(k, H)-KERNELS IN NEARLY TOURNAMENTS

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Abstract

Let H be a digraph, possibly with loops, D a digraph without loops, and $\rho : A(D) \to V(H)$ a coloring of $A(D)$ (D is said to be an H-colored digraph). If $W = (x_0, \ldots, x_n)$ is a walk in D, and $i \in \{0, \ldots, n-1\}$, then we say that there is an obstruction on x_i whenever $(\rho(x_{i-1}, x_i), \rho(x_i, x_{i+1})) \notin A(H)$ (when $x_0 = x_n$ the indices are taken modulo n). We denote by $O_H(W)$ the set $\{i \in \{0, \ldots, n-1\} : \text{there is an obstruction on } x_i\}$. The H-length of W, denoted by $l_H(W)$, is defined by $|O_H(W)|$ if W is closed or $|O_H(W)| + 1$ in the other case.

A (k, H)-kernel of an H-colored digraph D ($k \geq 2$) is a subset of vertices of D, say S, such that, for every pair of different vertices in S, every path between them has H-length at least k, and for every vertex $x \in V(D) \setminus S$ there exists an xS-path with H-length at most $k-1$. This concept widely generalize previous nice concepts such as kernel, k-kernel, kernel by monochromatic paths, kernel by properly colored paths, and H-kernel.

In this paper, we introduce the concept of (k, H)-kernel and we will study the existence of (k, H)-kernels in interesting classes of digraphs, called nearly tournaments, which have been large and widely studied due to its applications and theoretical results. We will show several conditions that guarantee the existence of a (k, H)-kernel in tournaments, r-transitive digraphs, r-quasi-transitive digraphs, multipartite tournaments, and local tournaments. As a consequence, previous results for k-kernels and kernels by alternating paths will be generalized, and some conditions for the existence of kernels by monochromatic paths and H-kernels in nearly tournaments will be shown.

Keywords: kernel, k-kernel, H-kernel, H-coloring, kernel by monochromatic paths, kernel by alternating paths.

2020 Mathematics Subject Classification: 05C15, 05C20, 05C69.

1. Introduction

A digraph D is *semicomplete* if every two different vertices in D are joined by at least one arc. An arc (u, v) in a digraph D is *symmetric* if $(v, u) \in A(D)$. A *tournament* is a semicomplete digraph without symmetric arcs. The class of tournaments is one of the most well-studied classes of digraphs with many deep and important results and applications, and provide a useful class of digraphs which allows to have a first approach to solve very difficult problems. As a consequence, several authors defined some new classes of digraphs which are generalizations of tournaments, commonly called nearly tournaments. For instance, the class of *local in-tournaments* (*local out-tournaments*) was introduced by Bang-Jensen in [8], as a digraph D such that for every $x \in V(D)$, the induced subdigraph of $N^-(x)$ (respectively $N^+(x)$) is a tournament. A *local tournament* is a digraph which is both local in-tournament and local out-tournament. Several results related with local tournaments have been proved. For a deeply study of local tournaments, see [5].

Other digraphs related to nearly tournaments are the multipartite tournaments. An arc (u, v) in a digraph D is *asymmetric* if (v, u) is not an arc in D. An *r-partite tournament* $(r \geq 2)$ is a digraph D such that $V(D)$ can be partitioned into r disjoint independent sets, and every two vertices in different classes are joined by an asymmetric arc. Multipartite tournaments were considered by Moon in [36], and several authors studied such class of digraphs. For a deeply study of multipartite tournaments, see [5].

A digraph D is *r-transitive* $(r \geq 2)$ if for every $\{u, v\} \subseteq V(D)$, whenever there exists a uv-path with length r, we have that $(u, v) \in A(D)$. When $r = 2$, an r-transitive digraph is called *transitive*. The class of r-transitive digraphs is one of the most studied nearly tournaments, due to their particularly nice structure, and several results related with such digraphs arose. For instance, see [5, 28, 31] and [40].

A digraph D is *quasi-transitive* if for every $\{u, v\} \subseteq V(D)$, whenever there exists a uv-path with length 2, we have that u and v are joined by an arc, that is, either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Quasi-transitive digraphs were introduced by Ghouila-Houri in [29] as a consequence of their relation with comparability graphs, and it is one of the most studied class of digraphs, probably the main reason is the characterization theorem showed by Bang-Jensen and Huang in [7]. Bang-Jensen [3] introduced the family of 3-quasi-transitive digraphs in the context of strong arc-locally semicomplete digraphs. A digraph D is *3-quasi-transitive* if for every $\{u, v\} \subseteq V(D)$, whenever there exists a uv-path with length 3, we have that u and v are joined by an arc. In an analogous way, a digraph D is *r-quasi-transitive* $(r \geq 2)$ if for every $\{u, v\} \subseteq V(D)$, whenever there exists a uv-path with length r, we have that u and v are joined by an arc. This class

was introduced by Galeana-Sánchez and Hernández-Cruz [32] in the context of k-kernels. For a deeply study of r-quasi-transitive digraphs, see [5].

The concept of *kernel* was introduced by von Neumann and Morgenstern in [37] as a subset S of vertices of a digraph D, such that for every pair of different vertices in S, there is no arc between them (that is, S is an *independent set*), and every vertex not in S has at least one out-neighbor in S (that is, S is an *absorbent set*). This concept has been deeply and widely studied by several authors due to a large amount of theoretical and practical applications, see for instance [12, 15, 16] and [18]. In [10] Chvátal proved that deciding if a digraph has a kernel is an NP-complete problem. As a consequence, several conditions that guarantee the existence of kernels has been showed. For instance, we have the following two classical results.

Theorem 1 (König [33]). *Every transitive digraph has a kernel.*

Theorem 2 (Duchet [17]). *If D is a digraph and every cycle in D has a symmetric arc, then D has a kernel.*

The concept of kernel has been generalized over the years. A subset S of vertices of D is said to be a *kernel by paths*, if for every $x \in V(D) \setminus S$, there exists an xS-path (that is, S is *absorbent by paths*) and, for every pair of different vertices $\{u, v\} \subseteq S$, there is no uv-path in D (that is, S is *independent by paths*). This concept was introduced by Berge in [9], and it is a well known result that every digraph has a kernel by paths [9] (see Corollary 2 on p. 311). The concept of (k, l)-kernel was introduced by Borowiecki and Kwaśnik in [34] as follows. If $k \geq 2$, a subset S of vertices of a digraph D is a *k-independent set*, if for every pair of different vertices in S, every walk between them has length at least k. If $l \geq 1$, we say that S is an *l-absorbent set* if for every $x \in V(D) \setminus S$ there exists an xS-walk with length at most l. If $k \geq 2$ and $l \geq 1$, a *(k, l)-kernel* is a subset of $V(D)$ which is both k-independent and l-absorbent. If $l = k - 1$, the (k, l)-kernel is called *k-kernel*. Notice that every 2-kernel is a kernel. A $(2, 2)$-kernel is called *quasi-kernel*, and it is known that every digraph has a quasi-kernel [11] (called semi-kernel by the authors). Several sufficient conditions for the existence of k-kernels in nearly tournaments have been proved. For instance, see [4, 30, 32] and [41].

Let H be a digraph, possibly with loops, D a digraph without loops, and $\rho : A(D) \to V(H)$ a coloring of $A(D)$ (D is said to be an *H-colored digraph*). A path W in D is an *H-path*, whenever the consecutive colors encountered on W form a walk in H (the concepts of *H-cycle* and *H-walk* are defined analogously). If $W = (x_0, \ldots, x_n)$ is an H-path, W is said to be an *$x_0 x_n$-H-path*. If $S \subseteq V(D)$ and $x_n \in S$, we say that W is an *$x_0 S$-H-path*. In [35] Linek and Sands introduced the concept of H-walk and their work was later considered by several authors, as example, [2, 13, 22] and [38]. A subset S of vertices of D is *absorbent by*

H-*paths*, if for every $x \in V(D) \setminus S$ there exists an xS-H-path; and S is said to be *independent by H-paths*, if for every pair of different vertices $\{u, v\} \subseteq S$, there is no uv-H-path between them. A *kernel by H-paths*, or simply *H-kernel*, is a subset of vertices of D that is both absorbent by H-paths and independent by H-paths. Several interesting kinds of kernels are particular cases of H-kernels, as example, kernels by paths, kernels by monochromatic paths, kernels by alternating paths, kernels by rainbow paths, and usual kernels. Several conditions on the existence of H-kernels have been showed, as example, see [22] and [23].

