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THE MATCHING EXTENDABILITY OF 7-CONNECTED MAXIMAL 1-PLANE GRAPHS

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Abstract

A graph is 1-planar if it can be drawn in the plane such that each edge is crossed at most once. A graph, together with a 1-planar drawing is called 1-plane. A graph is said to be $k(\geq 1)$-extendable if every matching of size k can be extended to a perfect matching. It is known that the vertex connectivity of a 1-plane graph is at most 7. In this paper, we characterize the k-extendability of 7-connected maximal 1-plane graphs. We show that every 7-connected maximal 1-plane graph with even order is k-extendable for $1 \leq k \leq 3$. And any 7-connected maximal 1-plane graph is not k-extendable for $4 \leq k \leq 11$. As for $k \geq 12$, any 7-connected maximal 1-plane graph with n vertices is not k-extendable unless $n = 2k$.

Keywords: perfect matching, 7-connected maximal 1-plane graph, matching extendability.

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1. Introduction

A *drawing* of a graph $G = (V, E)$ is a mapping D that assigns to each vertex in V a distinct point in the plane and to each edge uv in E a continuous arc connecting $D(u)$ and $D(v)$. When there is no ambiguity, we do not distinguish between a graph-theoretical object (such as a vertex or an edge) and its drawing. All drawings considered here are such that no edge crosses itself, no two edges cross more than once, and no two incident edges cross.

A drawing of a graph is a 1-*planar drawing* if each edge is crossed at most once. A graph is 1-*planar* if it has a 1-planar drawing. A graph together with a 1-planar drawing is a 1-*plane graph*. 1-planar (1-plane) graphs have been introduced in 1965 by Ringel [25]. One reason for interest in the class of 1-planar (1-plane) graphs is that they are closely related to the class of planar (plane) graphs. However, they have a number of qualitative differences. For instance, 1-planarity cannot be characterized in terms of forbidden minors [14]. For a graph with n vertices, it is possible to determine in linear time whether the graph is planar or not [15], while it is NP-complete to test whether a given graph is 1-planar [14], even for the graphs formed from planar graphs by adding a single edge [19]. In contrast to Fáry's theorem [17] for planar graphs, not every 1-planar graph has a 1-plane straight-line drawing [9]. More properties of 1-planar graphs are described in [2, 13, 28, 29]; see also [18] for a survey.

A graph is *maximal* 1-*planar* if we cannot add any edge from the complement so that the resulting graph is still 1-planar and simple. A graph is *maximal* 1-*plane* if we cannot add any edge to it so that the resulting drawing is still 1-plane and simple. It is obvious that a maximal 1-plane graph is not necessarily a maximal 1-planar graph. Maximal 1-planar (1-plane) graphs have also been studied extensively due to their interesting properties, including the recognition algorithm of maximal 1-plane graphs with a rotation system [10], edge density [3] and crossing number [20]. It is well-known that every 1-planar graph with n (≥ 3) vertices has at most $4n - 8$ edges (see [8, 21]). A 1-planar graph is said to be an *optimal* 1-*planar graph* if it has exactly $4n - 8$ edges.

In this paper, we are concerned with matching extendability for 1-planar graphs. Let G be a graph. A set $M \subseteq E(G)$ is a *matching* if no two edges from M share a vertex. A matching M is *perfect* if it covers $V(G)$ and M is *extendable* if G has a perfect matching containing M. Moreover, the graph G is said to be k-*extendable* if every matching of size k in G can be extended to a perfect matching. The study of matching extendability has more than 40 years' history. Among such research, Plummer [22] extensively studied matching extendability of the graphs on surfaces, and proved that no planar graph with minimum degree 3 is 3-extendable. In particular, 5-connected planar graphs have been investigated. Every 5-connected planar graph of even order is 2-extendable [24]. A graph G is said to be $E(m, n)$ if for every pair of disjoint matchings $M, N \subseteq E(G)$ of size m and n, respectively, there is a perfect matching F in G such that $M \subseteq F$ and $F \cap N = \emptyset$. Moreover, every 5-connected maximal planar graph of even order is $E(1, 3)$ [1].

