Discussiones Mathematicae
Graph Theory 44 (2024) 613–637
https://doi.org/10.7151/dmgt.2461

EDGE PRECOLORING EXTENSION OF TREES II

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Abstract

We consider the problem of extending and avoiding partial edge colorings of trees; that is, given a partial edge coloring φ of a tree T we are interested in whether there is a proper $\Delta(T)$-edge coloring of T that *agrees* with the coloring φ on every edge that is colored under φ; or, similarly, if there is a proper $\Delta(T)$-edge coloring that *disagrees* with φ on every edge that is colored under φ. We characterize which partial edge colorings with at most $\Delta(T) + 1$ precolored edges in a tree T are extendable, thereby proving an analogue of a result by Andersen for Latin squares. Furthermore we obtain some "mixed" results on extending a partial edge coloring subject to the condition that the extension should avoid a given partial edge coloring; in particular, for all $0 \leq k \leq \Delta(T)$, we characterize for which configurations consisting of a partial coloring φ of $\Delta(T) - k$ edges and a partial coloring ψ of $k + 1$ edges of a tree T, there is an extension of φ that avoids ψ.

Keywords: edge coloring, tree, precoloring extension, avoiding edge coloring.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

In this paper we are interested in extending partial edge colorings of graphs. A *partial edge coloring* (or *precoloring*) of a graph G is a coloring of some subset $E' \subseteq E(G)$. Unless otherwise stated, we shall assume that all (partial) edge colorings are proper. A partial t-edge coloring φ of G is called *extendable* if there is a proper t-edge coloring f of G such that $f(e) = \varphi(e)$ for every edge e that is colored under φ; f is called an *extension* of φ. We only consider extensions of a graph G with $\chi'(G)$ colors, unless otherwise stated, where $\chi'(G)$ as usual denotes the chromatic index of G.

The problem of extending precolorings have immediate applications in scheduling problems where some activities are prescheduled and we aim for a schedule of minimum "size" (i.e., using a minimum number of colors).

Edge precoloring extension problems seem to have been first considered for the balanced complete bipartite graphs $K_{n,n}$: these type of problems were already considered in the 1960s in connection with the problem of completing partial Latin squares and the well-known Evans' conjecture [9]. This conjecture suggests that any $n \times n$ partial Latin square with at most $n-1$ non-empty cells can be completed to an $n \times n$ Latin square, which is equivalent to asking for a proper n-edge coloring of $K_{n,n}$ that agrees with $n-1$ precolored edges. The conjecture was proved for large n by Häggkvist [12], and then for all n independently by Smetaniuk [15] and Andersen and Hilton [3]. Moreover, Andersen and Hilton [3] completely characterized $n \times n$ partial Latin squares with n non-empty cells that cannot be completed to a Latin square. In a follow-up paper Andersen [2] characterized which $n \times n$ partial Latin squares with $n+1$ non-empty cells are completable to Latin squares.

Kuhl and Denley [13] proved a "mixed" result in the same vein by showing that if P is a partial Latin square of order $4t$ with at most $t-1$ non-empty cells and Q is a partial Latin square of the same order that does not agree with P on any non-empty cell, then there is a completion of P that avoids Q, that is, the completion of P *disagrees* with Q on every non-empty cell of Q. Moreover, they conjectured that the same holds for partial Latin squares of any order $n \geq 4$ under the assumption that P has at most $n-2$ non-empty cells, which would be best possible.

Quite recently, motivated by results on completing partial Latin squares, questions on extending partial edge colorings of hypercubes were studied in [5]. In particular, a characterization of which partial edge colorings with at most d precolored edges are extendable to d-edge colorings of the d-dimensional hypercube Q_d was obtained, thereby establishing an analogue for hypercubes of the characterization by Andersen and Hilton [3] for partial Latin squares. Moreover, in [4] Casselgren *et al.* proved a "mixed" result by characterizing, for all

$1 \leq k \leq d$, for which configurations consisting of a partial coloring φ of $d-k$ edges and a partial coloring ψ of k edges, there is an extension of φ that avoids ψ.

In [6], we studied similar questions on edge precoloring extension of trees. In particular, we obtained an analogue for trees of the aforementioned result of Andersen and Hilton by characterizing exactly which precolorings of at most $\Delta(T)$ edges in a tree T are extendable to $\Delta(T)$-edge colorings of T. We also proved sharp conditions on when it is possible to extend a precoloring of a matching or a precoloring of a collection of connected subgraphs of a tree T to a $\Delta(T)$-edge coloring of T.