If $W = (x_0, \ldots, x_n)$ is a walk in an H-colored digraph D, and $i \in \{1, \ldots, n-1\}$ ($i \in \{0, \ldots, n-1\}$ when $x_0 = x_n$), we say that there is an *obstruction on x_i* if and only if $(\rho(x_{i-1}, x_i), \rho(x_i, x_{i+1})) \notin A(H)$ (indices are taken modulo n if $x_0 = x_n$). We denote by $O_H(W)$ the set $\{i \in \{0, \ldots, n-1\} :$ there is an obstruction on $x_i\}$. The *H-length of W*, denoted by $l_H(W)$, is defined as $l_H(W) = |O_H(W)|$ if W is closed or $l_H(W) = |O_H(W)| + 1$ otherwise. The H-length was primarily studied by Galeana-Sánchez and Sánchez-López in [22] for closed walks, and by Andenmatten, Galeana-Sánchez and Pach in [1] for open paths. Clearly, the usual length $l(W)$ coincides with the H-length $l_H(W)$, in the very particular case when $A(H) = \emptyset$. An open walk in an H-colored digraph is an *H-walk* if and only if it has H-length 1. A particular kind of H-coloring in graphs was studied by Szeider in [39] and, as a consequence, in [1] it was proved that under the assumption P $\neq$ NP, finding uv-paths of minimum H-length in H-colored graphs (H-colored digraphs) has no polynomial solution (although there is a polynomial algorithm to find uv-paths of minimum H-length for some H, see [1]).

Let D be an H-colored digraph and S a subset of vertices of D. If $l \geq 1$, we say that S is an *(l, H)-absorbent set*, if for every $v \in V(D) \setminus S$ there exists a vS-path whose H-length is at most l. If $k \geq 2$, we say that S is a *(k, H)-independent set*, if for every pair of different vertices in S, every path between them has H-length at least k. If $k \geq 2$ and $l \geq 1$, we say that S is a *(k, l, H)-kernel* if it is both (k, H)-independent and (l, H)-absorbent. If $l = k - 1$, a (k, l, H)-kernel is called *(k, H)-kernel*. It is straightforward to see that every H-kernel is a $(2, H)$-kernel, and every (k, l)-kernel is a (k, l, H)-kernel if H has no arcs nor loops. Since finding (k, l)-kernels in digraphs is an NP-complete problem, finding (k, l, H)-kernels in H-colored digraphs is also an NP-complete problem.

In this paper, we study the existence of (k, H)-kernels in H-colored nearly tournaments. For instance, we will prove that if D is an H-colored digraph, D has a (k, H)-kernel provided that:

 (i) $k \geq 3$ and D is a tournament;
 (ii) $k \geq r \geq 2$ and D is an r-transitive digraph;
(iii) $k \geq 4$ and D is a quasi-transitive digraph;
(iv) $k \geq 5$ and D is a 3-quasi-transitive digraph;

(v) $k \geq r \geq 2$, D is an r-quasi-transitive digraph, and every cycle with length $r + 1$ is an H-cycle;

(vi) $k \geq 5$ and D is an r-partite tournament $(r \geq 2)$;

(vii) D is a local in-tournament (out-tournament) and every cycle has H-length at most $k - 2$.

Previous results for k-kernels and kernels by alternating paths will be generalized. Also, some conditions for the existence of kernels by monochromatic paths and H-kernels in nearly tournaments will be shown.

2. Preliminaries

For terminology and notation not defined here, we refer the reader to [6]. Two vertices in a digraph are *adjacent* if there exist an arc between them. In this paper we write walk, path and cycle, instead of directed walk, directed path, and directed cycle, respectively. If $W = (x_0, \ldots, x_n)$ is a walk (path), we say that W is an $x_0 x_n$-*walk* ($x_0 x_n$-*path*). If $n \geq 2$ and $i \in \{1, \ldots, n - 1\}$, we say that x_i is an *internal vertex of* W. The *length* of W is the number n and it is denoted by $l(W)$. A cycle with length k is called k-*cycle*.

If $T_1 = (z_0, \ldots, z_n)$ and $T_2 = (w_0, \ldots, w_m)$ are walks in a digraph D, and $z_n = w_0$, we denote by $T_1 \cup T_2$ the walk $(z_0, \ldots, z_n = w_0, \ldots, w_m)$. Let $\{v_0, \ldots, v_n\}$ be a subset of vertices of D, and for every $i \in \{0, \ldots, n - 1\}$, T_i a $v_i v_{i+1}$-walk; we denote by $\bigcup_{i=0}^{n-1} T_i$ the concatenation of such walks. Given $W = (x_0, \ldots, x_n)$ a walk in D, and $\{i, j\} \subseteq \{0, \ldots, n - 1\}$ with $i < j$, we denote by (x_i, W, x_j) the walk $(x_i, x_{i+1}, \ldots, x_j)$. If S_1 and S_2 are two disjoint subsets of $V(D)$, a uv-walk in D is called an $S_1 S_2$-*walk* whenever $u \in S_1$ and $v \in S_2$. If $S_1 = \{x\}$ or $S_2 = \{x\}$, then we write $x S_2$-*walk* or $S_1 x$-*walk*, respectively. If there exists at least one uv-path in D, a uv-path with minimum length is called a uv-*geodesic*, and its length is denoted by $d_D(u, v)$.

An *m-colored digraph* is a digraph whose arcs are colored with m colors. If D is an m-colored digraph, a walk W in D is *monochromatic* (*alternating*) if every two consecutive arcs in W are colored alike (respectively, every two consecutive arcs in W have different colors). A *kernel by monochromatic paths* (*kernel by properly colored paths*) is a subset of $V(D)$, say S, such that (i) no two different vertices in S are joined by a monochromatic path (respectively, alternating path) and (ii) for every $w \in V(D) \setminus S$, there exists $x \in S$ and a wx-monochromatic path (respectively, a wx-alternating path).

Let D be an H-colored digraph and $k \geq 2$. We define the $(k - 1, H)$-*closure* of D, denoted by $C_H^{k-1}(D)$, as the digraph such that $V(C_H^{k-1}(D)) = V(D)$ and $(u, v) \in A\big(C_H^{k-1}(D)\big)$ if and only if there exists a uv-path in D with H-length at most $k - 1$. The following lemmas will be useful in what follows.

Lemma 3. *Let D be an H-colored digraph and $k \geq 2$. The following assertions hold.*

(a) *If W is an open walk in D, then $l_H(W) \leq l(W)$.*

(b) *If $\{u, v\} \subseteq V(D)$, and there exists a uv-walk with length at most $k - 1$ in D, then $(u, v) \in A\big(C_H^{k-1}(D)\big)$.*

(c) *If $\{u, v\} \subseteq V(D)$, and there exists a closed walk in D with length at most k, say W, such that $\{u, v\} \subseteq V(W)$, then (u, v) is a symmetric arc in $C_H^{k-1}(D)$.*

Proof. It follows from the definition of H-length that (a) holds. On the other hand, from Lemma 3(a), and the definition of $(k - 1, H)$-closure, we have that (b) holds. In order to show that (c) holds, notice that two different vertices in W are joined by a subpath of W with length at most $k - 1$. By applying Lemma 3(b), we can conclude (c). ∎

Lemma 4. *Let D be an H-colored digraph, and $W = (x_0, \ldots, x_n)$ a cycle in D. If $\{i, j\} \subseteq \{0, \ldots, n\}$ with $i < j$, and $W' = (x_i, W, x_j)$, then $O_H(W') \subseteq O_H(W)$.*

Proof. Clearly, Lemma 4 holds when $O_H(W') = \emptyset$. Now, suppose that $O_H(W') \neq \emptyset$ and let $t \in O_H(W')$. It follows from the definition of $O_H(W')$ that $t \in \{i + 1, \ldots, j - 1\}$ and $(\rho(x_{t-1}, x_t), \rho(x_t, x_{t+1})) \notin A(H)$. Hence, $t \in \{0, \ldots, n - 1\}$ and $(\rho(x_{t-1}, x_t), \rho(x_t, x_{t+1})) \notin A(H)$, which implies that $t \in O_H(W)$. Therefore, $O_H(W') \subseteq O_H(W)$. ∎

Given $k \geq 2$, a natural relation between (k, H)-kernels in H-colored digraphs and kernels in the $(k - 1, H)$-closure is showed in the following lemma.

Lemma 5. *Let D be an H-colored digraph and $k \geq 2$. D has a (k, H)-kernel if and only if $C_H^{k-1}(D)$ has a kernel.*

Proof. First we will show that if D has a (k, H)-kernel, say S, then S is a kernel in $C_H^{k-1}(D)$.

In order to show that S is an absorbent set in $C_H^{k-1}(D)$, consider $x \in V(C_H^{k-1}(D)) \setminus S$. Since S is a (k, H)-kernel in D, there exists $w \in S$, and an xw-path in D with H-length at most $k - 1$. It follows from the definition of $C_H^{k-1}(D)$ that $(x, w) \in A\big(C_H^{k-1}(D)\big)$, which implies that S is an absorbent set in $C_H^{k-1}(D)$.