Somewhat similarly, the maximal 1-plane graphs with the highest vertex connectivity, namely 7, are also what we care about in this paper. It should be noted that the study of matchings or matching extendability for 1-planar graphs is just getting started. In 2018, Fujisawa, Segawa, and Suzuki [12] investigated the matching extendability of optimal 1-planar graphs. The authors showed that every optimal 1-planar graph G of even order is 1-extendable. An edge in a 1-plane graph G is called a *crossing edge* if it crosses with another edge, and a *non-crossing edge* otherwise. If each edge of a cycle C is non-crossing, then the closed curve induced by the edges of C separates the plane into two regions, the bounded one (i.e., the interior of C) and the unbounded one (i.e., the exterior of C). A *connected component* (or *component* for short) of a graph is *odd* or *even* if it has an odd or even number of vertices, respectively. A cycle C is called a *barrier cycle* if each edge of C is a non-crossing edge, $G - V(C)$ consists of two odd components and each of the two regions separated by C contains an odd component of $G - V(C)$. The authors also showed that every optimal 1-planar graph G of even order is 2-extendable unless G contains a barrier 4-cycle, and every optimal 1-planar graph G of even order is 3-extendable unless G contains a barrier 6-cycle. However, except for optimal 1-planar graphs, there are still many open questions in the field. This is because a necessary condition for a graph G to be $k(\geq 1)$-extendable is that G be a graph that has a perfect matching. Unfortunately, we still know relatively little about which 1-planar (1-plane) graphs have a perfect matching. Hudák *et al.* [16] proved that every optimal 1-planar graph with even order and every 7-connected maximal 1-planar graph with even order have a perfect matching, respectively. Recently, Fabrici *et al.* [11] proved that every 4-connected maximal 1-planar graph is Hamiltonian. Therefore, every 4-connected maximal 1-planar graph with even order has a perfect matching. However if the "maximal" condition above is removed, 1-plane graphs of given connectivity may not have a perfect matching. Fujisawa *et al.* [12] found a 1-plane graph with connectivity 4 of even order that does not have a perfect matching. Biedl [5] subsequently proved that for any N, there exists a 5-connected 1-planar graph with $n \geq N$ vertices for which any matching has size at most $\frac{n-2}{2}$. It is still an open problem whether any 6-connected or 7-connected 1-planar graph has a perfect matching [7, 12].

In [12], it was shown that the connectivity of any optimal 1-planar graph G is either 4 or 6. Hence, there does not exist a 1-planar graph that is optimal and 7-connected. In this paper, we prove some combinatorial properties of 7-connected maximal 1-plane graphs. Furthermore, we characterize the k-extendability of

7-connected maximal 1-plane graphs.

The remainder of this paper is organized as follows. In Section 2, we give some necessary terminology, notations and lemmas. In Section 3, we provide some combinatorial properties of 7-connected maximal 1-plane graphs. Section 4 proves the main results of this paper (Theorems 16, 18 and 21).

2. Preliminaries

We consider here only finite simple graphs G with vertex set $V(G)$ and edge set $E(G)$, unless otherwise stated. The *degree* of a vertex v of G is denoted by $d_G(v)$, and the *minimum vertex-degree* of G is denoted by $\delta(G)$. We denote by $\bar{d}(G)$ the *average degree* of G, $\frac{1}{n}\sum_{v\in G} d_G(v)$, where n is the order of G. If S is a set of vertices, the *vertex-induced subgraph* $G[S]$ is the subgraph of G that has S as its set of vertices and contains all the edges of G that have both end-vertices in S. A *walk* is a finite sequence of edges that joins a sequence of vertices. A *trail* is a walk in which all edges are distinct. A *cycle* in a graph is a non-empty trail in that the only repeated vertices are the first and last vertices. A cycle that contains every vertex of a graph is called a *Hamiltonian cycle*. A graph is *Hamiltonian* if it contains a Hamiltonian cycle. Two paths connecting vertices u and v of a graph are *internally vertex-disjoint* if u and v are the only common vertices of the paths.

A *separating set* of a graph G is a set $S \subseteq V(G)$ such that $G - S$ has more than one connected component. The *connectivity* of G, written as $\kappa(G)$, is the minimum size of a vertex set S of G such that S is a separating set or $G - S$ has only one vertex. A graph G is *k-connected* if its connectivity is at least k.

