Discussiones Mathematicae
Graph Theory 44 (2024) 663–689
https://doi.org/10.7151/dmgt.2466

STRONG INCIDENCE COLOURING OF GRAPHS

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Abstract

An incidence of a graph G is a pair (v, e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v, e) and (w, f) of G are adjacent whenever (i) $v = w$, or (ii) $e = f$, or (iii) $vw = e$ or f. An incidence p-colouring of G is a mapping from the set of incidences of G to the set of colours $\{1, \ldots, p\}$ such that every two adjacent incidences receive distinct colours. Incidence colouring has been introduced by Brualdi and Quinn Massey in 1993 and, since then, studied by several authors.

In this paper, we introduce and study the strong version of incidence colouring, where incidences adjacent to the same incidence must also get distinct colours. We determine the exact value of — or upper bounds on — the strong incidence chromatic number of several classes of graphs, namely cycles, wheel graphs, trees, ladder graphs, square grids and subclasses of Halin graphs.

Keywords: strong incidence colouring, incidence colouring, tree, Halin graph, ladder graph, square grids, necklace, double star, wheel graph.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

All graphs considered in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively, by $\Delta(G)$ the maximum degree of G, by $N(v)$ the set of vertices adjacent to the vertex v and by $\mathrm{dist}_G(u,v)$ (respectively, $\mathrm{dist}_G(uv,wx)$) the distance between vertices u and v (respectively, edges uv and wx) in G.

An *incidence* of a graph G is a pair (v,e) where v is a vertex of G and e is an edge of G incident with v. Two incidences (v,e) and (w,f) of G are *adjacent* whenever (i) $v=w$, or (ii) $e=f$, or (iii) $vw=e$ or f.

An *incidence p-colouring* of G is a mapping from the set of incidences of G to the set of colours $\{1,\dots,p\}$ such that every two adjacent incidences receive distinct colours. The smallest p for which G admits an incidence p-colouring is the *incidence chromatic number* of G, denoted by $\chi_i(G)$. Incidence colourings were first introduced and studied by Brualdi and Quinn Massey [2]. Incidence colourings of various graph families have attracted much interest in recent years, see for instance [3, 4, 6, 7, 10, 11, 12].

A *strong edge p-colouring* of G is a mapping from the set of edges of G to the set of colours $\{1,\dots,p\}$ such that any two edges meeting at a common vertex, or being adjacent to the same edge of G, are assigned different colours. The smallest p for which G admits a strong edge p-colouring is the *strong chromatic index* of G, denoted by $\chi'_s(G)$.

The strong version of incidence colouring is defined in a similar way. A *strong incidence p-colouring* of a graph G is a mapping from the set of incidences of G to a finite set of colours $\{1,\dots,p\}$ such that any two incidences that are adjacent or adjacent to the same incidence receive distinct colours. The smallest p for which G admits a strong incidence p-colouring is the *strong incidence chromatic number*, denoted by $\chi_i^s(G)$.

Our paper is organised as follows. We first give some preliminary results in Section 2. We then study the strong incidence chromatic number of simple graph classes (stars, complete graphs, cycles, wheel graphs and trees) in Section 3, of ladder graphs in Section 4, of square grids in Section 5 and of subclasses of Halin graphs in Section 6. We finally propose some directions for future research in Section 7.

2. Preliminary Results

We list in this section some basic results on the strong incidence chromatic number of various graph classes.

The *square* G^2 of a graph G is the graph defined by $V(G^2)=V(G)$ and $uv\in E(G^2)$ if and only if $\mathrm{dist}_G(u,v)\le 2$. A colouring of G^2 is called a 2-*distance* *colouring* of G and the 2-*distance chromatic number* of G is denoted by $\chi_2(G)$.

For any graph G, the *incidence graph* of G, denoted by I_G, introduced in [1], is the graph whose vertices are the incidences of G, two incidences being joined by an edge whenever they are adjacent. Clearly, every incidence colouring of G is nothing but a proper vertex colouring of I_G, and every strong incidence colouring of G is nothing but a 2-distance colouring of I_G, so $\chi_i(G) = \chi(I_G)$ and $\chi_i^s(G) = \chi_2(I_G)$. Moreover, since every strong incidence colouring is an incidence colouring, we have $\chi_i(G) \leq \chi_i^s(G)$ for every graph G.

For every vertex v in a graph G, we denote by $A^-(v)$ the set of incidences of the form (v, vu), and by $A^+(v)$ the set of incidences of the form (u, uv) (see Figure 1). The incidences in $A^-(v)$ and $A^+(v)$ are sometimes called strong, respectively, weak. We thus have $|A^-(v)| = |A^+(v)| = \deg(v)$ for every vertex v. Every edge uv of G has two incidences (u, uv) and (v, vu). We will say that two incidences are *strongly adjacent* if they are either adjacent or adjacent to the same incidence. The following observation will be useful.

Observation 1. *For every incidence* (v, vu) *in a graph* G *with maximum degree* Δ*, the set of incidences that are strongly adjacent to* (v, vu) *is*

$$\bigcup_{w \in N(v) \setminus u} A^+(w) \cup \bigcup_{w \in N(v)} A^-(w) \cup \bigcup_{w \in N(u) \setminus v} A^-(w),$$

whose cardinality is at most $3\Delta^2 - 2\Delta$.

Indeed, the cardinality of the set of incidences that are strongly adjacent to (v, vu) (see Figure 2) is

$$\sum_{w \in N(v) \setminus u} \deg(w) + \sum_{w \in N(v)} \deg(w) + \sum_{w \in N(u) \setminus v} \deg(w) \leq 2\Delta(\Delta - 1) + \Delta^2.$$

Therefore, any partial strong incidence colouring with this number of colours can be extended to the entire graph.

For a given graph G, we let

$$\sigma(G) = \max_{uv \in E(G)} \left\{ 2\deg_G(v) + \deg_G(u) - 1 \right\}.$$

For every edge uv in $E(G)$, the incidences of the set $A^-(v) \cup A^+(v) \cup A^-(u)$, of cardinality $2\deg_G(v) + \deg_G(u) - 1$, are pairwise strongly adjacent, which means that they must be assigned distinct colours. Therefore, we have the following inequalities.

Proposition 2. *For every graph* G *with maximum degree* Δ, $\sigma(G) \leq \chi_i^s(G) \leq 3\Delta^2 - 2\Delta + 1$.

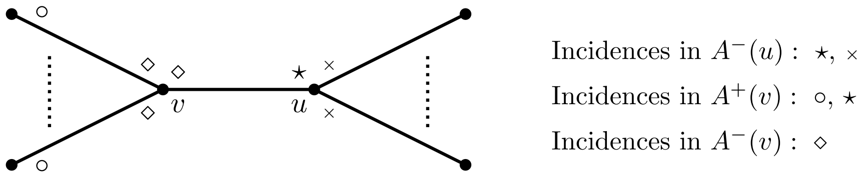


Figure 1. Incidences adjacent to the incidence (v, vu).

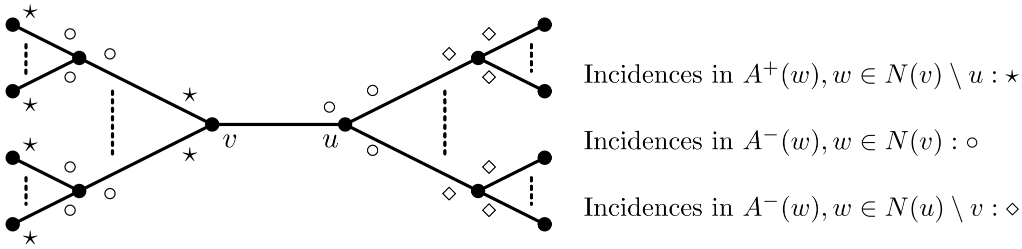


Figure 2. Incidences strongly adjacent to the incidence (v, vu).

In the following proposition we give an upper bound on the strong incidence chromatic number of a graph G as a function of its strong chromatic index.

Proposition 3. *For every graph G, $\chi_i^s(G) \leq 2\chi_s'(G)$.*

Proof. Let λ be a strong edge p-colouring of G. From λ, a strong incidence $2p$-colouring λ' is obtained using the set of $2p$ colours $\{1, 1', \ldots, p, p'\}$ as follows: for every edge $uv \in E(G)$, if $\lambda(uv) = k$, $k \in \{1, \ldots, p\}$, then $\lambda'(u, uv) = k$ and $\lambda'(v, vu) = k'$. Indeed, if $\lambda'(u, uv) = \lambda'(w, wx)$ for two incidences (u, uv) and (w, wx) of G, then $\lambda(uv) = \lambda(wx)$, which implies $\mathrm{dist}_G(uv, wx) \geq 3$, and thus $\mathrm{dist}_{I_G}((u, uv), (w, wx)) \geq 3$. ■

3. Simple Graph Classes

In this section, we determine the strong incidence chromatic number of stars, complete graphs, cycles, trees and wheel graphs.

We denote by S_n, $n \geq 1$, the star of order $n+1$; by K_n, $n \geq 1$, the complete graph of order n and by $K_{m,n}$, $m \geq n \geq 2$, the complete bipartite graph with parts of size m and n. In [2], Brualdi and Massey showed that $\chi_i(S_n) = n+1, \chi_i(K_n) = n$ and $\chi_i(K_{m,n}) = m + 2$, for all $m \geq n \geq 2$. Since all incidences of any graph in these classes of graphs are pairwise strongly adjacent, we have the following proposition.

Proposition 4. 1. *For every* $n \geq 1$, $\chi_i^s(S_n) = 2n$.

2. *For every* $n \geq 2$, $\chi_i^s(K_n) = n(n-1)$.
3. *For every* $m \geq n \geq 2$, $\chi_i^s(K_{m,n}) = 2nm$.

Let C_n, $n \geq 3$, denote the cycle of order n. Since $C_3 = K_3$, the result holds by Proposition 4 for $n = 3$. Suppose now $n \geq 4$ and observe that $I_{C_n} = C_{2n}^2$. Therefore, a strong incidence colouring of the cycle C_n is a 2-distance colouring of C_{2n}^2 that is nothing but a proper colouring of $(C_{2n}^2)^2 = C_{2n}^4$. We thus have the following result.

Proposition 5. *For every integer* $n \geq 3$, $\chi_i^s(C_n) = \chi(C_{2n}^4)$.

By setting $a = 4$, the following theorem gives the value of $\chi(C_{2n}^4)$.