Now, we will show that S is an independent set in $C_H^{k-1}(D)$. Proceeding by contradiction, suppose that there exists $\{u, v\} \subseteq S$ such that $(u, v) \in A\big(C_H^{k-1}(D)\big)$. It follows from the definition of $C_H^{k-1}(D)$ that there exists a uv-path with length at most $k - 1$ in D, which is not possible since S is a (k, H)-independent set in D. Hence, S is an independent set in $C_H^{k-1}(D)$. Therefore, if D has a (k, H)-kernel, then $C_H^{k-1}(D)$ has a kernel.

Now, we will show that if $C_H^{k-1}(D)$ has a kernel, say S', then S' is a (k, H)-kernel in D.

In order to show that S' is a $(k-1, H)$-absorbent set in D, consider $x \in V(D) \setminus S'$. Since S' is a kernel in $C_H^{k-1}(D)$, there exists $w \in S'$ such that $(x, w) \in A\big(C_H^{k-1}(D)\big)$. Hence, by the definition of $C_H^{k-1}(D)$, there exists an xw-path in D with H-length at most $k-1$, which implies that S' is a $(k-1, H)$-absorbent set in D.

Now, we will show that S' is a (k, H)-independent set in D. Proceeding by contradiction, suppose that there exists $\{u, v\} \subseteq S'$ and a uv-path in D, with H-length at most $k-1$. It follows from the definition of $C_H^{k-1}(D)$ that $(u, v) \in A\big(C_H^{k-1}(D)\big)$, which is not possible since S' is an independent set in $C_H^{k-1}(D)$. Hence, S' is a (k, H)-independent set in $C_H^{k-1}(D)$. Therefore, if $C_H^{k-1}(D)$ has a kernel, then D has a (k, H)-kernel. ∎

3. (k, H)-Kernels in Nearly Tournaments

In this section, we will show several conditions in H-colored nearly tournaments that guarantee the existence of (k, H)-kernels for certain values of k.

3.1. Tournaments and semicomplete digraphs

Theorem 6. *Let D be an H-colored semicomplete digraph. If $k \geq 3$, then D has a (k, H)-kernel.*

Proof. Let S be a quasi-kernel of D. Since D is semicomplete, then $S = \{x\}$ for some $x \in V(D)$. Clearly, S is a (k, H)-independent set in D. On the other hand, since for every vertex $w \in V(D) \setminus S$ it holds that $d_D(w, x) \leq 2$, it follows from Lemma 3(a) that S is a $(k-1, H)$-absorbent set in D. Therefore, S is a (k, H)-kernel in D. ∎

Theorem 7. *Let D be an H-colored semicomplete digraph in which every 3-cycle is an H-cycle. If $k \geq 2$, then D has a (k, H)-kernel.*

Proof. It follows from Theorem 6 that it only remains to show that D has a kernel by H-paths. Consider a quasi-kernel of D, say S. Since D is semicomplete, then $S = \{x\}$ for some $x \in V(D)$. Clearly, S is an independent set by H-paths. In order to show that S is an absorbent set by H-paths, consider $w \in V(D) \setminus S$. By the choice of S, we have that either $d_D(w, x) = 1$ or $d_D(w, x) = 2$. If $d_D(w, x) = 1$, then (w, x) is a wx-H-path. If $d_D(w, x) = 2$, consider a wx-geodesic in D, say (w, z, x). Since D is semicomplete, then w and x are joined by an arc in D and, since $d_D(w, x) = 2$, it follows that $(x, w) \in A(D)$. Hence, $C = (w, z, x, w)$ is a

3-cycle in D and, by hypothesis, C is an H-cycle. We can conclude that (w, z, x) is a wx-H-path. Therefore, S is an absorbent set by H-paths in D. ∎

From Theorems 6 and 7, it is straightforward to see the following corollaries.

Corollary 8. *If D is an H-colored tournament, then for every $k \geq 3$, D has a (k, H)-kernel.*

Corollary 9. *If D is an H-colored tournament, and every 3-cycle in D is an H-cycle, then for every $k \geq 2$, D has a (k, H)-kernel.*

Corollary 10. *Let D be an m-colored tournament such that every 3-cycles is monochromatic. Then D has a kernel by monochromatic paths.*

Proof. Let H be the digraph whose vertices are the colors represented in $A(D)$, and $(c, c') \in A(H)$ if and only if $c = c'$. Clearly, D is an H-colored digraph, and a walk in D is monochromatic if and only if it is an H-walk. Notice that every 3-cycle in D is an H-cycle.

It follows from Corollary 9 that D has an H-kernel, which is a kernel by monochromatic paths. ∎

Corollary 11. *If D is a tournament whose 3-cycles are alternating, then D has a kernel by alternating paths.*

Proof. Let H be the digraph whose vertices are the colors represented in $A(D)$, and $(c, c') \in A(H)$ if and only if $c \neq c'$. Clearly, D is an H-colored digraph, and a walk in D is alternating if and only if it is an H-walk. Notice that every 3-cycle in D is an H-cycle.

It follows from Corollary 9 that D has an H-kernel, which is a kernel by alternating paths. ∎

3.2. r-transitive digraphs

The following lemma will be useful in what follows.

Lemma 12 [20]**.** *Let D be an r-transitive digraph $(r \geq 2)$, and $\{u, v\} \subseteq V(D)$. If there exists a uv-path in D, then $d_D(u, v) \leq r - 1$.*

As a consequence of the previous lemma, we have the following results.

Theorem 13. *Let D be an H-colored r-transitive digraph $(r \geq 2)$. For every $l \geq r - 1$ and $k \geq 2$, D has a (k, l, H)-kernel.*

Proof. Let N be a kernel by paths in D. Since N is a path-independent set, then for every $k \geq 2$, N is a (k, H)-independent set. On the other hand, consider $x \in V(D) \setminus N$. Since N is a kernel by paths in D, there exists $w \in N$ and an

xw-path W. If W is an xw-geodesic, then $l(W) \leq r - 1$ (Lemma 12), which implies that $l_H(W) \leq r - 1$ (Lemma 3(a)). Hence, $l_H(W) \leq l$, concluding that N is an (l, H)-absorbent set. Therefore, N is a (k, l, H)-kernel. ∎

Corollary 14. *Let D be an H-colored r-transitive digraph. For every $k \geq r$, D has a (k, H)-kernel.*

Corollary 15. *Let D be an H-colored transitive digraph. If $k \geq 2$, then D has a (k, H)-kernel.*

Corollary 16 [20]**.** *If D is an r-transitive digraph ($r \geq 2$), then D has a k-kernel for every $k \geq 2$.*

Proof. Let $k \geq 2$ and H a digraph with order $|A(D)|$ and $A(H) = \emptyset$. We consider an H-coloring of D in which every two different arcs have different colors. Notice that for every walk W in D, $l_H(W) = l(W)$. It follows from Corollary 14 that D has a (k, H)-kernel, say S. It is straightforward to see that S is a k-kernel. ∎

3.3. Quasi-transitive digraphs.

The following lemma provides a nice structure on quasi-transitive digraphs, and will be useful to guarantee the existence of (k, H)-kernels for certain values of k in H-colored quasi-transitive digraphs.

Lemma 17 [6]**.** *Let D be a quasi-transitive digraph and $\{u, v\} \subseteq V(D)$ such that there exists a uv-path in D. If u and v are not adjacent in D, then there exists $\{x, z\} \subseteq V(D) \setminus \{u, v\}$ such that (u, x), (x, z), (z, v), (z, u) and (v, x) are arcs of D (see Figure 1).*
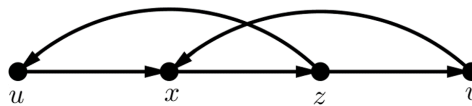

Figure 1.

As a consequence of the previous lemma, we have the following result.

Lemma 18. *Let D be an H-colored quasi-transitive digraph, and $k \geq 4$. If there exists a uv-walk in D, then either u and v are adjacent in D or (v, u) is a symmetric arc in $C_H^{k-1}(D)$.*

Proof. If (u, v) or (v, u) is an arc of D, we are done. Now, suppose that u and v are not adjacent in D. It follows from Lemma 17 that there exists $\{x, z\} \subseteq V(D) \setminus \{u, v\}$ such that (u, x), (x, z), (z, v), (z, u) and (v, x) are all arcs of D. Hence, (u, x, z, v) and (v, x, z, u) are paths in D with length 3. By Lemma 3(b), we can conclude that (u, v) is a symmetric arc in $C_H^{k-1}(D)$. ∎

Theorem 19. *Let D be an H-colored quasi-transitive digraph. If $k \geq 4$, then D has a (k, H)-kernel.*

Proof. First, we will show that every cycle in $C_H^{k-1}(D)$ has a symmetric arc, then, by applying Theorem 2 and Lemma 5, we will conclude that D has a (k, H)-kernel. Proceeding by contradiction, suppose that there exist a cycle in $C_H^{k-1}(D)$, say $C = (u_0, \ldots, u_n)$, with no symmetric arcs. Clearly, $n \geq 3$.