For any 1-plane drawing D of G, the *associated plane graph* $D^\times$ is the plane graph that is obtained from D by turning all crossings of D into new vertices of degree four. A vertex in $D^\times$ is called *false* if it corresponds to some crossing of D, and is *true* otherwise. A face or an edge of $D^\times$ is called *false* if it is incident with some false vertex, and is *true* otherwise.

Let C be a cycle in a 1-plane graph G so that no two edges of C cross each other. Thus, the closed curve induced by the edges of C separates the plane into two regions C_{int} and C_{out}. We call C a *conflict cycle* if both regions C_{int} and C_{out} contain one or more vertices of $G - V(C)$.

The following is a well-known fact giving the upper bound of the number of edges of 1-planar (1-plane) graphs.

Lemma 1 [8, 21]**.** *Let G be a 1-planar (1-plane) graph with at least 3 vertices. Then $|E(G)| \leq 4|V(G)| - 8$.*

Lemma 2 [6]**.** *Any simple 1-plane graph with minimum degree 7 has at least 24 vertices and the lower bound is tight (see Figure 1).*

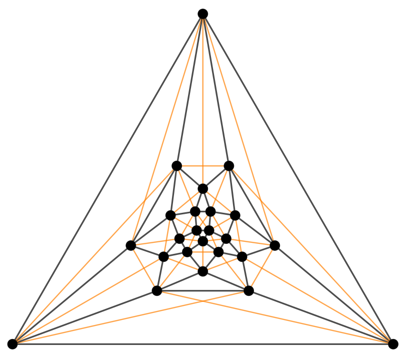

Figure 1. A 1-plane graph G with minimum degree 7 on 24 vertices.

Lemma 3 [27]**.** *Let G be a 4-connected planar graph. Then for any $u, v \in V(G)$, $G - \{u, v\}$ is Hamiltonian.*

The *degree* of a face in a plane graph is the number of edges in its boundary. A simple connected plane graph in which all faces have degree three is called a *triangulation*. A *separating cycle* C in a plane graph is a cycle so that both the interior of C as well as the exterior of C contain one or more vertices.

Lemma 4 [4]**.** *A triangulation is k-connected if and only if it has no separating cycle of length at most $k - 1$.*

The following two basic properties of k-extendable graphs were given by Plummer. They will be used in Theorems 18 and 21, respectively.

Lemma 5 [23]**.** *Let G be a graph of order $n \geq 2k + 2$ and $k \geq 1$. If G is k-extendable, then*

(i) *G is $(k - 1)$-extendable;*

(ii) *G is $(k + 1)$-connected.*

In [12], Fujisawa *et al.* generalized Lemma 2.3 in [24] which only discusses the case of $k = 1$. We will see that Lemma 6 is also a tool when we discuss the 2-extendability and 3-extendability for 7-connected maximal 1-plane graphs.

Lemma 6 [12]**.** *Let G be a k-extendable graph and $M = \{e_1, \ldots, e_{k+1}\}$ be a matching of G that is not extendable. Then there exists $S \subset V(G)$ such that*

(i) *$S \supset \bigcup_{i=1}^{k+1} V(e_i)$ and*

(ii) *$|S| = o(G - S) + 2k$,*

where $o(G - S)$ stands for the number of odd components of $G - S$.

3. Some Ppropertiess of 7-Connected Maximal 1-Plane Graphs

To prove our main theorems, we provide some combinatorial properties of 7-connected maximal 1-plane graphs in this section.

Lemma 7. *Let G be a 7-connected 1-plane graph. Then the minimum degree $\delta(G)$ is 7.*

Proof. By Lemma 1, we have $|E(G)| \leq 4|V(G)| - 8$. Then $\delta(G) \leq \lfloor \bar{d}(G) \rfloor \leq \frac{2(4|V(G)|-8)}{|V(G)|}$. Thus $\delta(G) \leq 7$. In converse, G has minimum degree at least 7, otherwise the neighbourhood of a vertex of degree less 7 yields a j-vertex-cut with $j < 7$, a contradiction. Thus $\delta(G) = 7$, as desired. ■

Lemma 8. *Any 7-connected maximal 1-plane simple graph G has at least 24 vertices, and the lower bound is tight (see Figure 1).*

Proof. By Lemma 7, the minimum degree of G is 7. Furthermore, we get $|V(G)| \geq 24$ by Lemma 2. It is not difficult to verify that G in Figure 1 is 7-connected and maximal. Thus, 24 is a sharp lower bound. ■

