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FAULT-TOLERANT IDENTIFYING CODES IN SPECIAL CLASSES OF GRAPHS

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Abstract

A detection system, modeled in a graph, is composed of "detectors" positioned at a subset of vertices in order to uniquely locate an "intruder" at any vertex. *Identifying codes* use detectors that can sense the presence or absence of an intruder within distance one. We introduce a fault-tolerant identifying code called a *redundant identifying code*, which allows at most one detector to go offline or be removed without disrupting the detection system. In real-world applications, this would be a necessary feature, as it would allow for maintenance on individual components without disabling the entire system. Specifically, we prove that the problem of determining the lowest cardinality of a redundant identifying code for an arbitrary graph is NP-hard, and we determine the bounds on the lowest cardinality for special classes of graphs, including trees, ladders, cylinders, and cubic graphs.

Keywords: domination, detection system, identifying-code, fault-tolerant, redundant-identifying-code, density.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

Let $G = (V(G), E(G))$ be a (simple) graph, with vertices $V(G)$ and edges $E(G)$, modelling a system or facility with detectors to recognize a possible problem, traditionally referred to as an "intruder". For example, the vertices of the graph can represent sections of a shopping mall, the intruder could be a shoplifter, and the detectors can be video surveillance equipment or motion, magnetic, or RFID sensors. The goal is to identify the exact location/vertex of the intruder by placing the minimum number of detectors in the facility/graph. To represent the capabilities of the sensor(s) placed at a point $v \in V(G)$, we associate each sensor, ρ, at location v with a detection region $R_\rho(v) \subseteq V(G)$, where ρ can detect the presence or absence of an intruder anywhere in $R_\rho(v)$. The vertex v itself is associated with a set of detection regions, $R(v)$, which is simply the set of $R_\rho(v)$ for each sensor ρ at position v. Note that when $|R(v)| = 1$, $R_\rho(v)$ has also been referred to as the "watching zone" of v in other papers, but with different notation [1].

Definition 1. Let G be a graph and $v \in V(G)$. The *open neighborhood* of v, denoted $N(v)$, is the set of all vertices adjacent to v, $\{w \in V(G) : vw \in E(G)\}$.

Definition 2. Let G be a graph and $v \in V(G)$. The *closed neighborhood* of v, denoted $N[v]$, is the set of all vertices adjacent to v as well as v itself, $N(v) \cup \{v\}$.

Many types of detection systems with various properties have been explored throughout the years. One such system is the *Locating-Dominating* (LD) *set*, where each detector can sense the presence of an intruder in its closed neighborhood but also has the ability to distinguish the vertex itself from its neighbors; that is, $R(v) = \{\{v\}, N(v)\}$ [14]. Another type of distinguishing set that has been explored is the *open-locating-dominating* (OLD)*set*, which is based on LD but removed the self-distinguishing property; that is, $R(v) = \{N(v)\}$ [12]. Of particular interest in this paper are *identifying codes* (ICs), where $R(v) = \{N[v]\}$ [10]. Over 470 papers have been published on these detection systems and other related concepts [11].

Detection systems are useful in modeling security systems and automated fault detection in networked systems; thus, it is often the case that we want some level of fault tolerance guaranteed in the system. Many different forms of fault tolerant detection systems have been explored, including the ability to correct false negative or false positive signals from the sensors. Identifying codes were introduced by Karpovsky *et al.* in 1998 [10]; in this paper, we will introduce *redundant identifying codes* (RED:ICs), which allow at most one detector to go offline or be removed without disrupting the system. To the best of our knowledge, this is the first paper to consider fault-tolerant identifying codes.

Definition 3. An *identifying code* $S \subseteq V(G)$ is a dominating set such that any two distinct vertices $u, v \in V(G)$ have $N[u] \cap S \neq N[v] \cap S$.

Definition 4. An identifying code $S \subseteq V(G)$ is a *redundant identifying code* (RED:IC) if, for each $v \in S$, the set $S \setminus \{v\}$ is an identifying code.

Detector-based systems commonly use terminology such as "dominated" or "distinguished", whose definitions vary depending on the sensors' capabilities. The following definitions are specifically for identifying codes and their fault-tolerant variants; assume that $S \subseteq V(G)$ is the set of detectors.

Definition 5. A vertex, $v \in V(G)$, is *k-dominated* if $|N[v] \cap S| = k$.

Definition 6. Two distinct vertices $u, v \in V(G)$ are said to be *k-distinguished* if $|(N[u] \cap S) \triangle (N[v] \cap S)| \geq k$.

We will also use terms such as "at least k-dominated" to denote l-dominated for some $l \geq k$.

Definition 7. A detector set, $S \subseteq V(G)$, is an IC if and only if each vertex is at least 1-dominated and all pairs are 1-distinguished.

Theorem 1. *A detector set, $S \subseteq V(G)$, is a* RED:IC *if and only if each vertex is at least* 2*-dominated and all pairs are* 2*-distinguished.*

In the remainder of this paper, two vertices are said to be "distinguished" if they meet the specific k-distinguished requirement for the type of set being discussed.

Theorem 1 was given by Slater [16] and proven more generally by Seo and Slater [13]; in this paper, it is specialized for RED:IC. Note that the requirements for Definition 7 and Theorem 1 are not satisfied by every graph. For instance, K_n for $n \geq 2$ does not support either of these requirements.

For finite graphs, we use the notations IC(G), and RED:IC(G) to denote the cardinality of the smallest possible such sets on graph G, respectively. For infinite graphs, we measure via the *density* of the subset, which is defined as the ratio of the size of the subset to the size of the whole set [9, 13]. Formally, for locally-finite (i.e., $B_r(v)$ finite for finite r) G, this is defined as $\limsup_{r\to\infty} \frac{|B_r(v)\cap S|}{|B_r(v)|}$ for any $v \in V(G)$, where $B_r(v) = \{u \in V(G) : d(u, v) \leq r\}$ denotes the ball with radius r around v. We use the notations IC%(G), and RED:IC%(G) to denote the lowest density of any possible such set on G [9, 13]. Note that density is also defined for finite graphs.

Figure 1 shows an example IC and RED:IC on G_8. In the IC set (a), we see that $N[v_1] \cap S = \{v_1, v_2\}$, $N[v_2] \cap S = \{v_1, v_2, v_3\}$, $N[v_6] \cap S = \{v_2\}$, and so on; each set has at least one item, so every vertex is at least 1-dominated.

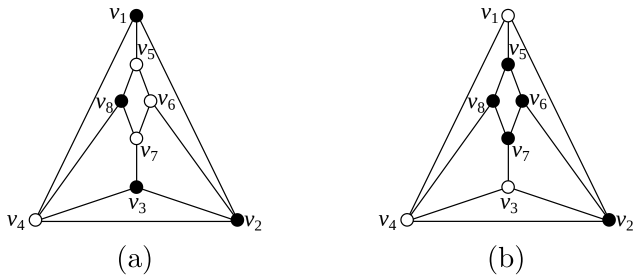


Figure 1. Optimal IC (a) and RED:IC (b) sets on G_8. Shaded vertices represent detectors.

