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OUTER CONNECTED DOMINATION IN MAXIMAL OUTERPLANAR GRAPHS AND BEYOND

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Abstract

A set S of vertices in a graph G is an outer connected dominating set of G if every vertex in $V \setminus S$ is adjacent to a vertex in S and the subgraph induced by $V \setminus S$ is connected. The outer connected domination number of G, denoted by $\tilde{\gamma}_c(G)$, is the minimum cardinality of an outer connected dominating set of G. Zhuang [Domination and outer connected domination in maximal outerplanar graphs, Graphs Combin. 37 (2021) 2679–2696] recently proved that $\tilde{\gamma}_c(G) \leq \lfloor \frac{n+k}{4} \rfloor$ for any maximal outerplanar graph G of order $n \geq 3$ with k vertices of degree 2 and posed a conjecture which states that G is a striped maximal outerplanar graph with $\tilde{\gamma}_c(G) = \lfloor \frac{n+2}{4} \rfloor$ if and only if $G \in \mathcal{A}$, where $\mathcal{A}$ consists of six special families of striped outerplanar graphs. We disprove the conjecture. Moreover, we show that the conjecture become valid under some additional property to the striped maximal outerplanar graphs. In addition, we extend the above theorem of Zhuang to all maximal $K_{2,3}$-minor free graphs without K_4 and all K_4-minor free graphs.

Keywords: maximal outerplanar graphs, outer connected domination, striped maximal outerplanar graphs.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

All graphs considered in this paper are finite and simple. For notation and terminology, we will typically follow [4]. Specifically, for a graph $G = (V, E)$, $|V|$ and $|E|$ are called the *order* and the *size* of G, respectively. A *neighbor* of a vertex v in the graph G is a vertex adjacent to v. The *open neighborhood* of v, denoted by $N_G(v)$, is the set of all neighbors of v. The *closed neighborhood* of v

is $N_G[v] = \{v\} \cup N_G(v)$. A *fan* F_n is a graph of order $n+1$ obtained by adding a vertex v to P_n with v adjacent to each vertex of P_n. For a set S of vertices in G, the subgraph of G induced by S is denoted by $G[S]$.

A graph G is *outerplanar* if it has an embedding in the plane such that all vertices belong to the boundary of its outer face. An outerplanar graph G is *maximal* if $G + uv$ is not outerplanar for any two non-adjacent vertices u and v of G. An inner face of a maximal outerplane graph G is an *internal triangle* if it is not adjacent to outer face. A maximal outerplane graph without internal triangles is called *striped*. A *minor* of a graph G is a graph which can be obtained from G by deleting vertices and deleting or contracting edges. Given a graph H, a graph G is H-minor free if no minor of G is isomorphic to H. An H-minor free graph G is *maximal* if $G + uv$ is not an H-minor free graph for any two non-adjacent vertices u and v of G. It is well known that a simple graph G is outerplanar if and only if G is both K_4-minor free and $K_{2,3}$-minor free (see [4]).

For a graph G, a set $S \subseteq V(G)$ is called a *dominating set* of G if each vertex $v \in V(G)\setminus S$ has a neighbor in S. The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. Furthermore, a dominating set S is an *outer connected dominating* set of G if the induced subgraph $G[V \setminus S]$ is connected. The *outer connected domination* number of G, denoted by $\tilde{\gamma}_c(G)$, is the minimum cardinality of an outer connected dominating set.

The concept of outer connected domination was introduced by Cyman [7]. More details and information can be found in [1, 2, 11, 13, 15]. Various types of domination of maximal outerplanar graphs were widely studied in literature, such as secure domination [3], partial domination [5], domination [6, 12, 14], total domination [8, 9], semipaired domination [10] and connected domination [18]. Very recently, Zhuang [17] proved the following theorem.

Theorem 1 [17]**.** *If G is a maximal outerplanar graph of order $n \geq 3$, and k is the number of vertices of degree 2 in G, then $\tilde{\gamma}_c(G) \leq \left\lfloor \frac{n+k}{4} \right\rfloor$.*

Theorem 2 [17]**.** *If G is a striped maximal outerplanar graph of order $n \geq 3$, then $\tilde{\gamma}_c(G) \leq \left\lfloor \frac{n+2}{4} \right\rfloor$.*

Let $\mathcal{A}$ is a set of well-defined striped maximal outerplanar graphs, which will defined in Section 3. Zhuang proposed the following conjecture.

Conjecture 3 [17]**.** *Let G be a striped maximal outerplanar graph. Then $\tilde{\gamma}_c(G) = \left\lfloor \frac{n+2}{4} \right\rfloor$ if and only if $G \in \mathcal{A}$.*

In this paper, we disprove Conjecture 3. Based on some structural properties of $K_{2,3}$-minor free graphs and K_4-minor free graphs, we extend Theorem 1 as follows.

Theorem 4. *Assume that G is a maximal $K_{2,3}$-minor free graph without K_4 or a maximal K_4-minor free graph of order $n \geq 3$. If k is the number of vertices of degree 2 in G, then* $\tilde{\gamma}_c(G) \leq \left\lfloor \frac{n+k}{4} \right\rfloor$.

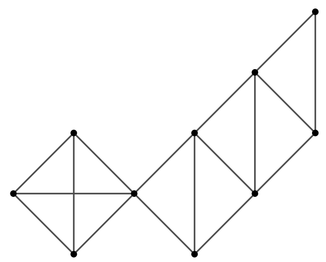

Figure 1. A maximal $K_{2,3}$-minor free graph.

The example illustrated in Figure 1 shows that the condition of K_4-free in Theorem 4 is necessary. One can see that it is maximal $K_{2,3}$-minor free and contains a subgraph isomorphic to K_4, but $\tilde{\gamma}_c(G) > \left\lfloor \frac{n+k}{4} \right\rfloor$.

Since $\left\lfloor \frac{n+k}{4} \right\rfloor < \left\lfloor \frac{n}{3} \right\rfloor$ when $k \leq \frac{n}{3}$, Theorem 4 improves the following result due to Zhuang for the case when $k \leq \frac{n}{3}$.

