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APPROXIMATE AND EXACT RESULTS FOR THE HARMONIOUS CHROMATIC NUMBER

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Abstract

Graph coloring is a fundamental topic in graph theory that requires an assignment of labels (or colors) to vertices or edges subject to various constraints. We focus on the harmonious coloring of a graph, which is a proper vertex coloring such that for every two distinct colors i, j at most one pair of adjacent vertices are colored with i and j. This type of coloring is edge-distinguishing and has potential applications in transportation networks, computer networks, airway network systems.

The results presented in this paper fall into two categories: in the first part of the paper we are concerned with the computational aspects of finding a minimum harmonious coloring and in the second part we determine the exact value of the harmonious chromatic number for some particular graphs and classes of graphs. More precisely, in the first part we show that finding a minimum harmonious coloring for arbitrary graphs is APX-hard and that the natural greedy algorithm is a $\Omega(\sqrt{n})$-approximation. In the second part, we determine the exact value of the harmonious chromatic number for all 3-regular planar graphs of diameter 3 and some cycle-related graphs.

Keywords: undirected graph, vertex coloring, harmonious coloring, harmonious chromatic number, regular graph, APX-hard.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

A key topic in the area of graph theory is represented by graph coloring. The proper vertex k-coloring is perhaps the most famous type of coloring and has many applications such as scheduling, pattern matching, exam timetabling, seating plans design (see [27]). There are numerous types of colorings, e.g., harmonious, graceful, metric, sigma, set, multiset (see [27] and the references therein).

In this paper we focus on harmonious colorings. We consider only finite undirected graphs $G(V, E)$, with $|V|$ vertices (or nodes) and $|E|$ edges. Given a graph G, we denote by $V(G)$ the set of vertices of G and by $E(G)$ the set of edges of G, respectively. Given a positive integer k, let $[k] = \{1, 2, \ldots, k\}$.

1.1. Preliminaries and previous work

The concept of harmonious coloring was proposed independently by Frank *et al.* [8] and by Hopcroft and Krishnamoorthy [10] and is defined below.

Definition 1 (Harmonious coloring)**.** Let G be a graph and $c : V(G) \to [k]$ be a proper vertex coloring of G. The coloring c is called *harmonious* if for every pair of distinct colors $i, j \in [k]$ there is at most one pair of adjacent vertices in G colored with i and j.

We are interested in finding the minimum number of colors required to have a valid harmonious coloring, that is to find the harmonious chromatic number of a graph, as defined next.

Definition 2 (The harmonious chromatic number)**.** The minimum positive integer k for which a graph G has a harmonious k-coloring, is called the *harmonious chromatic number of* G and is denoted by $h(G)$.

We can associate to a harmonious k-coloring c of G an edge coloring c' of G as follows: each edge uv is assigned the color $c'(uv) = \{c(u), c(v)\}$. A color $c'(uv)$ is a 2-element subset of the set of colors assigned to the vertices of G. In the resulting edge coloring c' all the edges are colored with distinct colors. Thus, it follows that $\binom{k}{2} \geq |E|$.

Note that the harmonious coloring is different than the harmonious labeling of a graph, introduced by Graham and Sloane [9]. In a harmonious labeling c of an undirected graph G the colors of vertices are elements of $\mathbb{Z}_k$ (set of integers modulo k) and the induced edge-coloring c' is defined as $c'(uv) = (c(u) + c(v))(\text{mod } k)$.

1.1.1. Known results related to the computational complexity of the harmonious coloring problem

Hopcroft and Krishnamoorthy [10] show that the harmonious coloring problem for arbitrary graphs is NP-complete. Moreover, determining whether a graph has

a harmonious coloring using at most k colors is known to be NP-complete even for trees [7], split graphs [2], interval graphs [2, 3] and several other classes of graphs [1, 2, 3, 6, 7, 12]. Polynomial time algorithms are known for some special classes of graphs [18], the most important being for trees of bounded degree [5].

A recent paper that deals with the computational aspects of harmonious coloring is [14]. In this paper, the authors list the classes of graphs for which the harmonious coloring is known to be NP-hard.

Kolay *et al.* [14] study the parameterized complexity of the harmonious coloring problem under various parameters such as solution size, above or below known guaranteed bounds, and vertex cover number of the graph.

1.1.2. Previous results for harmonious chromatic number on particular classes of graphs

Concerning the exact value of the harmonious chromatic number of a graph, there are only a few graphs for which the precise value of the harmonious chromatic number is known. The harmonious chromatic number of the path with n vertices P_n has been determined by Lu [15], and of cycles C_n by Mitchem [19]. The harmonious chromatic number of a class of caterpillars with at most one vertex of degree more than 2 (paths, stars, shooting stars, and comets), and an upper bound of the harmonious chromatic number of 3-regular caterpillars were found by Mansuri *et al.* [25]. Harmonious coloring has been studied for distance degree regular graphs of diameter 3 and several particular classes of graphs such as Parachute, Jellyfish, Gear, and Helm graph by Huilgol and Sriram [11].

The harmonious chromatic number for the central graph, middle graph, and total graph of some families of graphs was studied in various papers: prism graph by Mansuri *et al.* [16]; flower graph, belt graph, rose graph and steering graph by Muthumari and Umamamheswari [20]; snake derived architecture by Selvi [24]; Jahangir graph by Selvi and Azhaguvel [23]; star graph by Rajam and Pauline [22], and double star graph by Vernold *et al.* [26].