For brevity, let $\triangle_{a,b} = (N[a] \cap S)\triangle(N[b] \cap S)$. We see that $\triangle_{v_1,v_2} = \{v_3\}$, $\triangle_{v_1,v_3} = \{v_1, v_3\}$, $\triangle_{v_5,v_7} = \{v_1, v_3\}$, and so on; each has at least one item, so all vertex pairs are 1-distinguished. Therefore, Definition 7 yields that (a) is an IC. We can perform a similar analysis on the vertices of (b) to see that all vertices are at least 2-dominated and all pairs are 2-distinguished, so by Theorem 1 it is a RED:IC. It can be shown that no smaller sets with these requirements exist on this graph (Corollary 1 from Section 3 can be used to demonstrate this). Thus, $\mathrm{IC}(G_8) = 4$ and $\mathrm{RED{:}IC}(G_8) = 5$. If we would prefer to use densities, we also have that $\mathrm{IC\%}(G_8) = \frac{1}{2}$ and $\mathrm{RED{:}IC\%}(G_8) = \frac{5}{8}$.

In the following section, we prove that the problem of determining the minimum cardinality of RED:IC(G) for an arbitrary graph G is NP-hard. In Section 3 we show existence criteria of the redundant identifying code for general graphs, and determine the lower bound on the minimum density of RED:IC, which is a tight bound when n is even. In Section 4 we explore several special classes of graph—including ladders, cylinders, trees, and cubic graphs—and find lower and upper bounds of RED:IC(G), some tight bounds, and several extremal families of graphs with minimum and maximum value.

2. NP-Hardness of RED:IC

It has been shown that many graphical parameters related to detection systems, such as finding the cardinality of the smallest IC, LD, or OLD sets, are NP-hard problems [2, 3, 5, 12]. We will now prove that the problem of determining the smallest RED:IC set is also NP-hard. For additional information about NP-hardness, see Garey and Johnson [8].

3-SAT
INSTANCE: Let X be a set of N variables. Let ψ be a conjunction of M clauses, where each clause is a disjunction of three literals from distinct variables of X.

QUESTION: Is there an assignment of values to X such that ψ is true?

Redundant Identifying Code (RED-IC)
INSTANCE: A graph G and integer K with $2 \leq K \leq |V(G)|$.
QUESTION: Is there a RED:IC set S with $|S| \leq K$? Or equivalently, is RED:IC$(G) \leq K$?

Theorem 2. *The* RED-IC *problem is NP-complete.*

Proof. Clearly, RED-IC is NP, as every possible candidate solution can be generated nondeterministically in polynomial time (specifically, $O(n)$ time), and each candidate can be verified in polynomial time using Theorem 1. To complete the proof, we will now show a reduction from 3-SAT to RED-IC.

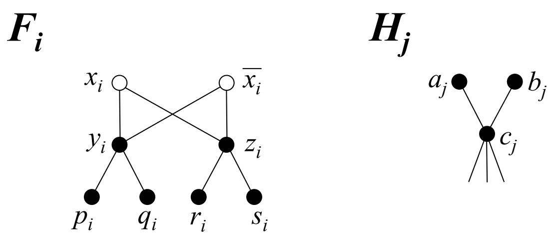


Figure 2. Variable and clause graphs.

Let ψ be an instance of the 3-SAT problem with M clauses on N variables. We will construct a graph, G, as follows. For each variable x_i, create an instance of the F_i graph (Figure 2); this includes a vertex for x_i and its negation $\overline{x_i}$. For each clause c_j of ψ, create a new instance of the H_j graph (Figure 2). For each clause $c_j = \alpha \vee \beta \vee \gamma$, create an edge from the c_j vertex to α, β, and γ from the variable graphs, each of which is either some x_i or $\overline{x_i}$; for an example, see Figure 3. The resulting graph has precisely $8N + 3M$ vertices and $8N + 5M$ edges, and can be constructed in polynomial time. To complete the problem instance, we define $K = 7N + 3M$.

Suppose $S \subseteq V(G)$ is a RED:IC on G with $|S| \leq K$. By Theorem 1, every vertex must be at least 2-dominated; thus, we require at least $6N+3M$ detectors, as shown by the shaded vertices in Figure 2. Additionally, in each F_i we see that y_i, p_i and z_i, r_i are not distinguished unless $\{x_i, \overline{x_i}\} \cap S \neq \emptyset$. Thus, we find that $|S| \geq 7N + 3M = K$, implying that $|S| = K$, so $|\{x_i, \overline{x_i}\} \cap S| = 1$ for each $i \in \{1, \ldots, N\}$. For each H_j, we see that a_j and c_j are not distinguished unless c_j is adjacent to at least one additional detector vertex. As no more detectors may be added, it must be that each c_j is now dominated by one of its three neighbors in the F_i graphs; therefore, ψ is satisfiable.

For the converse, suppose we have a solution to the 3-SAT problem ψ; we will show that there is a RED:IC, S, on G with $|S| \leq K$. We construct S by first including all of the $6N + 3M$ vertices needed for 2-domination. Then, for each variable, x_i, if x_i is true, then we let the vertex $x_i \in S$; otherwise, we let $\overline{x_i} \in S$. Thus, the fully-constructed S has $|S| = 7N + 3M = K$. Because we selected each $x_i \in S$ or $\overline{x_i} \in S$ based on a satisfying truth assignment for ψ, each c_j must be adjacent to at least one additional detector vertex from the F_i graphs. Also, by the hypothesis that the literals of a clause come from distinct variables (otherwise that clause is either not a valid 3-SAT clause or is a tautology and can be omitted from ψ), for every i, either x_i or $\overline{x_i}$ is adjacent to at least one additional detector vertex c_j in H_j, so x_i and $\overline{x_i}$ are distinguished. Similarly, it can be shown that all other vertex pairs are distinguished, so S is a RED:IC for G with $|S| \leq K$, completing the proof. ■

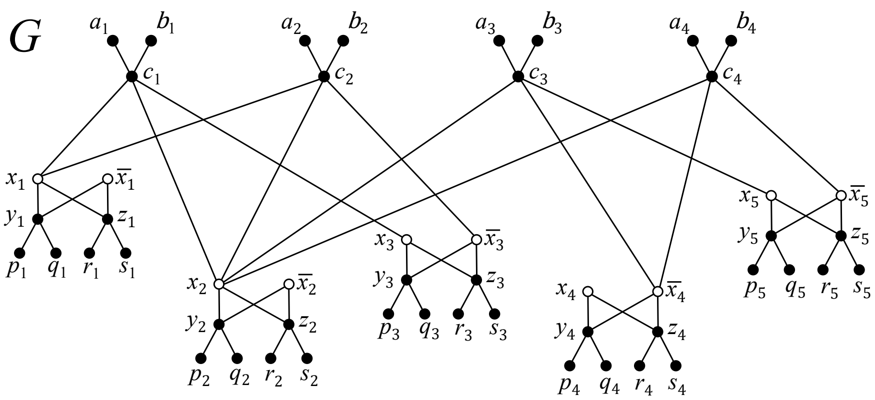


Figure 3. Construction of G with $N = 5$, $M = 4$, and $K = 47$ from $\psi = (x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee \overline{x_3}) \wedge (x_2 \vee \overline{x_4} \vee x_5) \wedge (x_2 \vee \overline{x_4} \vee \overline{x_5})$.

3. Existence of RED:IC and Bounds on RED:IC(G)

Definition 8. If $u \in V(G)$ has $N(u) = \{v\}$, then u is called a *leaf vertex* and v is called a *support vertex*.

Definition 9. If a vertex, v, is neither a leaf nor support vertex, it is called a *pure interior vertex*.

