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RESISTANCE IN REGULAR CLASS TWO GRAPHS

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Abstract

A well-known theorem of Vizing separates graphs into two classes: those which admit proper Δ-edge-colourings, known as class one graphs; and those which do not, known as class two graphs. Class two graphs do admit proper $(\Delta + 1)$-edge-colourings. In the context of snarks (class two cubic graphs), there has recently been much focus on parameters which are said to measure how far the snark is from being 3-edge-colourable, and there are thus many well-known lemmas and results which are widely used in the study of snarks. These parameters, or so-called measurements of uncolourability, have thus far evaded consideration in the general case of k-regular class two graphs for $k > 3$. Two such measures are the resistance and vertex resistance of a graph. For a graph G, the (vertex) resistance of G, denoted as $(r_v(G))$ $r(G)$, is defined as the minimum number of (vertices) edges which need to be removed from G in order to render it class one. In this paper, we generalise some of the well-known lemmas and results to the k-regular case. For the main result of this paper, we generalise the known fact that $r(G) = r_v(G)$ if G is a snark by proving the following bounds for k-regular G: $r_v(G) \leq r(G) \leq \lfloor \frac{k}{2} \rfloor r_v(G)$. Moreover, we show that both bounds are best possible for any even k.

Keywords: class two graphs, resistance, vertex resistance.

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1. Introduction

As is well-known, the edge chromatic number of a graph is either Δ or $\Delta+1$ where Δ is the maximum degree of the graph in question, by Vizing's theorem [16, 17]. Such graphs are referred to as cubic class one and cubic class two graphs, respectively. Cubic class two graphs are more commonly known as snarks. Snarks have long been of particular interest in graph theory, for many reasons. Contemporary efforts to understand the complexity of snarks consists largely of the consideration of parameters which measure how far the snarks are from being uncolourable, or so-called "measurements of uncolourability". See for instance [1,6,8,10,11,13–15], as well as more recently [2, 3, 9]. To our best knowledge, these parameters have not been studied in the general class two case. We believe that the study of these parameters in the general case will provide insights into the structure of class two graphs in general (as it has in snarks), as well as further illuminate the uniqueness of snarks themselves.

A *semi-graph* G is a pair $G = (V, E)$ which consists of a set of vertices $V = V(G)$ and a set $E = E(G)$ where E is a multiset of 2-element and singleton subsets of V. In $E(G)$, the 2-element sets are called *edges* (as expected) while the 1-element sets are called *semi-edges*. Note that if E contains no 1-element subsets, then G is simply a *graph*. We denote the edge $\{u, v\}$ as uv and the semi-edge $\{u\}$ as (u). Furthermore, we define the *join* between two semi-edges (u) and (v) as the removal of semi-edges (u) and (v), and the addition of the edge uv. A semi-edge (u) and a vertex v may also join to form an edge uv, with semi-edge (u) being removed. The degree of a vertex v in a semi-graph G is defined as the combined total number of edges and semi-edges incident with v. Essentially, semi-edges behave like edges except that they are associated with one vertex instead of two, with each vertex having at most one semi-edge. For our purposes in this paper, we only consider graphs and semi-graphs which are simple. That is, they contain no loops or parallel edges.

Let G be a semi-graph. A k-edge-colouring, f, of G is a mapping from the set of edges and semi-edges of G to a set of k colours. That is, $f : E \longrightarrow \{1, \ldots, k\}$. f is a *proper* k-edge-colouring of G if no two adjacent elements in E are mapped to the same colour. By Vizing's theorem, if G is a graph and f is a proper colouring, then the smallest possible value of k is Δ or $\Delta + 1$, where Δ is the maximum degree of any vertex in G. If the smallest possible value of k is Δ, then we say that G is class one, or Δ-edge-colourable. Otherwise we say that G is class two, or $(\Delta + 1)$-edge-colourable. Vizing's theorem is easily seen to be applicable to semi-graphs as well. Given a k-edge-colouring f, we call the set $f^{-1}(i)$ a colour class, for each $i \in \{1, \ldots, k\}$. A vertex v is *conflicting* with regard to f if more than one of the edges incident to v are mapped to the same colour. A colour c is *missing* at v with regard to f if there exists no edge incident to v which is assigned c by f, otherwise it is *present* at v with regard to f.