Theorem 6 (Prowse and Woodall [8])**.** *Let* n *and* a *be positive integers such that* $n \geq 2a$ *and* $n = q(a+1)+r$, *with* $q > 0$ *and* $0 \leq r \leq a$. *Then* $\chi(C_n^a) = a+1+\lceil r/q \rceil$.

Using Proposition 5 and Theorem 6, we can infer the value of the strong incidence chromatic number of any cycle.

Theorem 7. *Let* n *be a positive integer such that* $n \geq 3$ *and* $2n = 5q + r$, *with* $q > 0$ *and* $0 \leq r \leq 4$. *Then* $\chi_i^s(C_n) = 5 + \lceil r/q \rceil$.

We now determine the value of $\chi_i^s(W_n)$, where W_n, $n \geq 3$, is the wheel graph of order $n+1$, obtained from C_n by adding a universal vertex. It is easy to observe that $\chi_i(W_n) = n + 1$ for every $n \geq 3$. Indeed, since the square of W_n is the complete graph K_{n+1} and thanks to the relation between the incidence colouring of G and the coloring of the square of G [12], we get

$$n + 1 = \Delta(W_n) + 1 \leq \chi_i(W_n) \leq \chi(W_n^2) = \chi(K_{n+1}) = n + 1.$$

Using Proposition 4 and Theorem 7, we can determine the strong incidence chromatic number of wheel graphs.

Theorem 8. *Let* n *be a positive integer such that* $n \geq 3$ *and* $2n = 5q + r$, *with* $q > 0$ *and* $0 \leq r \leq 4$. *Then* $\chi_i^s(W_n) = 5 + 2n + \lceil r/q \rceil$.

Proof. Let T denote the spanning subgraph of W_n isomorphic to S_n. Since every incidence in T is strongly adjacent to every incidence not in T, we get $\chi_i^s(W_n) = \chi_i^s(C_n) + \chi_i^s(S_n)$ and the result follows from Proposition 4 and Theorem 7. ■

We finally determine the strong incidence chromatic number of trees.

Theorem 9. *If* G *is a tree then* $\chi_i^s(G) = \max_{uv \in E(G)}\{2\deg_G(v) + \deg_G(u) - 1\} = \sigma(G)$.

Proof. By Proposition 2, we have $\chi_i^s(G) \geq \sigma(G)$. The other direction is proved by induction on $|V(G)|$. If G is a star, then $\sigma(G) = 2n$ and the result follows from Proposition 4. We can thus assume that G is not a star, so $|V(G)| \geq 4$. Let u be a vertex of G of degree $p + 1 \geq 2$ having only one neighbour, denoted by u', which is not a leaf, and let $A = \{v_1, \ldots, v_p\}$ be the set of p leaves that are neighbours of u. Let λ be a strong incidence colouring of $G \setminus A$. Now, observe that each incidence of the form (u, uv_i), $1 \leq i \leq p$, has at most

$$2\deg_{G\setminus A}(u') = 2\deg_G(u') \leq \sigma(G) - \deg_G(u) + 1$$

strongly adjacent incidences in $G \setminus A$, and each incidence of the form (v_i, v_iu), $1 \leq i \leq p$, has at most

$$\deg_{G\setminus A}(u') + 1 = \deg_G(u') + 1 \leq \sigma(G) - 2\deg_G(u) + 2$$

strongly adjacent incidences in $G \setminus A$, so λ can be extended to a strong incidence colouring of G, starting by colouring the incidences of the form (u, uv_i), $1 \leq i \leq p$, and then, the incidences of the form (v_i, v_iu), $1 \leq i \leq p$. This concludes the proof. ■

4. Ladder Graphs

The *ladder graph*, denoted by L_h, is obtained from two paths of order h, $h \geq 1$, $P_h = v_1 \cdots v_h$ and $P'_h = v'_1 \cdots v'_h$ by adding the edges $v_iv'_i$, $1 \leq i \leq h$. In the following theorem, we give the value of $\chi_i^s(L_h)$.

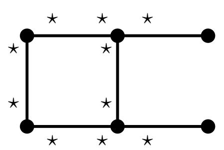


Figure 3. The graph L.

Theorem 10. *For every integer $h \geq 3$, $\chi_i^s(L_h) = 10$.*

Proof. Note that when $h \geq 3$, L_h contains a subgraph isomorphic to the graph L (see Figure 3) which contains 10 pairwise strongly adjacent incidences (marked with a star), implying $\chi_i^s(L_h) \geq 10$ for every $h \geq 3$. To complete the proof, it suffices to give a strong incidence 10-colouring λ of L_h. Such a colouring can be obtained as follows (see Figure 4 for the case $h = 5$).

1. We sequentially colour the incidences of the path P_h using the pattern 123456.
2. We sequentially colour the incidences of the path P'_h using the pattern 456123.

3. For every i, $1 \leq i \leq h$, we set
   - $\lambda(v'_i, v'_i v_i) = 7$ and $\lambda(v_i, v_i v'_i) = 8$, if i is odd,
   - $\lambda(v'_i, v'_i v_i) = 9$ and $\lambda(v_i, v_i v'_i) = 10$) if i is even.

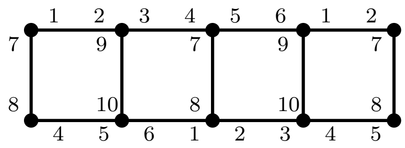


Figure 4. A strong incidence 10-colouring of L_5.

The so-obtained colouring is clearly a strong incidence colouring of L_h. This concludes the proof. ■

5. Square Grids

The square grid $G_{m,n}$ is the graph defined by the set of vertices $V(G_{m,n}) = \{v_{i,j} \mid 1 \leq i \leq m,\ 1 \leq j \leq n\}$ and the set of edges $E(G_{m,n}) = \{(v_{i,j}, v_{i',j'}) \mid |i - i'| + |j - j'| = 1\}$. We denote by P_i the path $v_{i,1} \cdots v_{i,n}$ and by P'_j the path $v_{1,j} \cdots v_{m,j}$. In the following theorem, we give the value of $\chi_i^s(G_{m,n})$.

Theorem 11. *For every integers m and n, $m \geq n \geq 2$, $\chi_i^s(G_{m,n}) = 12$.*

Proof. Note that when $m \geq n \geq 2$, $G_{m,n}$ contains a subgraph isomorphic to the graph R (see Figure 5) which contains 12 pairwise strongly adjacent incidences (marked with a star), implying $\chi_i^s(G_{m,n}) \geq 12$ for every $m \geq n \geq 2$. To complete the proof, it suffices to give a strong incidence 12-colouring of $G_{m,n}$ as follows.

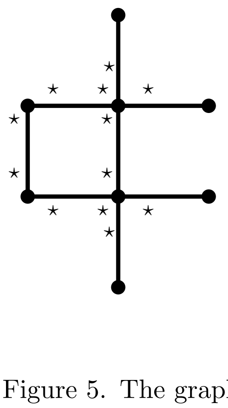


Figure 5. The graph R.

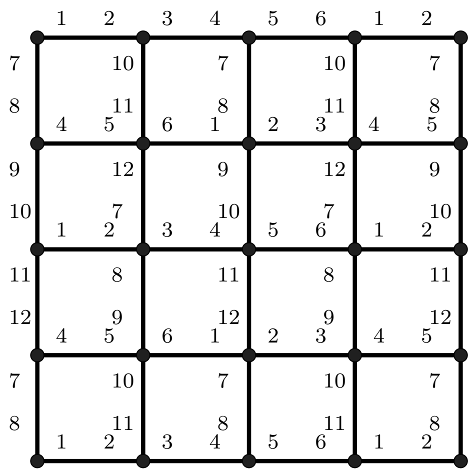


Figure 6. A strong incidence 12-colouring of $G_{5,5}$.

1. For every i, $1 \leq i \leq m$, we sequentially colour the incidences of the path P_i using the pattern 1.2.3.4.5.6 if i is odd, and the pattern 4.5.6.1.2.3 otherwise.

2. For every j, $1 \leq j \leq n$, we sequentially colour the incidences of the path P_j using the pattern 7.8.9.10.11.12 if j is odd, and the pattern 10.11.12.7.8.9 if i is even.

The so-obtained colouring is clearly a strong incidence colouring of $G_{m,n}$. This concludes the proof. ∎

6. Subclasses of Halin Graphs

Recall first that a *Halin graph* H is a planar graph obtained from a tree of order at least 4 with no vertex of degree 2, by adding a cycle connecting all its leaves [5]. We call this cycle the *outer cycle of* H. The subgraph T obtained by deleting all the edges of the outer cycle of H is thus a tree, called the *internal tree of* H.

In this section, we determine the exact value of — or upper bounds on — the strong incidence chromatic number of every Halin graph whose internal tree is either a comb or a double star.

6.1. Halin graphs whose internal tree is a comb

A tree is called a $(3,1)$-tree if the degree of each non-leaf vertex is 3. A caterpillar is a tree T such that, after deleting all its leaves, the remaining graph is a simple path called the *spine of* T. A *comb* is a caterpillar which is also a (3,1)-tree. It is easy to see that every Halin graph whose internal tree is a comb is a cubic

Halin graph. In particular, if the spine has one vertex then this is the complete graph K_4.

For every integer $h \geq 1$, we construct a Halin graph H_h of order $2h+2$ whose internal tree T_h is a comb, using the construction given in [9]. Let $P_h = v_1v_2\cdots v_h$ be the spine of T_h. We denote by ℓ_1 and ℓ'_1 (respectively, ℓ_h and ℓ'_h) the two leaves of v_1 (respectively, v_h), by ℓ_i the unique leaf of v_i, $2 \leq i \leq h-1$, and by C_h the outer cycle of H_h.

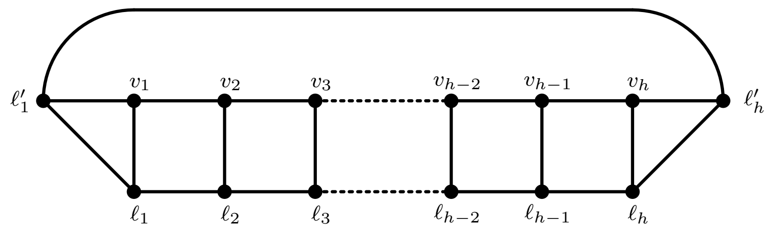


Figure 7. The graph N_h.