Next, we present a couple of known results related to graphs of diameter 2. Recall that the *distance* $d(u, v)$ between two vertices is the length of a shortest $u - v$ path in a graph $G(V, E)$, and the *diameter* $diam(G)$ is the largest distance between any two vertices of G.

Theorem 3 (folklore)**.** *Any graph G with n vertices and diameter 2 has the harmonious chromatic number n.*

Among the most known graphs of diameter two are individual graphs like complete bipartite graph $K_{3,3}$, Wagner graph, Moser spindle graph, Golden-Harary graph, Fritsch graph, Petersen graph, house graph, prism graph Y_3, octahedron graph, and some classes of graphs like cographs, the friendship graphs, the fan graphs, the wheel graphs.

1.2. Our results

In this paper, we show the following results. In Section 2 we tackle the harmonious coloring problem from the computational point of view. More precisely, we show that the harmonious coloring problem cannot be approximated within a factor of $1.17 - \epsilon$, assuming $P \neq NP$ and within a factor $4/3 - \epsilon$, assuming the Unique Games Conjecture, $\forall \epsilon > 0$. We prove our hardness results by generalizing the NP-hardness reduction of Hopcroft and Krishnamoorthy [10]. We also show why the natural greedy algorithm (that colors vertices one by one and assigns the smallest color possible) is not a good approximation.

Then, in Section 3 we determine exact values of the harmonious chromatic number for particular classes of graphs, like $(3, 3)$-regular planar and some families of cycle-related graphs. Some of these results are obtained using a backtracking based computer program.

2. Computational Results on Harmonious Coloring

In this section, we aim to tackle the computational complexity of harmonious coloring.

2.1. Hardness of approximation of harmonious coloring on general graphs

In this subsection, we show that the harmonious coloring APX-hard or that it does not admit a polynomial time approximation scheme. In other words, there exists a constant c such that the harmonious coloring number on general graphs cannot be approximated within a factor of c.

Theorem 4. *There exists a constant $c < 1.17$ such that the harmonious coloring problem cannot be approximated within a factor of c, unless $P = NP$. Moreover, if we assume the Unique Games Conjecture, the harmonious coloring problem cannot be approximated within a factor of $4/3 - \epsilon$ for any $\epsilon > 0$.*

Proof. We show our result via a reduction from the Independent Set problem. Our reduction is a simple modification of the reduction of Hopcroft and Krishnamoorthy [10]. Given a graph $G = (V, E)$ for which we aim to find an independent set with $k \leq |V|$ elements, we can construct in polynomial time an instance of the harmonious coloring problem for a graph with two connected components G' and G''. The first component G' has vertex set $V \cup \{v_1, v_2, v_3\}$. The set of edges of $E(G')$ is obtained by adding at $E(G)$ edges between every vertex of G and v_1, v_2, and v_3, respectively, and edges $\{v_1, v_2\}$, $\{v_2, v_3\}$, $\{v_1, v_3\}$. The second component G'' is a clique with $|V|$ vertices.

Observe that G' cannot be harmoniously colored with less than $|V|+3$ colors, since it has diameter at most 2 (Theorem 3).

The claim is that this two-component graph can be harmoniously colored with $2|V|+3-k$ colors if and only if G has an independent set of size k.

Assume first that G has an independent set X of size k. We define a harmonious coloring for the two-component graph as follows: color vertices of G' with distinct colors; then color $|X|$ vertices of G'' with the colors used for the vertices of X in G' and the rest of the vertices of G'' with $|V|-|X|$ new colors. The obtained coloring is harmonious and uses $|V|+3+|V|-|X| = 2|V|+3-|X|$ colors.

Conversely, assume that the two-component graph has a harmonious coloring with $2|V|+3-k$ colors. For $k=1$ there is an independent set of size k in G. Assume $k \geq 2$. For the vertices in component G' exactly $|V|+3$ distinct colors are used (Theorem 3). We have $|V|-k$ unused colors left only for vertices in G''. Since G'' is a clique, vertices from G'' have distinct colors. It follows that there are k colors used both for vertices in G' and G''. By the definition of a harmonious coloring, it follows that in G' these vertices form an independent set. This independent set is also an independent set in G, since vertices v_1, v_2, v_3 are pairwise adjacent and adjacent to all the vertices in G.

Let $0 < s < c \leq \frac{1}{2}$ be constants and let $GapIS(c,s)$ be a "promise gap problem" where an n-vertex graph is given with the promise that either it contains an independent set of size cn or contains no independent set of size sn and the algorithmic task is to distinguish between the two cases. According to our reduction, we have that if $GapIS(c,s)$ is NP-hard, then the harmonious coloring is NP-hard to approximate within

$$\frac{2|V|+3-s|V|}{2|V|+3-c|V|}.$$

Thus, harmonious coloring is NP-hard to approximate within $\frac{2-s}{2-c}+\epsilon$, for some $\epsilon > 0$.