Definition 10 [7]**.** Two distinct vertices $u, v \in V(G)$ are said to be *twins* if $N[u] = N[v]$ (*closed twins*) or $N(u) = N(v)$ (*open twins*).

From Theorem 1, we know that each pair of vertices must be 2-distinguished; if u and v are closed twins, then they cannot be distinguished, so no RED:IC exists. Further, if u and v are open twins, then they must both be detectors in order to be distinguished. We also see that all support and leaf vertices must be detectors in order to 2-dominate the leaves; if a support vertex is not at least 4-dominated, then it will not be distinguished from its leaves. Thus, we arrive at the following.

Observation 1. RED:IC *exists only if there are no closed twins and every support vertex, v, has $deg(v) \geq 3$.*

Observation 2. *If S is a* RED:IC *and u and v are open twins, then $\{u, v\} \subseteq S$.*

Observation 3. *There is no graph with* RED:IC$(G) \leq 3$.

Observation 4. *The smallest graphs with* RED:IC(G) *are $K_{1,3}$ and C_4, each with* RED:IC$(G) = 4$.

Theorem 3. *Let G be connected with $n \geq 4$.* RED:IC *exists if and only if there are no closed twins, every support vertex has at least degree three, and every triangle $abc \in G$ has $|N[a] \triangle N[b]| \geq 2$.*

Proof. Let $S = V(G)$ be a set of detectors; because G is connected and $|V(G)| \geq 2$, every vertex is at least 2-dominated. We will show that each $v \in V(G)$ is distinguished from every other vertex $u \in V(G)$. If $uv \notin E(G)$, then u and v are distinguished by themselves; otherwise, we assume $uv \in E(G)$. By hypothesis, u and v are not closed twins; without loss of generality, let $w_1 \in N(u) \setminus N[v]$. If v is a leaf, then $deg(u) \geq 3$ by hypothesis, so u and v are distinguished by the neighbors of u different from v; otherwise there exists $w_2 \in N(v) \setminus \{u\}$. If $w_2 \notin N(u)$, then u and v are distinguished by w_1 and w_2; otherwise uvw_2 is a triangle, so by hypothesis u and v are 2-distinguished, meaning S is a RED:IC. For the converse, suppose one of the properties is not met. If there are closed twins or there is a support vertex with degree at most 2, then by Observation 1, no RED:IC exists. Finally, if there is a triangle $abc \in G$ with $|N[a] \triangle N[b]| \leq 1$, then a and b cannot be distinguished, so no RED:IC exists, completing theproof. ∎

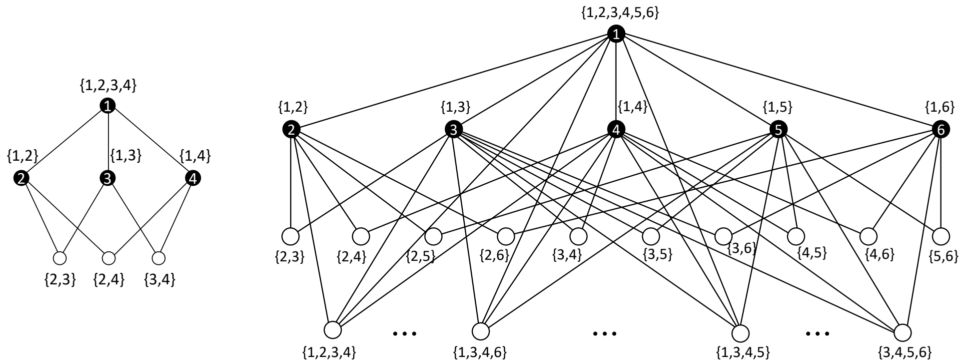
Based on Theorem 1, we can easily construct an algorithm to test if a connected graph G with $n \geq 4$ has a RED:IC set: simply check that for any $u, v \in V(G)$, $|N[u] \triangle N[v]| \geq 2$, which can be done in $\mathcal{O}(m\Delta(G))$ time in the worst case if the graph input is an adjacency list. From Theorem 3, if G is triangle-free, we need only ensure that support vertices have at least degree three and that there are no closed twins; if the input is a sorted adjacency list, this can be done in $\mathcal{O}(n\Delta(G))$ time by iterating over each vertex and storing the closed neighborhoods in a set. Finally, if G is a tree, we need only check that every support vertex has at least degree three, which can be done in $\mathcal{O}(n)$ time if the input is an adjacency list.

Next, we consider a lower bound on the value of RED:IC(G). We start by analyzing the maximum size of a graph with IC$(G) = k$. We know that every vertex must be at least 1-dominated and all pairs must be 1-distinguished; this means that the "codeword" of each $v \in V(G)$, $N[v] \cap S$, must be distinct and non-empty. Thus, the set of all non-empty subsets of the total k detectors represent valid codewords, giving us the following result.

Observation 5. *If* IC $(G) \leq k$, *then* $|V(G)| \leq 2^k - 1$.

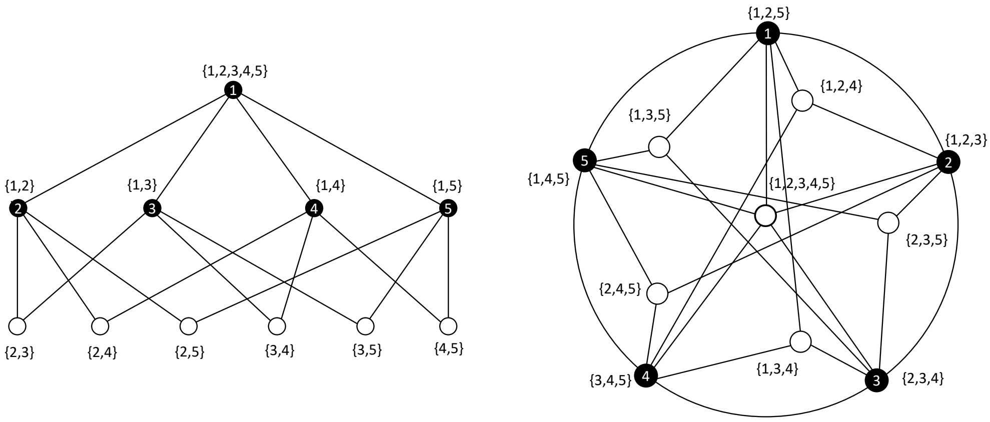
Theorem 4. *If* RED:IC$(G) \leq k$, *then* $|V(G)| \leq 2^{k-1} - 1$.

Proof. Suppose we have a RED:IC, $S \subseteq V(G)$, with $|S| \leq k$. Thus, by definition, there exists an IC S' with $|S'| \leq k - 1$. Observation 5 gives us that $|V(G)| \leq 2^{k-1} - 1$. ■



(a) (b)

Figure 4. Two members of an extremal family of graphs with largest n, where $n = 7$ for $k = 4$ (a) and $n = 31$ for $k = 6$ (b).



(a) (b)

Figure 5. Two constructions of graphs with large n (here, $n = 11$) for $k = 5$.