Let $\mathcal{H}_h^c$ be the set of all Halin graphs whose internal tree is a comb of order $2h+2$. A Halin graph H_h such that $C_h = \ell'_1\ell_1\ell_2\cdots\ell_h\ell'_h\ell'_1$ is called a *necklace*. We denote by N_h the (unique) necklace of order $2h+2$ (see Figure 7). Observe that $\mathcal{H}_h^c = \{N_h\}$ for every h, $1 \leq h \leq 3$.

It is easy to see that all incidences of N_1 are pairwise strongly adjacent. Therefore, $\chi_i^s(N_1) \geq 12$. For N_2, the incidences of the set

$$A^-(v_1) \cup A^+(v_1) \cup A^-(v_2) \cup \{(\ell'_1, \ell'_1\ell'_2), (\ell'_2, \ell'_2\ell'_1),\ (\ell_1, \ell_1\ell_2), (\ell_2, \ell_2\ell_1)\},$$

of cardinality 12, are pairwise strongly adjacent. Hence, we have $\chi_i^s(N_2) \geq 12$. Also, the cardinality of the set of the incidences of N_3 is 24. Therefore, if we colour the incidences of N_3 with 11 colours, then at least two colours must be repeated at least three times or at least one colour must be repeated at least four times. It is tedious but not difficult to check that this is not possible. Hence, $\chi_i^s(N_3) \geq 12$. Strong incidence 12-colourings of N_h, $1 \leq h \leq 3$, are given in Figure 8.

Suppose now $h \geq 4$. In [9], Shiu, Lam and Tam proved the following theorem.

Theorem 12 (Shiu, Lam and Tam [9])**.** *If* $H \in \mathcal{H}_h^c$, $h \geq 4$, *then* $6 \leq \chi'_s(H) \leq 8$.

By Proposition 3 and Theorem 12, we get $\chi_i^s(H) \leq 16$, for every graph $H \in \mathcal{H}_h^c$, $h \geq 4$. We prove that if H is not a necklace then this bound can be decreased to 14.

Theorem 13. *If* $H \in \mathcal{H}_h^c \setminus \{N_h\}$, $h \geq 4$, *then* $11 \leq \chi_i^s(H) \leq 14$.

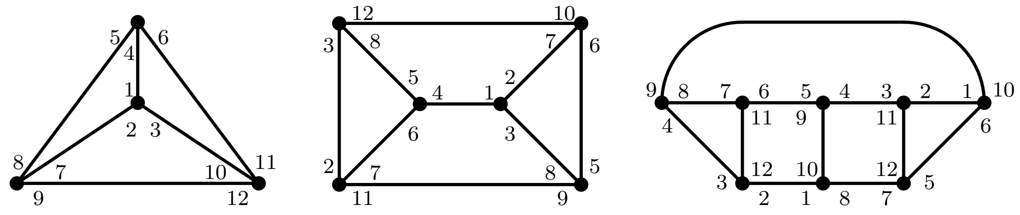


Figure 8. Strong incidence 12-colourings of N_1, N_2 and N_3.

Proof. Let $H \in \mathcal{H}_h^c \backslash \{N_h\}$. By exchanging if necessary the leaves ℓ_1 and ℓ'_1, or ℓ_h and ℓ'_h, we can assume that H has either the form depicted in Figure 9(a) or the form depicted in Figure 9(b), where the edges $v_i\ell_i, 3 \leq i \leq h-2$, may be either upward or downward. In both cases, H contains a subgraph isomorphic to the graph F (see Figure 10) which contains 11 pairwise strongly adjacent incidences (marked with a star), implying $\chi_i^s(H) \geq 11$.

We now construct a strong incidence 14-colouring λ of H assuming that H has either the form depicted in Figure 9(a) or the form depicted in Figure 9(b). Observe that in both cases the incidence $(v_2, v_2\ell_2)$ (respectively, $(v_{h-1}, v_{h-1}l_{h-1})$) is not strongly adjacent with the incidence $(\ell'_1, \ell'_1\ell_1)$ (respectively, $(\ell'_h, \ell'_h\ell_h)$). Such a colouring can be obtained as follows (see Figure 11).

- We colour the incidences of the $(\ell'_1 - \ell'_h)$-path (which contains the path P_h) sequentially, from (ℓ'_1, ℓ'_1v_1) to (ℓ'_h, ℓ'_hv_h) using the pattern 12345.
- For every integer i, $1 \leq i \leq h$, we set $\lambda(v_i, v_i\ell_i) = 6$ if i is odd and $\lambda(v_i, v_i\ell_i) = 7$ otherwise.
- We colour circularly the incidences of the form $(\ell_i, \ell_iv_i), 1 \leq i \leq h$ according to their order in the outer cycle C_h by alternating the colours 8 and 9.
- We now colour the incidences of the outer cycle C_h according to the value of h mod 5.

$h = 5k$, $k \geq 1$ (see Figure 11(a) for the case $h = 5$). We first exchange the colours of the two incidences $(v_2, v_2\ell_2)$ and (v_2, v_2v_1), and the colours of the two incidences $(v_{h-1}, v_{h-1}v_h)$ and $(v_{h-1}, v_{h-1}\ell_{h-1})$. We then set $\lambda(\ell'_1, \ell'_1\ell_1) = \lambda(v_2, v_2\ell_2)$, $\lambda(\ell_1, \ell_1\ell'_1) = \lambda(v_2, v_2v_3)$, $\lambda(\ell'_h, \ell'_h\ell_h) = \lambda(v_{h-1}, v_{h-1}\ell_{h-1})$ and $\lambda(\ell_h, \ell_h\ell'_h) = \lambda(v_{h-1},\ v_{h-1}v_{h-2})$. We finally sequentially colour the uncoloured incidences of C_h, starting from $(\ell_1, \ell_1\ell_2)$, using the pattern 10.11.12.13.14. If $\ell_2\ell'_h \in E(H)$ then we set $\lambda(v_2, v_2\ell_2) = \lambda(\ell'_1, \ell'_1\ell_r)$, $r \neq 1$. If $\ell'_1\ell_{h-1} \in E(H)$ then we set $\lambda(v_{h-1}, v_{h-1}\ell_{h-1}) = \lambda(\ell'_h, \ell'_h\ell_m)$, $m \neq h$.

$h = 5k+1$, $k \geq 1$ (see Figure 11(b) for the case $h = 6$). We first set $\lambda(\ell'_1, \ell'_1\ell_1) = \lambda(v_2, v_2v_3)$. We then sequentially colour the uncoloured incidences of C_h, starting from $(\ell_1, \ell_1\ell'_1)$, using the pattern 10.11.12.13.14.

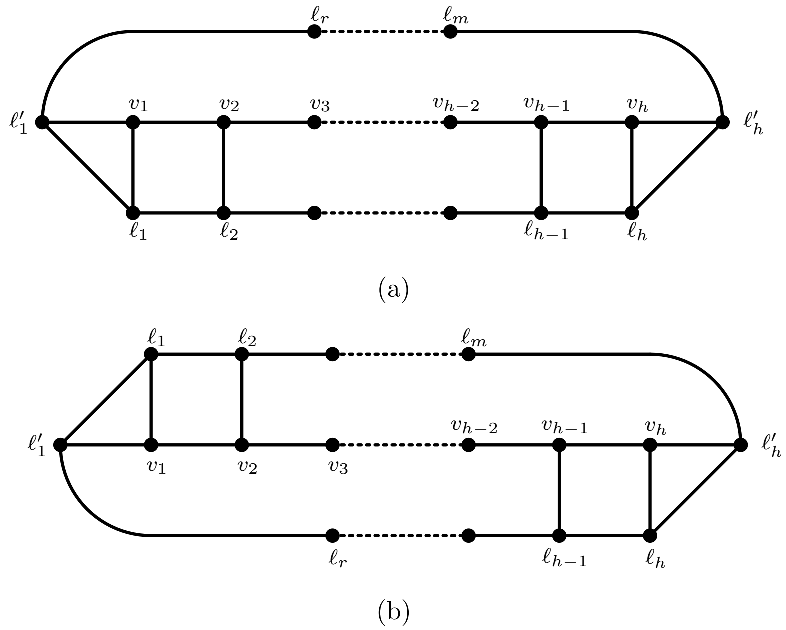


Figure 9. Each graph $H \in \mathcal{H}_h^c \setminus \{N_h\}$ has the form (a) or the form (b).

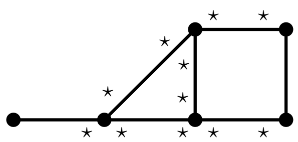


Figure 10. The graph F.

$h = 5k+2$, $k \geq 1$ (see Figure 11(c) for the case $h = 7$). We first exchange the colours of the two incidences $(v_2, v_2\ell_2)$ and (v_2, v_2v_1). We then set $\lambda(\ell'_1, \ell'_1\ell_1) = \lambda(v_2, v_2\ell_2)$, $\lambda(\ell_1, \ell_1\ell'_1) = \lambda(v_2, v_2v_3)$ and $\lambda(\ell'_h, \ell'_h\ell_h) = \lambda(v_{h-1}, v_{h-1}v_{h-2})$. We finally sequentially colour the uncoloured incidences of C_h, starting from $(\ell_1, \ell_1\ell_2)$, using the pattern 10.11.12.13.14.

$h = 5k+3$, $k \geq 1$ (see Figure 11(d) for the case $h = 8$). We sequentially colour the incidences of C_h, starting from $(\ell'_1, \ell'_1\ell_1)$, using the pattern 10.11.12.13.14.

$h = 5k + 4$, $k \geq 0$ (see Figure 11(e) for the case $h = 4$). We first exchange the colours of the two incidences $(v_2, v_2\ell_2)$ and (v_2, v_2v_1). We then set $\lambda(\ell'_1, \ell'_1\ell_1) = \lambda(v_2, v_2\ell_2)$ and $\lambda(\ell_1, \ell_1\ell'_1) = \lambda(v_2, v_2v_3)$. We finally sequentially colour the uncoloured incidences of C_h, starting from $(\ell_1, \ell_1\ell_2)$, using the pattern 10.11.12.13.14.