The best gap known is of Dinur and Safra [4] and has $GapIS(1-2^{-1/d}-\epsilon,\epsilon)$ for $d \geq 2$. Thus, for $d=2$, we have that the harmonious coloring is hard to approximate within $\frac{2}{1+\frac{1}{\sqrt{2}}} \approx 1.17$, unless $P=NP$. Then, according to Khot and Regev [13], assuming the Unique Games Conjecture we have $GapIS(1/2-\epsilon,\epsilon)$. Thus, assuming the Unique Games Conjecture, the harmonious coloring problem is hard to approximate within a factor of $4/3-\epsilon$. ■

2.2. The natural greedy algorithm is an $\Omega(\sqrt{n})$-approximation

A natural greedy algorithm to harmoniously color a graph is as follows. Process the vertices arbitrarily and color each vertex with the smallest available color,

i.e., the smallest color that keeps the coloring up to this step harmonious. In this section, we show that this greedy algorithm is a $\Omega(\sqrt{n})$-approximation even in the case of trees, where n is the number of nodes in the tree. The result is stated in the next theorem.

Theorem 5. *There exists a tree T with $n = N(N-1)$ vertices that has a harmonious coloring with $2N-2$ colors and is colored by the greedy algorithm with $(N-1)^2+1$ colors for a certain ordering of its vertices.*

Proof. The tree T, illustrated in Figure 1 is defined as follows. The root a_0 has $N-1$ children $a_1, \ldots, a_{N-1}$. Each of the $N-2$ nodes $a_2, \ldots, a_{N-1}$ has only one children. We term the children of the node a_i with b_i. Then, each of the nodes $b_2, \ldots b_{N-1}$ has $N-1$ children. We denote the $N-1$ children of the node b_i as $c_i^1, c_i^2, \ldots, c_i^{N-1}$. Tree T has $n = N+N-2+(N-1)(N-2) = N(N-1)$ vertices.

The greedy algorithm colors the root a_0 with 1, a_1 with 2, and the nodes $a_2, \ldots, a_{N-1}$ with colors $3, 4, \ldots, N$. Then, each of the nodes $b_2, \ldots, b_{N-1}$ have color 2. Finally, each of the nodes c_i^j have a distinct color, which results in a total of $N + (N-1)(N-2) = (N-1)^2 + 1$ colors.

A coloring with $2N-2$ is as follows. The root a_0 is colored with 1 and the vertices $a_2, \ldots, a_{N-1}$ with colors $2, 3, \ldots, N$. In turn, the nodes $b_2, \ldots, b_{N-1}$ are colored with colors $N+1, N+2, \ldots, 2N-2$. Finally, for every $2 \le i \le N-1$ the nodes c_i^j with $1 \le j \le N-1$ are colored with the colors from the set $\{1, 2, \ldots, N\}$ different than the color of a_i.

Therefore, the greedy algorithm has an approximation factor of $\Omega(N) = \Omega(\sqrt{n})$. ■

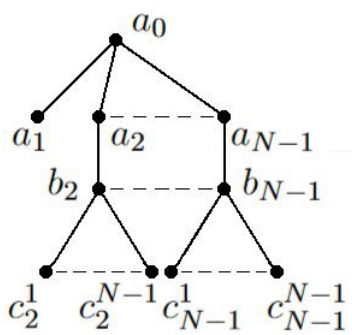

Figure 1. Counterexample for the greedy algorithm.

3. Exact Value of the Harmonious Chromatic Number for Some Particular Graphs, and Classes of Graphs

In this section, we determine the harmonious chromatic number for some families of graphs like regular graphs and cycle-related graphs. We remind that the results for the graphs of diameter 2 are presented in Section 1.1.2.

3.1. 3-regular graphs of diameter 3

First, recall the definition of a regular graph.

Definition 6. A connected graph G is a *regular graph* if every vertex of G has the same number of neighbors, so every vertex has the same degree. A regular graph with vertices of degree r is called an *r-regular graph* or *regular graph of degree r*.

Theorem 3 refers to any graphs of diameter two, including r-regular graphs. For example, octahedron is a 4-regular graph of diameter 2 (Figure 2), Wagner graph (Figure 3) and Petersen graph (Figure 4) are 3-regular graphs of diameter 2, hence they have the harmonious chromatic number n.

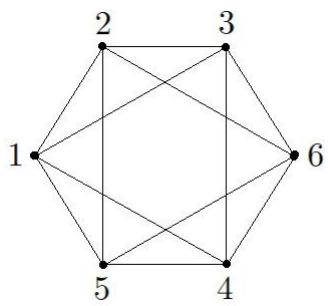

Figure 2. A harmonious 6-coloring of octahedron graph.

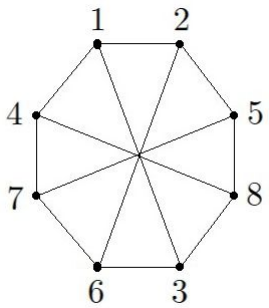

Figure 3. A harmonious 8-coloring of Wagner graph.

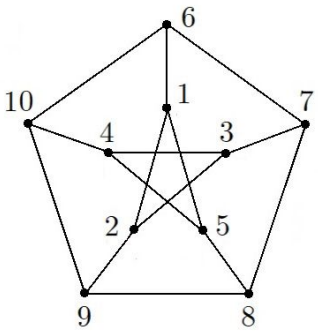

Figure 4. A harmonious 10-coloring of Petersen graph.

A graph with maximum degree Δ and diameter *diam* is called a $(\Delta, diam)$-graph. We determine the harmonious chromatic number for all $(3,3)$-regular planar graphs, and for well known $(3,3)$-regular non-planar graphs. McKay and Royle [17] give a list of 3-regular graphs of diameter 3.