Proposition 9. *Let G be a 7-connected 1-plane graph. Then G does not contain any conflict 3-cycle.*

Proof. Suppose that there exists a conflict 3-cycle C in G. Let u, v be two vertices which lie the inside and outside of $G - C$, respectively. We see that u is connected to v by at most 6 internally vertex-disjoint paths (see Figure 2), and thus G is at most 6-connected by Menger's Theorem. This contradicts the 7-connectivity of G. ■

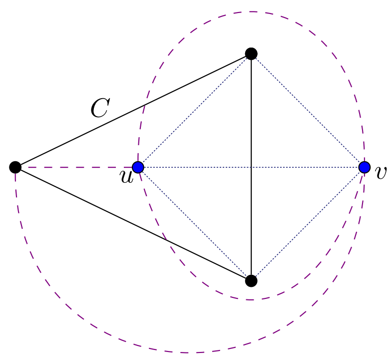

Figure 2. Connecting two vertices u and v by at most 6 internally vertex-disjoint paths.

We define the *plane skeleton* $S(G)$ of a 1-plane graph G to be the subgraph of G containing all non-crossing edges of G. A 1-plane graph G is *near optimal*, if

(i) any face of a plane skeleton $S(G)$ is either triangular or quadrangular, (ii) any quadrangular face bounded by $v_0v_1v_2v_3v_0$ of $S(G)$ contains the unique crossing point created by a pair of crossing edges v_0v_2 and v_1v_3 and (iii) no two triangular faces of $S(G)$ share any edge. A 1-plane graph is said to be *locally optimal* if there are four non-crossing edges v_1v_2, v_2v_3, v_3v_4, v_4v_1 for any pair of edges v_1v_3 and v_2v_4 that cross each other. Clearly, any near optimal 1-plane graph is locally optimal.

Lemma 10 [26]**.** *Every* 5*-connected maximal* 1*-plane graph* G *is near optimal.*

Lemma 11. *Let* G *be a* 7*-connected maximal* 1*-plane graph. By arbitrarily removing one edge from each pair of crossing edges in* G*, we obtain a spanning* 4*-connected subgraph of* G *that is a triangulation.*

Proof. Let G' be a graph obtained by removing one edge from each pair of crossing edges in G. Since G is near optimal by Lemma 10, G' is a triangulation. Moreover, G' is 3-connected by Lemma 4. Suppose that there exits a separating 3-cycle $C = abca$ in G'. Denote by F_1 and F_2 two components of $G' - C$. We have that F_i ($i \in \{1, 2\}$) and F_{3-i} cannot lie inside and outside of $G' - C$, respectively, for otherwise C is a conflict 3-cycle of G, which would contradict Proposition 9. Combining this with Lemma 4 for $k = 4$, G' is 4-connected, as desired. ■

In [12], the authors introduced three operations on locally optimal 1-plane graph that still preserve the local optimality.

Lemma 12 [12]**.** *Let* G *be a locally optimal* 1*-plane graph. Then each of the following operations on* G *preserve the local optimal* 1*-planarity:* (a) *deleting a vertex;* (b) *contracting a non-crossing edge* (*and deleting any multiple edges that may arise*); (c) *deleting a crossing edge.*

Let G be a graph and S be a subset of $V(G)$. As in [12], we also consider a bipartite graph $B(G, S)$ as follows: (i) remove all even components of $G - S$, (ii) shrink each odd component of $G - S$ to a separate vertex and delete any multiple edges thus formed, and (iii) delete the edges joining vertices in S. For example, let G be the graph on the left of Figure 3 and let $S = \{u, v, w\}$. F_1, F_2 are the even components of $G - S$ and F_3, F_4 and F_5 are the odd components of $G - S$. First we remove the even components F_1 and F_2 (delete all vertices in F_1 and F_2 together with their incident edges). We shrink odd components F_3, F_4 and F_5 into separate vertices x, y and z, respectively. We then delete all multiple edges formed in the process of shrinking F_3, F_4 and F_5. Finally, deleting all the edges uw and vw in $E(G[S])$ gives the bipartite graph $B(G, S)$, see the right of Figure 3.

Combining this with the above three operations, the authors got the following result in [12].