In this paper, we continue our study of questions on extending edge precolorings of trees. First, we prove an analogue for trees of the result of Andersen [2] by characterizing exactly which precolorings of at most $\Delta(T)+1$ edges in a tree T are extendable to $\Delta(T)$-edge colorings of T. Next, we use this result for proving a characterization, for all $1 \leq r \leq \Delta(T)+1$, for which configurations consisting of a precoloring φ of a tree T with $\Delta(T)+1-r$ colored edges, and a precoloring ψ of T with r colored edges, there is an extension of φ that avoids ψ.

On a slightly different note, Cropper *et al.* [7] proved necessary and sufficient conditions for the existence of an edge list multicoloring of a tree, which we were made aware of while preparing this paper. Since a precoloring extension problem has a natural interpretation as a list coloring problem, the results in this paper could, at least in principle, be deduced from results therein. Nevertheless, our focus, and approach, in this paper (as well as in [6]) is different, and we have not made any attempt to apply their results.

2. Extending Precolorings of a Tree T with $\Delta(T)+1$ Precolored Edges

We shall use standard terminology for edge colorings. Let φ be a proper t-edge coloring of G. If $\varphi(e)=i$, then we say that e is *φ-colored i*. For a vertex $v \in V(G)$, we say that a color i *appears at v under φ* if there is an edge e incident to v with $\varphi(e)=i$. Moreover, if $1 \leq a,b \leq t$, then a path P in G is called *(a,b)-colored under φ* if the edges of P are alternately colored a and b. We also say that such a path is *bicolored under φ*.

In all the above definitions, we often leave out the explicit reference to a coloring φ, if the coloring is clear from the context.

In [6] we proved that a proper precoloring of at most $\Delta(T)$ edges in a tree T is always extendable unless the precoloring φ satisfies any of the following conditions:

(C1) there is an uncolored edge uv in T such that u is incident with edges of $k<\Delta(T)$ distinct colors and v is incident to $\Delta(T)-k$ edges colored with $\Delta(T)-k$

other distinct colors (so uv is adjacent to edges of $\Delta(T)$ distinct colors);

(C2) there is a vertex u of degree $\Delta(T)$ that is incident with edges of $\Delta(T) - k$ distinct colors $c_1, \dots, c_{\Delta(T)-k}$, and k vertices $v_1, \dots, v_k$, where $1 \leq k < \Delta(T)$, such that for $i = 1, \dots, k$, uv_i is uncolored but v_i is incident with an edge colored by a fixed color $c \notin \{c_1, \dots, c_{\Delta(T)-k}\}$;

(C3) there is a vertex u of degree $\Delta(T)$ such that every edge incident with u is uncolored but there is a fixed color c satisfying that every edge incident with u is adjacent to another edge colored c;

(C4) $\Delta(T) = 2$ and there are two precolored edges using the same color if they are at even distance, and using different colors if they are at odd distance.

For a tree T with $\Delta(T) \geq 2$, the set of all colorings satisfying the corresponding condition above is denoted by C_i for $i = 1, 2, 3, 4$, and we set $C = \bigcup C_i$. The following is a main result of [6].

Theorem 2.1. *Let T be a forest with maximum degree $\Delta(T)$. If φ is a proper $\Delta(T)$-edge precoloring of T with at most $\Delta(T)$ precolored edges and $\varphi \notin C$, then φ is extendable to a proper $\Delta(T)$-edge coloring of T.*

An immediate consequence of this theorem is the following.

Corollary 2.2. *If T is a forest with maximum degree $\Delta(T)$, then any partial edge coloring with at most $\Delta(T) - 1$ precolored edges is extendable to a proper $\Delta(T)$-edge coloring of T.*

Here, we shall prove that a proper precoloring of at most $\Delta(T)+1$ edges in a tree T is always extendable unless the precoloring φ satisfies any of the following four conditions:

(R1) φ satisfies any of the conditions (C1), (C2), (C3) or (C4);

(R2) there are two uncolored adjacent edges uv and uw in T such that u is incident with edges of $\Delta(T) - 3$ distinct colors $c_1, \dots, c_{\Delta(T)-3}$, and both v and w are incident to two edges colored with two other distinct colors $c_{\Delta(T)-2}, c_{\Delta(T)-1}$ (so both uv and uw are adjacent to edges of $\Delta(T) - 1$ distinct colors);

(R3) $\Delta(T) = 3$, there are two uncolored adjacent edges uv and uw with $d(v) = d(w) = 3$ in T, and there is a color c, such that every edge incident with v or w except uv and uw, is either uncolored but adjacent to an edge colored c, or colored by a color distinct from c;

(R4) $\Delta(T) = 3$ and there is an uncolored edge uv with $d(u) = d(v) = 3$, and there are two colors c and c' in T such that every edge incident with u except uv is either uncolored but adjacent to an edge colored c or colored by some color distinct from c, and every edge incident with v except uv is either uncolored but adjacent to an edge colored c' or colored by some color distinct from c'.