Let G be a k-regular semi-graph. Some so-called measurements of uncolourability of cubic graphs which are of interest in this paper are the following. The resistance of G, denoted as $r(G)$, defined as the $\min\{|f^{-1}(i)| : f$ is a proper $(k + 1)$-edge-colouring of G$\}$. That is, the minimum number of edges that can be removed from a graph such that the resulting graph is k-edge-colourable. The vertex resistance of G, denoted as $r_v(G)$, defined as the minimum number of conflicting vertices in a k-edge-colouring of G. That is, the minimum number of vertices that can be removed from a graph such that the resulting graph is k-edge-colourable.

If f is a $(k+1)$-edge-colouring of a k-regular graph G such that no two adjacent edges are assigned the same colour by f except possibly for edges coloured i, where i is the only colour $\in \{1, \ldots, k\}$ such that $|f^{-1}(i)| = r(G)$, then we call f a *minimal* colouring. In general, we will use colour sets $\{1, \ldots, k\}$ and $\{0, 1, \ldots, k\}$ for class one and class two graphs, respectively. We will assume $f^{-1}(0) = r(G)$ for a minimal colouring f of G. Given an edge-colouring f of G, if $f(e) = 0$ for some edge $e \in G$, then we call e a *conflicting edge* with regard to f. (Note that no semi-edges need ever be conflicting). That is to say, in a minimal colouring f of G, there are $r(G)$ conflicting edges and they need not be independent. All other colour classes form matchings of G. A path which has edges coloured alternately with colours a and b with regard to f we call an *ab-path*.

There are many well-known lemmas and propositions which are widely used in the study of the cubic case. In this paper, we generalise some such lemmas and propositions to the k-regular case. In particular, we generalise the well-known and ubiquitously used Parity Lemma. We also prove that resistance can never equal one, for any k-regular G, just as in the cubic case. In the main result of this paper, we generalise the known fact that $r(G) = r_v(G)$ if G is a snark by proving the following bounds for k-regular G: $r_v(G) \le r(G) \le \left\lfloor \frac{k}{2} \right\rfloor r_v(G)$. Moreover, we construct graphs which prove that both bounds are best possible for any even k.

2. Some Useful Lemmas

We begin with our first useful lemma. This lemma can very much be regarded as a generalisation of Lemma 2.3 in [14], which highlights properties of minimal colourings in snarks.

Lemma 1. *Let G be a k-regular graph and let f be a minimal colouring of G. Let $e = uv$ be a conflicting edge in G with regard to f. Then there exists distinct colours $a, b \in \{1, \ldots, k\}$ such that a is present at v and missing at u, and b is present at u and missing at v. Moreover, the ab-path starting at v terminates at u.*

Proof. Since G is k-regular and u is incident to a conflicting edge, there must be at least one colour in $\{1, \ldots, k\}$ which is missing at u, say a. If u is missing at v as well, then e could be properly coloured with a, which contradicts that f is a minimal colouring. Therefore, a is present at v. Similarly, b is present at u and missing at v.