Theorem 5 [17]. *If G is a maximal K_4-minor free graph of order $n \geq 3$, then* $\tilde{\gamma}_c(G) \leq \left\lfloor \frac{n}{3} \right\rfloor$.

Corollary 6. *If G is a maximal K_4-minor free graph of order $n \geq 3$ with k vertices of degree 2, then*

$$\tilde{\gamma}_c(G) \leq \begin{cases} \left\lfloor \frac{n+k}{4} \right\rfloor & \text{if } k \leq \frac{n}{3}, \\ \left\lfloor \frac{n}{3} \right\rfloor & \text{otherwise.} \end{cases}$$

In Section 3, we will show that for a striped maximal outerplanar graph $G \notin \mathcal{G}$, $\tilde{\gamma}_c(G) = \left\lfloor \frac{n+2}{4} \right\rfloor$ if and only if $G \in \mathcal{A}$, where $\mathcal{G}$ is defined later.

2. Maximal $K_{2,3}$-Minor Free and K_4-Minor Free Graphs

We say that a graph G is *k-colored* if the vertices of G are colored by at most k-colors such that each vertex has a different color from any of its adjacent vertices. A C_4-*multicoloring* of a graph G is a 4-coloring of G that assigns four distinct colors to the vertices of every cycle of length 4. A 2-*tree* is defined recursively as follows. A single edge is a 2-tree. Any graph obtained from a 2-tree by adding a new vertex and making it adjacent to the end vertices of an existing edge is also a 2-tree. It is well known that the maximal K_4-minor free graphs are exactly the 2-trees.

Now, we will give some results on maximal K_4-minor free graph and maximal $K_{2,3}$-minor free graph which are useful in what follows.

Lemma 7 [16]**.** *Let G be a connected graph. Then G is a maximal $K_{2,3}$-minor free graph if and only if the following holds.*

(1) *Each block of G is either a K_4 or is a maximal outerplanar graph not isomorphic to $K_4 - e$.*

(2) *If two blocks share a common vertex, then at least one of them is a K_4.*

Lemma 8 [16]**.** *Every maximal $K_{2,3}$-minor free G admits a C_4-multicoloring.*

Lemma 9 [16]**.** *Let G be a maximal $K_{2,3}$-minor free with $\delta(G) \geq 2$. If $R \subseteq V(G)$ contains all vertices of some given color, then R dominates all vertices of $V(G)$, except possibly those of degree* 2.

Let c be a k-coloring of vertices of a graph G. For a set $S \subseteq V(G)$, $c(S) = \{c(v) : v \in S\}$ denotes the colors which appear on the vertices in S under c.

Lemma 10 [16]**.** *A maximal K_4-minor free graph G with order $n \geq 4$ admits a C_4-multicoloring such that $\left|c(N_G(v))\right| = 3$ for any vertex v with $d_G(v) \geq 3$.*

Lemma 11 [16]**.** *Assume that G is a maximal K_4-minor free graph with a C_4-multicoloring c satisfying the property as described in the previous Lemma* 10*. If $R \subseteq V(G)$ contains all vertices of some given color, then R dominates all vertices of $V(G)$, except possibly those of degree* 2.

Next, we are ready to give the proof of Theorem 4.

Proof. Let G be a maximal $K_{2,3}$-minor free and K_4-free graph or a maximal K_4-minor free graph satisfying the assumption of Theorem 4. It is clear that every maximal K_4-minor free graph of order at least 3 has minimum degree 2. By Lemma 7, one can see that $\delta(G) \geq 2$ if G is a maximal $K_{2,3}$-minor free and K_4-free graph of order at least 3. Thus, $\delta(G) \geq 2$. Let $S = \{v_1, v_2, \ldots, v_k\}$ be the set of vertices of G having degree 2 and u_i be one of the two vertices adjacent to v_i for each $i \in \{1, 2, \ldots, k\}$. We prepare a set of additional vertices $S' = \{v'_1, v'_2, \ldots, v'_k\}$ and construct a graph G' with $V(G') = V(G) \cup S'$ and $E(G') = E(G) \cup \{v'_1v_1, v'_2v_2, \ldots, v'_kv_k\} \cup \{v'_1u_1, v'_2u_2, \ldots, v'_ku_k\}$.

It is clear that G' is a maximal $K_{2,3}$-minor free and K_4-free graph if G is a maximal $K_{2,3}$-minor free and K_4-free graph, and G' a maximal K_4-minor free graph if G is maximal K_4-minor free graph. So, by Lemmas 8 and 10, G' has a C_4-multicoloring c. Let V_t denote the set of vertices assigned color t, where $t \in \{1, 2, 3, 4\}$. Choosing one suitable color class, say V_1, such that

$|V_1| \leq \left\lfloor \frac{|G'|}{4} \right\rfloor = \left\lfloor \frac{n+k}{4} \right\rfloor$. Note that every vertex of S has degree 3 in G'. By Lemmas 9 and 11, V_1 dominates all vertices of G.

By Lemma 7, we known that if G' is a maximal $K_{2,3}$-minor free and K_4-free graph, then G' is a maximal outerplanar graph. Note that a maximal outerplanar graph is 2-tree. It is well known that the maximal K_4-minor free graphs are exactly the 2-trees. Thus, G' is a 2-tree. Let $G_i = G_{i-1} - v_i$ for $i = 1, 2, \ldots, n + k - 3$, where v_i is a vertex of degree 2 in G_{i-1} and $G_0 = G'$. It is clear that G_{n-k+3} is a K_3.

Let $X = V_2 \cup V_3 \cup V_4$. Next, we show that the subgraph $G'[X]$ is connected. Equivalently, we only need to show that if $G'[V(G_i) \cap X]$ is connected, then $G'[V(G_{i-1}) \cap X]$ is also connected. Suppose that $G'[V(G_i) \cap X]$ is connected. If $c(v_i) = 1$, then $G'[V(G_i) \cap X] = G'[V(G_{i-1}) \cap X]$. Otherwise, $c(v_i) \in \{2, 3, 4\}$. Clearly, at most one neighbor of v_i is colored by 1 in G_{i-1}. It follows that $G'[V(G_{i-1}) \cap X]$ is connected. Therefore, $G'[X]$ is connected.