It follows from Lemma 18 that, for every $i \in \{0, \ldots, n-1\}$, u_i and u_{i+1} are adjacent in D (indices are taken modulo n). Moreover, since C has no symmetric arcs in $C_H^{k-1}(D)$, we can conclude that for every $i \in \{0, \ldots, n-1\}$, $(u_i, u_{i+1}) \in A(D)$, which implies that C is a cycle in D. Let $q = \max\{i \in \{1, \ldots, n-1\} : (u_0, u_i) \in A(D)\}$.

Notice that $q \leq n-2$, otherwise, (u_{n-1}, u_0) is a symmetric arc in D, which implies that C has a symmetric arc in $C_H^{k-1}(D)$, contradicting the choice of C. Hence, (u_0, u_q, u_{q+1}) is a path in D. Since D is a quasi-transitive digraph, we have that either $(u_{q+1}, u_0) \in A(D)$ or $(u_0, u_{q+1}) \in A(D)$. By the choice of q, it follows that $(u_{q+1}, u_0) \in A(D)$. We can conclude that (u_{q+1}, u_0, u_q) is a path in D with length $2 \leq k-1$, and, by Lemma 3(b), we have that $\left(u_{q+1}, u_q\right) \in A\left(C_H^{k-1}(D)\right)$, contradicting the choice of C. Therefore, every cycle in $C_H^{k-1}(D)$ has a symmetric arc.

It follows from Theorem 2 that $C_H^{k-1}(D)$ has a kernel, which implies that D has a (k, H)-kernel (Lemma 5). ∎

Corollary 20 [19]**.** *If D is a quasi-transitive digraph, then for every $k \geq 4$, D has a k-kernel.*

Proof. Let $k \geq 4$ and H be a digraph with order $|A(D)|$ and $A(H) = \emptyset$. We consider an H-coloring of D in which every two different arcs of D have different colors. Notice that for every walk W in D, $l_H(W) = l(W)$. It follows from Theorem 19 that D has a (k, H)-kernel, say S. It is straightforward to see that S is a k-kernel of D. ∎

3.4. 3-quasi-transitive digraphs

The following lemma for 3-quasi-transitive digraphs will be useful.

Lemma 21 [20]**.** *Let D be a 3-quasi-transitive digraph, and $\{u, v\} \subseteq V(D)$ such that there exists a uv-walk in D. The following assertions hold.*

 (a) *If $d(u, v) = 3$ or $d(u, v) \geq 5$, then $d(v, u) = 1$.*
 (b) *If $d(u, v) = 4$, then $d(v, u) \leq 4$.*

By the previous lemma, we have the following corollary.

Corollary 22. *Let D be an H-colored 3-quasi-transitive digraph, $k \geq 5$, and $(u, v) \in A\big(C_H^{k-1}(D)\big)$. If (u, v) is an asymmetric arc in $C_H^{k-1}(D)$, then $d_D(u, v) \leq 2$.*

Proof. Proceeding by contradiction, suppose that $d_D(u, v) \geq 3$. It follows from Lemma 21 that there exists a vu-path with length at most 4 in D. By Lemma 3(b), we can conclude that $(v, u) \in A\big(C_H^k(D)\big)$, which is not possible since (u, v) is an asymmetric arc in $C_H^{k-1}(D)$. Hence, $d_D(u, v) \leq 2$. ∎

A more elaborated proof will show that every H-colored 3-quasi-transitive digraph has a (k, H)-kernel for every $k \geq 5$.

Theorem 23. *Let D be an H-colored 3-quasi-transitive digraph. For every $k \geq 5$, D has a (k, H)-kernel.*

Proof. First, we will show that every cycle in $C_H^{k-1}(D)$ has a symmetric arc, then, by applying Theorem 2 and Lemma 5, we will conclude that D has a (k, H)-kernel. Proceeding by contradiction, suppose that there exist a cycle in $C_H^{k-1}(D)$, say $C = (u_0, \ldots, u_n)$, with no symmetric arcs. Clearly, $n \geq 3$.

For every $i \in \{0, \ldots, n-1\}$, we consider a $u_i u_{i+1}$-geodesic in D, say W_i (indices are taken modulo n). It follows from Corollary 22 that for every $i \in \{0, \ldots, n-1\}$, $l(W_i) \leq 2$. Let $C' = \bigcup_{i=0}^{n-1} W_i$, and assume that $C' = (x_0, \ldots, x_l)$. Notice that C' is a closed walk in D, and $x_0 = x_l = u_0 = u_n$.

Claim 1. *For every $i \in \{l-3, l-2, l-1, l\}$, $(u_1, x_i) \notin A(D)$.*

Proceeding by contradiction, suppose that there exists $i \in \{l-3, l-2, l-1, l\}$, such that $(u_1, x_i) \in A(D)$. Since C is asymmetric, we have that (u_0, u_1) is an asymmetric arc in $C_H^{k-1}(D)$, then $i \neq l$. It follows that $(u_1, x_i) \cup (x_i, C', x_0)$ is a $u_1 u_0$-walk in D with length at most 4, which implies that $(u_1, u_0) \in A\big(C_H^{k-1}(D)\big)$ (Lemma 3(b)), contradicting the choice of C, and the claim holds.

On the other hand, since $u_1 \in V(C')$, then u_1 has at least one out-neighbor in $V(C')$, and let $q = \max\{i \in \{1, \ldots, l\} : (u_1, x_i) \in A(D)\}$. The following assertions will be useful:

(i) $q \leq l - 4$.
   It follows from Claim 1.
(ii) For every $t \in \{q+1, q+2, q+3, q+4\}$, $x_q \neq x_t$.
   It follows from the choice of q.
(iii) For every $t \in \{q, q+1, q+2, q+3, q+4\}$, $u_1 \neq x_t$.

If $q < l - 4$, then by the choice of q, we have that $u_1 \neq x_i$ for every $i \in \{q, q+1, q+2, q+3, q+4\}$. If $q = l - 4$ (that is, $x_{q+4} = x_l$), then by the choice

of q, we have that $u_1 \neq x_i$ for every $i \in \{q, q+1, q+2, q+3\}$ and, since $u_0 = x_l$, we have that $u_1 \neq x_l$.

It follows from (ii) and (iii) that $(u_1, x_q, x_{q+1}, x_{q+2})$ is a path in D. Since D is a 3-quasi-transitive digraph, then either $(u_1, x_{q+2}) \in A(D)$ or $(x_{q+2}, u_1) \in A(D)$. By the choice of q, we have that $(x_{q+2}, u_1) \in A(D)$.

Claim 2. *There exists $t \in \{0, \dots, n-1\}$ such that $x_{q+1} = u_t$ and $x_{q+3} = u_{t+1}$.*

First, we will show that (x_q, x_{q+1}, x_{q+2}) has no two consecutive vertices of C, such a property will be called (*). Proceeding by contradiction, suppose that (x_q, x_{q+1}, x_{q+2}) has two consecutive vertices of C, say u_r and u_{r+1}. Since $(u_1, x_q, x_{q+1}, x_{q+2}, u_1)$ is a 4-cycle in D, it follows from Lemma 3(c) that u_r and u_{r+1} are joined by a symmetric arc in $C_H^{k-1}(D)$, which is not possible by the choice of C. Therefore, property (*) holds.

On the other hand, from the choice of C', we can conclude that there exists $t \in \{0, \dots, n-1\}$ such that $A(W_t) \cap \{(x_q, x_{q+1}), (x_{q+1}, x_{q+2})\} \neq \emptyset$. Remark that $l(W_t) \leq 2$. It follows from property (*) that $x_{q+1} = u_t$ and $x_{q+3} = u_{t+1}$, and the claim holds (see Figure 2).
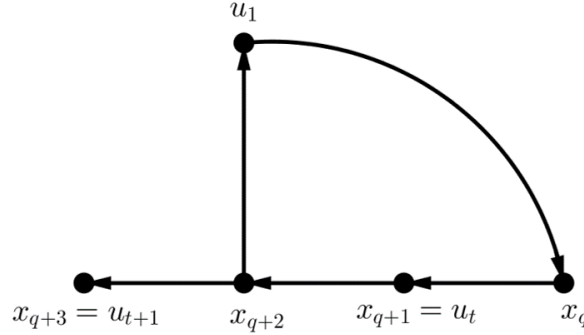

Figure 2.