Now, we will show how to construct a family of extremal graphs with largest n for a given number of detectors, $k =$ RED:IC(G). For $k = 2j$, start with a star graph, $K_{1,k-1}$, where every vertex is a detector. We then add an additional

$\binom{k}{2} - (k-1)$ non-detectors which are adjacent to a distinct pair of detectors (keeping in mind there are already $k-1$ detector vertices dominated by exactly two detectors including themselves), an additional $\binom{k}{4}$ non-detectors which are adjacent to distinct sets of 4 detectors, an additional $\binom{k}{6}$ non-detectors which are adjacent to distinct sets of 6 detectors, and so on through $\binom{k}{k-2}$ (as $\binom{k}{k}$ was already created at the beginning). Because only even numbers of detector neighbors were chosen, every vertex will be at least 2-dominated and 2-distinguished. Then, the total number of vertices is thus $\binom{k}{2} + \binom{k}{4} + \cdots + \binom{k}{k} = 2^{k-1} - 1$ (see Equation 1 below). We see that this value matches the theoretical upper bound established by Theorem 4. This construction yields an infinite family of extremal graphs with largest n for any even value of RED:IC, k; example graphs for $k = 4$ and $k = 6$ are shown in Figure 4.

$$\begin{aligned} 2^n = (1+1)^n &= \sum_{k=0}^{n} \binom{n}{k} \\ 0 = (1-1)^n &= \sum_{k=0}^{n} \binom{n}{k} (-1)^k. \end{aligned} \tag{1}$$

For $k = 2j+1$, we again start with a star graph, $K_{1,k-1}$, where every vertex is a detector. We add non-detectors in a similar fashion to the case when $k = 2j$, starting with $\binom{k}{2} - (k-1)$ non-detectors, but ending with $\binom{k}{k-3}$ (as non-detectors representing $\binom{k}{k-1}$ will not be distinguished from the $\binom{k}{k}$ detector). This construction is shown for $k = 5$ in Figure 5(a). Then, the total number of vertices is $\binom{k}{2} + \binom{k}{4} + \cdots + \binom{k}{k-3} + \binom{k}{k} = 2^{k-1} - k$ (see Equation 1).

Alternatively, for $k = 2j+1$, we can start with a cycle on k vertices, C_k, where every vertex is a detector. Add an additional $\binom{k}{3} - k$ non-detectors that are adjacent to distinct set of three detectors. Add $\binom{k}{5}$ non-detectors so that they are adjacent to distinct set of 5 detectors. Add $\binom{k}{7}$ non-detectors so that they are adjacent to distinct set of 7 detectors, and so on through $\binom{k}{k}$. An example of this construction for $k = 5$ is given by Figure 5(b). Because only odd numbers of detector neighbors were chosen, every vertex will be at least 2-dominated and 2-distinguished. Then, the total number of vertices is $\binom{k}{3} + \binom{k}{5} + \cdots + \binom{k}{k-2} + \binom{k}{k} = 2^{k-1} - k$.

There is no RED:IC for a complete graph, as every vertex is a closed twin with any other vertex. There are $p = \lfloor \frac{n}{2} \rfloor$ disjoint pairs of closed twins in K_n. For any of the p pairs of twins, u and v, we can remove the uv edge; this makes them no longer closed twins with one another, and does not affect other vertices. If n is even, removing the p edges corresponding to the p disjoint pairs of twins results in a complete multipartite graph with p parts, each of size 2. By Observation 2, this graph must necessarily have RED:IC$(G) = n$ because every vertex is an open twin with some other vertex, and G has the maximum number of edges that RED:IC allows, as only the necessary p edges were removed.

From Theorem 4, we see that if S is a RED:IC of size k, then $n \leq 2^{k-1} - 1$, from which we see that $\log_2(n+1) + 1 \leq k$. This gives us the following corollary.

Corollary 1. *If G has a* RED:IC*, then* $\lceil \log_2(n+1) \rceil + 1 \leq \text{RED : IC}(G) \leq n$*.*

4. Special Classes of Graphs

Observation 6. *If S is a* RED:IC *set, then every degree* 3 *support vertex u has $N[u] \subseteq S$.*

Observation 7. *If S is a* RED:IC *set and vuw is a path in G where u and w have degree* 2*, then $v \in S$.*

From Observation 1, finite paths do not have RED:IC. From Observation 7, we see that the infinite path and cycles with $n \geq 4$ require all vertices to be detector vertices, hence RED:IC%$(G) = 1$.

4.1. Trees

Because a tree on $n \geq 4$ vertices is closed-twin free and triangle free, we arrive at the following corollary via Theorem 3.

Corollary 2. *Let T be a tree with $n \geq 4$.* RED:IC *exists if and only if every support vertex, v, has $deg(v) \geq 3$.*

Characterization of the extremal trees with RED:IC$(T_n) = n$

By the requirement of 2-domination, any $T = K_{1,n-1}$, has RED:IC$(T) = n$. In fact, from Figure 6, we see that any tree T of order $4 \leq n \leq 8$ which admits RED:IC has RED:IC$(T) = n$. We will now characterize these extremal trees.

Theorem 5. *If T is a tree of order $n \geq 4$ with a* RED:IC*, then* RED:IC$(T) = n$ *if and only if each vertex, $v \in V(G)$, is a leaf vertex, is a support vertex, is adjacent to a degree* 3 *support vertex, or belongs to a path vuw where u and w have degree* 2*.*

Proof. Clearly, if T satisfies the four properties in the theorem statement, then RED:IC$(T) = n$, in particular thanks to Observations 6 and 7. For the converse, suppose that some $v \in V(T)$ does not satisfy any of the four properties; we will show that $V(T) \setminus \{v\}$ is a RED:IC. By hypothesis, we know a RED:IC exists, which by Observation 1 implies that all support vertices have at least degree 3.

Because v is not a leaf vertex, $deg(v) \geq 2$; let $w \in N(v)$, let T' be the subtree of $T - v$ containing w, and let $n' = |V(T')|$. We will show that $S' = V(T')$ is a RED:IC for T', meaning $V(T) \setminus \{v\}$ is a RED:IC for the original graph, T. Because v is not a support vertex, $deg(w) \geq 2$; let $z \in N(w) \setminus \{v\}$. If $deg(w) = deg(z) = 2$, we contradict that v does not satisfy Observation 7; thus, we assume that $deg(w) \geq 3$ or $deg(z) \neq 2$. Suppose $deg(w) = 2$; then require $deg(z) \neq 2$. If $deg(z) = 1$, then w is a support vertex which does not have at least degree 3, a contradiction; otherwise, we assume that $deg(z) \geq 3$. We see that $n' \geq 4$ and every support vertex in the restricted graph T' has at least degree 3, so Corollary 2 yields that $V(T')$ is a RED:IC on T', and we are done. Otherwise, $deg(w) \geq 3$. If $deg(w) \geq 4$, we again see that $n' \geq 4$ and all support vertices in T' have at least degree 3, so we are done; thus, we can assume that $deg(w) = 3$. If w is a support vertex, then we contradict that v does not satisfy Observation 6; thus, we assume that w is not a support vertex, meaning for every $z_i \in N(w), deg(z_i) \geq 2$. Thus, we again see that $n' \geq 4$ and all support vertices in T' maintain degree at least 3, so we are done. Therefore, in any case, we find that $S' = V(T')$ is a RED:IC for T', so $V(T) \setminus \{v\}$ is a RED:IC for T, completing the proof. ∎

Lower bound on RED:IC(T_n) for finite trees

As we will see later in Theorem 11 in the cubic graphs subsection, the infinite 3-regular tree has RED:IC%$(T_\infty) = \frac{4}{7} \approx 0.5714$. However, for finite trees the lower bound on RED:IC(T) is much higher as shown in the next theorem.