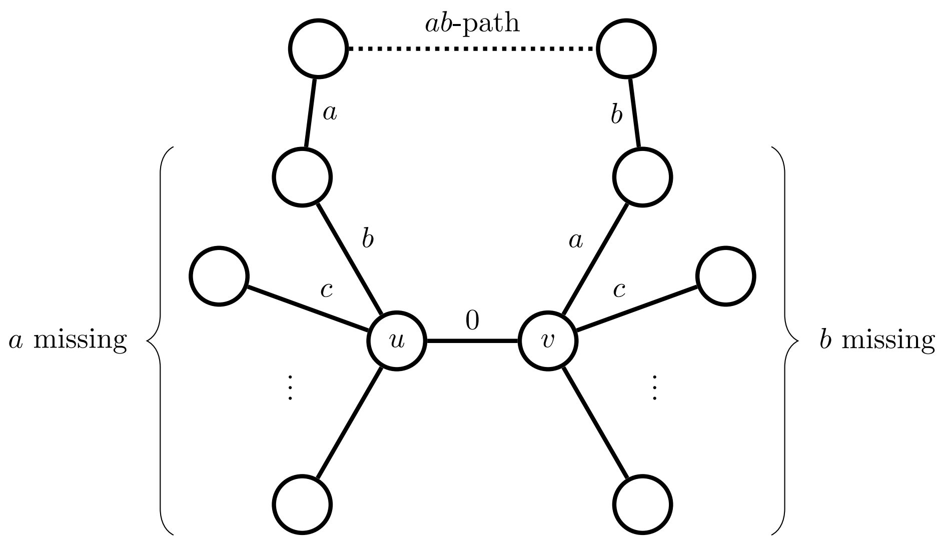

Figure 1. An ab-path from v to u.

Consider the ab-path starting at v. If this path terminates at any other vertex other than u, then we could swap the colours on the edges of the path so that colour a is now missing at both u and v. e could then be coloured a, which contradicts that f is minimal. Therefore, the ab-path starting at v terminates at u (see Figure 1). ∎

The Parity Lemma is probably the most widely used tool in the study of edge-colourability of cubic graphs. The lemma has been stated in many different ways in the literature. Here, we present a version in terms of semi-graphs and semi-edges.

Lemma 2 (The Parity Lemma for cubic graphs)**.** *Let G be a cubic semi-graph with m semi-edges and let f be a proper* 3*-edge-colouring of G. If m_i equals the number of semi-edges coloured i by f for $i = \{1, 2, 3\}$, then*

$$m_1 \equiv m_2 \equiv m_3 \equiv m \pmod 2.$$

The generalisation of the Parity Lemma to the k-regular case is lesser known, perhaps due to not being as useful since the result is not as strong. The difference is that in the general case, the total number of semi-edges need not have the same parity as each m_i. Nonetheless, researchers have made some use it, whether explicitly or implicitly. See Lemma 2.1 in [12] for a version which deals with regular graphs of odd degree which the authors attribute to [7], and Lemma 3.4

in [4] for an essentially equivalent result to the one we present here. The version we present is again in terms of semi-graphs and semi-edges.

Lemma 3 (The Parity Lemma for regular graphs)**.** *Let G be a k-regular semi-graph with m semi-edges and let f be a proper k-edge-colouring of G. If m_i equals the number of semi-edges coloured i by f for $i = \{1, \dots, k\}$, then*

$$m_1 \equiv m_2 \equiv \cdots \equiv m_k \pmod 2.$$

Proof. The number of vertices in G which are incident to an edge coloured c is equal to two times the number of edges coloured c plus the number of semi-edges coloured c. Since every colour is incident to every vertex exactly once, we have that the order of G is equal to two times the number of edges coloured c plus one times the number of semi-edges coloured c, for any $c \in \{1, \dots, k\}$. This of course has the same parity as the number of semi-edges coloured c. Thus, if m_i and m_j have differing parity for some distinct colours i and j, then the order of G is both odd and even, a contradiction. Therefore, $m_1 \equiv m_2 \equiv \cdots \equiv m_k \pmod 2$. ∎

It is well-known that $r(G) > 1$ if G is a snark. As a corollary to the above Parity Lemma, we now prove that $r(G) > 1$ for any k-regular graph, as in the cubic case.