Proposition 7. *For a $(3,3)$-regular graph G the minimum number of colors for a harmonious coloring is 7.*

Proof. Let $G(V, E)$ be a $(3, 3)$-regular graph. Then, obviously, $|V(G)| \geq 8$. Let c be a harmonious coloring of G. If all colors are distinct, then at least 8 colors are used. Otherwise, there are two distinct vertices v, and u with $c(u) = c(v)$. Then, vertices from $N(u) \cup N(v)$ must have distinct colors, different from $c(u)$. Since $c(u) = c(v)$, we have $d(u, v) \geq 3$ and $N(u) \cap N(v) = \emptyset$. It follows that there are 6 vertices in $N(u) \cup N(v)$, all having distinct colors, different from $c(u)$, hence at least 7 colors are used. ■

Pratt [21] establishes that the smallest 3-regular planar graph of diameter 3 has 8 vertices and the largest 3-regular planar graph of diameter 3 has 12 vertices. The number of non-isomorphic planar $(3, 3)$-regular graphs with 8 vertices is 3, with 10 vertices is 6, and with 12 vertices is 2. Note that an r-regular graph with r odd must have an even number of vertices (Handshaking lemma).

Figure 5 displays all the $(3, 3)$-regular planar graphs with 8 vertices. Figure 6 displays all $(3, 3)$-regular planar graphs with 10 vertices. Figure 7 displays the two $(3, 3)$-regular planar graphs with 12 vertices.

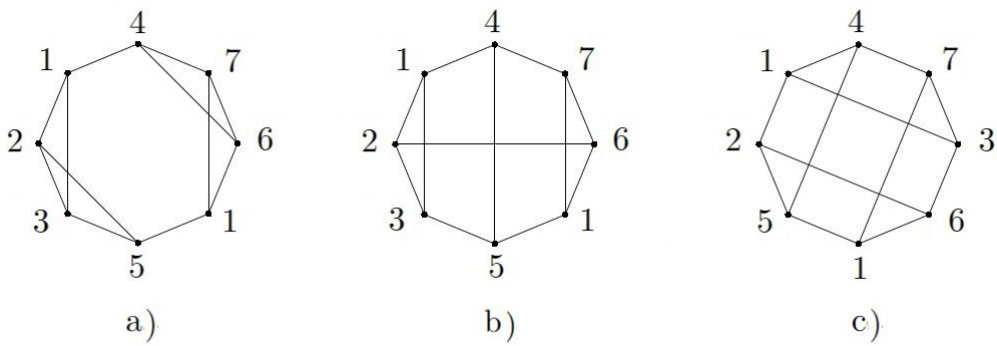

Figure 5. Harmonious 7-colorings of all (3,3)-regular planar graphs with 8 vertices (the color 1 is repeating).

Proposition 8. *For* $(3, 3)$*-regular graphs with* 8 *or* 10 *vertices the harmonious chromatic number is* 7*.*

Proof. From Proposition 7, the number of colors for a harmonious coloring of a $(3, 3)$-regular graph with 8 or 10 vertices is at least 7. Then, to prove the result, it suffices to provide 7-harmonious colorings for these graphs. In Figure 5 we present all planar $(3, 3)$-regular graphs with 8 vertices along with a 7-harmonious coloring of each of them and in Figure 6 we present harmonious colorings with 7 colors for each planar $(3, 3)$-regular graphs with 10 vertices. ■

Theorem 9. *The harmonious chromatic number for the only* 2 *planar* $(3, 3)$*-graphs with* 12 *vertices is* 8*.*

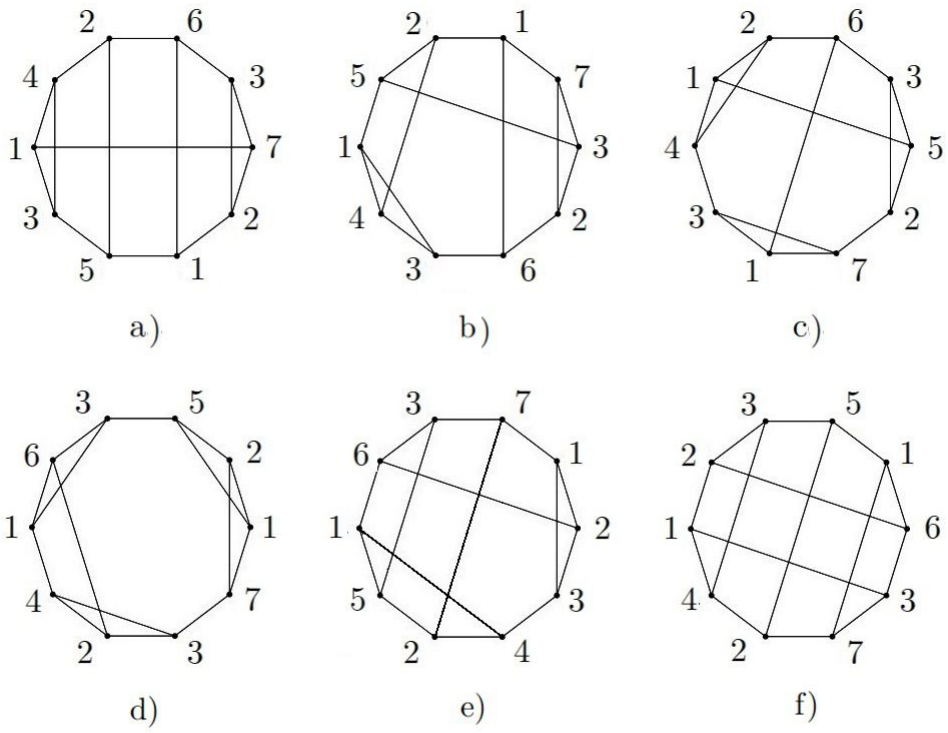

Figure 6. Harmonious 7-colorings of all $(3,3)$-regular planar graphs with 10 vertices (the colors 1, 2, and 3 are repeating).