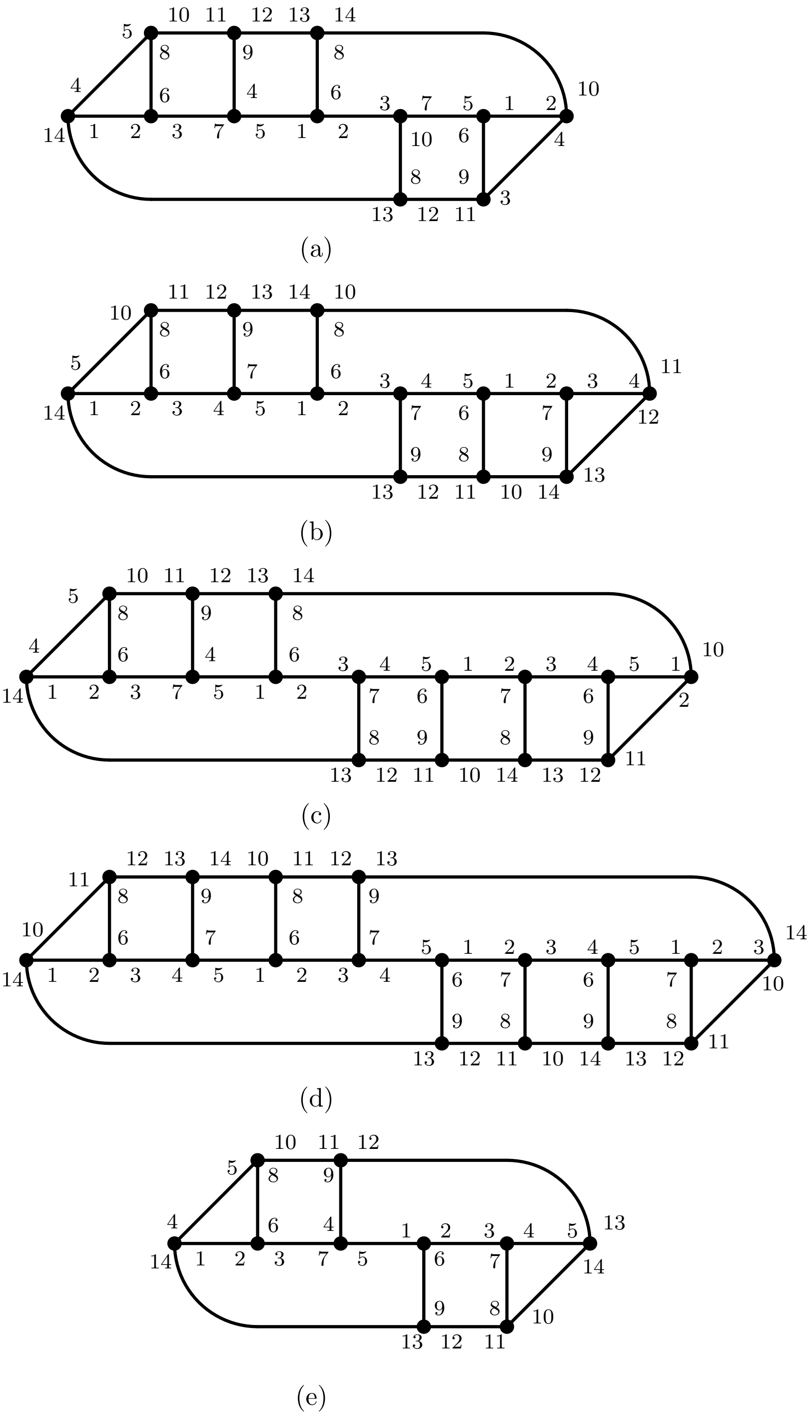


Figure 11. Strong incidence 14-colourings of some graphs in $\mathcal{H}_h^c \setminus \{N_h\}$, $4 \leq h \leq 8$.

In each case, the so-obtained colouring is clearly a strong incidence colouring of $G \in \mathcal{H}_h^c$. This completes the proof. ■

We now determine the value of the strong incidence chromatic number of necklaces.

Theorem 14. *For every necklace N_h, $h \geq 1$, we have*

$$\chi_i^s(N_h) = \begin{cases} 12 & \textit{if } h = 1, 2, 3, 5, \\ 11 & \textit{otherwise.} \end{cases}$$

Proof. Similarly as in the proof of Theorem 13, N_h contains a subgraph isomorphic to the graph F (see Figure 10), which implies $\chi_i^s(N_h) \geq 11$ for every $h \geq 1$. The values of $\chi_i^s(N_h)$, $1 \leq h \leq 3$, were given in the beginning of the subsection.

If $h = 5$, the cardinality of the set of the incidences of N_5 is 36. Therefore, if we colour the graph with 11 colours, then at least one color is repeated more that four times, or at least three colors are repeated four times. We will prove that at most two colors can be repeated four times, and no color can be repeated more than four times, which will imply $\chi_i^s(N_5) \geq 12$.

The set of incidences of N_5 can be partitioned into four sets, the set of incidences marked with a star, the set of incidences marked with a diamond, the set of incidences marked with a circle and the set of incidences marked with a cross (see Figure 12(a)). Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of any incidence of N_5 can be repeated at most four times. Moreover, any colour repeated four times must be used exactly once in each of these sets. We now prove that among the colours of the incidences marked with a star in Figure 12(a), only two colours can be repeated four times.

• The incidence $(\ell'_1, \ell'_1\ell'_5)$.
The set of the incidences that are not strongly adjacent with this incidence can be partitioned into the two sets $\{(v_2, v_2\ell_2), (v_2, v_2v_3), (\ell_2, \ell_2v_2), (\ell_2, \ell_2\ell_3), (v_3, v_3v_2), (\ell_3, \ell_2\ell_3)\}$ and $\{(v_3, v_3\ell_3), (v_3, v_3v_4), (\ell_3, \ell_3v_3), (\ell_3, \ell_3\ell_4), (v_4, v_4v_3), (v_4, v_4\ell_4), (v_4, v_4v_5), (\ell_4, \ell_4\ell_3), (\ell_4, \ell_4v_4), (\ell_4, \ell_4\ell_5)\}$. Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of the incidence $(\ell'_1, \ell'_1\ell'_5)$ can be repeated only two more times.

• The incidence (ℓ'_1, ℓ'_1v_1).
The set of the incidences that are not strongly adjacent with this incidence can be partitioned into the two sets $\{(\ell_2, \ell_2v_2), (\ell_2, \ell_2\ell_3), (v_3, v_3v_2), (v_3, v_3\ell_3), (v_3, v_3v_4), (\ell_3, \ell_3\ell_2), (\ell_3, \ell_3v_3), (\ell_3, \ell_3\ell_4)\}$ and $\{(v_4, v_4v_3), (v_4, v_4\ell_4), (v_4, v_4v_5), (\ell_4, \ell_4\ell_3), (\ell_4, \ell_4v_4), (\ell_4, \ell_4\ell_5), (v_5, v_5v_4), (v_5, v_5\ell_5), (\ell_5, \ell_5\ell_4), (\ell_5, \ell_5v_5),\}$. Since, in each of these sets, all incidences are pairwise strongly adjacent, the colour of the incidence (ℓ'_1, ℓ'_1v_1) can be repeated only two more times.

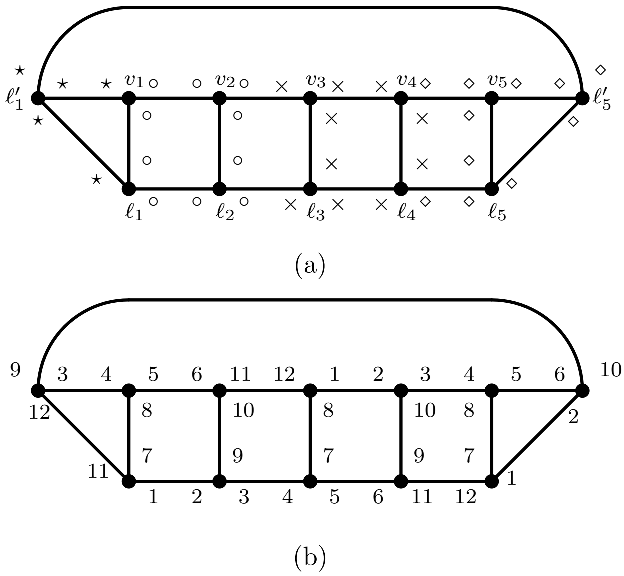


Figure 12. The necklace N_5 (for the proof of Theorem 14).

• By symmetry, the case of the incidence $(\ell_1', \ell_1'\ell_1)$ is similar to the case of the incidence $(\ell_1', \ell_1'v_1)$.

Based on the above, only the colours of the two incidences $(\ell_1, \ell_1\ell_1')$ and $(v_1, v_1\ell_1')$ can be repeated four times. Hence, $\chi_i^s(N_5) \geq 12$. A strong incidence 12-colouring of N_5 is given in Figure 12(b), so $\chi_i^s(N_5) = 12$.

Finally, if $h = 4$ or $h \geq 6$ then it suffices to construct a strong incidence 11-colouring of N_h. Such a colouring can be obtained as follows (see Figure 13). We consider two cases, depending on the parity of h.

• h is even.

We first colour the subgraph S_h induced by the set of vertices $\{v_1, \ldots, v_h, \ell_1, \ldots, \ell_h\}$ (which is isomorphic to the ladder graph L_h), as in the proof of Theorem 10.

We then modify the colouring of the subgraph S_h and we complete the colouring of N_h according to the value of $h \bmod 6$:

∗ $h = 6k$, $k \geq 1$ (see Figure 13(a) for the case $h = 6$). We set $\lambda(\ell_1, \ell_1\ell_1') = 6$, $\lambda(\ell_1', \ell_1'\ell_1) = 10$, $\lambda(\ell_1', \ell_1'v_1) = 9$, $\lambda(v_1, v_1\ell_1') = 11$, $\lambda(\ell_1', \ell_1'\ell_h') = 3$, $\lambda(\ell_h', \ell_h'\ell_1') = 2$, $\lambda(\ell_h', \ell_h'v_h) = 7$, $\lambda(\ell_h', \ell_h'\ell_h) = 8$, $\lambda(v_h, v_h\ell_h') = 11$, $\lambda(\ell_h, \ell_h\ell_h') = 5$.

∗ $h = 6k+2$, $k \geq 1$ (see Figure 13(b) for the case $h = 8$). We set $\lambda(\ell_1, \ell_1\ell_2) = 11$, $\lambda(\ell_1, \ell_1\ell_1') = 6$, $\lambda(\ell_1', \ell_1'\ell_1) = 10$, $\lambda(\ell_1', \ell_1'v_1) = 9$, $\lambda(v_1, v_1\ell_1') = 3$, $\lambda(\ell_1', \ell_1'\ell_h') = 1$, $\lambda(\ell_h', \ell_h'\ell_1') = 2$, $\lambda(\ell_h', \ell_h'v_h) = 7$, $\lambda(\ell_h', \ell_h'\ell_h) = 8$, $\lambda(v_h, v_h\ell_h') = 6$, $\lambda(\ell_h, \ell_h\ell_h') = 3$.