Corollary 4. *Let G be a k-regular class two graph with $k \geq 3$. Then $r(G) > 1$.*

Proof. Since G is class two, we have $r(G) > 0$. Suppose that $r(G) = 1$. Then there exists a minimal colouring f of G such that there is exactly one conflicting edge $e = uv$. We split uv into two semi-edges and colour both of them properly with regard to f. Necessarily, $f((u)) = a \neq b = f((v))$, for some distinct $a, b \in \{1, \dots, k\}$. Let m_i be the number of semi-edges coloured i for each $i \in \{1, \dots, k\}$. Then m_a and m_b are odd, but every other m_i is even, contradicting the Parity Lemma. Therefore, $r(G) > 1$. ∎

3. Main Result

It is well-known, and perhaps counter-intuitive, that $r(G) = r_v(G)$ if G is cubic. Emphasising the uniqueness of snarks, we prove lower and upper bounds for $r(G)$ in terms of $r_v(G)$ for k-regular G and, furthermore, show that both bounds are best possible for any even k.

Theorem 5. *Let G be a k-regular class two graph. Then $r_v(G) \leq r(G) \leq \left\lfloor \frac{k}{2} \right\rfloor r_v(G)$.*

Proof. Let f be a minimal colouring of G. For each of the $r(G)$ conflicting edges, we need to remove at most one incident vertex, to render a k-edge-colourable graph. Therefore, $r_v(G) \leq r(G)$.

Let f be a k-edge-colouring of G with $r_v(G)$ conflicting vertices. For each conflicting vertex with regard to f, we recolour a minimum amount of incident edges with 0 instead, such that the only conflicting vertices with regard to f involve adjacent conflicting edges, that is, vertices with more than one edge coloured 0. For each conflicting edge e, if it is possible to adjust f so that e can be coloured properly with some colour in $\{1, \ldots, k\}$, then we adjust f and colour e properly.

Now, let u be a vertex incident to a conflicting edge. Say u is incident to m conflicting edges in total. Let $\{uv_1, \ldots, uv_m\}$ be the set of conflicting edges incident to u. There must then be at least m distinct colours in $\{1, \ldots, k\}$ which are missing at u. Say these are $\{a_1, \ldots, a_m\}$. If some a_j is missing at some v_i, then we could properly colour the edge uv_i with a_j instead of 0, which contradicts our edge-colouring f. Thus we may assume that $a_1, \ldots, a_m$ are all present at each v_i. Since each v_i is incident to at least one conflicting edge, there must be some colour in $\{1, \ldots, k\}$ which is missing at each v_i. Say b_i is missing at v_i for each i. If the a_1b_i-path starting at v_i does not terminate at u, then we could swap the colours on this path and properly colour uv_i with a_i. This again contradicts our edge-colouring f. Thus we may assume that the a_1b_i-path starting at v_i terminates at u for each i. If $b_i = b_j$ for some distinct i and j, then both the a_1b_i-path starting at v_i and the a_1b_j-path starting at v_j must terminate at u. Otherwise, we could swap the colours on one of these paths, and colour the corresponding conflicting edge properly instead. Therefore, all b_i are distinct. If b_i is also missing at u for some i, then we could colour uv_i with b_i instead, again contradicting our edge-colouring f. Thus we may assume that all b_i are present at u (see Figure 2).

We now have that there are m distinct colours present at u, and at least a further m distinct colours missing at u. Since there are k colours in total, it follows that $m \leq \left\lfloor \frac{k}{2} \right\rfloor$. Therefore, for each of the $r_v(G)$ conflicting vertices with regard to the initial k-edge-colouring, we can find an edge-colouring with at most $\left\lfloor \frac{k}{2} \right\rfloor$ conflicting edges incident to each of the $r_v(G)$ vertices, and there exists no two adjacent edges which are both coloured i for $i \in \{1, \ldots, k\}$. It follows that $r(G) \leq \left\lfloor \frac{k}{2} \right\rfloor r_v(G)$. ■

We now prove that these bounds are best possible for any even k. To prove that the upper bound is best possible for any even k, we present the case of the complete graph on $k+1$ vertices, K_{k+1}. It turns out that $r(K_{k+1}) = \frac{k}{2} r_v(K_{k+1}) = \frac{k}{2}$. To prove that the lower bound is best possible, we construct a k-regular graph G with $r(G) = r_v(G) = k$.