For $i = 1, 2, 3, 4$, we denote by R_i the set of all partial colorings of a tree T, $\Delta(T) \geq 2$, satisfying the corresponding condition above, and we set $R = \bigcup R_i$. Let us briefly explain why φ is not extendable if it is a precoloring of T with exactly $\Delta(T) + 1$ precolored edges and $\varphi \in R$.

Suppose first that the precoloring φ satisfies the first condition (R1). Since any precoloring contained in C is not extendable, there is no proper $\Delta(T)$-edge coloring of T that agrees with φ. If φ, on the other hand, satisfies condition (R2), then any extension of φ satisfies that color $c_{\Delta(T)} \in \{1, \ldots, \Delta(T)\} \setminus \{c_1, \ldots, c_{\Delta(T)-1}\}$ must appear on both uv and uw. However, such a $\Delta(T)$-edge coloring cannot be proper, since this implies that u is incident with two edges colored $c_{\Delta(T)}$.

Suppose now that φ satisfies condition (R3). If f is a $\Delta(T)$-edge coloring that is an extension of φ, then f satisfies that the color c must appear on both uv and uw. Thus, f cannot be proper. We leave to the reader to verify that φ is not extendable if it satisfies condition (R4). Our main result is the following.

Theorem 2.3. *Let T be a forest with maximum degree $\Delta(T)$. If φ is a proper $\Delta(T)$-edge precoloring of T with at most $\Delta(T) + 1$ precolored edges and $\varphi \notin R$, then φ is extendable to a proper $\Delta(T)$-edge coloring of T.*

Proof. The proof of Theorem 2.3 proceeds by induction on $|E(T)|$. The statement is trivial for any forest with at most two edges; thus we assume that T is a forest with $|E(T)| \geq 3$ and that the theorem holds for any forest T', with $|E(T')| < |E(T)|$, that is, any proper partial coloring φ' of at most $\Delta(T') + 1$ edges is extendable to a proper $\Delta(T')$-edge coloring unless $\varphi' \in R$, and consider a proper partial coloring φ of T with exactly $\Delta(T) + 1$ precolored edges; for the case of fewer precolored edges, Theorem 2.1 yields the result.

If $\Delta(T) = 1$, then the statement trivially holds. Now assume that $\Delta(T) = 2$; then every component of T is a path and exactly three edges of T are precolored under φ. Let us first assume that all precolored edges have the same color. Let e_1, e_2, e_3 be the precolored edges of T. We may assume that all the precolored edges are in the same component because otherwise if every component contains at most one precolored edge, then we are done by Corollary 2.2; else, if there is a component that contains two precolored edges and $\varphi \notin C_4$, then we are done by Theorem 2.1.

Without loss of generality we can assume that e_2 is contained in a path from e_1 to e_3. If the distance between e_1 and e_2, or e_2 and e_3 is even, then $\varphi \in C_4$; otherwise since every component is a path, we can color the edges of T by colors 1 and 2 alternatively. A similar argument applies in the case when two different colors appear on the precolored edges.

Let us now consider the case when $\Delta(T) \geq 3$. If T has an uncolored pendant edge e such that $\Delta(T-e) = \Delta(T)$, then the restriction of φ to $T-e$ is extendable by the induction hypothesis; thus φ is extendable to a proper $\Delta(T)$-edge coloring of T. Therefore we assume that T has no such pendant edge e. Note that, since $\Delta(T)+1$ edges are precolored, this implies that T contains only one unique vertex v of maximum degree if $\Delta(T) \geq 4$, and at most two vertices u and v of maximum degree if $\Delta(T) = 3$.

Without loss of generality we shall assume that color 1 is used at least as frequently as any other color, and let E_i be the set of all edges colored i. Thus E_1 contains at least two edges. Our general proof strategy is to show that there is a matching $\mathcal{M}$ in T containing all edges precolored 1, no other precolored edges, and covering every vertex of degree $\Delta(T)$, subject to the condition that the restriction of the coloring φ to $T - \mathcal{M}$ not being in C; we can then deduce the result from the induction hypothesis. We shall need to consider many cases, and first we consider the case when $\Delta(T) = 3$.