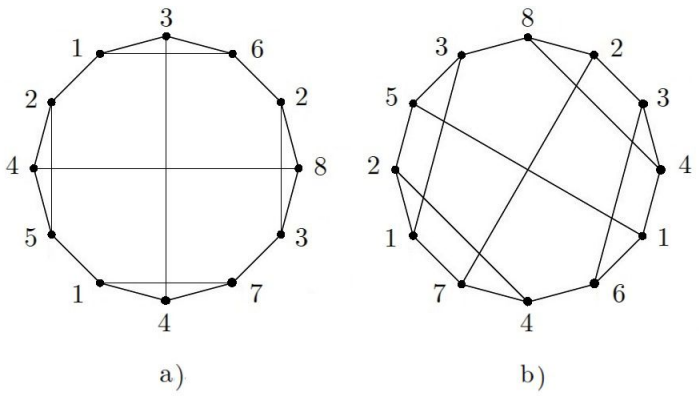

Figure 7. Harmonious 8-coloring for the two (3,3)-regular planar graphs with 12 vertices (the colors 1, 2, 3, and 4 are repeating).

Proof. Figure 7 shows a harmonious 8-coloring of the truncated tetrahedron graph and a harmonious 8-coloring of the second $(3,3)$-regular planar graph with 12 vertices.

Using a computer program, we proved that these graphs cannot be colored harmoniously with fewer colors, by exhaustively trying all the possible harmonious colorings with 7 colors. Our program is based on the classical backtracking schema: we color vertices one by one in increasing order of their index, and at one step we verify that there are no conflicts for the color c assigned to the current vertex by considering the colors of the neighbors of all vertices previous colored with c. ■

The source code of the program used in the proof of Theorem 9 is available at https://github.com/veruxy/Harmonious-coloring.

The $(3,3)$-regular non-planar graphs can have more than 12 vertices. Although we could not classify all $(3,3)$-regular graphs according to their harmonious chromatic number, we fully explore the planar graphs from this category and provide a tool — a computer program — to explore the harmonious coloring of these graphs when the number of vertices is small enough.

3.2. Some families of cycle-related graphs

In previous work by Huilgol and Sriram [11], the value of the harmonious chromatic number for some graphs generated from a cycle, like wheel graph W_n an n-cycle with each vertex connected with an extra vertex), gear graph G_n (an n-wheel graph with an extra vertex between each pair of adjacent vertices on the perimeter of W_n), and Helm graph H_n (an n-wheel graph with a pendant edge attached to each vertex on the perimeter of W_n) are determined. These graphs have n vertices on a cycle connected to a central vertex, and then $\Delta = n$. The harmonious chromatic number is $h(W_n) = h(G_n) = h(H_n) = n + 1$.

There are several interesting cycle-related graphs of diameter 2, like double wheel graph and flower graph (obtained from helm graph by joining every pendant with the central vertex). Each of these graphs has the harmonious chromatic number equal to their order (Theorem 3).

Next, we determine the exact values of the harmonious chromatic number of other families of cycle-related graphs of diameter greater than 2, namely sunflower graph, sun graph, closed sun graph, and lollipop graph.

The *sunflower graph* Sf_n is obtained from an n-wheel graph W_n with set of vertices $\{v_0, v_1, v_2, \ldots, v_n\}$ by adding n vertices u_i, $1 \leq i \leq n$, and joining each new vertex u_i with two adjacent vertices v_i, v_{i+1}, $1 \leq i \leq n-1$, and u_n with v_n and v_1. Thus, Sf_n has $2n+1$ vertices, and $4n$ edges. The degree for each vertex of Sf_n: $d(v_0) = n$, $d(v_i) = 5$, and $d(u_i) = 2$, where $1 \leq i \leq n$.

Theorem 10. *The sunflower graph Sf_n has $h(Sf_n) = 7$, for $3 \leq n \leq 4$, $h(Sf_n) = 8$ for $5 \leq n \leq 6$, and $h(Sf_n) = n + 1$, for $n \geq 7$.*

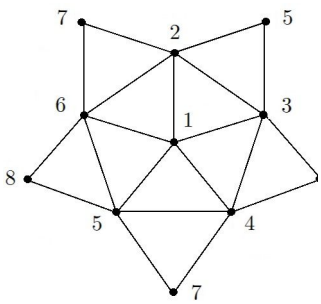

Figure 8. A harmonious 8-coloring of sunflower graph Sf_5.