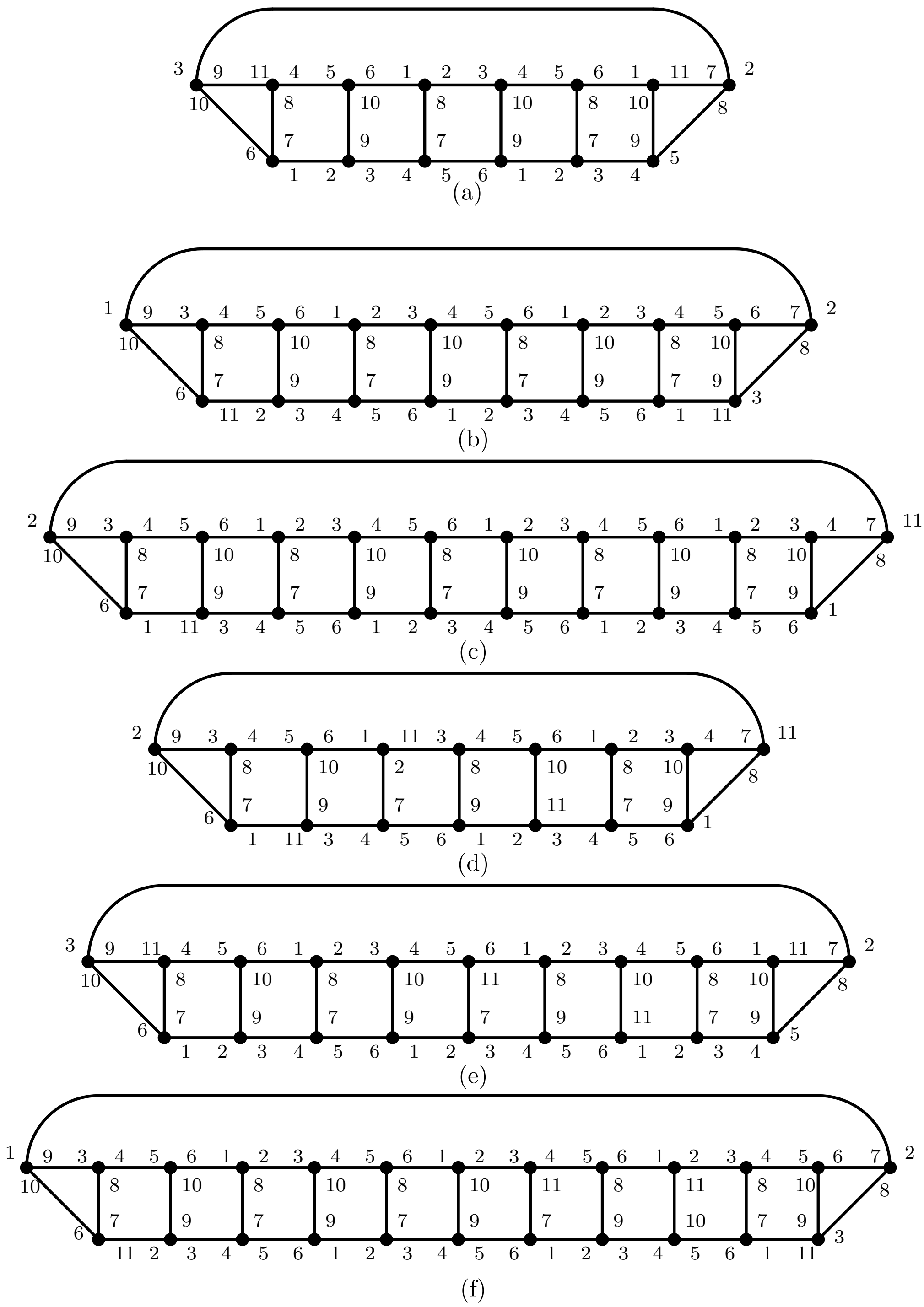


Figure 13. Strong incidence 11-colourings of N_h, $6 \le h \le 11$.

$*$ $h = 6k+4$, $k \geq 0$ (see Figure 13(c) for the case $h = 10$). We set $\lambda(\ell_2, \ell_2\ell_1) = 11$, $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1 v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 3$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 2$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 11$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 4$, $\lambda(\ell_h, \ell_h\ell'_h) = 1$.

• h is odd.

We first colour the subgraph S_h induced by the vertices $\{v_1, \ldots, v_h, \ell_1, \ldots, \ell_h\}$ (which is isomorphic to the ladder graph L_h), as in the proof of Theorem 10, and we make the following modifications.

We set $\lambda(v_{h-4}, v_{h-4}\ell_{h-4}) = 11$, $\lambda(v_{h-3}, v_{h-3}\ell_{h-3}) = 8$, $\lambda(v_{h-1}, v_{h-1}\ell_{h-1}) = 8$, $\lambda(\ell_{h-1}, \ell_{h-1}v_{h-1}) = 7$, $\lambda(v_h, v_h\ell_h) = 10$, $\lambda(\ell_h, \ell_h v_h) = 9$.

We modify the colouring of the subgraph S_h and we complete the colouring of N_h according to the value of $h \mod 6$:

$*$ $h = 6k+1$, $k \geq 1$ (see Figure 13(d) for the case $h = 7$). We set $\lambda(\ell_2, \ell_2\ell_1) = 11$, $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1 v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 3$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 2$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 11$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 4$, $\lambda(\ell_h, \ell_h\ell'_h) = 1$, $\lambda(v_{h-2}, v_{h-2}\ell_{h-2}) = 10$, $\lambda(\ell_{h-2}, \ell_{h-2}v_{h-2}) = 11$. if $h = 7$ the we exchange the colours of the incidences (v_3, v_3v_4) and $(v_3, v_3\ell_3)$.

$*$ $h = 6k+3$, $k \geq 1$ (see Figure 13(e) for the case $h = 9$). We set $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1 v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 11$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 3$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 2$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 11$, $\lambda(\ell_h, \ell_h\ell'_h) = 5$, $\lambda(v_{h-2}, v_{h-2}\ell_{h-2}) = 10$, $\lambda(\ell_{h-2}, \ell_{h-2}v_{h-2}) = 11$.

$*$ $h = 6k+5$, $k \geq 1$ (see Figure 13(f) for the case $h = 11$). We set $\lambda(\ell_1, \ell_1\ell_2) = 11$, $\lambda(\ell_1, \ell_1\ell'_1) = 6$, $\lambda(\ell'_1, \ell'_1\ell_1) = 10$, $\lambda(\ell'_1, \ell'_1 v_1) = 9$, $\lambda(v_1, v_1\ell'_1) = 3$, $\lambda(\ell'_1, \ell'_1\ell'_h) = 1$, $\lambda(\ell'_h, \ell'_h\ell'_1) = 2$, $\lambda(\ell'_h, \ell'_h v_h) = 7$, $\lambda(\ell'_h, \ell'_h\ell_h) = 8$, $\lambda(v_h, v_h\ell'_h) = 6$, $\lambda(\ell_h, \ell_h\ell'_h) = 3$, $\lambda(v_{h-2}, v_{h-2}\ell_{h-2}) = 11$, $\lambda(\ell_{h-2}, \ell_{h-2}v_{h-2}) = 10$.

In each case, the so-obtained colouring is clearly a strong incidence colouring of N_h. This completes the proof. ■

6.2. Halin graphs whose internal tree is a double star

The *double star*, denoted by $S_{m,n}$, $m \geq n \geq 2$, is the graph obtained from the stars S_m and S_n by adding an edge joining the central vertex v of S_m to the central vertex u of S_n. The Halin graph $HD_{m,n}$ (see Figure 14) is the Halin graph whose internal tree is the double star $S_{m,n}$ and whose outer cycle is $u_1 \cdots u_n v_m \cdots v_1 u_1$. We denote by P the path $v_1 \cdots v_m$ and by P' the path $u_1 \cdots u_n$.

It is easy to see that for every graph $HD_{m,n}$, $m \geq n \geq 2$, the incidences in the set

$$A^-(v) \cup A^+(v) \cup A^-(u) \cup \{(v_1, v_1u_1)\},$$

of cardinality $2\deg(v) + \deg(u) = \sigma(HD_{m,n}) + 1$, are pairwise strongly adjacent. Therefore, we have the following inequality.

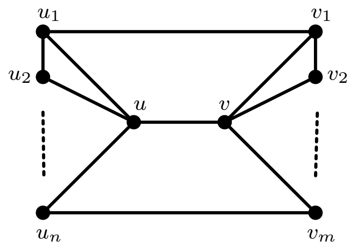


Figure 14. The Halin graph $HD_{m,n}$.

Proposition 15. *For every two integers m and n, $m \geq n \geq 2$, $\chi_i^s(HD_{m,n}) \geq 2m + n + 3 = \sigma(HD_{m,n}) + 1$.*

We first define a partial colouring λ of $HD_{m,n}$, for every $m \geq n \geq 2$, as follows.

- The incidences $(v, vu), (v, vv_1), (v, vv_2), \ldots, (v, vv_m)$ are coloured with the colours $1, 2, 3, \ldots, m + 1$, respectively.
- The incidences $(u, uv), (u, uu_1), (u, uu_2), \ldots, (u, uu_n)$ are coloured with the colours $m + 2, m + 3, m + 4, \ldots, m + n + 2$, respectively.
- The incidences $(v_1, v_1v), (v_2, v_2v), \ldots, (v_m, v_mv)$ are coloured with the colours $m + n + 3, m + n + 4, \ldots, 2m + n + 2$, respectively.
- The incidence (v_1, v_1u_1) is coloured with the colour $2m + n + 3$.

In the next lemmas, we will extend λ to a colouring of $HD_{m,n}$, according to the values of m and n.