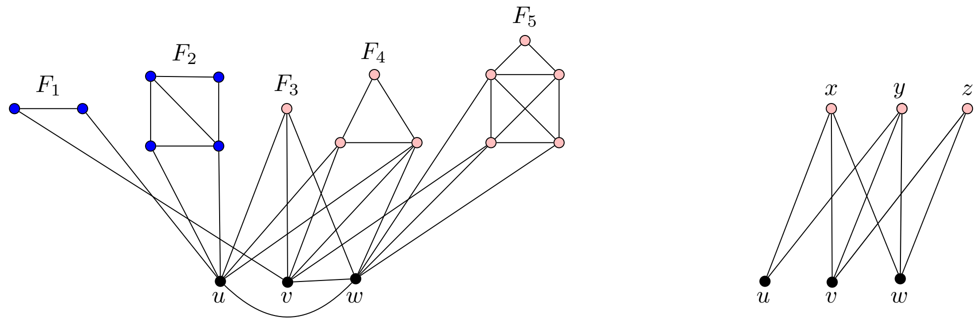

Figure 3. A graph G on the left and $B(G, S)$ on the right.

Proposition 13 [12]**.** *Let G be an optimal 1-planar graph and S be a subset of vertices of G. Then $B(G, S)$ is planar.*

We noticed that the proof of Proposition 13 in [12] only uses the property that the optimal 1-planar graph is locally optimal. That is to say, if the"optimal" condition is reduced to "locally optimal", $B(G, S)$ is also planar.

Lemma 14. *Let G be a locally optimal 1-plane graph and S be a subset of vertices of G. Then $B(G, S)$ is planar.*

Proof. As analyzed above, the proof of Proposition 13 in Reference [12] can still be applied to locally optimal 1-plane graphs. For the sake of completeness and comprehension for the reader, we sketch the idea here. First, we remove all even components of $G - S$. Next, we contract each non-crossing edge and remove all multiple edges appearing in any odd component of $G - S$. By (a) and (b) of Lemma 12, the resulting graph is locally optimal 1-plane as well. We then claim that any odd component X_i can be shrunk into one vertex by shrinking non-crossing edges and deleting some edges in S. Without loss of generality, we assume that $|V(X_1)| \geq 2$. Then X_1 contains a crossing edge uv. Let xy be the edge which crosses uv. Then we can find four non-crossing edges ux, xv, vy, yu since the graph we have is locally optimal. Since each edge is a crossing edge in X_1, we have $\{x, y\} \in S$. Then the edge $xy \in E(G[S])$, and here we delete xy. Thus uv can continue to be contracted. By (c) of Lemma 12, the resulting graph is a locally optimal 1-plane graph as well.

By repeating the above contractions and deletions of edges, we obtain a locally optimal 1-plane graph G' whose vertex set is $S \cup X$, where each vertex of X corresponds to an odd component of $G - S$. Notice that the edge between X_i and S and the edge between X_j $(j \neq i)$ and S do not cross, because G is locally optimal. At the end of the procedure, we delete the edges joining vertices in S to obtain $B(G, S)$. Then we see that any crossing is eliminated, and thus $B(G, S)$ is planar. ∎

Corollary 15. *Let G be a 5-connected maximal 1-plane graph and S be a subset of vertices of G. Then $B(G, S)$ is planar.*

Proof. By Lemma 10, G is near optimal, and furthermore, G is locally optimal. Thus $B(G, S)$ is planar by Lemma 14. ∎

4. THE MATCHING EXTENDABILITY OF 7-CONNECTED MAXIMAL 1-PLANE GRAPHS

This section proves the main results of this paper.

Theorem 16. *For every integer k where $1 \leq k \leq 3$, every 7-connected maximal 1-plane graph G of even order is k-extendable.*

Proof. Let uv be an arbitrary edge of G. By Lemma 11, G has a spanning 4-connected plane subgraph T. Then it follows from Lemma 3 that $T - \{u, v\}$ has a Hamiltonian cycle. Since $T - \{u, v\}$ has an even number of vertices, we obtain a perfect matching M of $T - \{u, v\}$, and thus $M \cup \{uv\}$ is a perfect matching of G. Thus G is 1-extendable.