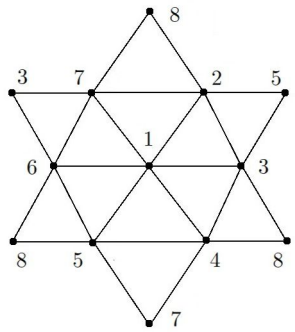

Figure 9. A harmonious 8-coloring of sunflower graph Sf_6.

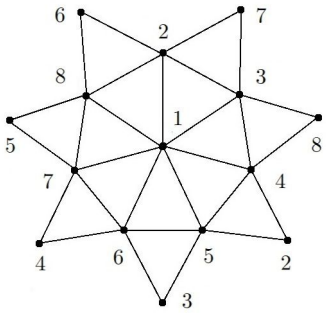

Figure 10. A harmonious 8-coloring of sunflower graph Sf_7.

Proof. The sunflower graph Sf_3 has diameter 2, and thus, for Theorem 3, $h(Sf_3) = 7$. For $4 \leq n \leq 6$ we used our computer program described in proof of Theorem 9 to obtain the harmonious chromatic number (see Figure 8 and Figure 9).

The sunflower graph Sf_n, with $n \geq 7$, has the harmonious chromatic number $h(Sf_n) \geq h(W_n) = n + 1$. In order to prove that equality holds, we describe a harmonious coloring for Sf_n with $n + 1$ colors. Color the central vertex v_0 with color 1; then color the vertices v_i, $1 \leq i \leq n$, on the cycle with colors in order in set $C = \{2, 3, \ldots, n + 1\}$, clockwise, and assign to the vertices u_i colors in set C, clockwise, starting from the vertex u_2, situated at distance 3 from the vertex v_1 previously colored with 2 (Figure 10). ■

The *sun graph* S_n is obtained from the complete graph K_n, with vertices denoted $v_1, v_2, \ldots, v_n$ and n new vertices $u_1, u_2, \ldots, u_n$, each connected with two adjacent vertices on an outer cycle of K_n, more precisely vertex u_i is adjacent to v_i and v_{i+1}, for every $1 \leq i \leq n - 1$, and u_n is adjacent to v_n and v_1. Thus, the sun graph S_n has $2n$ vertices, and $n(n - 1)/2 + 2n$ edges.

Theorem 11. *The sun graph S_n, $n \geq 3$, has $h(S_n) = n + 2$ if n is even, and $h(S_n) = n + 3$ if n is odd.*

Proof. In a harmonious coloring of S_n vertices $v_1, \ldots, v_n$ of the clique must have distinct colors. Denote these colors $1, \ldots, n$. Since $d(u_i, v_j) \leq 2$ for every $1 \leq i, j \leq n$, it follows that colors $1, \ldots, n$ cannot be used for vertices $u_1, \ldots, u_n$. Moreover, since $d(u_i, u_{i+1}) = 2$ for every $1 \leq i \leq n - 1$ and $d(u_n, u_1) = 2$, it follows that, if n is even at least 2 new colors are needed for vertices $u_1, \ldots, u_n$

and if n is odd at least 3 new colors are needed. Hence

$$h(S_n) \geq \begin{cases} n+2, & \text{if } n \text{ even,} \\ n+3, & \text{if } n \text{ odd.} \end{cases}$$

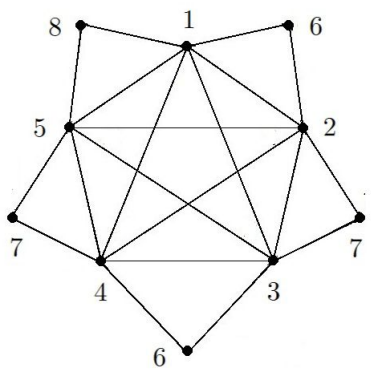

Figure 11. A harmonious 8-coloring of sun graph S_5.

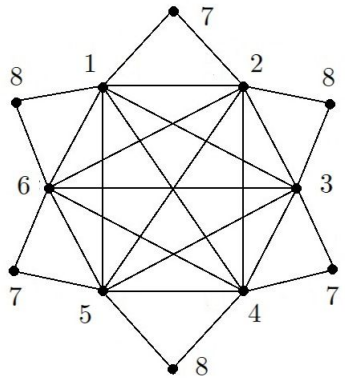

Figure 12. A harmonious 8-coloring of sun graph S_6.

The lower bound can be achieved for the following coloring, hence equality holds.

- For n even, let $c(v_i) = i$ for every $1 \leq i \leq n$, $c(u_j) = n+1$ if j is odd and $c(u_j) = n+2$ if j is even for $1 \leq j \leq n$ (Figure 12);
- For n odd, let $c(v_i) = i$ for every $1 \leq i \leq n$, $c(u_j) = n+1$ if j is odd and $c(u_j) = n+2$ if j is even for $1 \leq j \leq n-1$ and $c(u_n) = n+3$ (Figure 11). ■

The *closed sun graph* $\overline{\mathrm{S_n}}$ is the graph S_n with edges between vertices u_i, u_{i+1}, where $1 \leq i < n$, and between u_n and u_1. Thus, $\overline{\mathrm{S_n}}$ has $2n$ vertices and $n(n-1)/2 + 3n$ edges. Then, $d(v_i) = n+1$, and $d(u_i) = 4$.

Theorem 12. *The closed sun graph* $\overline{\mathrm{S_n}}$ *has* $h(\overline{\mathrm{S_n}}) = 2n$, *for* $n \leq 5$ *and* $h(\overline{\mathrm{S_n}}) = n + h(C_n)$, *for* $n > 5$.