Finally, let $V_1 \cap S' = \{v'_{i_1}, v'_{i_2}, \ldots, v'_{i_k}\}$ and $V'_1 = (V_1 - S') \cup \{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\}$. We can see that V_1 dominated $V(G)$ and so does V'_1. Thus, V'_1 is an outer connected dominating set of G satisfying $|V'_1| = |V_1| \leq \left\lfloor \frac{n+k}{4} \right\rfloor$. ■

3. Striped Maximal Outerplanar Graphs

Lemma 12 [12]**.** *Let G be a striped maximal outerplanar graph of order $n > 4$. Then for any vertex $v \in V(G)$ of degree 2, the degrees of two neighbors of v are 3 and l ($l \geq 4$), respectively.*

By Lemma 12, for a striped maximal outerplanar graph G of order $n \geq 6$ and a vertex $u \in V(G)$ adjacent to a 2-degree vertex and a 3-degree vertex, we define a graph G^u and the DELETED VERTEX SEQUENCE $(v_1, v_2, \ldots)$ as follows.

Procedure CREATE GRAPH (G, u);
input: a graph G and a vertex $u \in V(G)$.
begin
  $i := 1; S := \emptyset$;
  $T := \{w : w \in N[u], d_{G-S}(w) = 2\}$;
  while $(t \neq \emptyset)$
  begin
  select a vertex $v \in T$;
  $S := S \cup \{v\}; v_i := v; i := i + 1$;
  $T := \{w : w \in N[u], d_{G-S}(w) = 2\}$;
  end
  $G^u := G - S$;
end.

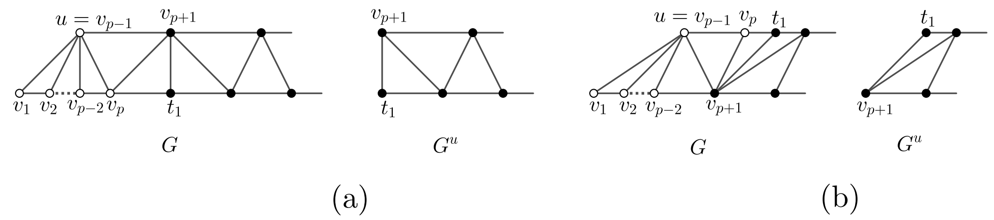

Figure 2. The case of G^u with $n-p$ vertices.

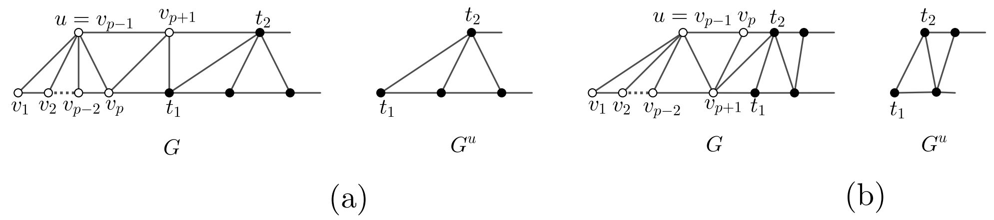

Figure 3. The case of G^u with $n-p-1$ vertices.

In fact, G^u is a subgraph of G obtained by repeatedly removing vertices of degree 2 in $N[u]$ from G, and v_i is the ith removal of the procedure CREATE GRAPH. See Figures 2 and 3. Let $d(u) = p$. If $n = p+1$, then G is a p-fan, G^u is isomorphic to K_2 and $\tilde{\gamma}_c(G) = 1$. If $n \geq p+2$, by Lemma 12, it can be seen that $u = v_{p-1}$ and the DELETED VERTEX SEQUENCE is $(v_1, v_2, \ldots, v_{p-1}(= u), v_p)$ or $(v_1, v_2, \ldots, v_{p-1}(= u), v_p, v_{p+1})$. Thus G^u has $n-p$ or $n-p-1$ vertices. Furthermore, if $n = p+2$ or $p+3$, then G^u is isomorphic to K_2. If $n \geq p+4$, then G^u is still a striped maximal outerplanar graph.

Next, we define a family $\mathcal{G}$ of striped maximal outerplanar graphs G with the following properties.

(1) $d_G(v) \geq 7$ for each vertex v adjacent to a 2-degree vertex.

(2) G^v has $n-p$ vertices and $v_{p-2}v_{p+1} \in E(G)$, as illustrated in Figure 2(b), where G^v is the graph arising from CREATE GRAPH corresponding to the vertex v.

Theorem 13. *Let $G \notin \mathcal{G}$ be a striped maximal outerplanar graph of order $n \geq 8$ and u be a vertex of G that adjacent to a 2-degree vertex and a 3-degree vertex. Then $\tilde{\gamma}_c(G) = \tilde{\gamma}_c(G^u) + 1$.*

Proof. It is clear that the result is true for $n \leq d(u) + 3$. So, $n \geq d(u) + 4$. Let $d_G(u) = p$ and $N(u) = \{v_1, v_2, \ldots, v_p\}$. Let $(v_1, v_2, \ldots)$ be the DELETED VERTEX SEQUENCE of the CREATE GRAPH. Let S_1 be a minimum outer connected dominating set of G^u. Suppose that $|V(G^u)| = |V(G)| - p$ and

$v_{p-2}v_{p+1} \in E(G)$ as shown in Figure 2(b). Since $G \notin \mathcal{G}$, $4 \le d_G(u) \le 6$. Let

$$S' = \begin{cases} S_1 \cup \{u\} & \text{if } v_{p+1} \notin S_1, \\ S_1 \cup \{v_2\} & \text{if } v_{p+1} \in S_1. \end{cases}$$

It is clear that S' is an outer connected dominating set of G. Therefore, $\tilde{\gamma}_c(G) \le |S'| = |S_1| + 1 = \tilde{\gamma}_c(G^u) + 1$.