Now, we will show that $(x_q, x_{q+1}, \dots, x_{q+4})$ is a path in D. Remark that $x_q \notin \{x_{q+1}, x_{q+2}, x_{q+3}, x_{q+4}\}$ (because of (ii)). In order to show that $x_{q+1} \notin \{x_{q+2}, x_{q+3}, x_{q+4}\}$, notice that, since $(x_{q+1}, x_{q+2}) \in A(D)$, then $x_{q+1} \neq x_{q+2}$. Since $u_t = x_{q+1}$ and $u_{t+1} = x_{q+3}$ (Claim 2), then $x_{q+1} \neq x_{q+3}$. Now, $x_{q+1} \neq x_{q+4}$, otherwise, $(u_{t+1}, u_t) \in A\big(C_H^{k-1}(D)\big)$, which is not possible by the choice of C. In order to show that $x_{q+2} \notin \{x_{q+3}, x_{q+4}\}$, we have that $x_{q+2} \neq x_{q+3}$ because they are joined by an arc in D. Moreover, we can see that $x_{q+2} \neq x_{q+4}$, otherwise, $(u_{t+1}, x_{q+2}, u_1, x_q, u_t)$ is a $u_{t+1}u_t$-walk in D with length at most 4, and by Lemma 3(b), $(u_{t+1}, u_t) \in A\big(C_H^{k-1}(D)\big)$, which is not possible by the choice of C. Finally, $x_{q+3} \neq x_{q+4}$ because they are joined by an arc in D. Hence, $(x_q, x_{q+1}, \dots, x_{q+4})$ is a path in D.

Since D is a 3-quasi-transitive digraph, then we have that either $(x_q, x_{q+3}) \in A(D)$ or $(x_{q+3}, x_q) \in A(D)$. If $(x_{q+3}, x_q) \in A(D)$, then (x_{q+3}, x_q, x_{q+1}) is a $u_{t+1}u_t$-path in D with length at most $k - 1$, which implies that $(u_{t+1}, u_t) \in A\big(C_H^{k-1}(D)\big)$, contradicting the choice of C. Hence, $(x_q, x_{q+3}) \in A(D)$. It follows that $(u_1, x_q, x_{q+3}, x_{q+4})$ is a 3-path in D, concluding that either $(u_1, x_{q+4}) \in A(D)$ or $(x_{q+4}, u_1) \in A(D)$. By the choice of q, we have that $(x_{q+4}, u_1) \in A(D)$. In that case, $(x_{q+3}, x_{q+4}, u_1, x_q, x_{q+1})$ is a $u_{t+1}u_t$-path in D with length at most $k - 1$, concluding that $(u_{t+1}, u_t) \in A\big(C_H^{k-1}(D)\big)$ (Lemma 3(b)), contradicting the choice of C (see Figure 3).
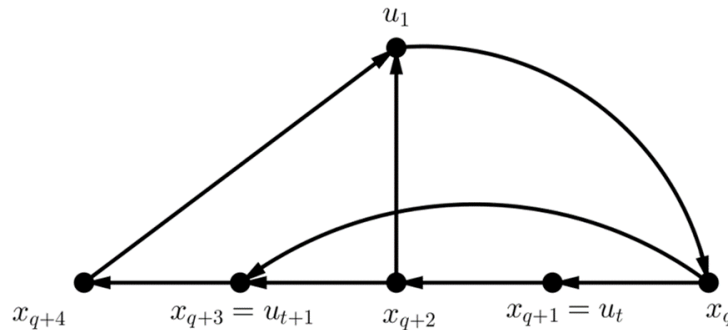

Figure 3.

Therefore, every cycle in $C_H^{k-1}(D)$ has a symmetric arc, concluding that $C_H^{k-1}(D)$ has a kernel (Theorem 2), which implies that D has a (k, H)-kernel (Lemma 5). ∎

As a direct consequence of Theorem 23 we have the following corollary. An analogous result can be found in [20].

Corollary 24 [20]. *If D is a 3-quasi-transitive digraph, then D has a k-kernel for every $k \geq 5$.*

Proof. Let $k \geq 5$ and H a digraph with order $|A(D)|$ and $A(H) = \emptyset$, and consider an H-coloring of D in which every two different arcs have different colors. Notice that for every walk W in D, $l_H(W) = l(W)$. It follows from Theorem 23 that D has a (k, H)-kernel, say S. It is straightforward to see that S is a k-kernel of D. ∎

It follows from Theorems 19 and 23 that we have the following conjecture.

Conjecture 25. *If D is an H-colored r-quasi-transitive digraph $(r \geq 2)$, then D has a (k, H)-kernel for every $k \geq r + 2$.*

3.5. r-quasi-transitive digraphs

Theorem 27 will show that, under certain conditions on the cycles of an H-colored r-quasi-transitive digraph, it is possible to guarantee the existence of (k, H)-kernels in such digraphs. First, we have the following result.

Theorem 26. *Let D be an H-colored r-quasi-transitive digraph $(r \geq 2)$, such that every cycle in D with length $r + 1$ is an H-cycle, and let $\{u, v\} \subseteq V(D)$. If $d_D(u, v) \geq r$ and T is a uv-geodesic in D, then T is an H-path.*

Proof. Let $T = (u = x_0, \ldots, x_n = v)$. If $n = r$, it follows from the fact that D is an r-quasi-transitive digraph that either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Since $d_D(u, v) \geq r$, then we have that $(v, u) \in A(D)$. Hence, $C' = T \cup (v, u)$ is a cycle with length $r + 1$. By hypothesis, C' is an H-cycle, which implies that T is an H-path in D.

Now, suppose that $n > r$. In order to show that T is an H-path, we will show that T has no obstructions. Let $i \in \{1, \ldots, n - 1\}$, and consider the following cases.

Case 1. $i \in \{1, \ldots, n - r\}$. In this case, notice that $(x_{i-1}, x_i, \ldots, x_{i+r-1})$ is a path with length r. Since D is an r-quasi-transitive digraph, then either $(x_{i-1}, x_{i+r-1}) \in A(D)$ or $(x_{i+r-1}, x_{i-1}) \in A(D)$. It follows from the fact that T is a uv-geodesic that $(x_{i-1}, x_{i+r-1}) \notin A(D)$, which implies that $(x_{i+r-1}, x_{i-1}) \in A(D)$. Hence, $(x_{i-1}, x_i, \ldots, x_{i+r-1}, x_{i-1})$ is a cycle with length $r + 1$, which is an H-cycle by hypothesis. Therefore, $(\rho(x_{i-1}, x_i), \rho(x_i, x_{i+1})) \in A(H)$.

Case 2. $i \in \{n - r + 1, \ldots, n - 1\}$. In this case, notice that $(x_{n-r}, \ldots, x_{i-1}, x_i, x_{i+1}, \ldots, x_n)$ is a path with length r. Since D is an r-quasi-transitive digraph, then either $(x_{n-r}, x_n) \in A(D)$ or $(x_n, x_{n-r}) \in A(D)$. It follows from the fact that T is a uv-geodesic that $(x_{n-r}, x_n) \notin A(D)$, which implies that $(x_n, x_{n-r}) \in A(D)$. Hence, $(x_{n-r}, x_{n-r+1}, \ldots, x_n, x_{n-r})$ is a cycle with length $r + 1$, which is an H-cycle by hypothesis. Therefore, $(\rho(x_{i-1}, x_i), \rho(x_i, x_{i+1})) \in A(H)$

By the previous cases, we conclude that T is an H-path. ∎

Theorem 27. *Let D be an H-colored r-quasi-transitive digraph $(r \geq 2)$, such that every cycle with length $r + 1$ is an H-cycle. For every $k \geq r$, D has a (k, H)-kernel.*

Proof. First, we will show that for every $k \geq r$, $C_H^{k-1}(D)$ is a transitive digraph. Then, by applying Theorem 1, we can conclude that $C_H^{k-1}(D)$ has a kernel and, by Lemma 5, D has a (k, H)-kernel.

Consider (u, v) and (v, w) in $A\big(C_H^{k-1}(D)\big)$ with $u \neq w$. It follows that there exists a uv-path in D with H-length at most $k - 1$, say T_1, and there exists a vw-path in D with H-length at most $k - 1$, say T_2. Hence, $T_1 \cup T_2$ is a uw-walk in D. Now, consider a uw-geodesic in D, say T. If $l(T) \leq k - 1$, then

$(u, w) \in A\big(C_H^{k-1}(D)\big)$ (Lemma 3(b)). If $l(T) \geq k$, then by Theorem 26 we have that T is a uw-H-path, which implies that $(u, w) \in A\big(C_H^{k-1}(D)\big)$.

Therefore, $C_H^{k-1}(D)$ is a transitive digraph, which implies that $C_H^{k-1}(D)$ has a kernel (Theorem 1). We can conclude that D has a (k, H)-kernel (Lemma 5). ∎

As a consequence of the previous theorem, we have the following corollaries.