Theorem 6. *If T is a tree of order $n \geq 4$ with* RED:IC*, then* $\left\lceil \frac{4}{5}(n+1) \right\rceil \leq \mathrm{RED:IC}(T) \leq n$.

Proof. Suppose S is a RED:IC for T and let $j = |V(G) - S|$ be the number of non-detectors; then, $n = |S| + j$. Because T is acyclic and each non-detector must be at least 2-dominated, we know that there must be at least $j + 1$ connected components of detectors. Because S must be a RED:IC on the graph induced by S, each connected component of detectors must have a RED:IC, meaning the minimum size of each detector component is four. Thus, $|S| \geq 4j + 4$; by rearranging terms, we see that $|S| \geq 4(n - |S|) + 4$, from which we find that $|S| \geq \frac{4}{5}(n + 1)$. We know that RED:IC$(T) \in \mathbb{N}$, so we can strengthen this to $|S| \geq \left\lceil \frac{4}{5}(n+1) \right\rceil$, completing the proof. ∎

Figure 6 shows optimal RED:ICs on all trees of order $n \leq 10$ for which RED:IC exists. Table 1 provides a summarized view of the number of trees with a given RED:IC value for each $n \in [11, 17]$. Figure 7 gives examples of extremal trees on $n \geq 4$ vertices with RED:IC$(T_n) = \left\lceil \frac{4}{5}(n+1) \right\rceil$. In general, we see that when there are j non-detectors, there must be $p = j + 1$ detector components

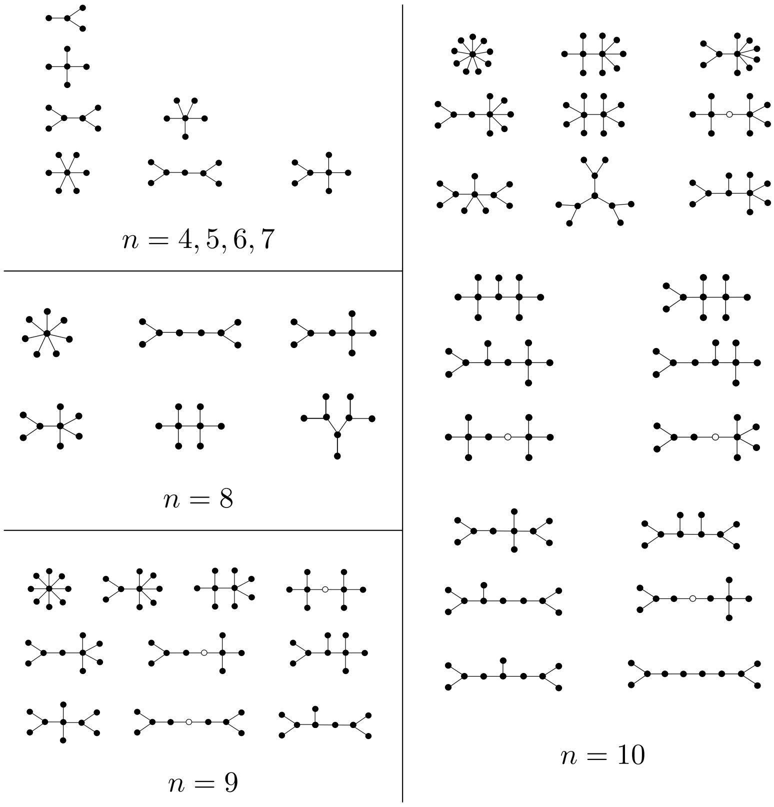


Figure 6. All trees of order $n \leq 10$ with RED:IC.

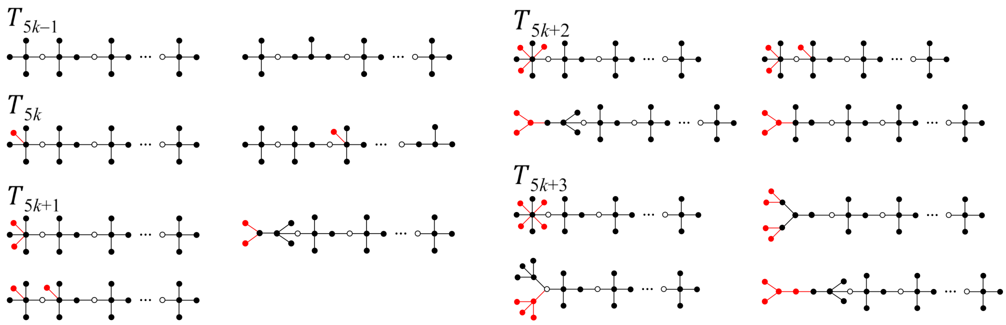


Figure 7. Extremal trees with RED:IC$(T) = \left\lceil \frac{4}{5}(n+1) \right\rceil$. Red vertices denote "excess" detectors.

(or up to $p = j + 2$ if $n = 5k + 3$)—each detector component can be selected from Figure 6 on at most 8 vertices—and the total number of detectors must be $4p + [(n + 1) \bmod 5]$; edges between non-detectors and detectors can be added arbitrarily so long as the result is a tree. Note that this is essentially the same as starting with an extremal tree on $5k - 1$ vertices, as in Figure 7, and adding an additional $(n+1) \bmod 5$ "excess" detectors, so long as the placements do not cause RED:IC to no longer exist.

n	11	12	13	14	15	16	17
trees	235	551	1301	3159	7741	19320	48629
with RED:IC	39	82	167	360	766	1692	3726
RED:IC$(T_n) = n - 2$	0	0	0	13	29	96	287
RED:IC$(T_n) = n - 1$	10	24	64	130	323	744	1731
RED:IC$(T_n) = n$	29	58	103	217	414	852	1708

Table 1. Numeric results for RED:ICs on trees.

4.2. Ladders and cylinders

The infinite ladder

Theorem 7. *The infinite ladder graph has* RED:IC%$(P_2 \square P_\infty) = \frac{2}{3}$.

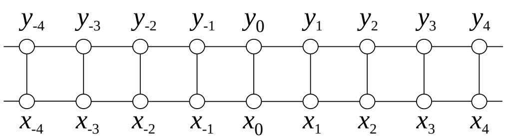


Figure 8. Ladder graph labeling.

Proof. Figures 9(c) and (d) give a family of cylinders with RED:IC%$(G) = \frac{2}{3}$; each of these solutions can be tiled infinitely to produce a RED:IC on $P_2 \square P_\infty$ with density $\frac{2}{3}$. To prove this is optimal, we need only show that $\frac{2}{3}$ is a lower bound for the minimum density. To proceed, we will look at an arbitrary non-detector vertex $x \notin S$ and show that we can associate at least two detectors with x. For $v \in V(G)$, let $R_6(v) = N[v] \cup \{u \in V(G) : |N(u) \cap N(v)| = 2\}$. For this association argument, we enforce that x can only be associated with detector vertices within $R_6(x)$; specifically, we employ partial ownership of detectors, so a detector vertex $v \in S$ contributes $\frac{1}{k}$, where $k = |R_6(v) \cap \overline{S}|$, toward the required total of two detectors.