Proof. For $n \leq 5$ we have $h(\overline{\mathrm{S_n}}) = 2n$, since in this case $\overline{\mathrm{S_n}}$ has diameter 2. For $n > 5$, vertices of the clique must be colored with n distinct colors and these colors cannot be used for any vertex from the outer cycle C_n, since a vertex from the outer cycle is at distance at most 2 from any vertex of the clique; hence we have $h(\overline{\mathrm{S_n}}) \geq n + h(C_n)$. To prove that equality holds, we consider the following coloring for $\overline{\mathrm{S_n}}$ (Figure 13, Figure 14), which can be easily verified that is harmonious.

- First color with $1, \ldots, n$ the vertices of the clique K_n;

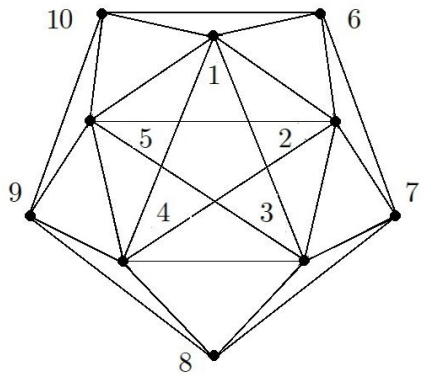

Figure 13. A harmonious 10-coloring of closed sun graphs $\overline{S_5}$.

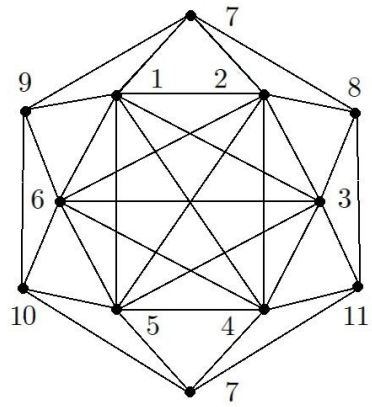

Figure 14. A harmonious 11-coloring of closed sun graphs $\overline{S_6}$.

- Then consider a harmonious coloring for the outer cycle C_n with $h(C_n)$ colors, using colors from $n+1$ to $n+h(C_n)$. ■

Let G, H be two connected graphs and consider one vertex from each of these two graphs: $a \in V(G)$, $b \in V(H)$. Denote by $(G,a) \odot (H,b)$ the graph obtained from the union of graphs G and H by identifying vertices a and b. We will call this operation vertex-union.

For two positive numbers $n \geq 3$, $m \geq 2$ *Lollipop graph* $L_{n,m}$ is the vertex-union $(K_n, u) \odot (P_m, v)$ where u is any vertex of a clique K_n and v is a degree 1 vertex of a path P_m.

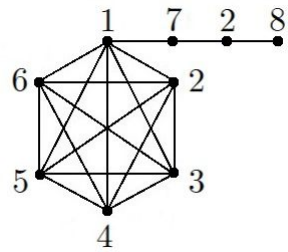

Figure 15. A harmonious 8-coloring of lollipop graph $L_{6,4}$.

Theorem 13. *Let $n \geq 3$ and $m \geq 2$ and t be the minimum natural number such that $m \leq 1 + nt + \frac{t(t-1)}{2}$. The Lollipop graph $L_{n,m}$ has the harmonious chromatic number $h(L_{n,m}) = n + t$ in the following cases.*

- *t Is even and n is odd;*

- *t and n are even and $m \leq 1 + nt + \frac{t(t-1)}{2} - \frac{t}{2}$;*
- *t is odd, n is even and and $m \leq 1 + nt + \frac{t(t-1)}{2} - (n-2)$;*
- *t and n are odd and $m \leq 1 + nt + \frac{t(t-1)}{2} - \left(n - 2 + \max\left(\frac{t-(n-2)}{2}, 0\right)\right)$,*

otherwise $h(L_{n,m}) = n + t + 1$.

Proof. In this proof for a complete graph K_r we denote the vertices by $1, \ldots, r$. Also, for $n \leq r$ we denote by $\langle [n] \rangle$ the clique induced in K_r by vertices $1, \ldots, n$.

Let $k = nt + \frac{t(t-1)}{2}$.

Let $r = h(L_{n,m})$ and let c be an r-harmonious coloring of $L_{n,m}$. The n vertices of the clique of $L_{n,m}$ must have distinct colors. Assume without loss of generality, that these colors are $1, \ldots, n$. Then, in the complete graph K_r, according to the coloring c, the colors of the vertices of the clique in $L_{n,m}$ correspond to a clique with n vertices $1, \ldots, n$ in K_r and the colors of the vertices from the path P_m of $L_{n,m}$ correspond to a trail (possible closed) with m vertices in $K_r - E(\langle [n] \rangle)$ (obtained from K_r by removing all the edges between vertices $1, \ldots, n$) starting with a vertex from $1, \ldots, n$.

Conversely, if in a clique K_r with $r \geq n$ there exists a trail with m vertices in $K_r - E(\langle [n] \rangle)$ starting with a vertex from $1, \ldots, n$ (assume without loss of generality it starts from vertex 1), then $L_{n,m}$ has an r-harmonious coloring. It follows that the harmonious chromatic number of $L_{n,m}$ is the minimum r with such property.