Theorem 6. *Let k be an even integer. Then* $r(K_{k+1}) = \frac{k}{2} r_v(K_{k+1}) = \frac{k}{2}$.

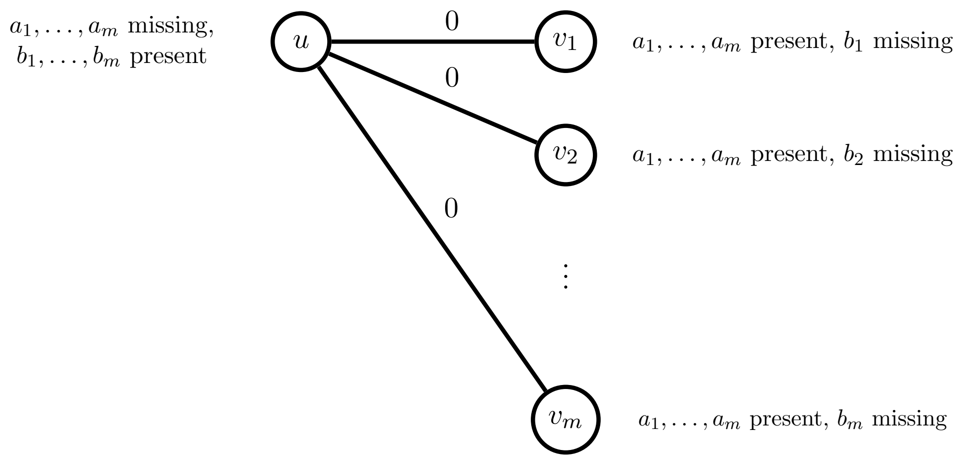

Figure 2. A conflicting vertex u with m conflicting edges.

Proof. It is well-known that any complete graph with odd order is class two, and any complete graph with even order is class one [5]. It follows that the removal of any one vertex from K_{k+1} renders a class one graphs. Therefore, $r_v(K_{k+1}) = 1$.

Let f be a minimal colouring of K_{k+1}. Since each non-zero colour class of f forms a matching, we have that each non-zero colour class has order at most $\frac{k}{2}$. Therefore, the k non-zero colours colour at most $k\frac{k}{2}$ edges. Since K_{k+1} has size $\frac{k(k+1)}{2}$, this implies that there are at least $\frac{k(k+1)}{2} - k\frac{k}{2} = \frac{k}{2}$ conflicting edges. Therefore, $r(K_{k+1}) \geq \frac{k}{2}$. By Theorem 5, it follows that $r(K_{k+1}) = \frac{k}{2}$. ■

Remark 7. Let k be an even integer. We can deduce from the proofs of Theorem 5 and Theorem 6 that there exists a minimal colouring of K_{k+1} with $\frac{k}{2}$ conflicting edges such that all conflicting edges are incident to the same vertex.

Theorem 8. *For every even integer $k > 3$, there exists a k-regular graph G_k with $r(G_k) = r_v(G_k) = k$.*

Proof. We construct the required graph as follows. Consider the graph K_{k+1} with $\frac{k}{2} - 1$ edges removed, all of which are incident to the same vertex. Denote this graph as H. By Remark 7 and Theorem 6, H is class two and has $r(H) = 1$. H has $\frac{k}{2} - 1$ vertices of degree $k - 1$, and one vertex of degree $\frac{k}{2} + 1$. Take k copies of H to form a disconnected graph G_k with components $H_1, \ldots, H_k$, and colour each component with the same minimal colouring. We now add semi-edges to each vertex in G_k with degree less than k so that G_k is k-regular. Note that for some vertices we will have to add $\frac{k}{2} - 1$ semi-edges, for others we will have to add one semi-edge, and the rest already have degree k. Colour all semi-edges accordingly by extending the minimal colouring of each component. Each component will then have its semi-edges coloured the same set of colours. Given that there are an even number of components, we have an even number of semi-edges coloured the same for each distinct colour. It is then easy to see that semi-edges of the same colour can simply be joined, and the colouring maintained, so that G_k is now connected. Note there may be many possible ways of joining the semi-edges. The colouring of G_k now has k conflicting edges, one in each H_i. Clearly, having less than k conflicting edges in a different colouring of G_k is impossible since we would then have at least one copy of some H_i which is properly coloured, a contradiction. Therefore, $r(G_k) = k$. Similarly, having less than k conflicting vertices in a k-colouring of G_k is impossible since again we would then have a copy of some H_i which has no conflicting vertex, and is then properly coloured, a contradiction. Therefore, $r_v(G_k) = r(G_k) = k$ (see Figure 3 and 4 for examples). ■