In what following, we consider the remaining case that $|V(G^u)| = |V(G)| - p$ and $v_{p-2}v_p \in E(G)$ as shown in Figure 2(a) or $|V(G^u)| = |V(G)| - p - 1$ as shown in Figure 3. One can see that $S_1 \cup \{u\}$ is an outer connected dominating set of G. Therefore, $\tilde{\gamma}_c(G) \le |S_1| + 1 = \tilde{\gamma}_c(G^u) + 1$.

Next, we shall show that $\tilde{\gamma}_c(G^u) \le \tilde{\gamma}_c(G) - 1$. Let S be a minimum outer connected dominating set of G. Note that S contains at least one vertex of $N[v_1]$ and $N[v_i] \subseteq N[u]$ for any $i \in \{1, 2, \ldots, l-2\}$.

We can distinguish four cases as follows.

Case 1. $|V(G^u)| = |V(G)| - p$ and $v_{p-2}v_p \in E(G)$ (see Figure 2(a)). The degree of v_p in $G - \{v_1, v_2, v_{p-2}, u\}$ is 2 and $N_G[u] \cup N_G[v_p] \subseteq N_G[u] \cup N_G[v_{p+1}]$. We consider a set S' with

$$S' = \begin{cases} (S \setminus \{v_1, v_2, \ldots, v_{p-2}\}) \cup \{u\} & \text{if } v_p \notin S, \\ (S \setminus \{v_1, v_2, \ldots, v_{p-2}, v_p\}) \cup \{t_1\} & \text{if } v_p \in S. \end{cases}$$

Case 2. $|V(G^u)| = |V(G)| - p$ and $v_{p-2}v_{p+1} \in E(G)$ (see Figure 2(b)). Since $G \notin \mathcal{G}$ and $|V(G^u)| = |V(G)| - p$, we have $4 \le d_G(u) \le 6$ and $d_G(t_1) = 3$. We consider a set S' with

$$S' = \begin{cases} (S \setminus \{v_1, v_2, \ldots, v_{p-2}\}) \cup \{u\} & \text{if } \{v_p, v_{p+1}\} \not\subseteq S; \\ (S \setminus \{v_1, v_3, \ldots, v_{p-2}\}) \cup \{v_2\} & \text{if } v_{p+1} \in S \text{ and } v_p \notin S; \\ (S \setminus \{v_1, v_2, \ldots, v_{p-2}, v_p\}) \cup \{u, t_1\} & \text{if } v_{p+1} \notin S \text{ and } v_p \in S. \end{cases}$$

Case 3. $|V(G^u)| = |V(G)| - p - 1$ and $v_p t_1 \in E(G)$ (see Figure 3(a)). We consider a set S' with

$$S' = \begin{cases} (S \setminus \{v_1, v_2, \ldots, v_{p-2}\}) \cup \{u\} & \text{if } \{v_p, v_{p+1}\} \not\subseteq S; \\ (S \setminus \{v_1, v_2, \ldots, v_{p-2}, v_p, v_{p+1}\}) \cup \{u, t_2\} & \text{if } v_p \in S \text{ and } d_G(t_2) = 3 \text{ or} \\ & v_{p+1} \in S \text{ and } d_G(t_2) = 3; \\ (S \setminus \{v_1, v_2, \ldots, v_{p-2}, v_p, v_{p+1}\}) \cup \{u, t_1\} & \text{if } v_p \in S \text{ and } d_G(t_2) \ge 4 \text{ or} \\ & v_{p+1} \in S \text{ and } d_G(t_2) \ge 4. \end{cases}$$

Case 4. $|V(G^u)| = |V(G)| - p - 1$ and $v_p t_2 \in E(G)$ (see Figure 3(b)). We consider a set S' with

$$S' = \begin{cases} (S \setminus \{v_1, v_2, \ldots, v_{p-2}\}) \cup \{u\} & \text{if } \{v_p, v_{p+1}\} \not\subseteq S; \\ (S \setminus \{v_1, v_2, \ldots, v_{p-2}, v_p, v_{p+1}\}) \cup \{u, t_1\} & \text{if } v_p \in S \text{ and } d_G(t_1) = 3 \text{ or} \\ & v_{p+1} \in S \text{ and } d_G(t_1) = 3; \\ (S \setminus \{v_1, v_2, \ldots, v_{p-2}, v_p, v_{p+1}\}) \cup \{u, t_2\} & \text{if } v_p \in S \text{ and } d_G(t_1) \ge 4 \text{ or} \\ & v_{p+1} \in S \text{ and } d_G(t_1) \ge 4. \end{cases}$$

In all cases, it can be easily seen that S' is a minimum outer connected dominating set of G and $S' \setminus \{u\} \subseteq V(G^u)(S' \setminus \{v_2\} \subseteq V(G^u))$. It is clear that $S' \setminus \{u\}$ is a minimum outer connected dominating set of G^u. Thus, $\tilde{\gamma}_c(G^u) \leq \tilde{\gamma}_c(G) - 1$. ■

Now, we are ready to give the construction of a family graphs $\mathcal{A}$ which was construct by Zhuang in [17]. Each graph in Figures 4 and 5 is called "basic graph", and each graph depicted in Figure 6 is called "gadget". For each of those graphs, there are some dashed edges whose end vertices marked with letter x and y, respectively, or x' and y', respectively. We call the former "xy-edge", and the latter "$x'y'$-edge". Each graph of $\mathcal{A}$ is recursively constructed from one of the basic graphs, and some of the gadgets. Next, we introduce an operations as follows. At first, each of the basic graphs is denoted by G_0.

Operation 1. In step i $(i \geq 1)$, we select an xy-edge of $(i-1)$th gadget of G_{i-1}, identify one of the $x'y'$-edge of the ith gadget and the xy-edge we selected to a single edge such that the vertex x corresponding to the vertex x', and the vertex y corresponding the vertex y', we denote the resulting graph by G_i.

Note that in Operation 1, each G_i is a striped maximal outerplanar graph. Next, for $1 \leq i \leq 6$, let $\mathcal{A}_i$ be the family of striped maximal outerplanar graph G defined as follows and let $\mathcal{A} = (\bigcup_{i=1}^{6} \mathcal{A}_i) \cup \{K_3\}$.