Lemma 16. *For every integer $m \geq 2$,*

$$\chi_i^s(HD_{m,2}) = \begin{cases} \sigma(HD_{m,2}) + 4 & \textit{if } m = 2, \\ \sigma(HD_{m,2}) + 3 & \textit{otherwise.} \end{cases}$$

Proof. It is easy to see that if $m = 2$ then the incidences of the set

$$A^-(v) \cup A^+(v) \cup A^-(u) \cup \{(v_1, v_1u_1), (v_2, v_2u_2), (u_1, u_1v_1), (u_2, u_2v_2)\},$$

of cardinality 12, are pairwise strongly adjacent. Therefore, $\chi_i^s(HD_{2,2}) \geq 12$. If $m = 3$, then each colour of the set $\{1, 2, 3, 4, 5, 8, 9, 10, 11\}$ of cardinality 9 is forbidden on the incidences of P, due to the partial colouring λ. To colour the four incidences of the path P, we can use the two colours 6 and 7, and we have to use two additional colours 12 and 13. Therefore, $\chi_i^s(HD_{3,2}) \geq 13$. A strong incidence 12-colouring of $HD_{2,2}$ and a strong incidence 13-colouring of $HD_{3,2}$ are given in Figure 15. Suppose now $m \geq 4$. Observe that each colour of the set $\{1, \ldots, m+1, m+2, m+5, \ldots, 2m+4\}$ of cardinality $2m+2$ is forbidden on the incidences of P, due to the partial colouring λ. Therefore, $\chi_i^s(HD_{m,2}) \geq 2m+7 = \sigma(HD_{m,2})+3$. We now give a strong incidence $2m+7$-colouring by extending λ to a colouring of $HD_{m,2}$. To colour the incidences of the path P, we can use the three colours $m+3, m+4$ and $2m+5$, and we have to use two additional colours $2m+6$ and $2m+7$.

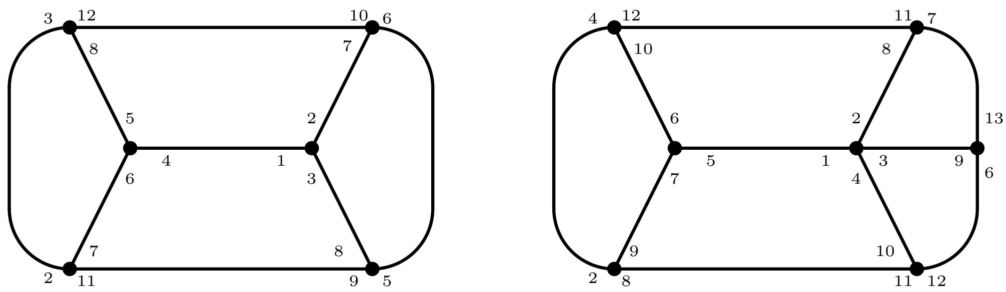


Figure 15. Strong incidence colourings of $HD_{2,2}$ and $HD_{3,2}$.

• We will sequentially colour the path P, starting with the incidence (v_1, v_1v_2), according to the value of $m \mod 5$, as follows.

$m = 5k$, $k \geq 1$ (see Figure 16(a) for the case $m = 5$). We use the pattern $(2m+7)(m+4)(m+3)(2m+6)(2m+5)$.

$m = 5k+1$, $k \geq 1$ (see Figure 16(b) for the case $m = 6$). We use the pattern $(2m+6)(m+4)(m+3)(2m+7)(2m+5)$.

$m = 5k+2$, $k \geq 1$ (see Figure 16(c) for the case $m = 7$). We use the pattern $(m+4)(m+3)(2m+6)(2m+7)(2m+5)$.

$m = 5k+3$, $k \geq 1$ (see Figure 16(d) for the case $m = 8$). We use the pattern $(2m+7)(m+4)(m+3)(2m+6)(2m+5)$.

$m = 5k+4$, $k \geq 1$ (see Figure 16(e) for the case $m = 9$). We use the pattern $(2m+7)(2m+6)(m+3)(m+4)(2m+5)$.

• We then set $\lambda(v_m, v_mu_n) = 2m+5$ if $m = 5k+3$, and $\lambda(v_m, v_mu_n) = 2m+6$ otherwise.

• We finally complete the colouring of $HD_{m,2}$ by assigning the colours $3, 4, 2m+4, m+5, 2m+3$ and $m+6$ to the incidences $(u_1, u_1u_2), (u_2, u_2u_1), (u_1, u_1u)$, $(u_2, u_2u), (u_1, u_1v_1)$ and (u_2, u_2v_m), respectively.

In each case, the so-obtained colouring is clearly a strong incidence colouring of $HD_{m,2}$. This completes the proof. ∎

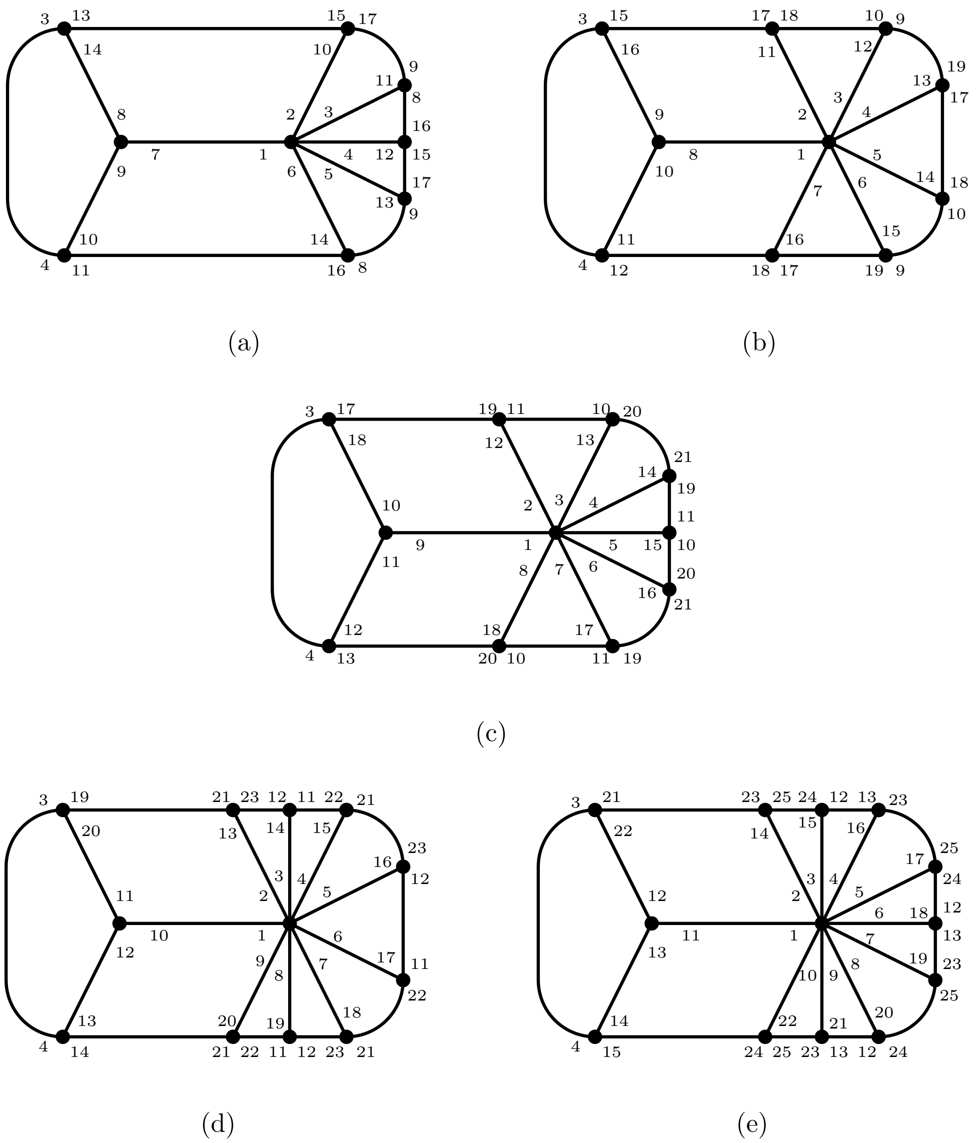


Figure 16. Strong incidence colourings of $HD_{m,2}$, $5 \leq m \leq 9$.

Lemma 17. *For every integer $m \geq 3$,*

$$\chi_i^s(HD_{m,3}) = \begin{cases} \sigma(HD_{m,3}) + 3 & \text{if } m = 3 \text{ or } m = 4, \\ \sigma(HD_{m,3}) + 2 & \text{otherwise.} \end{cases}$$

Proof. If $m = 3$ then we consider two cases.

• If $\lambda(v_3, v_3u_3) = 12$ then each colour of the set $\{1, 2, 3, 4, 5, 9, 10, 11, 12\}$ is forbidden on the incidences of the set $\{(u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_1, u_1v_1), (u_3, u_3v_3)\}$, because these incidences are strongly adjacent to all the incidences already coloured. We now consider two subcases.

If we colour the incidences of the set $\{(u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_1, u_1v_1), (u_3, u_3v_3)\}$ with five distinct colours then we must use two additional colours 13 and 14. Therefore, $\chi_i^s(HD_{3,3}) \geq 14$.

If we colour the incidences of the set $\{(u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_1, u_1v_1), (u_3, u_3v_3)\}$ with at most four colours then we have $\lambda(u_1, u_1v_1) = (u_3, u_3v_3) = 13$, because only incidences (u_1, u_1v_1) and (u_3, u_3v_3) are not strongly adjacent. Thus each colour of the set $\{1, 2, 3, 4, 5, 9, 10, 11, 12, 13\}$ of cardinality 10 is forbidden on the incidences of P, due to the partial colouring λ. To colour the four incidences of the path P we can use the three colours $6, 7$ and 8, and we have to use an additional colour 14. Therefore, $\chi_i^s(HD_{3,3}) \geq 14$.

• If $\lambda(v_3, v_3u_3) = 13$ then each colour of the set $\{1, 2, 3, 4, 5, 9, 10, 11, 12, 13\}$ of cardinality 10 is forbidden on the incidences of P, due to the partial colouring λ. To colour the four incidences of the path P we can use the three colours $6, 7$ and 8, and we have to use an additional colour 14. Therefore, $\chi_i^s(HD_{3,3}) \geq 14$.

If $m = 4$ then we again consider two cases.

• If $\lambda(v_4, v_4u_3) = 14$ then each colour of the set $\{1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 14\}$ of cardinality 11 is forbidden on the incidences of P, due to the partial colouring λ. To colour the incidences of the path P we can use the three colours $7, 8$ and 9, and we have to use two additional colours 15 and 16. Therefore, $\chi_i^s(HD_{4,3}) \geq 16$.

• If $\lambda(v_4, v_4u_3) = 15$ then we consider two subcases.