Suppose that G is not 2-extendable. Since G is 1-extendable, as we proved above, there exists a set $S \subset V(G)$ that satisfies (i), (ii) of Lemma 6 for $k = 1$. By (i) of Lemma 6, we have $|S| \geq 4$. We now consider the bipartite graph $B(G, S)$. From the process of construction of $B(G, S)$, $|V(B(G, S))| - |S| = o(G - S)$. Furthermore, by (ii) of Lemma 6, $o(G - S) = |S| - 2$. Thus $|V(B(G, S))| - |S| = |S| - 2$. Since G is 7-connected, any odd component of $G - S$ is adjacent to at least 7 vertices in S. From the process of construction of $B(G, S)$, for edges between an odd component of $G - S$ and S, only multiple edges (if arise) are deleted when the odd component is shrunk to a vertex. Thus any vertex of $B(G, S) - S$ is adjacent to at least 7 vertices in S. Therefore,

$$|E(B(G, S))| \geq 7(|V(B(G, S)) - |S|) = 7(|S| - 2) = 7|S| - 14. \tag{1}$$

Moreover, it follows from Corollary 15 that $B(G, S)$ is planar. Using Euler's formula on bipartite planar graphs, one has

$$\begin{aligned} |E(B(G, S))| &\leq 2|V(B(G, S))| - 4 \\ &= 2(|S| + |V(B(G, S))| - |S|) - 4 \\ &= 2(|S| + |S| - 2) - 4 \\ &= 4|S| - 8. \end{aligned} \tag{2}$$

Combining (1) with (2), we have $|S| \leq 2$, which contradicts the fact that $|S| \geq 4$. Thus G is 2-extendable.

For the case $k = 3$, we use arguments similar to those for $k = 2$. For brevity, some details of the proof have been omitted. Suppose that G is not 3-extendable. As G is 2-extendable, we can take $S \subset V(G)$ which satisfies (i), (ii) of Lemma 6 for $k = 2$. We consider similarly the bipartite graph $B(G,S)$. By (i) and (ii) of Lemma 6, we have $|S| \geq 6$ and $|V(B(G,S))| - |S| = |S| - 4$. Since G is 7-connected, each vertex of $B(G,S) - S$ is adjacent to at least seven vertices in S. Therefore,

$$|E(B(G,S))| \geq 7(|S| - 4) = 7|S| - 28. \tag{3}$$

Moreover, by Corollary 15, $B(G,S)$ is planar. Thus one has

$$\begin{aligned} |E(B(G,S))| &\leq 2|V(B(G,S))| - 4 \\ &= 2(|S| + |V(B(G,S)) - |S|) - 4 \\ &= 2(|S| + |S| - 4) - 4 \\ &= 4|S| - 12. \end{aligned} \tag{4}$$

Combining (3) with (4), we have $|S| \leq 5$, which contradicts the fact that $|S| \geq 6$. This concludes the proof. ■

The following corollary is obtained immediately from Theorem 16.

Corollary 17. *Every 7-connected maximal 1-plane graph G of even order has a perfect matching.*

Theorem 18. *For every integer k where $4 \leq k \leq 11$, any 7-connected maximal 1-plane graph G of even order is not k-extendable.*

Proof. We first consider the case of $k = 4$. By Lemma 7, we let v be a vertex of degree 7 and assume that the neighbors of v in clockwise order are v_0,v_1,v_2,$\ldots$, v_6. Let D be a 1-plane drawing chosen arbitrarily and $D^\times$ be the associated plane graph of D.

We first give the following two claims.

Claim 19. *The vertex v is incident with exactly one true triangular face and six false triangular faces in $D^\times$.*

Proof. Since G is near optimal by Lemma 10, $D^\times$ is a triangulation. Thus v is incident with exactly 7 triangular faces in $D^\times$. Note that any two false vertices are not adjacent. Then since $d_G(v)$ is odd, v is incident with at least one true triangular face, say vv_0v_1v in $D^\times$. Furthermore we claim that vv_0v_1v is a unique true triangular face incident with v. By (iii) of the definition of near optimal 1-plane graphs, we assume that vv_0 and vv_1 are incident with false triangular faces vz_1v_0v and vz_2v_1v, respectively. By (ii) of the definition of near optimal 1-plane graphs, vz_1v_0v lies in a quadrangular face of the plane skeleton $S(G)$, and z_1 is the unique crossing point in $vv_0v_6v_5v$. And similarly vz_2v_1 lies in a quadrangular face of $S(G)$, say $vv_1v_2v_3v$. So vv_3v_4v cannot be a true triangular face. Otherwise vv_4v_5v is also a true triangular face. Here we see that faces vv_3v_4v and vv_4v_5v share vv_4. This contradicts (iii) of the definition of near optimal 1-plane graphs. Similarly, vv_4v_5v is not a true triangular face. Thus vv_4 can only be a crossing edge, as shown in Figure 4. Thus we proved Claim 19. $\square$