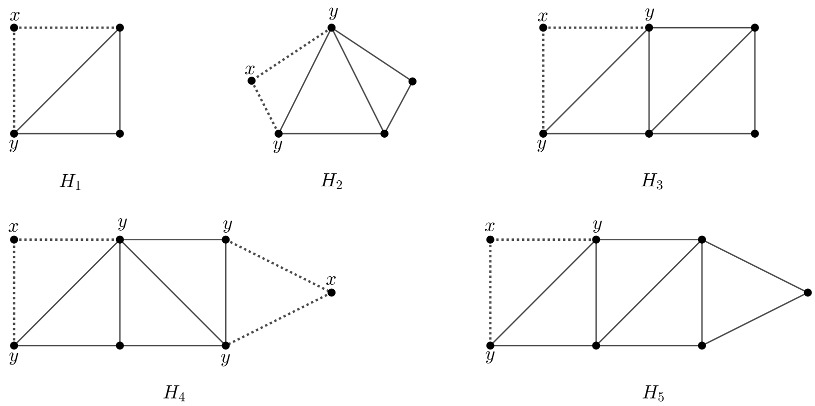

Figure 4. Basic graphs (1).

$\mathcal{A}_1 = \{G : G$ is obtained from H_3 by a finite sequence of Operation 1, each corresponding gadget is H_3', except t gadgets $(0 \leq t \leq 3)$ which belong to $\{H_4', H_4'', H_5'\}\} \cup \{H_3\}$.

$\mathcal{A}_2 = \{G : G$ is obtained from H_3 by a finite sequence of Operation 1, each corresponding gadget is H_3', except one gadget which belongs to $\{H_6', H_7', H_8', H_9', H_{10}', H_{11}'\}\}$.

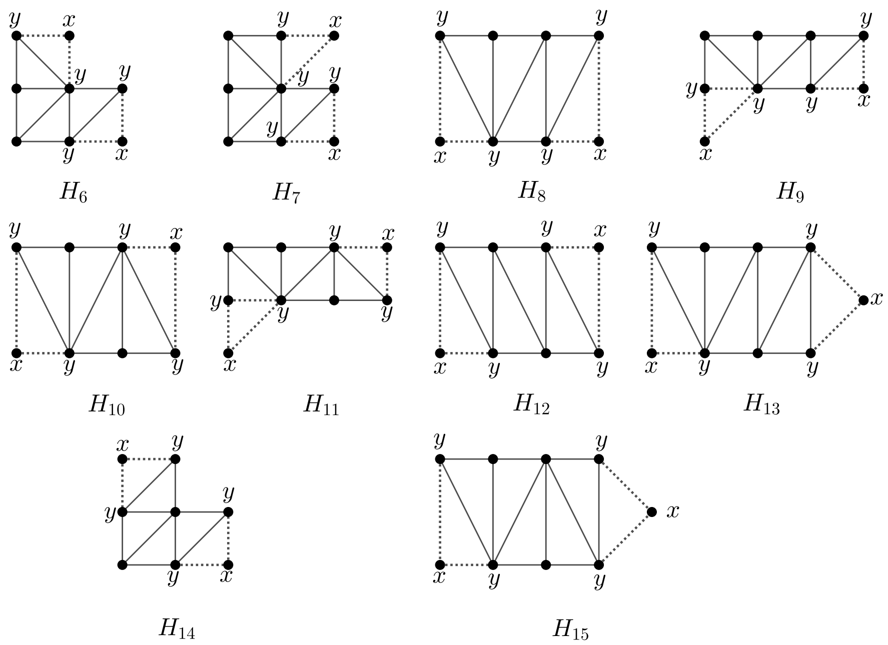

Figure 5. Basic graphs (2).

$\mathcal{A}_3 = \{G : G$ is obtained from H_3 by a finite sequence of Operation 1, each corresponding gadget is H_3', except two gadgets P_1 and P_2, where $P_1 \in \{H_4', H_4'', H_5'\}$ and $P_2 \in \{H_6', H_8', H_{10}'\}\}$.

$\mathcal{A}_4 = \{G : G$ is obtained from one of H_4 and H_5 by a finite sequence of Operation 1, and each corresponding gadget is H_3', except possibly one gadget which belongs to $\{H_6', H_8', H_{10}'\}$, or possibly at most two gadgets which belong to $\{H_4', H_4'', H_5'\}\} \cup \{H_4, H_5\}$.

$\mathcal{A}_5 = \{G : G$ is obtained from one of H_1, H_6, H_8, H_{10}, H_{12} and H_{14} by a finite sequence of Operation 1, each corresponding gadget is H_3', except possibly one gadget which belongs to $\{H_4', H_4'', H_5'\}\} \cup \{H_1, H_6, H_8, H_{10}, H_{12}, H_{14}\}$.

$\mathcal{A}_6 = \{G : G$ is obtained from one of H_2, H_7, H_9, H_{11}, H_{13} and H_{15} by a finite sequence of Operation 1, and each corresponding gadget is $\{H_3'\}\} \cup \{H_2, H_7, H_9, H_{11}, H_{13}, H_{15}\}$.

In Figure 7, we give an example G obtained from basic graph H_3 by one Operation 1, the corresponding gadget is H_3'. Hence $G \in \mathcal{A}_1$.

In Figure 8, we give two counterexamples to Conjecture 3. Since for the graph G of Figure 8(a) with $|V(G)| \equiv 0 \pmod 4$, but $G \notin \mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{A}_4 \cup \mathcal{A}_5$. However, we can show that if $G \notin \mathcal{G}$ is a striped maximal outerplanar graph, Conjecture 3 is true.