If we colour the incidences of the set $\{(u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_1, u_1v_1), (u_3, u_3v_4)\}$ without using an additional colour then we have $\lambda(u_1, u_1v_1) = 15$ or $\lambda(u_1, u_1u) = 15$, because the incidences (u_2, u_2u), (u_3, u_3u) and (u_3, u_3v_4) are strongly adjacent with (v_4, v_4u_3). In both cases, each colour of the set $\{1, 2, 3, 4, 5, 6, 10, 11, 12, 13, 15\}$ of cardinality 11 is forbidden on the incidences of P, due to the partial colouring λ. To colour the incidences of the path P we can use the four colours $7, 8, 9$ and 14, and we have to use an additional colour 16. Therefore, $\chi_i^s(HD_{4,3}) \geq 16$.

If we colour the incidences of the set $\{(u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_1, u_1v_1), (u_3, u_3v_4)\}$ using an additional colour then $\chi_i^s(HD_{4,3}) \geq 16$.

A strong incidence 14-colouring of $HD_{3,3}$ and a strong incidence 16-colouring of $HD_{4,3}$ are given in Figure 17. Suppose now $m \geq 5$. Observe that each colour of the set $\{1, \ldots, m+1, m+2, m+6, \ldots, 2m+5\}$ of cardinality $2m+2$ is forbidden on the incidences of P, due to the partial colouring λ. Therefore, $\chi_i^s(HD_{m,3}) \geq 2m+7 = \sigma(HD_{m,3}) + 2$. We now give a strong incidence $2m+7$-colouring by extending λ to a colouring of $HD_{m,3}$. To colour the incidences of

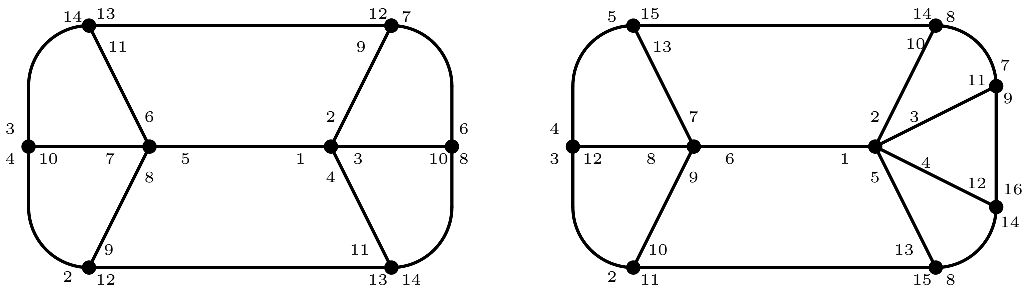


Figure 17. Strong incidence colourings of $HD_{3,3}$ and $HD_{4,3}$.

the path P, we can use the four colours $m+3, m+4, m+5$ and $2m+6$, and we have to use an additional colour $2m+7$.

• We will sequentially colour the incidences of path P, starting from the incidence (v_1, v_1v_2), according to the value of $m \mod 5$, as follows.

$m = 5k$, $k \geq 1$ (see Figure 18(a) for the case $m = 5$). We use the pattern $(m+4)(m+5)(m+3)(2m+7)(2m+6)$.

$m = 5k+1$, $k \geq 1$ (see Figure 18(b) for the case $m = 6$). We use the pattern $(2m+7)(m+3)(m+4)(m+5)(2m+6)$.

$m = 5k+2$, $k \geq 1$ (see Figure 18(c) for the case $m = 7$). We use the pattern $(m+4)(m+3)(2m+7)(m+5)(2m+6)$.

$m = 5k+3$, $k \geq 1$ (see Figure 18(d) for the case $m = 8$). We use the pattern $(2m+7)(m+5)(m+3)(m+4)(2m+6)$.

$m = 5k+4$, $k \geq 1$ (see Figure 18(e) for the case $m = 9$). We use the pattern $(m+4)(2m+7)(m+3)(m+5)(2m+6)$.

• We then set $\lambda(v_m, v_mu_n) = 2m+6$ if $m = 5k+3$, and $\lambda(v_m, v_mu_n) = 2m+7$ otherwise.

• We finally complete the colouring of $HD_{m,3}$ by assigning the colours $5, 4, 3, 2, 2m+5, 2m+4, m+6, m+8$ and $m+7$ to the incidences $(u_1, u_1u_2), (u_2, u_2u_1), (u_2, u_2u_3), (u_3, u_3u_2), (u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_1, u_1v_1)$ and (u_3, u_3v_m), respectively.

In each case, the so-obtained colouring is clearly a strong incidence colouring of $HD_{m,3}$. This completes the proof. ■

Lemma 18. *For every integer $m \geq 4$,*

$$\chi_i^s(HD_{m,4}) = \begin{cases} \sigma(HD_{m,4}) + 1 & \text{if } m \equiv 3 \pmod 5, \\ \sigma(HD_{m,4}) + 2 & \text{otherwise.} \end{cases}$$

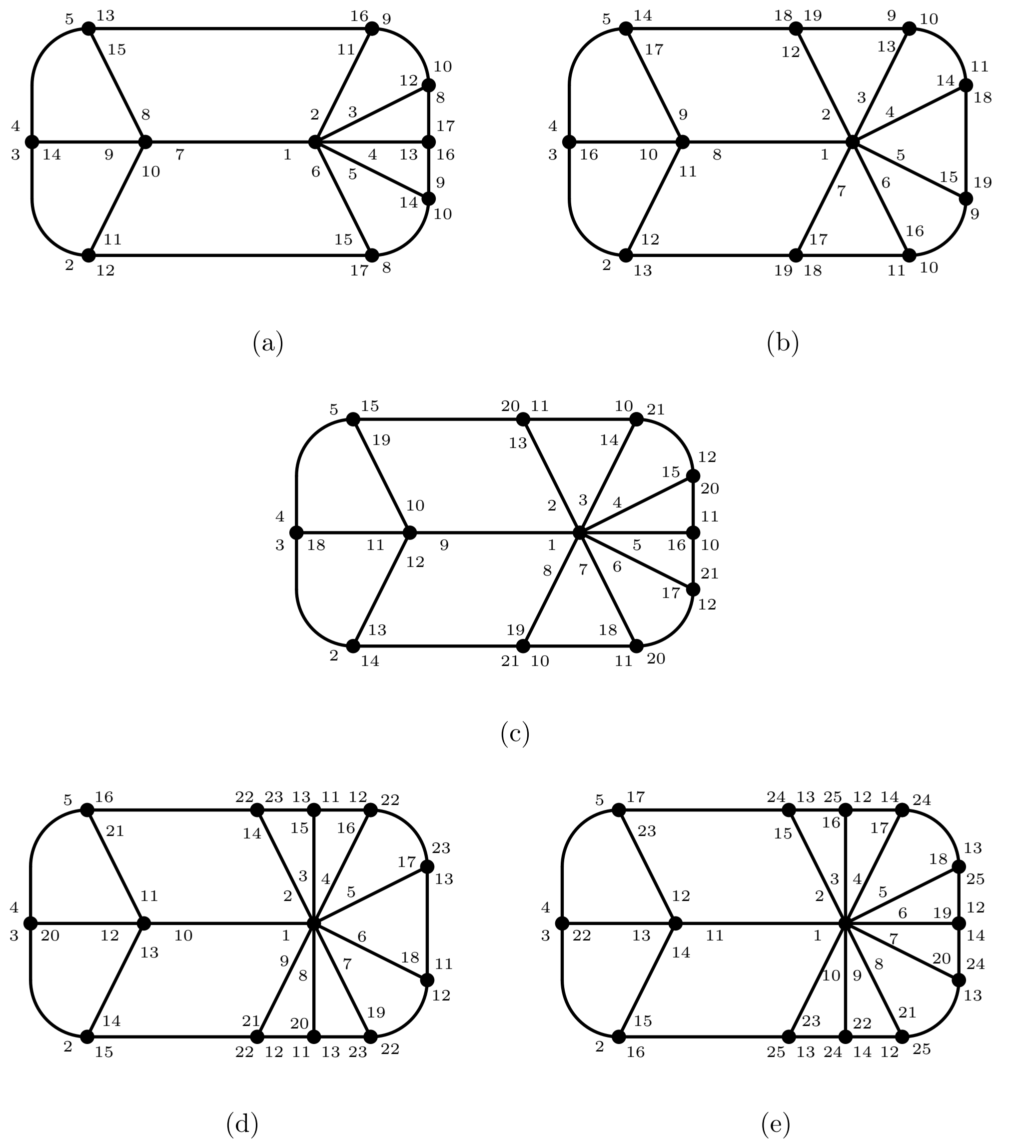


Figure 18. Strong incidence colourings of $HD_{m,3}$, $5 \leq m \leq 9$.

Proof. If $m = 4$ then we consider two cases.

• If $\lambda(v_4, v_4u_4) = 15$ then each colour of the set $\{1, 2, 3, 4, 5, 6, 11, 12, 13, 14, 15\}$ of cardinality 11 is forbidden on the incidences of P, due to the partial colouring λ. To colour the four incidences of the path P we can use the four colours $7, 8, 9$ and 10, and we have to use an additional colour 16. Therefore, $\chi_i^s(HD_{4,4}) \geq 16$.

• If $\lambda(v_4, v_4u_4) = 16$ then $\chi_i^s(HD_{4,4}) \geq 16$.

If $m = 5$ then we again consider two cases.

• If $\lambda(v_5, v_5u_4) = 17$ then each colour of the set $\{1, 2, 3, 4, 5, 6, 7, 12, 13, 14, 15, 16, 17\}$ of cardinality 13 is forbidden on the incidences of P, due to the partial colouring λ. To colour the four incidences of the path P we can use use the four colours $8, 9, 10$ and 11, and we have to use an additional colour 18. Therefore, $\chi_i^s(HD_{5,4}) \geq 18$.

• If $\lambda(v_5, v_5u_4) = 18$ then $\chi_i^s(HD_{5,4}) \geq 18$.