To begin, we will say that $x = x_0$, using the labeling convention shown in Figure 8. Suppose that $y_0 \notin S$; then $y_{-1}, y_1 \in S$ to 2-dominate y_0 and $x_{-1}, x_1 \in S$ to 2-dominate x_0. We see that x_1 and y_1 are not distinguished, so we need $x_2, y_2 \in S$; by symmetry, $x_{-2}, y_{-2} \in S$. Then x receives a $\frac{1}{2}$ contribution from each of $\{x_{-1}, y_{-1}, x_1, y_1\}$, and we are done. Otherwise, we can assume

$y_0 \in S$; suppose that $y_1 \notin S$. We require $x_1, x_2 \in S$ to 2-dominate x_1, and $y_{-1} \in S$ to 2-dominate y_0. Vertices x_1 and x_2 are not distinguished, so we need $x_3, y_2 \in S$, and by symmetry $x_{-1}, y_{-2} \in S$. Then x receives at least $\frac{1}{2}$ from each of $\{x_1, y_0, x_{-1}, y_{-1}\}$, and we are done. Otherwise, we can assume $y_1 \in S$ and by symmetry $y_{-1} \in S$, in addition to $y_0 \in S$ that we showed previously. Vertex x_0 must be 2-dominated; without loss of generality let $x_1 \in S$. We see that $\{x_2, y_2\} \subseteq \overline{S}$ would cause x_1 and y_1 to not be distinguished, so x_1 and y_1 each contribute at least $\frac{1}{2}$ to x. If $x_{-1} \in S$, then y_0 contributes the final $\frac{1}{1}$ and we are done, so we assume that $x_{-1} \notin S$; then y_0 contributes $\frac{1}{2}$ and we need only another $\frac{1}{2}$ to have a total of two. We require $x_{-2} \in S$ to 2-dominate x_{-1}, and $y_{-2} \in S$ to distinguish y_{-1} and y_0. Then y_{-1} contributes the final $\frac{1}{2}$, completing the proof. ∎

Theorem 8. *Let F and H be disjoint graphs with* RED:ICs *S_F and S_H, respectively. Let $G = F + H + E_{FH}$ where E_{FH} is a set of disjoint edges between F and H. Then $S = S_F \cup S_H$ is a* RED:IC *for G.*

Proof. We will show that S satisfies Theorem 1. First, the existence of RED:ICs S_F and S_H ensures that every vertex in $V(G)$ is at least 2-dominated. Next, let $u, v \in V(G)$ be two distinct vertices. Suppose $u, v \in V(F)$. Then u, v are 2-distinguished in F by S_F. Because E_{FH} cannot add any edges within F (only between F and H), it must be that u and v are still 2-distinguished by S in G.

Otherwise, without loss of generality assume $u \in V(F)$ and $v \in V(H)$. By hypothesis, vertex u must be 2-dominated in F by S_F, and similarly v must be 2-dominated in H by S_H. Thus, there exist $x \in N(u) \cap V(F) \cap S$ and $y \in N(v) \cap V(H) \cap S$. Suppose $uv \in E_{FH} \subseteq E(G)$. Because the edges in E_{FH} must be disjoint and $uv \in E_{FH}$, it must be that $uy \notin E(G)$ and $vx \notin E(G)$; thus, u and v are 2-distinguished by $\{x, y\} \subseteq S$ in G. Now, we assume that $uv \notin E(G)$. If $\{u, v\} \subseteq S$, then u and v will be 2-distinguished by u and v themselves; otherwise, without loss of generality assume $u \notin S$. Because u must be at least 2-dominated in F by S_F, we now know that there exists $z \in (N(u) \cap V(F) \cap S) \setminus \{x\}$. If $vx \notin E_{FH}$ and $vz \notin E_{FH}$, then u and v will be 2-distinguished by x and z; otherwise without loss of generality let $vx \in E_{FH}$. Because $vx \in E_{FH}$, it must be that $vz \notin E_{FH}$. If $uy \notin E_{FH}$, then u and v will be 2-distinguished by y and z; otherwise, we also assume $uy \in E_{FH}$. Finally, because v must be at least 2-dominated in H by S_H, it must be that there exists $w \in (N[v] \cap V(H) \cap S) \setminus \{y\}$; note that it is possible that $w = v$. Because $uy \in E_{FH}$, it must be that $uw \notin E_{FH}$. Thus, u and v are 2-distinguished by z and w, completing the proof. ∎

Theorem 9. *For $j \geq 4$, we have* RED:IC$(P_2 \square P_j)$ =RED:IC $(P_2 \square C_j) = \left\lceil \frac{2}{3} n \right\rceil$.

Proof. Suppose for a contradiction that S is a RED:IC on $P_2 \square P_j$ with $|S| < \left\lceil \frac{2}{3} n \right\rceil$. Then Theorem 8 would allow us to repeatedly create duplicates of $P_2 \square P_j$

(with duplicated detectors) and connect them end-to-end to produce a RED:IC on $P_2 \square P_\infty$ with density strictly less than $\frac{2}{3}$. This contradicts Theorem 7; thus, RED:IC$(P_2 \square P_j) \geq \left\lceil \frac{2}{3}n \right\rceil$. For the cylinder, $P_2 \square C_j$, we see that any maximal ladder subgraph is spanning and can be used as a tile to construct the infinite path, so we similarly find that RED:IC$(P_2 \square C_j) \geq \left\lceil \frac{2}{3}n \right\rceil$. Figure 9 gives an infinite family of RED:ICs on finite ladders and cylinders which achieve the lower bound of $\left\lceil \frac{2}{3}n \right\rceil$, with $k \geq 1$ for the general construction, completing the proof. ■

Although Theorem 9 holds for all $j \geq 4$, there are two graphs where $j = 3$, $P_2 \square P_3$ and $P_2 \square C_3$, which do not fall under the general trend. We find that RED:IC$(P_2 \square P_3) = \frac{2}{3}n$ as expected, but RED:IC$(P_2 \square C_3) = n$, breaking the general pattern.