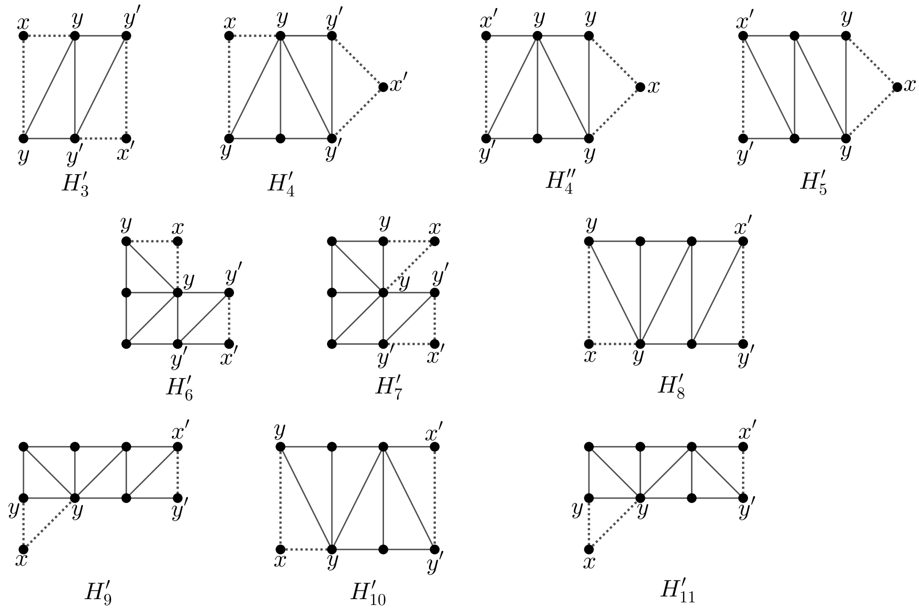

Figure 6. Gadgets.

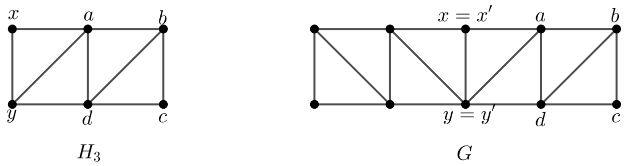

Figure 7. Example of $\mathcal{A}$.

Theorem 14. *Let $G \notin \mathcal{G}$ be a striped maximal outerplanar graph. Then, $\tilde{\gamma_c}(G) = \left\lfloor \frac{n+2}{4} \right\rfloor$ if and only if $G \in \mathcal{A}$.*

Proof. The sufficiency is easy to verify. So we prove the necessity only. If $n \leq 7$, then G is K_3 or isomorphism to one of the graphs shown in Figure 5. That is, $G \in \mathcal{A}$. Thus $n \geq 8$. By Lemma 12, let u be a vertex of G that adjacent to a 2-degree vertex and a 3-vertex and let $d(u) = p$ and $N(u) = \{v_1, v_2, \ldots, v_p\}$, where $p \geq 4$ and $d(v_1) = 2$. It is clear that $n \geq d(u) + 2$. If $n \leq n + 1$, then $G \cong F_p$, where $p \leq 4$. Which contradicts the fact that $n \geq 8$. Suppose that $d(u)+2 \leq n \leq d(u)+3$ or $n \geq d(u)+4$ and $G^u = 3$. One can see that $\tilde{\gamma_c}(G) = 2$. Then we have $n = 8, 9$. It follows that G is isomorphic to one of the ten graphs in Figure 6.

Thus, we only consider $n \geq d(u) + 4$ and $|G^u| \geq 4$. Assume that for any

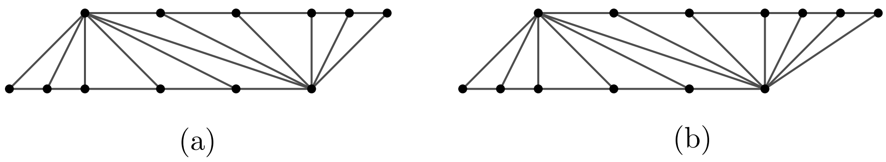

Figure 8. Counterexamples of Conjecture 3.

striped maximal outerplanar graph of order $n < n'$, the result holds. It is clear that

$$\tilde{\gamma_c}(G) = \left\lfloor \frac{n+2}{4} \right\rfloor = \begin{cases} \lfloor \frac{n}{4} \rfloor & \text{if } n \equiv 0, 1 \pmod 4, \\ \lceil \frac{n}{4} \rceil & \text{otherwise.} \end{cases}$$

In what following, we will consider four cases.

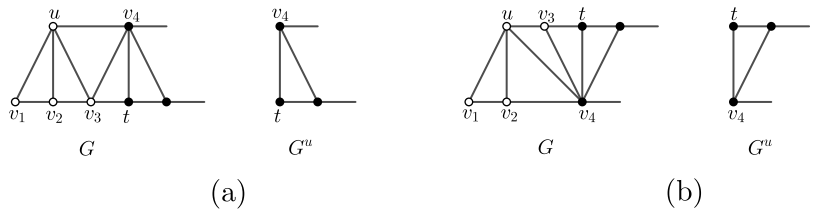

Figure 9. (v_1, v_2, u, v_3) is the DELETED VERTEX SEQUENCE of the procedure CREATE GRAPH.

Case 1. $|G| = 4k$, where $k \geq 2$. Then G is a graph satisfying $\tilde{\gamma_c}(G) = k$. By Theorem 13, $\tilde{\gamma_c}(G^u) = \tilde{\gamma_c}(G) - 1 = k - 1$. Note that G^u is also a striped maximal outerplanar graph with $|G^u| \geq 4$. If $|G^u| = n - d(u)$, then $\tilde{\gamma_c}(G^u) \leq \left\lfloor \frac{n-d(u)}{4} \right\rfloor$ when $|G^u| \equiv 0, 1 \pmod 4$ and $\tilde{\gamma_c}(G^u) \leq \left\lceil \frac{n-d(u)}{4} \right\rceil$ when $|G^u| \equiv 2, 3 \pmod 4$. Similarly, if $|G^u| = n - d(u) - 1$, then $\tilde{\gamma_c}(G^u) \leq \left\lfloor \frac{n-d(u)-1}{4} \right\rfloor$ when $n \equiv 0, 1 \pmod 4$ and $\tilde{\gamma_c}(G^u) \leq \left\lceil \frac{n-d(u)-1}{4} \right\rceil$ when $n \equiv 2, 3 \pmod 4$. It is easy to verify that $d(u) = 4, 5, 6$ when $|G^u| = n - d(u)$ and $d(u) = 4, 5$ when $|G^u| = n - d(u) - 1$. We can distinguish three subcases as follows.