Corollary 28. *If D is an r-quasi-transitive digraph ($r \geq 2$) with no cycle of length $r + 1$, then for every $k \geq r$, D has a k-kernel.*

Proof. Let $k \geq r$ and H a digraph with order $|A(D)|$ and $A(H) = \emptyset$. We consider an H-coloring of D in which every two different arcs have different colors. Notice that for every walk W in D, $l_H(W) = l(W)$. It follows from Theorem 27 that D has a (k, H)-kernel, say S. It is straightforward to see that S is a k-kernel of D. ∎

Corollary 29. *Let D be an H-colored quasi-transitive digraph such that every 3-cycle is an H-cycle. For every $k \geq 2$, D has a (k, H)-kernel.*

Corollary 30. *If D is an m-colored quasi-transitive digraph in which every 3-cycle is monochromatic, then D has a kernel by monochromatic paths.*

Proof. Let H be the digraph whose vertices are the colors represented in $A(D)$, and $(u, v) \in A(H)$ if and only if $u = v$. Clearly, D is an H-colored digraph, and a walk in D is an H-walk if and only if it is monochromatic. It follows from Corollary 29 that D has a $(2, H)$-kernel, say S. It is straightforward to see that S is a kernel by monochromatic paths. ∎

Corollary 31 [14]. *If D is an m-colored quasi-transitive digraph in which every 3-cycle is alternating, then D has a kernel by alternating paths.*

Proof. Let H be the digraph whose vertices are the colors represented in $A(D)$ and $(u, v) \in A(H)$ if and only if $u \neq v$. Clearly, D is an H-colored digraph, and a walk in D is an H-walk if and only if it is alternating. It follows from Corollary 29 that D has a $(2, H)$-kernel, say S. It is straightforward to see that S is a kernel by alternating paths. ∎

3.6. Multipartite tournaments

The following lemma will be useful in order to show that every H-colored multipartite tournament has a (k, H)-kernel, for certain values of k.

Lemma 32. *Let D be an H-colored r-partite tournament ($r \geq 2$), $k \geq 5$, and $(u, v) \in A\big(C_H^{k-1}(D)\big)$. If (u, v) is an asymmetric arc in $C_H^{k-1}(D)$, then $d_D(u, v) \leq 2$.*

Proof. Let $\{S_1, \ldots, S_r\}$ be the partition of $V(D)$ into independent sets. Proceeding by contradiction, suppose that $d_D(u, v) \geq 3$, and let $W = (u = x_0, \ldots, x_n = v)$ be a uv-geodesic in D. If $u \in S_i$ and $v \in S_j$ with $i \neq j$, then either $(u, v) \in A(D)$ or $(v, u) \in A(D)$. Since $d_D(u, v) \geq 3$, we conclude that $(v, u) \in A(D)$, which implies that $(v, u) \in A\big(C_H^{k-1}(D)\big)$, contradicting the assumption on (u, v). Hence, there exists $t \in \{1, \ldots, r\}$ such that $\{u, v\} \subseteq S_t$. Consider the following cases.

Case 1. $l(W) = 3$. Since D is an r-partite tournament, then either $(x_1, v) \in A(D)$ or $(v, x_1) \in A(D)$. It follows from the fact that W is a uv-geodesic that $(x_1, v) \notin A(D)$, which implies that $(v, x_1) \in A(D)$. A similar argument shows that $(x_2, u) \in A(D)$. Hence, (v, x_1, x_2, u) is a walk in D with length $3 \leq k - 1$. By Lemma 3(b) we have that $(v, u) \in A\big(C_H^{k-1}(D)\big)$, contradicting the assumption on (u, v), and the claim holds.

Case 2. $l(W) \geq 4$. First, we will show that for every $i \in \{2, \ldots, n - 2\}$, $x_i \in S_t$. Proceeding by contradiction, suppose that there exists $i \in \{2, \ldots, n-2\}$ such that $x_i \notin S_t$. It follows from the fact that D is an r-partite tournament that either $(u, x_i) \in A(D)$ or $(x_i, u) \in A(D)$. Since W is a uv-geodesic, then $(u, x_i) \notin A(D)$, which implies that $(x_i, u) \in A(D)$. A similar argument shows that $(v, x_i) \in A(D)$. Hence, (v, x_i, u) is a path in D with length $2 \leq k - 1$ and, by Lemma 3(b), we conclude that $(v, u) \in A\big(C_H^{k-1}(D)\big)$, contradicting our assumption on (u, v). Therefore, for every $i \in \{2, \ldots, n-2\}$, $x_i \in S_t$. Since S_t is an independent set, we have that $n = 4$.

On the other hand, as $x_1 \notin S_t$ and D is an r-partite tournament, then we have that either $(v, x_1) \in A(D)$ or $(x_1, v) \in A(D)$. Since W is a uv-geodesic, then $(x_1, v) \notin A(D)$, which implies that $(v, x_1) \in A(D)$. By applying an analogous argument on x_3 and u, we have that $(x_3, u) \in A(D)$. Hence, (v, x_1, x_2, x_3, u) is a vu-path with length at most $k - 1$ and, by Lemma 3(b), we conclude that $(v, u) \in A\big(C_H^{k-1}(D)\big)$, contradicting the assumption on (u, v)

By the previous cases, we have that $d_D(u, v) \leq 2$. ∎

Theorem 33. *Let D be an H-colored r-partite tournament $(r \geq 2)$. If $k \geq 5$, then D has a (k, H)-kernel.*

Proof. First, we will show that every cycle in $C_H^{k-1}(D)$ has a symmetric arc, then, by applying Theorem 2 and Lemma 5, we will conclude that D has a (k, H)-kernel. Proceeding by contradiction, suppose that there exists a cycle in $C_H^{k-1}(D)$, say $C = (u_0, \ldots, u_n)$, with no symmetric arcs. For every $i \in \{0, \ldots, n - 1\}$, consider a $u_i u_{i+1}$-geodesic in D, say W_i (indices are taken modulo n). It follows from Lemma 32 that for every $i \in \{0, \ldots, n - 1\}$, $l(W_i) \leq 2$. Let $C' = \bigcup_{i=0}^{n-1} W_i$ and assume that $C' = (x_0, \ldots, x_l)$. Notice that C' is a closed walk and $x_0 = x_l = u_0 = u_n$.

Claim 1. *For every* $i \in \{l - 3, l - 2, l - 1, l\}$, $(u_1, x_i) \notin A(D)$.

Proceeding by contradiction, suppose that $(u_1, x_i) \in A(D)$ for some $i \in \{l - 3, l - 2, l - 1, l\}$. Since $x_l = u_0$, we have that $i \neq l$. It follows that $(u_1, x_i) \cup (x_i, C', x_0)$ is a $u_1 u_0$-walk in D with length at most $k - 1$, which implies that $(u_1, u_0) \in A\big(C_H^{k-1}(D)\big)$ (Lemma 3(b)), contradicting the choice of C.

On the other hand, since $u_1 \in V(C')$, then u_1 has at least one out-neighbor in C. Consider $q = \max\{i \in \{1, \ldots, l\} : (u_1, x_i) \in A(D)\}$. Notice that $q \leq l - 4$ (Claim 1). In order to present our proof more compact, we will show the following claims.

Claim 2. $(x_{q+2}, u_1) \in A(D)$.

Proceeding by contradiction, suppose that $(x_{q+2}, u_1) \notin A(D)$. By the choice of q, we have that $(u_1, x_{q+2}) \notin A(D)$, which implies that there exists $m \in \{1, \ldots, r\}$ such that $\{u_1, x_{q+2}\} \subseteq S_m$. It follows from the fact that D is an r-partite digraph, that either $(x_{q+3}, u_1) \in A(D)$ or $(u_1, x_{q+3}) \in A(D)$. By the choice of q, we conclude that $(x_{q+3}, u_1) \in A(D)$. Notice that $(x_q, x_{q+1}, x_{q+2}, x_{q+3}, u_1, x_q)$ is a 5-cycle in D. Since every $u_i u_{i+1}$-geodesic has length at most 2, then $(x_q, x_{q+1}, x_{q+2}, x_{q+3})$ has at least two consecutive vertices in C, say u_t and u_{t+1}. It follows from Lemma 3(c) that (u_t, u_{t+1}) is a symmetric arc in $C_H^{k-1}(D)$, which is not possible by the choice of C (see Figure 4). Hence, the claim holds.
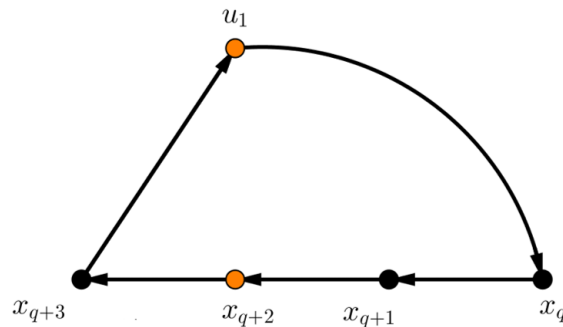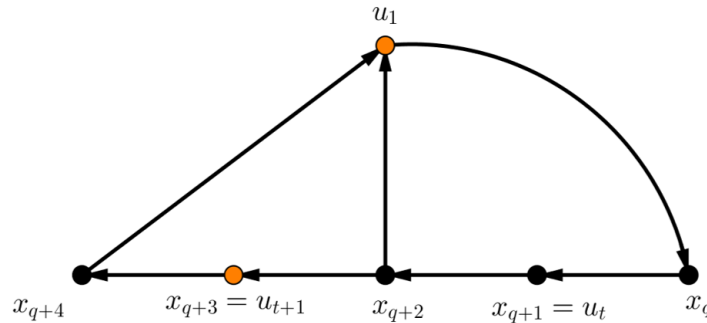

Figure 4.