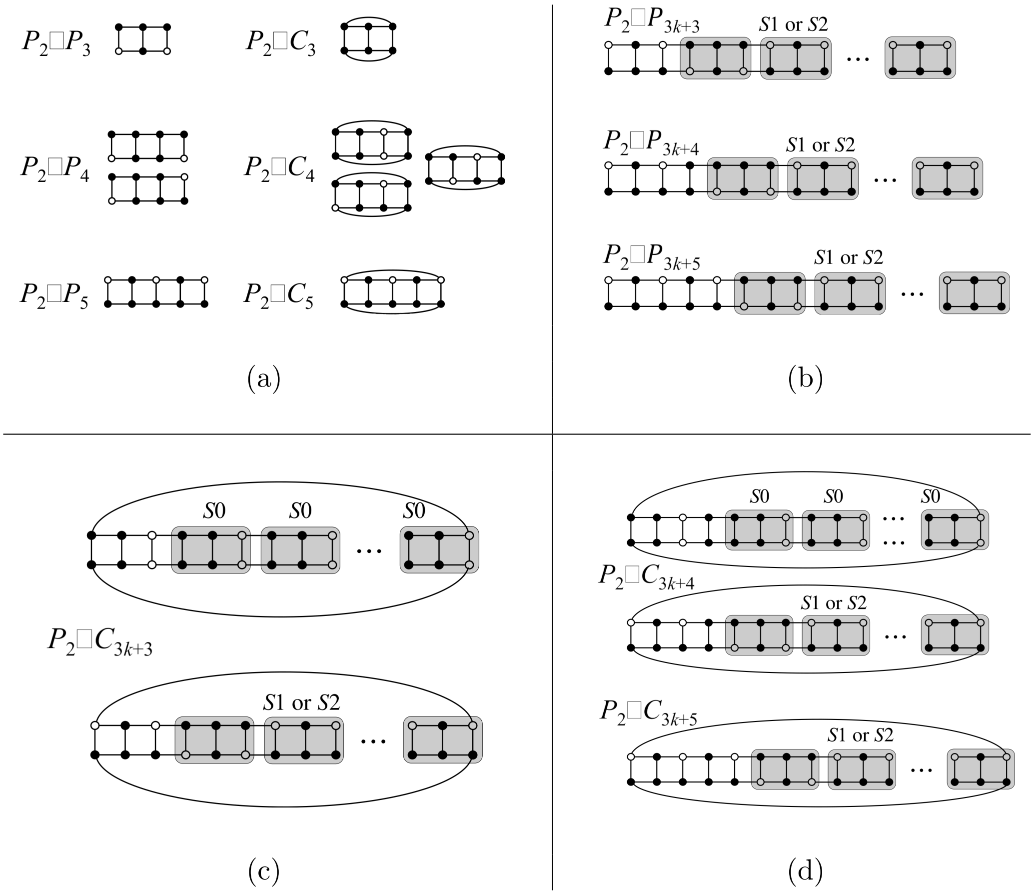


Figure 9. A set of optimal RED:ICs on finite ladders and cylinders. $S0$, $S1$, and $S2$ refer to $P_2 \square P_3$ tiles (disjoint subgraphs) which can be repeated. Base cases on $n = 8$ and $n = 10$ are given by (a).

Theorem 10. *For a finite torus* $C_i \square C_j$, RED:IC$(C_i \square C_j) \geq \lceil \frac{2}{5} n \rceil$.

Proof. Let $G = C_i \square C_j$. G is 4-regular, meaning for any $x \in V(G)$, $|N[x]| = 5$; thus, the 2-domination requirement of RED:IC implies that RED:IC%$(G) \geq \frac{2}{5}$. Therefore, RED:IC$(G) \geq \lceil \frac{2}{5} n \rceil$. ■

4.3. Hypercubes

Let $Q_n = P_2^n$, where G^n denotes repeated application of the $\square$ operator, be the hypercube in n dimensions. If S is a RED:IC on Q_n for $n \geq 2$, then Theorem 8 would allow us to duplicate the vertices to produce a new RED:IC of size $2|S|$ on $Q_{n+1} = Q_n \square P_2$; thus, RED:IC%$(Q_n)$ is a non-increasing sequence in terms of n. We have found that RED:IC%$(Q_5) = \frac{3}{8}$, which serves as an upper bound for the minimum density of RED:IC sets in larger hypercubes. Figure 10 shows a RED:IC set for each of the hypercubes on $n \leq 5$ dimensions. From programmatic analysis, we believe these to be optimal RED:ICs.

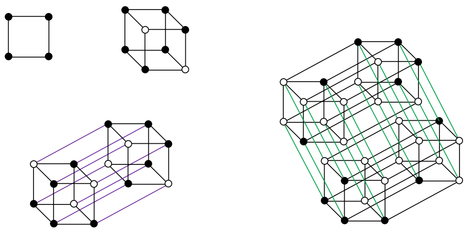


Figure 10. RED:ICs for Q_n with $n \leq 5$.

4.4. Cubic Graphs

Observation 8. RED:IC *exists for all closed-twin-free cubic graphs.*

Observation 9. *On a cubic graph,* RED:IC *exists if and only if* IC *exists.*

Lower bound on RED:IC(G) for cubic

As introduced by Slater [15], for a dominating set $S \subseteq V(G)$ of G and a vertex $v \in S$, let $sh(v) = \sum_{u \in N[v]} 1/|N[u] \cap S|$ denote the *share* of v; i.e., v's contribution to the domination of its neighbors. Because S is a dominating set, each k-dominated vertex contributes $\frac{1}{k}$ precisely k times to its neighbors' share values; thus, $\sum_{v \in S} sh(v) = n$, implying that the inverse of the average share is equal to the density of S in $V(G)$. Therefore, an upper bound on the average share (over all detectors) can be reciprocated to give a lower bound for the density. This technique has been proven to work even in the case of infinite graphs.

As a shorthand, we will let σ_A denote $\sum_{k \in A} \frac{1}{k}$ for some sequence of single-character symbols, A. Thus, $\sigma_a = \frac{1}{a}$, $\sigma_{ab} = \frac{1}{a} + \frac{1}{b}$, and so on. We also let $dom(v) = |N[v] \cap S|$ denote the *domination number* of some vertex $v \in V(G)$.

Theorem 11. *If G is a cubic graph, then* RED:IC%$(G) \geq \frac{4}{7} \approx 0.5714$.

Proof. Let S be a RED:IC for G, let $x \in S$ be an arbitrary detector, and let $N(x) = \{a, b, c\}$. Among the three vertices a, b, and c, we have at most one edge, as otherwise we create closed-twins and a RED:IC would not exist. Suppose $ab \in E(G)$. We know that there exists $z_1 \in N(a) \setminus N[b]$ and $z_2 \in N(b) \setminus N[a]$, as otherwise we create closed-twins. In order to distinguish x and a we require $c, z_1 \in S$; and by symmetry to distinguish x and b we require $c, z_2 \in S$. If $a \in S$ or $b \in S$ then $sh(x) \le \sigma_{3332} = \frac{1}{3} + \frac{1}{3} + \frac{1}{3} + \frac{1}{2} = \frac{3}{2} < \frac{7}{4}$, and we are done; otherwise $a, b \notin S$. To distinguish x and c, we require $dom(c) = 4$, so $sh(x) \le \sigma_{4222} = \frac{7}{4}$ and we are done. Otherwise, by symmetry, we can assume that there are no edges among a, b, and c. Suppose $dom(x) = 2$; let $a \in S$ and $b, c \notin S$. As seen in the previous case, to distinguish x and a we require $dom(a) = 4$, so $sh(x) \le \sigma_{4222}$ and we are done. Similarly, if $dom(x) = 4$, then we are done, which leaves the last remaining case: $dom(x) = 3$. Let $a, b \in S$ and $c \notin S$. If $dom(a) \ge 3$ or $dom(b) \ge 3$, then $sh(x) \le \sigma_{3322}$ and we are done; otherwise $dom(a) = dom(b) = 2$. We see that x and a are not distinguished, a contradiction. Thus, in any case $sh(x) \le \frac{7}{4}$, giving a lower bound of RED:IC%$(G) \ge \frac{4}{7}$ and completing the proof. ■

RED:IC on the infinite 3-regular tree

Theorem 12. *The infinite cubic tree, T, has* RED:IC%$(T) = \frac{4}{7}$.