Subcase 1.1. $|G^u| = n - d(u)$ and $d(u) = 4$. Note that $|G^u| \equiv 0 \pmod 4$. By the induction hypothesis, $G^u \in \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_5\}$. More precisely, there are four possibilities for G^u.

(1) G^u is obtained from H_3 by a finite sequence of Operation 1, and following gadget is H_3', except two gadgets which belong to $\{H_4', H_4'', H_5'\}$. Without loss of generality, let v_1, v_2, u, v_3 be the DELETED VERTEX SEQUENCE of the procedure CREATE GRAPH, we have the situations depicted in Figure 9. We

can see that $E(G) \setminus E(G^u) = \{v_1u, v_2u, v_3u, v_4u, v_1v_2, v_2v_3, v_3v_4, v_3t\}$ or $E(G) \setminus E(G^u) = \{v_1u, v_2u, v_3u, v_4u, v_1v_2, v_2v_4, v_3v_4, v_3t\}$, where t is the neighbor of v_4 of degree 2 in G^u. One can see that G is obtained from the basic graph G^u by Operation 1, and the gadget is H_3'. So $G \in \mathcal{A}_1$.

(2) G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H_3', except one gadget which belongs to $\{H_6', H_8', H_{10}'\}$.

(3) G^u is obtained from one of H_4 and H_5 by a finite sequence of Operation 1, and following gadget is H_3', except one gadget which belongs to $\{H_4', H_4'', H_5'\}$.

(4) G^u is obtained from one of H_1, H_6, H_8, H_{10}, H_{12} and H_{14} by a finite sequence of Operation 1, and each following gadget is H_3'.

For the three cases (2), (3) and (4), by the similar argument of (1), G is obtained from G^u by one Operation 1, and the gadget is H_3'. So, $G \in \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_5\}$.

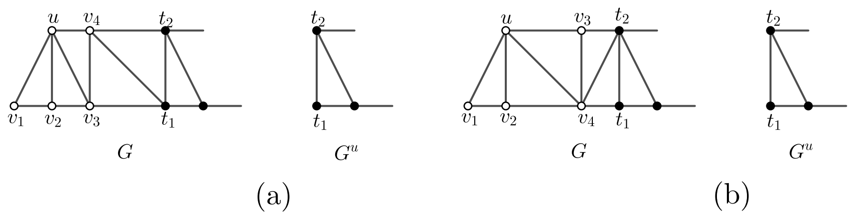

Figure 10. (v_1, v_2, u, v_3, v_4) is the DELETED VERTEX SEQUENCE of the procedure CREATE GRAPH.

Subcase 1.2. $|G^u| = n - d(u)$ and $d(u) = 5$, or $|G^u| = n - d(u) - 1$ and $d(u) = 4$. Note that $|G^u| \equiv 3 \pmod 4$. By the induction hypothesis, $G^u \in \{\mathcal{A}_1, \mathcal{A}_4\}$. More precisely, G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H_3', except one gadget which belong to $\{H_4', H_4'', H_5'\}$ or G^u is obtained from one of H_4 and H_5 by a finite sequence of Operation 1, and each following gadget is H_3'. In the former case, by the similar argument of Case 1.1, G is obtained from G^u by Operation 1, and the gadget is H_4'. In the latter case, without loss of generality, let v_1, v_2, u, v_3, v_4 be the DELETED VERTEX SEQUENCE of the procedure CREATE GRAPH, we have the situations depicted in Figure 10. We can see that $E(G) \setminus E(G^u) = \{v_1u, v_2u, v_3u, v_4u, v_1v_2, v_2v_3, v_3v_4, v_3t_1, v_4t_1v_4t_2\}$ or $E(G) \setminus E(G^u) = \{v_1u, v_2u, v_3u, v_4u, v_1v_2, v_2v_4, v_3v_4, v_3t_2, v_4t_1, v_4t_2\}$. In either case, G is obtained from G^u by one Operation 1, and the gadget is one of H_5' and H_4''. So $G \in \mathcal{A}_1 \cup \mathcal{A}_4$.

Subcase 1.3. $|G^u| = n - d(u)$ and $d(u) = 6$, or $|G^u| = n - d(u) - 1$ and $d(u) = 5$. It is easy to see that $|G^u| \equiv 2 \pmod 4$. By the induction hypothesis, $G^u \in \mathcal{A}_1$. More precisely, G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H_3'.

In the former case, by the similar argument (1) of Case 1.1, it means that G is obtained from G^u by one Operation 1, and the gadget is H'_6. So $G \in \mathcal{A}_2$. In the latter case, similar to the argument as in Case 1.2, G is obtained from G^u by one Operation 1, and the gadget is one of H'_8 and H'_{10}. So $G \in \mathcal{A}_2$.

Case 2. $|G| = 4k + 1$, where $k \geq 2$. Then G is a graph satisfying $\tilde{\gamma}_c(G) = k$. Similar to the argument as in Case 1, we have $d(u) = 4, 5, 6, 7$ when $|G^u| = n - d(u)$ and $d(u) = 4, 5, 6$ when $|G^u| = n - d(u) - 1$. We distinguish four subcases as follows.

Subcase 2.1. $|G^u| = n - d(u)$ and $d(u) = 4$. Note that $|G^u| \equiv 1 \pmod 4$. By the induction hypothesis, $G^u \in \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\}$. More precisely, there are seven possibilities for G^u.

(1) G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3, except three gadgets which belong to $\{H'_4, H''_4, H'_5\}$.

(2) G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3, except one gadget which belongs to $\{H'_7, H'_9, H'_{11}\}$.

(3) G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3, except two gadgets P_1 and P_2, where $P_1 \in \{H'_4, H''_4, H'_5\}$ and $P_2 \in \{H'_6, H'_8, H'_{10}\}$.

(4) G^u is obtained from one of H_4 and H_5 by a finite sequence of Operation 1, and following gadget is H'_3, except one gadget which belongs to $\{H'_6, H'_8, H'_{10}\}$.