Claim 3. *There exists* $t \in \{0, \ldots, n - 1\}$ *such that* $x_{q+1} = u_t$ *and* $x_{q+3} = u_{t+1}$.

First, we will show that (x_q, x_{q+1}, x_{q+2}) has no two consecutive vertices of C, such a property will be called (*). Proceeding by contradiction, suppose that (x_q, x_{q+1}, x_{q+2}) has two consecutive vertices of C, u_s and u_{s+1}. Since $(u_1, x_q, x_{q+1}, x_{q+2}, u_1)$ is a 4-cycle in D, it follows from Lemma 3(c) that u_s and u_{s+1} are joined by a symmetric arc in $C_H^{k-1}(D)$, which is not possible by the choice of C. Therefore, property (*) holds.

On the other hand, from the choice of C', we can conclude that there exists $t \in \{0, \dots, n-1\}$ such that $A(W_t) \cap \{(x_q, x_{q+1}), (x_{q+1}, x_{q+2})\} \neq \emptyset$. Remark that $l(W_t) \leq 2$. It follows from property (*) that $x_{q+1} = u_t$ and $x_{q+3} = u_{t+1}$, and the claim holds.

Now, we consider the following cases.

Case 1. $\{u_1, x_{q+3}\} \subseteq S_m$ *for some* $m \in \{1, \dots, r\}$. In this case, we have that u_1 and x_{q+4} are adjacent in D, and by the choice of q, we can conclude that $(x_{q+4}, u_1) \in A(D)$. Hence, $(x_{q+3}, x_{q+4}, u_1, x_q, x_{q+1})$ is a $u_{t+1}u_t$-walk in D with length at most 4, which implies that $(u_{t+1}, u_t) \in A(C_H^{k-1}(D))$ (Lemma 3(b)), contradicting the choice of C (see Figure 5).

Figure 5.

Case 2. $u_1 \in S_m$ *and* $x_{q+3} \in S_{m'}$ *for some* $\{m, m'\} \subseteq \{1, \dots, r\}$, *and* $m \neq m'$. In this case, we have that u_1 and x_{q+3} are adjacent in D, and by the choice of q, we conclude that $(x_{q+3}, u_1) \in A(D)$. Hence, $(x_{q+3}, u_1, x_q, x_{q+1})$ is a $u_{t+1}u_t$-walk in D with length at most 3, which implies that $(u_{t+1}, u_t) \in A(C_H^{k-1}(D))$ (Lemma 3(b)), contradicting the choice of C (see Figure 6).

Therefore, we have that every cycle in $C_H^{k-1}(D)$ has a symmetric arc, which implies that $C_H^{k-1}(D)$ has a kernel (Theorem 2), concluding that D has a (k, H)-kernel (Lemma 5). ∎

As a direct consequence of Theorem 33 we have the following corollary. An analogous result can be found in [21].

Corollary 34 [21]**.** *If D is an r-partite tournament, then for every $k \geq 5$, D has a k-kernel.*

Proof. Let $k \geq 5$ and H a digraph with order $|A(D)|$ and $A(H) = \emptyset$. We consider an H-coloring of D in which every two different arcs have different colors. Notice that for every walk W in D, $l_H(W) = l(W)$. It follows from Theorem 33 that
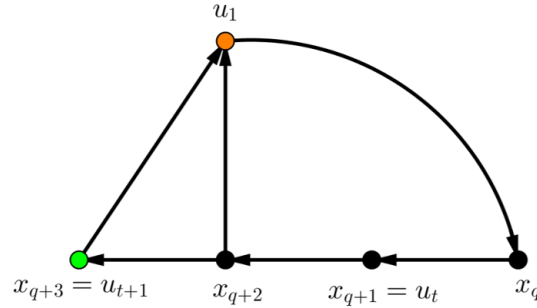
Figure 6.

D has a (k,H)-kernel, say S. It is straightforward to see that S is a k-kernel in D. ∎

3.7. Local tournaments

The following lemma will be useful in what follows.

Lemma 35 [8]**.** *Every pair of vertices in each strong component of a local in-tournament (out-tournament) lie on a cycle.*

As a consequence of the previous lemma, we have the following result for H-colored local in-tournaments.

Theorem 36. *Let D be an H-colored local in-tournament. If every cycle in D has H-length at most $k-2$ ($k \geq 2$), then D has a (k,H)-kernel.*

Proof. First, we will show that every cycle in $C_H^{k-1}(D)$ has a symmetric arc, then, by applying Theorem 2 and Lemma 5, we will conclude that D has a (k,H)-kernel.

Let $C = (u_0,\ldots,u_n)$ be a cycle in $C_H^{k-1}(D)$. For every $i \in \{0,\ldots,n-1\}$, we consider a $u_i u_{i+1}$-path with H-length at most $k-1$, say T_i (indices are taken modulo n). Clearly, $C' = \bigcup_{i=0}^{n-1} T_i$ is a closed walk in D containing u_0 and u_1, which implies that u_0 and u_1 lie in the same strong component of D. It follows from Lemma 35 that there exists a cycle in D, say C_0, such that $\{u_0, u_1\} \subseteq V(C_0)$. Suppose that $C_0 = (u_1 = x_0,\ldots,u_0 = x_t,\ldots,x_l)$. If $C'_0 = (u_1, C_0, u_0)$, then $|O_H(C'_0)| \leq |O_H(C_0)|$ (Lemma 4). By hypothesis, we conclude that $|O_H(C'_0)| \leq k-2$, which implies that $l_H(C'_0) \leq k-1$. Hence, $(u_1, u_0) \in A(C_H^{k-1}(D))$, concluding that C has a symmetric arc in $C_H^{k-1}(D)$.

Therefore, we have that every cycle in $C_H^{k-1}(D)$ has a symmetric arc, which implies that $C_H^{k-1}(D)$ has a kernel (Theorem 2), concluding that D has a (k,H)-kernel (Lemma 5). ∎

Theorem 37. *Let D be an H-colored local out-tournament. If every cycle in D has H-length at most $k-2$ ($k \geq 2$), then D has a (k, H)-kernel.*

Proof. An analogous proof as in Theorem 36 will show Theorem 37. ∎

Corollary 38. *Let D be an H-colored local tournament. If every cycle in D has H-length at most $k-2$ ($k \geq 2$), then D has a (k, H)-kernel.*

Corollary 39. *Let D be a local in-tournament and $k \geq 4$. If every cycle in D has length at most $k-2$, then D has a k-kernel.*

Proof. Let $k \geq 4$ and H a digraph with order $|A(D)|$ and $A(H) = \emptyset$. We consider an H-coloring of D in which every two different arcs have different colors. Notice that for every walk W in D, $l_H(W) = l(W)$. Since every cycle in D has length at most $k-2$, then every cycle in D has H-length at most $k-2$. It follows from Theorem 36 that D has a (k, H)-kernel, say S. It is straightforward to see that S is a k-kernel in D. ∎

Corollary 40. *Let D be a local out-tournament and $k \geq 4$. If every cycle in D has length at most $k-2$, then D has a k-kernel.*

4. A Brief Note on (k, H)-Panchromatic Digraphs

Let D be a digraph and $k \geq 2$; we say that D is a (k, H)-*panchromatic digraph* if for every digraph H (possibly with loops), and every H-coloring of D, D has a (k, H)-kernel. Previous work on panchromaticity in digraphs can be found in [24, 25, 26] and [27]. As a direct consequence of the results proved in this paper, we have that the following nearly tournaments are (k, H)-panchromatic for certain values of k.