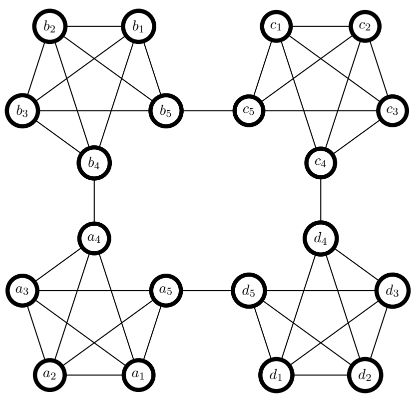

Figure 3. An example of G_4, as constructed in the proof of Theorem 8 with $r(G_4) = r_v(G_4) = 4$.

4. Further Considerations

As mentioned in the introduction, there are quite a few parameters which, in some way or the other, 'measure' how far a snark is from being 3-edge-colourable. Of these, resistance and vertex resistance are possibly the most natural and intuitive to consider. Resistance and vertex resistance also have obvious analogous definitions in the k-regular case. Certainly, other parameters might also be worthy of investigation in the k-regular case. For example, let G be a snark: the oddness of G which is defined as the minimum number of odd components in a 2-factor of G; the flow-resistance of G which is defined as the minimum number of zero edges in a 4-flow of G (it is known that a cubic graph admits a proper 3-edge-colouring if

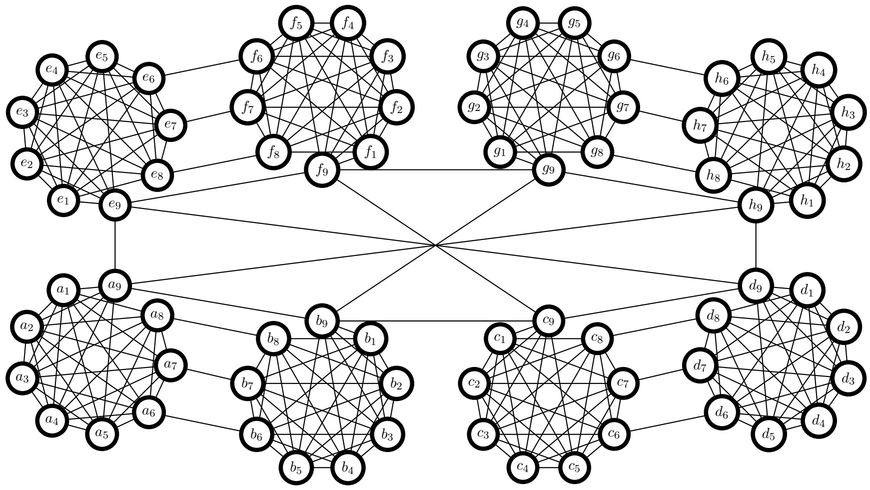

Figure 4. An example of G_8, as constructed in the proof of Theorem 8 with $r(G_8) = r_v(G_8) = 8$.

and only if it admits a nowhere zero flow); and the edge-reducibility of G which is defined as the minimum number of edge-reductions on G required to render a 3-edge-colourable cubic graph (an edge-reduction is the removal of an edge and consequent suppression of 2-valent vertices). These parameters, among others which are not mentioned here, are not necessarily easily defined in the k-regular case (if even possible), but such an endeavour may lead to further insights into class two graphs in general.

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