Let t be the smallest number such that $|E(L_{n,m})| = |E(K_n)| + |E(P_m)| \leq E(K_{n+t})$, that is such $m - 1 \leq nt + \frac{t(t-1)}{2} = k$. Then $h(L(K_{n,m})) \geq n + t$ and equality holds only if the following property is satisfied: there exists a trail with m vertices in $K_{n+t} - E(\langle [n] \rangle)$ starting with a vertex 1.

In $K_{n+t} - E(\langle [n] \rangle)$ vertices $1, \ldots, n$ have degree t and vertices $n+1, \ldots, n+t$ have degree $n+t-1$. In order to have a trail with m vertices starting with vertex 1 in $K_{n+t} - E(\langle [n] \rangle)$, the largest subgraph of this graph that has an Eulerian trail must have at least $m - 1$ edges and all vertices of this subgraph must have even degree with at most 2 exceptions; if there are vertices of odd degree in this subgraph, then vertex 1 must be one of them, thus at least $t - 1$ of vertices $n + 1, \ldots, n + t$ have even degree in this subgraph.

We consider four cases, according to the parity of n and m.

Case 1. If t is even and n is odd, then $K_{n+t} - E(\langle [n] \rangle)$ is Eulerian, hence it has an Eulerian cycle. This cycle includes a trail with m vertices starting from vertex 1, hence in this case $h(L(K_{n,m})) = n + t$.

Case 2. If t is even and n is even, in order to have a subgraph in $K_{n+t} - E(\langle [n] \rangle)$ with all vertices from $n + 1$ to $n + t$ of even degree with at most one exception, then we must remove at least $\frac{t}{2}$ edges, hence $m - 1$ must be at most

$k - \frac{t}{2}$. We can obtain such a subgraph by removing edges $(n+1, n+2)$, $(n+3, n+4), \ldots, (n+t-1, n+t)$. This subgraph is Eulerian, hence is has a trail with m vertices starting from vertex 1. It follows that if $m - 1 \leq k - \frac{t}{2}$, then we have $h(L(K_{n,m}) = n + t$. Otherwise, $h(L(K_{n,m})) \geq n + t + 1$ and equality holds, since by adding a new vertex to K_{n+t} and joining it with $n+1, \ldots, n+t$ we obtain an Eulerian subgraph of $K_{n+t+1} - E(\langle [n] \rangle)$ with at least m edges.

Case 3. If t is odd and n is even, then, to have a subgraph with an Eulerian trail, we must remove edges such that at least $n-2$ vertices from $1, \ldots, n$ have even degree. Since these vertices are pairwise nonadjacent, we must remove at least $n-2$ edges. For example if we remove $(3, n+1), (4, n+1), \ldots, (n, n+1)$ we obtain a subgraph with an Eulerian trail with one extremity in 1.

Hence, in this case, if $m - 1 \leq k - (n-2)$, then we have $h(L(K_{n,m})) = n + t$, otherwise, as in Case 2, $h(L(K_{n,m})) = n + t + 1$.

Case 4. If t is odd and n is odd, consider two subcases.

Subcase 4.1. If $t \geq n - 2$, as in Case 2, we must remove at least $\frac{n+t}{2} - 1$ edges in order to have a subgraph with an Eulerian trail. We can remove the edges: $(i, n+i-2)$ for $3 \leq i \leq n$, and $(n+n-1, n+n), (n+n+1, n+n+2), \ldots, (n+t-1, n+t)$ and obtain the desired subgraph, hence in this case if $m - 1 \leq k - \left(\frac{n+t}{2} - 1\right) = k - \left(n - 2 + \frac{t-n+2}{2}\right)$ we have $h(L(K_{n,m})) = n + t$, otherwise, as in Case 2, we have $h(L(K_{n,m})) = n + t + 1$.

Subcase 4.2. If $t < n - 2$, as in Case 3, we must remove at least $n - 2$ edges such that at least $n-2$ vertices from $1, \ldots, n$ became of even degree. For example, we remove the edges $(i, n+i-2)$ for $3 \leq i \leq t+1$ and the edges $(i, n+t)$ for $t+2 \leq i \leq n$ and obtain a subgraph with an Eulerian trail from vertex 1. Hence, in this case, if $m - 1 \leq k - (n-2)$, then we have $h(L(K_{n,m})) = n + t$, otherwise $h(L(K_{n,m}) = n + t + 1$. ■

4. Conclusions and Future Work

In this paper, we studied the harmonious chromatic number, which is a proper vertex coloring such that for every two distinct colors i, j at most one pair of adjacent vertices are colored with i and j.

We showed that finding a minimum harmonious coloring for arbitrary graphs is APX-hard, the natural greedy algorithm is a $\Omega(\sqrt{n})$-approximation. In the second part of our paper, we determined the exact value of the harmonious chromatic number for all 3-regular planar graphs of diameter 3 and some cycle-related graphs.

We state an open problem related to the approximability of the harmonious chromatic number.

Open Question 1. Does there exist a constant factor approximation algorithm for the harmonious chromatic number on arbitrary graphs?

Finally, we list a couple of classes of cycle-related graphs for which it is interesting to find the exact value of the harmonious chromatic number: square graph, tadpole or dragon graph, barbell graph, diamond snake, total graph of path, total graph of cycle.

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