Theorem 41. *If D is a digraph, then D is (k, H)-panchromatic provided that:*
  (i) *(Theorem 6) D is semicomplete and $k \geq 3$;*
 (ii) *(Corollary 15) D is transitive and $k \geq 2$;*
(iii) *(Corollary 14) D is r-transitive ($r \geq 2$) and $k \geq r$;*
 (iv) *(Theorem 19) D is quasi-transitive and $k \geq 4$;*
  (v) *(Theorem 23) D is 3-quasi-transitive and $k \geq 5$;*
 (vi) *(Theorem 33) D is an r-partite tournament ($r \geq 2$) and $k \geq 5$.*

Acknowledgments

References

[1] C. Andenmatten, *H-distance, H-A-kernel and in-state splitting in H-colored graphs and digraphs* (MSc Thesis supervised by H. Galeana-Sánchez and J. Pach, École Polytechnique Fédérale de Lausanne, Switzerland, 2019).

[2] P. Arpin and V. Linek, *Reachability problems in edge-colored digraphs*, Discrete Math. **307** (2007) 2276–2289.
https://doi.org/10.1016/j.disc.2006.09.042

[3] J. Bang-Jensen, *The structure of strong arc-locally semicomplete digraphs*, Discrete Math. **283** (2004) 1–6.
https://doi.org/10.1016/j.disc.2004.01.011

[4] J. Bang-Jensen, Y. Guo, G. Gutin and L. Volkmann, *A classification of locally semicomplete digraphs*, Discrete Math. **167−168** (1997) 101–114.
https://doi.org/10.1016/S0012-365X(96)00219-1

[5] J. Bang-Jensen and G. Gutin, Classes of Directed Graphs (Springer, 2018).
https://doi.org/10.1007/978-3-319-71840-8

[6] J. Bang-Jensen and G. Gutin, Digraphs: Theory, Algorithms and Applications, Second Edition (Springer, London, 2000).
https://doi.org/10.1007/978-1-84800-998-1

[7] J. Bang-Jensen and J. Huang, *Quasi-transitive digraphs*, J. Graph Theory **20** (1995) 141–161.
https://doi.org/10.1002/jgt.3190200205

[8] J. Bang-Jensen, J. Huang and E. Prisner, *In-tournament digraphs*, J. Combin. Theory Ser. B **59** (1993) 267–287.
https://doi.org/10.1006/jctb.1993.1069

[9] C. Berge, Graphs and Hypergraphs (North-Holland Publishing Co., Amsterdam, 1985).

[10] V. Chvátal, On the computational complexity of finding a kernel, Report CRM300 (Centre de Recherches Mathématiques, Université de Montréal, 1973).

[11] V. Chvátal and L. Lovasz, *Every directed graph has a semi-kernel*, in: Hypergraph Seminar, Part II: Graphs, Matroids, Designes, C. Berge and D. Ray-Chandhuri (Ed(s)), (Lecture Notes in Mathematics **411**, Springer-Verlag, 1974) 175.
https://doi.org/10.1007/BFb0066192

[12] N. Creignou, *The class of problems that are linearly equivalent to satisfiability or a uniform method for proving NP-completeness*, Theoret. Comput. Sci. **145** (1995) 111–145.
https://doi.org/10.1016/0304-3975(94)00182-I

[13] P. Delgado-Escalante, H. Galeana-Sánchez and S. Arumugam, *Restricted domination in arc-colored digraphs*, AKCE Int. J. Graphs Comb. **11** (2014) 95–104.

[14] P. Delgado-Escalante, H. Galeana-Sánchez and E. O'Reilly-Regueiro, *Alternating kernels*, Discrete Appl. Math. **236** (2018) 153–164.
https://doi.org/10.1016/j.dam.2017.10.013

[15] Y. Dimopoulos and V. Magirou, *A graph theoretic approach to default logic*, Inform. Comput. **112** (1994) 239–256.
https://doi.org/10.1006/inco.1994.1058

[16] Y. Dimopoulos and A. Torres, *Graph theoretical structures in logic programs and default theories*, Theoret. Comput. Sci. **170** (1996) 209–244.
https://doi.org/10.1016/S0304-3975(96)80707-9

[17] P. Duchet, *Graphes noyau-parfaits*, Ann. Discrete Math. **9** (1980) 93–101.
https://doi.org/10.1016/S0167-5060(08)70041-4

[18] A.S. Fraenkel, *Combinatorial game theory foundations applied to digraph kernels*, Electron. J. Combin. **4 (2)** (1997) #R10.
https://doi.org/10.37236/1325

[19] H. Galeana-Sánchez and C. Hernández-Cruz, *k-kernels in generalizations of transitive digraphs*, Discuss. Math. Graph Theory **31** (2011) 293–312.
https://doi.org/10.7151/dmgt.1546

[20] H. Galeana-Sánchez and C. Henrández-Cruz, *k-kernels in k-transitive and k-quasi-transitive digraphs*, Discrete Math. **312** (2012) 2522–2530.
https://doi.org/10.1016/j.disc.2012.05.005

[21] H. Galeana-Sánchez, C. Hernández-Cruz and S. Arumugam, *k-kernels in multipartite tournaments*, AKCE Int. J. Graphs Comb. **8** (2011) 181–198.

[22] H. Galeana-Sánchez and R. Sánchez-López, *H-kernels and H-obstructions in H-colored digraphs*, Discrete Math. **338** (2015) 2288–2294.
https://doi.org/10.1016/j.disc.2015.05.021

[23] H. Galeana-Sánchez and R. Sánchez-López, *Richardson's theorem in H-coloured digraphs*, Graphs Combin. **32** (2016) 629–638.
https://doi.org/10.1007/s00373-015-1609-3

[24] H. Galeana-Sánchez and R. Strausz, *On panchromatic digraphs and the panchromatic number*, Graphs Combin. **31** (2015) 115–125.
https://doi.org/10.1007/s00373-013-1367-z

[25] H. Galeana-Sánchez and R. Strausz, *On panchromatic patterns*, Discrete Math. **339** (2016) 2536–2542.
https://doi.org/10.1016/j.disc.2016.03.014

[26] H. Galeana-Sánchez and M. Tecpa-Galván, *$\mathscr{H}$-panchromatic digraphs*, AKCE Int. J. Graphs Comb. **17** (2020) 303–313.
https://doi.org/10.1016/j.akcej.2019.05.005

[27] H. Galeana-Sánchez and M. Toledo, *New classes of panchromatic digraphs*, AKCE Int. J. Graphs Comb. **12** (2015) 261–270.
https://doi.org/10.1016/j.akcej.2015.11.006

[28] P. Garcia-Vázquez and C. Hernández-Cruz, *Some results on 4-transitive digraphs*, Discuss. Math. Graph Theory **37** (2017) 117–129.
https://doi.org/10.7151/dmgt.1922

[29] A. Ghouila-Houri, *Caractérisation des graphes non orientés dont on peut orienter les de maniere a obtenir le graphe d'une relation d'ordre*, C.R. Acad. Sci. Paris **254** (1962) 1370–1371.

[30] S. Heard and J. Huang, *Disjoint quasi-kernels in digraphs*, J. Graph Theory **58** (2008) 251–260.
https://doi.org/10.1002/jgt.20310

[31] C. Hernández-Cruz, *3-transitive digraphs*, Discuss. Math. Graph Theory **32** (2012) 205–219.
https://doi.org/10.7151/dmgt.1613

[32] C. Hernández-Cruz and H. Galeana-Sánchez, *k-kernels in k-transitive and k-quasi-transitive digraphs*, Discrete Math. **312** (2012) 2522–2530.
https://doi.org/10.1016/j.disc.2012.05.005

[33] D. König, Theorie der Endlichen und Unendlichen Graphen (Reprinted from AMS Chelsea Publishing Company, Providence, Rhode Island, 1950).

[34] M. Kwaśnik, On (k, l)-kernels in graphs and their products, Ph.D. Thesis (Technical University of Wrocław, Wrocław, 1980).

[35] V. Linek and B. Sands, *A note on paths in edge-colored tournaments*, Ars Combin. **44** (1996) 225–228.

[36] J.W. Moon, Topics on Tournaments (Holt, Rinehart and Winston, New York, 1968).

[37] J. von Neumann and O. Morgenstern, Theory of Games and Economic Behavior (Princeton University Press, Princeton, 1944).

[38] K.B. Reid, *Monotone reachability in arc-colored tournaments*, Congr. Numer. **146** (2000) 131–141.

[39] S. Szeider, *Finding paths in graphs avoiding forbidden transitions*, Discrete Appl. Math. **126** (2003) 261–273.
https://doi.org/10.1016/S0166-218X(02)00251-2

[40] R. Wang, $(k-1)$-*kernels in strong k-transitive digraphs*, Discuss. Math. Graph Theory **35** (2015) 229–235.
https://doi.org/10.7151/dmgt.1787

[41] R. Wang and H. Zhang, $(k+1)$-*kernels and the number of k-kings in k-quasi-transitive digraphs*, Discrete Math. **338** (2015) 114–121.
https://doi.org/10.1016/j.disc.2014.08.009