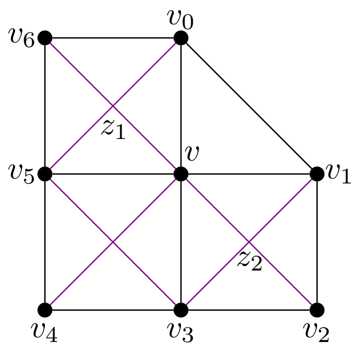

Figure 4. $G[v \cup N_G(v)]$.

Since vv_0v_1v is a true triangular face, by (iii) of the definition of near optimal 1-plane graphs, v_0v_1 is incident with a triangular false face. So v_0v_1 is incident with a quadrangular face bounded by $v_0v_1uwv_0$ of the plane skeleton $S(G)$ that contains the unique crossing point created by a pair of crossing edges v_0u and v_1w.

Claim 20. *Neither u nor w is v_i for $2 \leq i \leq 6$.*

Proof. Recall that a cycle C of a 1-plane graph G is called a conflict cycle if no two edges of C cross each other and each of the two regions separated by C contains at least one vertex of $G - V(C)$. According to the symmetry of v_0 and v_1, it is sufficient for us to consider the following three cases.

Case 1. $u = v_k$ for $2 \leq k \leq 3$. For $2 \leq k \leq 3$, we can find that $vv_0u(= vv_0v_k)v$ is a conflict 3-cycle of G, which contradicts Proposition 9.

Case 2. $u = v_5$. We see that $v_0u(= v_0v_5)$ and v_1w cross each other, and v_0u is crossed by vv_6. This contradicts the 1-planarity of G.

Case 3. $u = v_6$. If $u = v_6$, then $v_0u(= v_0v_6)$ is crossed by v_1w. This contradicts the fact that v_0v_6 is a non-crossing edge by Claim 19.

Therefore, Claim 20 holds. $\square$

Here we notice that the set of vertices $\{v_0, w, v_1, v_2, v_3, v_4, v_5, v_6\}$ forms an 8-cycle
$v_0wv_1v_2v_3v_4v_5v_6v_0$. We choose a matching $M = \{v_0w, v_1v_2, v_3v_4, v_5v_6\}$. M cannot be extended since v cannot be saturated by any perfect matching of G that contains M. Thus G is not 4-extendable.

Suppose that G is k-extendable for $5 \leq k \leq 11$. By Lemma 8, $|V(G)| \geq 24$. Thus $|V(G)| \geq 2k+2$ and we see that G is $(k-1)$-extendable by Lemma 5(i). Hence, G is 4-extendable, a contradiction. ■

Theorem 21. *For every integer $k \geq 12$, any 7-connected maximal 1-plane graph G with even order n is not k-extendable unless $n = 2k$.*

Proof. If $n < 2k$, then we cannot find k matching edges of G. Thus G is not k-extendable. If $n = 2k$, then any k independent edges themselves form a perfect matching of G by Corollary 17. As for $n \geq 2k+2$, if G is k-extendable for $k \geq 12$, then the vertex connectivity of G is at least 13 by Lemma 5, a contradiction. ■

5. Remarks

In this paper, we have characterized the k-extendability of 7-connected maximal 1-plane graphs. The matching extendability that we consider here is classical. What about some general versions of matching extension, like $E(m,n)$-extendability or $[k,n]$-extendability, for 7-connected maximal 1-plane graphs or optimal 1-planar graphs? We also noticed that a bottleneck in the study of matchings or matching extendability for 1-plane graphs is how to remove the maximality condition in our theorems. That is to say, the following problem still remains open: Does every 7-connected 1-plane graph of even order have a perfect matching ([7, 12])?

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