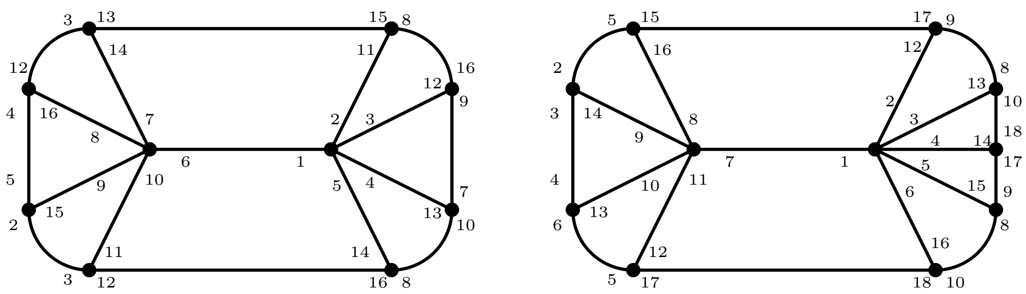
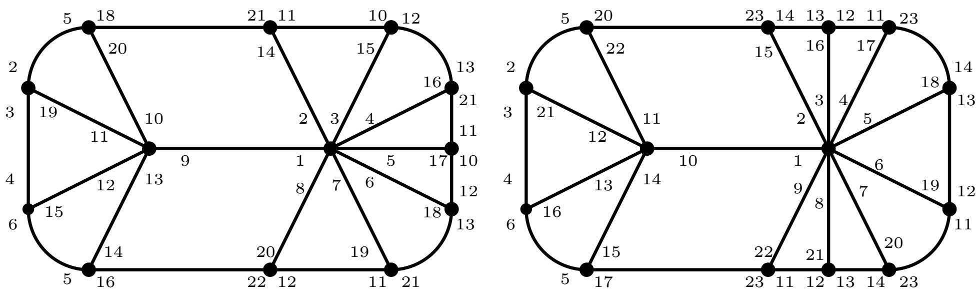


Figure 19. Strong incidence colourings of $HD_{4,4}$ and $HD_{5,4}$.

A strong incidence 16-colouring of $HD_{4,4}$ and a strong incidence 18-colouring of $HD_{5,4}$ are given in Figure 19. Suppose now $m \geq 5$. Observe that each colour of the set $\{1, \ldots, m+1, m+2, m+7, \ldots, 2m+6\}$ of cardinality $2m+2$ is forbidden on the incidences of P and the incidence (v_m, v_mu_n), due to the partial colouring λ. Since the colouring of the path P requires at least five colours, we have $\chi_i^s(HD_{m,4}) \geq 2m+7 = \sigma(HD_{m,4})+1$. We now give a strong incidence $(2m+7)$-colouring by extending λ to a colouring of $HD_{m,4}$. Since the incidence (v_m, v_mu_n) is strongly adjacent to all the incidences already coloured except the incidence (v_m, v_mu_n), we have $\lambda(v_m, v_mu_n) \geq 2m + 7$, if $\lambda(v_m, v_mu_n) = 2m + 7$, then we can colour the path P with the colours $m + 3, m + 4, m + 5, m + 6$ and $2m + 7$ if and only if $m \equiv 3 \pmod 5$. We will sequentially colour the incidences of the path P, starting from the incidence (v_1, v_1v_2), as well as the incidence (v_m, v_mu_n), according to the value of m mod 5, as follows.

• $m \neq 5k + 3$ (see Figure 20(a) for the case $m = 7$). We use the pattern $(m+4)(m+3)(m+5)(m+6)(2m+7)$ for P and we set $\lambda(v_m, v_mu_n) = 2m+8$.

• $m = 5k+3$, $k \geq 1$ (see Figure 20(b) for the case $m = 8$). We use the pattern $(m+6)(m+5)(m+4)(m+3)(2m+7)$ for P and we set $\lambda(v_m, v_mu_n) = 2m+7$.



(a) (b)

Figure 20. Strong incidence colourings of $HD_{7,4}$ and $HD_{8,4}$.

We finally colour the remaining incidences of $HD_{m,4}$ by assigning the colours $5, 2, 3, 4, 6, 5, 2m+6, 2m+5, m+8, m+7, 2m+4$ and $m+9$ to the incidences (u_1, u_1u_2), (u_2, u_2u_1), (u_2, u_2u_3), (u_3, u_3u_2), (u_3, u_3u_4), (u_4, u_4u_3), (u_1, u_1u), (u_2, u_2u), (u_3, u_3u), (u_4, u_4u), (u_1, u_1v_1) and (u_4, u_4v_m), respectively. In each case, the so-obtained colouring is clearly a strong incidence colouring of $HD_{m,4}$. This completes the proof. ■

Lemma 19. *For every integer* $m \geq 5$, $\chi_i^s(HD_{m,5}) = \sigma(HD_{m,5}) + 1$.

Proof. By Proposition 15, $\chi_i^s(HD_{m,5}) \geq 2m + 8 = \sigma(HD_{m,5}) + 1$. To complete the proof, we give a strong incidence $(2m+8)$-colouring by extending λ to a colouring of $HD_{m,5}$ as follows.

• We first sequentially colour the incidences of the path P, starting from the incidence (v_1, v_1v_2), according to the value of $m \mod 5$ as follows.

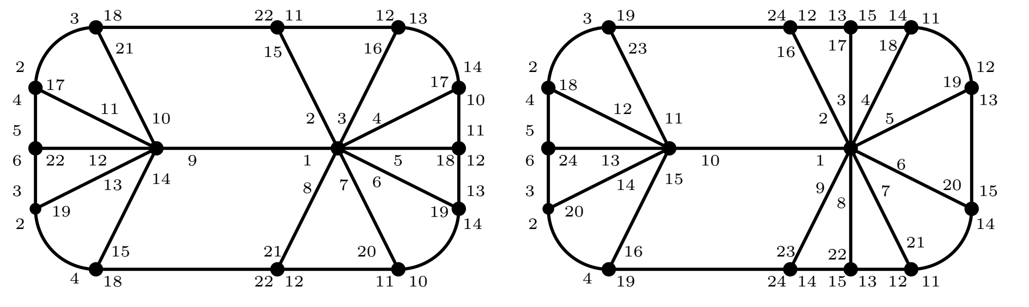
$m \neq 5k+3$, $k \geq 1$ (see Figure 21(a) for the case $m = 7$). We use the pattern $(m+4)(m+5)(m+6)(m+7)(m+3)$.

$m = 5k+3$, $k \geq 1$ (see Figure 21(b) for the case $m = 8$). We use the pattern $(m+4)(m+5)(m+7)(m+6)(m+3)$.

• We then sequentially colour the incidences of the path P', starting from the incidence (u_1, u_1u_2), using the pattern 32456.

• We finally colour the remaining incidences of $HD_{m,5}$ by assigning the colours $2m+7, m+10, 2m+8, m+12, m+8, m+11, m+11$ and $2m+8$ to the incidences $(u_1, u_1u), (u_2, u_2u)$, $(u_3, u_3u), (u_4, u_4u)$, (u_5, u_5u) $(u_1, u_1v_1), (u_5, u_5v_m)$ and (v_m, v_mu_n), respectively.

In each case, the so-obtained colouring is clearly a strong incidence colouring of $HD_{m,5}$. This completes the proof. ■



(a) (b)

Figure 21. Strong incidence colourings of $HD_{7,5}$ and $HD_{8,5}$.

Lemma 20. *For every two integers m and n, $m \geq n \geq 6$, $\chi_i^s(HD_{m,n}) = \sigma(HD_{m,n}) + 1$.*

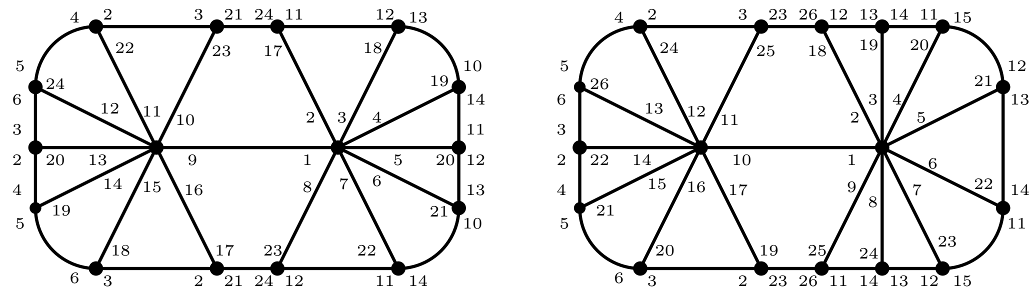


Figure 22. Strong incidence colourings of $HD_{7,7}$ and $HD_{8,7}$.

Proof. By Proposition 15, $\chi_i^s(HD_{m,n}) \geq 2m + n + 3 = \sigma(HD_{m,n}) + 1$. To complete the proof, we give a strong incidence $(2m+n+3)$-colouring by extending λ to a colouring of $HD_{m,n}$ as follows (see Figure 22).

• We first sequentially colour the incidences of the path P, starting from the incidence (v_1, v_1v_2), using the pattern $(m + 4)(m + 5)(m + 6)(m + 3)(m + 7)$.

• We then sequentially colour the incidences of the path P', starting from the incidence (u_1, u_1u_2), using the pattern 32456.

• We finally set $\lambda(u_i, u_iu) = 2m + n + 3 - i$, for every $i, i \in \{1, 2, 4, \ldots, n\}$, $\lambda(u_3, u_3u) = 2m+n+3$, $\lambda(u_n, u_nv_m) = \lambda(u_1, u_1v_1) = 2m+n$ and $\lambda(v_m, v_mu_n) = 2m + n + 3$.

The so-obtained colouring is clearly a strong incidence colouring of $HD_{m,n}$. This completes the proof. ■

Putting together Lemmas 16, 17, 18, 19 and 20, we finally get the following theorem. Recall that $\sigma(HD_{m,n}) = 2m + n + 2$.

Theorem 21. *For every two integers m and n, $m \geq n \geq 2$,*

$$\chi_i^s(HD_{m,n}) = \begin{cases} \sigma(HD_{m,n}) + 4 & \text{if } n = 2 \text{ and } m = 2, \\ \sigma(HD_{m,n}) + 3 & \text{if } n = 2 \text{ and } m \neq 2, \\ & \text{or } (n,m) \in \{(3,3),(3,4)\}, \\ \sigma(HD_{m,n}) + 2 & \text{if } n = 3 \text{ and } m \notin \{3,4\}, \\ & \text{or } n = 4 \text{ and } m \not\equiv 3 \pmod 5, \\ \sigma(HD_{m,n}) + 1 & \text{otherwise.} \end{cases}$$

7. Discussion

In this paper, we have introduced and studied the strong version of incidence colouring. We have determined the exact value of — or upper bounds on — the strong incidence chromatic number of several classes of graphs, namely cycles, wheel graphs, trees, ladder graphs and some subclasses of Halin graphs. We leave as open problems the following questions.

1. What is the best possible upper bound on the strong incidence chromatic number of graphs with bounded maximum degree? In particular, what about graphs with maximum degree 3?
2. What is the best possible upper bound on the strong incidence chromatic number of Halin graphs?
3. What is the best possible upper bound on the strong incidence chromatic number of d-degenerated graphs?

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Received 30 September 2021
Revised 4 July 2022
Accepted 7 July 2022
Available online 3 August 2022