AP, then it must be because of n-partitions $\pi = (\lambda, \dots, \lambda)$ that are not realisable in G. Since every path P on $n-1$ vertices of G has the property that the last vertex of G neighbours every second vertex of P, it is easy to see (through, e.g., the picking process) that realisations of π in G exist whenever $\lambda \neq 2$. Meanwhile, it can be noticed that G does not admit perfect matchings, thus no realisations of $(2, \dots, 2)$. Thus, the condition on $\alpha(G)$ in Theorem 2 is an important one.

• Second, consider, as G, any graph obtained in the following way. Let $n_1, n_2, n_3 \geq 1$ be integers, and let G be obtained from the disjoint union of three complete graphs C_1, C_2, and C_3 of order n_1, n_2, and n_3, respectively, by adding a new universal vertex x (thus joined to all vertices of C_1, C_2, and C_3). Note that G is not 2-connected, as x is a cut vertex. Also, we have $n = n_1 + n_2 + n_3 + 1$, and every three pairwise independent vertices u, v, and w of G must satisfy, say, $u \in V(C_1)$, $v \in V(C_2)$, and $w \in V(C_3)$, from which we deduce that $\overline{\sigma_3}(G) = d^*(u,v,w) = n_1 + n_2 + n_3 - 1 = n - 2$.

Now consider any case of G where $n_1 \equiv 4 \bmod 6$, $n_2 \equiv 5 \bmod 6$, and $n_3 \equiv 2 \bmod 6$. Under those conditions, note that n is a multiple of 6, thus a multiple of 2 and 3. On the one hand, note that G admits perfect matchings (the vertices of C_1 can be matched together, similarly for those of C_3, while the remaining vertices, of C_2 and x, can be matched together). On the other hand, note that G admits no realisation of $(3, \dots, 3)$ (as, in a realisation, due to the conditions on n_1, n_2, and n_3, the vertices of C_2 and x would have to be covered by parts of size 3, making it impossible for the remaining vertices, of C_1 and C_3, to be covered by parts of size 3 inducing connected graphs, since x is a cut vertex). This example shows that the 2-connectivity condition in the statement of Theorem 2 cannot be dropped out.

• Third, consider, as G, the graph obtained by starting from four disjoint edges u_1v_1, u_2v_2, u_3v_3, and u_4v_4, and then adding two new independent vertices x and y joined to all u_i's and v_i's. Note that G has order 10, is 2-connected, and admits perfect matchings (one is $\{xu_1, yv_1, u_2v_2, u_3v_3, u_4v_4\}$). Also, any three pairwise independent vertices u, v, and w of G do not include x and y, and are thus pairwise independent u_i's and v_i's. From this, we deduce that $d^*(u,v,w) = 9-2 = 7 = n-3$, and thus $\overline{\sigma_3}(G) = n-3$. However, G is not AP, as it admits no realisation of $(4,3,3)$ (to see this is true, note first that, if such a realisation existed, then x and y would belong to different parts, while the last part would induce a connected subgraph of order 3 or 4; this is impossible since $G-\{x,y\}$ has four connected components of order 2). While this example does no generalise well to larger graphs (because of the $\overline{\sigma_3}$ condition), it shows that the requirement on $\overline{\sigma_3}(G)$ in the statement of Theorem 2 cannot be just lowered to $n-3$ right away, even under the other conditions on G.

Let us add also that our proof of Theorem 2 yields a polynomial-time algorithm to decide whether a 2-connected n-graph G with $\overline{\sigma_3}(G) \geq n-2$ is AP: just determine, in polynomial time, the size of a largest matching of G (which can be done through Edmonds' Blossom Algorithm [5]).

We end up with a few words on Theorem 1, claimed by Brandt. As mentioned in the introductory section, the only remains on this result are the title ("Finding Vertex Decompositions in Dense Graphs") of an invited talk given in 2013, and the corresponding abstract, which can be found online, and which we report now for the sake of keeping track of it.

Abstract (Brandt, CID 2013)**.** *A graph $G = (V,E)$ is called arbitrarily vertex decomposable, if for any partition $\pi = (n_1, n_2, \dots, n_k)$ of $|V| = n = n_1 + n_2 + \cdots + n_k$ into positive integers there is a decomposition $V = V_1 \cup V_2 \cup \cdots \cup V_k$ with $|V_i| = n_i$ such that the graph induced by V_i is connected for all i. The decision problem whether G admits a π-decomposition is known as a notoriosly hard problem, while the complexity status of the decision problem whether G is arbitrarily vertex decomposable is not known in general. The problem is not even known to be in NP.*

Let $\mathcal{G}$ be the class of graphs G where the degree sum of any set of three independent vertices of G is at least n. We show that if $G \in \mathcal{G}$ satisfies the two necessary conditions of being connected and having a matching on $\lfloor \frac{n}{2} \rfloor$ edges, then G is arbitrarily vertex decomposable. This gives a polynomial time algorithm deciding whether $G \in \mathcal{G}$ is arbitrarily vertex decomposable (just determine the size of a largest matching of G and determine whether it is connected). The proof is algorithmic and gives an algorithm that finds a decomposition for any partition π in time $\mathcal{O}(n^3)$. As a starting point of this algorithm we present another algorithm that finds in a connected graph in $\mathcal{G}$ either a hamiltonian path or a very long cycle with certain additional properties.

It would be a bit daring to try to retrieve Brandt's proof from this abstract only, but we can at least make some guess. In the very last part of the abstract, Brandt mentions finding long paths or cycles in connected n-graphs G with $\sigma_3(G) \geq n$, which is indeed the usual way to proceed to establish such results, as illustrated by the results from [9, 13, 14] and our proof of Theorem 2. This high-level description actually reminds us of a result of Momege, who proved the following:

Theorem 7 (Momege [15]). *If G is a connected n-graph with $\sigma_3(G) \geq n$, then either G is traceable, or any longest cycle of G is dominating.*

Recall that a cycle C of a graph G is *dominating* if every edge of G is incident to a vertex of C. Note that we used a sort of similar result in our proof of Theorem 2, when we proved that R must be a stable set. The next step in our proof was then proving that $|R|$ is small, which seems less immediate to achieve under the condition that $\sigma_3(G) \geq n$. So, it might be that Theorem 1 can be proved by first making use of Theorem 7, but, if it can, we are not sure, however, what the next step would be.

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