Proof. Theorem 11 gives us a lower bound of RED:IC%$(T) \ge \frac{4}{7}$. The figure given in Figure 11 gives a RED:IC, S, on T. We see that every detector vertex, $x \in S$, has $sh(x) = \sigma_{4222} = \frac{7}{4}$, meaning the density of S in T is $\frac{4}{7}$, completing the proof. ■

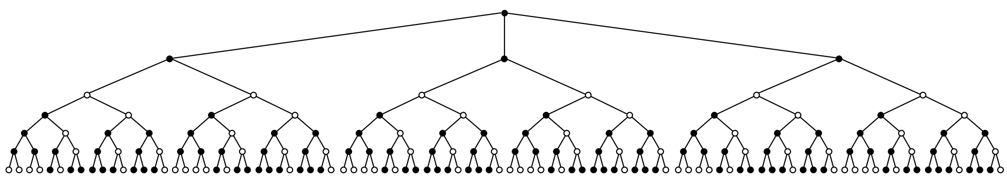


Figure 11. RED:IC%$(T) \le \frac{4}{7}$.

RED:IC on the infinite hexagonal grid

For the hexagonal grid, HEX, the tiling of the solution given in Figure 12 contains $\frac{2}{3}$ of the vertices as detectors; thus, we have RED:IC%(HEX) $\le \frac{2}{3}$. The lower bound is from Theorem 11.

Theorem 13. *For the infinite hexagonal grid,*HEX, $\frac{4}{7} \le$RED:IC%(HEX) $\le \frac{2}{3}$.

For comparison to IC, note that the bounds of $\frac{5}{12} \leq$ IC%(HEX) $\leq \frac{3}{7}$ were proven by Cukierman and Yu [6] (lower bound) and Cohen *et al.* [4] (upper bound).

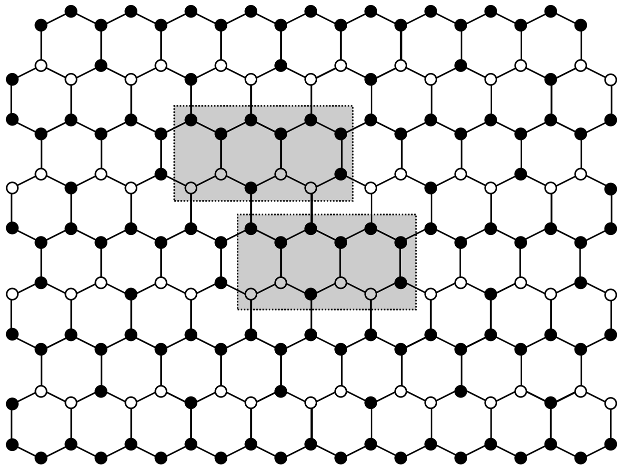


Figure 12. RED:IC%(HEX) $\leq \frac{2}{3}$.

Extremal cubic graphs with lower bound

Let G_{14} be the subgraph shown in Figure 13; G_{14} contains four "loose" edges which may go to arbitrary vertices inside or outside of the subgraph (so long as the result is cubic). We see that each vertex in G_{14} is at least 2-dominated, and it can be shown that each pair of vertices is 2-distinguished regardless of the specific incidence of the loose edges. For example, vertex pair a, e is distinguished by $\{b, f\}$, vertex pair b, e is distinguished by $\{b, c\}$, vertex pair e, g is distinguished by $\{a, c\}$, and so on. Thus, we see that G_{14} has a RED:IC of size 8. From Theorem 11, we know that a cubic graph must have RED:IC%$(G) \geq \frac{4}{7}$; thus, any cubic graph constructed using (only) copies of G_{14} will have the minimum density of $\frac{4}{7}$. Copies of the G_{14} subgraph can be connected in a ring to create an infinite family of cubic graphs which have the extremal value of RED:IC%$(G) = \frac{4}{7}$. This construction is shown in Figure 14.

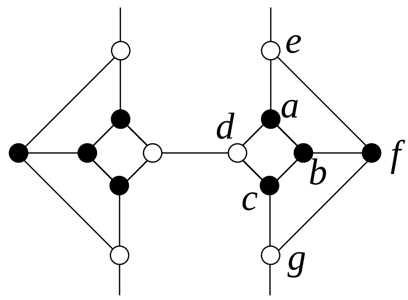


Figure 13. Cubic subgraph on 14 vertices requiring 8 detectors.

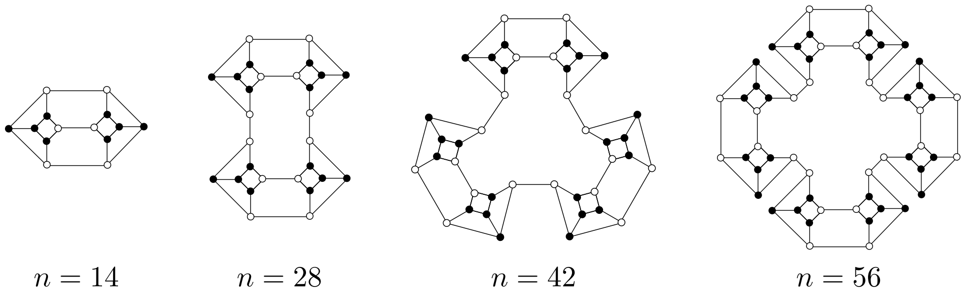


Figure 14. Infinite family of cubic graphs with RED:IC$(G) = \frac{4}{7}$.

Extremal cubic graphs with upper bound

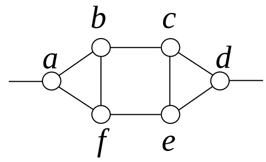


Figure 15. Cubic subgraph on 6 vertices requiring 6 detectors.

Let G_6 be the subgraph on 6 vertices from Figure 15; G_6 has two "loose" edges which extend out from a and d to any external vertex, so long as the entire graph is cubic. We see that vertices b and f can only be distinguished by having $\{c, e\} \subseteq S$, and by symmetry $\{b, f\} \subseteq S$. If G is composed exclusively of disjoint copies of G_6 (allowing loose edges to overlap), then each vertex like a or d must be connected to another vertex like a or d, as all other vertices already have degree three. We see that to distinguish a and b, we require the vertex adjacent to a by its loose edge to be a detector, so by symmetry all vertices like a and d must be detectors. Thus, all vertices in G_6 must be detectors. Copies of the G_6 subgraph can be connected in a ring to form an infinite family of cubic graphs with the extremal value RED:IC%$(G) = 1$, as shown in Figure 16.

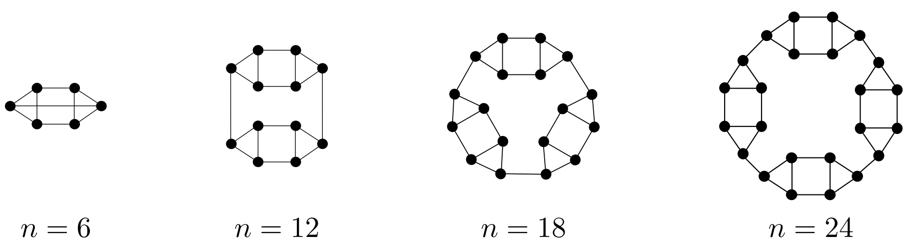


Figure 16. Infinite family of cubic graphs with RED:IC$(G) = n$.

Table 2 gives a summary of results for the number of cubic graphs on up to 20 vertices which has a given value for RED:IC(G).

n	6	8	10	12	14	16	18	20
cubic graphs	2	5	19	85	509	4060	41301	510489
with RED:IC	2	4	14	63	386	3189	33586	427277
lowest RED:IC(G)	6	6	6	8	8	10	11	12
highest RED:IC(G)	6	6	8	12	12	14	18	18

Table 2. Results on RED:ICs for finite (connected) cubic graphs.

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