(5) G^u is obtained from one of H_4 and H_5 by a finite sequence of Operation 1, and following gadget is H'_3, except two gadgets which belong to $\{H'_4, H''_4, H'_5\}$.

(6) G^u is obtained from one of H_1, H_6, H_8, H_{10}, H_{12} and H_{14} by a finite sequence of Operation 1, and each following gadget is H'_3, except one gadget which belongs to $\{H'_4, H''_4, H'_5\}$.

(7) G^u is obtained from one of H_2, H_7, H_9, H_{11}, H_{13} and H_{15} by a finite sequence of Operation 1, and each following gadget is H'_3.

Analogous to the argument as in Case 1, G is obtained from G^u by one Operation 1, and the gadget is H'_3. So, $G^u \in \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5, \mathcal{A}_6\}$.

Subcase 2.2. $|G^u| = n - d(u)$ and $d(u) = 5$, or $|G^u| = n - d(u) - 1$ and $d(u) = 4$. Note that $|G^u| \equiv 0 \pmod 4$. By the induction hypothesis, $G^u \in \{\mathcal{A}_1, \mathcal{A}_2, \mathcal{A}_4, \mathcal{A}_5\}$. More precisely, there are four possibilities for G^u.

(1) G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3, except two gadgets which belong to $\{H'_4, H''_4, H'_5\}$.

(2) G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3, except one gadget which belongs to $\{H'_6, H'_8, H'_{10}\}$.

(3) G^u is obtained from one of H_4 and H_5 by a finite sequence of Operation 1, and following gadget is H'_3, except one gadget which belongs to $\{H'_4, H''_4, H'_5\}$.

(4) G^u is obtained from one of H_1, H_6, H_8, H_{10}, H_{12} and H_{14} by a finite sequence of Operation 1, and each following gadget is H'_3.

It is easy to see that G is obtained from G^u by one Operation 1, and the gadget is one of H'_4, H''_4 and H'_5. So, $G \in \{\mathcal{A}_1, \mathcal{A}_3, \mathcal{A}_4, \mathcal{A}_5\}$.

Subcase 2.3. $|G^u| = n-d(u)$ and $d(u) = 6$, or $|G^u| = n-d(u)-1$ and $d(u) = 5$. Note that $|G^u| \equiv 3 \pmod 4$. By the induction hypothesis, $G^u \in \{\mathcal{A}_1, \mathcal{A}_4\}$. More precisely, there are two possibilities for G^u.

(1) G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3, except one gadget which belong to $\{H'_4, H''_4, H'_5\}$.

(2) G^u is obtained from one of H_4 and H_5 by a finite sequence of Operation 1, and each following gadget is H'_3.

In either case, one can see that G is obtained from G^u by one Operation 1, and the gadget is one of H'_6, H'_8 and H'_{10}. So, $G \in \{\mathcal{A}_3, \mathcal{A}_4\}$.

Subcase 2.4. $|G^u| = n - d(u)$ and $d(u) = 7$, or $|G^u| = n - d(u) - 1$ and $d(u) = 6$. Note that $|G^u| \equiv 2 \pmod 4$. By the induction hypothesis, $G^u \in \mathcal{A}_1$. More precisely, G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3. In the former case, one can see that G is obtained from G^u by one Operation 1, and following gadget is H'_7. So $G \in \mathcal{A}_2$. In the latter case, G is obtained from G^u by one Operation 1, and following gadget is one of H'_9 and H'_{11}. So $G \in \mathcal{A}_2$.

Case 3. $|G| = 4k+2$, where $k \geq 2$. Then G is a graph satisfying $\tilde{\gamma}_c(G) = k+1$. Similar to the previous discussion, it is easy to verify that $d(u) = 4$ when $|G^u| = n - d(u)$. Note that $|G^u| \equiv 2 \pmod 4$. By the induction hypothesis, $G^u \in \mathcal{A}_1$. More precisely, G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3. Analogous to the argument as in Case 1, it is easy to see that G is obtained from G^u by one Operation 1, and the gadget is H'_3. So, $G \in \mathcal{A}_1$.

Case 4. $|G| = 4k+3$, where $k \geq 2$. Then G is a graph satisfying $\tilde{\gamma}_c(G) = k+1$. Similar to the previous discussion, it is easy to verify that $d(u) = 4, 5$ when $|G^u| = n - d(u)$ and $d(u) = 4$ when $|G^u| = n - d(u) - 1$. We distinguish three subcases as follows.

Subcase 4.1. $|G^u| = n - d(u)$ and $d(u) = 4$. Note that $|G^u| \equiv 3 \pmod 4$. By the induction hypothesis, $G^u \in \{\mathcal{A}_1, \mathcal{A}_4\}$. More precisely, there are 2 possibilities for G^u.

(1) G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H'_3, except one gadget which belong to $\{H'_4, H''_4, H'_5\}$.

(2) G^u is obtained from one of H_4 and H_5 by a finite sequence of Operation 1, and each following gadget is H'_3.

Analogous to the previous argument, G is obtained from G^u by one Operation 1, and the gadget is H'_3. So, $G^u \in \{\mathcal{A}_1, \mathcal{A}_4\}$.

Subcase 4.2. $|G^u| = n - d(u)$ and $d(u) = 5$. Note that $|G^u| \equiv 2 \pmod 4$. By the induction hypothesis, $G^u \in \mathcal{A}_1$. More precisely, G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H_3'. Thus, it is easy to see that G is obtained from G^u by one Operation 1, and the gadget is H_4'. So $G \in \mathcal{A}_1$.

Subcase 4.3. $|G^u| = n - d(u) - 1$ and $d(u) = 4$. Note that $|G^u| \equiv 2 \pmod 4$. By the induction hypothesis, $G^u \in \mathcal{A}_1$. More precisely, G^u is obtained from H_3 by a finite sequence of Operation 1, and each following gadget is H_3'. One can see that G is obtained from G^u by one Operation 1, and the gadget is one of H_4'' and H_5'. So $G \in \mathcal{A}_1$. ■

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