Case 1. $\Delta(T) = 3$. We shall distinguish between two different cases, whether T contains only one unique vertex v or two vertices u and v of maximum degree 3. Note that since T contains four precolored edges and color 1 is used at least as frequently as any other color, $|E_2| + |E_3| \leq 2$.

Case 1.1. *T contains a unique vertex v of degree* 3. Suppose first that T contains a unique vertex v of maximum degree 3 and that some edge of E_1 is incident with v. If the restriction of φ to $T - E_1$ does not satisfy (C4), then by the induction hypothesis, the restriction of φ to $T - E_1$ is extendable to a proper edge coloring using colors $2, 3$; hence φ is extendable.

If the restriction of φ to $T-E_1$ satisfies (C4), then since $|E_2|+|E_3| \leq 2$, there are exactly two precolored edges e_1 and e_2 precolored 2 or 3, at even distance if they have the same color, or at odd distance if they have different colors. We pick an uncolored edge e that is not adjacent to any edge from E_1 which is contained in a path from e_1 to e_2; since there is a unique vertex of degree 3, and $\varphi \notin C_1 \cup C_2$, there is such an edge e.

We assign the color 1 to e and consider the graph $T - (E_1 \cup \{e\})$. Since this graph only contains two precolored edges, the restriction of φ to $T - (E_1 \cup \{e\})$ is not in C_4. Thus, since $\Delta(T - (E_1 \cup \{e\}) = 2$, it is extendable to a proper edge coloring of $T - (E_1 \cup \{e\})$ by Theorem 2.1. Hence, φ is extendable.

Assume now that no edge from E_1 is incident with v. Then we pick an uncolored edge e incident to v that is not adjacent to any edge from E_1; since $\varphi \notin C_1 \cup C_2 \cup C_3$, there is such an edge e. Now, if the restriction of φ to $T - (E_1 \cup \{e\})$ does not satisfy (C4), then we color e with the color 1, and apply the induction hypothesis to the restriction of φ to $T - (E_1 \cup \{e\})$ to obtain an extension of φ; otherwise, if the restriction of φ to $T - (E_1 \cup \{e\})$ satisfies (C4), then we need to consider two different cases depending on whether v is contained in the path P between the two precolored edges e_1 and e_2 in $T-(E_1\cup\{e\})$, or not.

Subcase A. *v is contained in the path P from e_1 to e_2.* Let us first assume that v is contained in P. Since no edge from E_1 is contained in P, we can pick an uncolored edge e' incident to v that is not adjacent to any edge from E_1, but is contained in P; since v is the only vertex of degree 3, and $\varphi \notin C_1\cup C_2\cup C_3$, there exists such an edge e'. Then, in this case, the restriction of φ to $T-(E_1\cup\{e'\})$ does not satisfy (C4). By coloring e' by color 1, and applying the induction hypothesis to $T-(E_1\cup\{e'\})$, we obtain an extension of φ.

Subcase B. *v is not contained in the path P from e_1 to e_2.* Let us now assume that v is not contained in P. Since the restriction of φ to $T-(E_1\cup\{e\})$ satisfies (C4), we pick an uncolored edge e' that is not adjacent to any edge from $E_1\cup\{e\}$, and which is contained in P; since $e_1, e_2 \notin E_1$ and v is not contained in P, there is such an edge e'. Finally, we may color the edges e and e' by the color 1, and apply the induction hypothesis to $T-(E_1\cup\{e,e'\})$ to obtain an extension of φ.

Case 1.2. *T contains two vertices u and v of degree* 3. We shall consider some different subcases.

Case 1.2.1. *E_1 covers both u and v.* Suppose first that E_1 covers both u and v. If the restriction of φ to $T-E_1$ does not satisfy (C4), then by the induction hypothesis, the restriction of φ to $T-E_1$ is extendable to a proper edge coloring using colors $2,3$; hence φ is extendable. If the restriction of φ to $T-E_1$ satisfies (C4), then there are two precolored edges e_1 and e_2, at even distance if they have the same color, or at odd distance if they have different colors. As before, our strategy is to pick an uncolored edge e that is not adjacent to any edge from E_1, and which is contained in the path P from e_1 and e_2; we will prove that since $\varphi \notin R$, there is such an edge e.

If e_1 and e_2 are at distance at least 5, then we can certainly pick such an edge e, and if they are at distance 4, then there is such an edge unless $\varphi \in R_3$; on the other hand if e_1 and e_2 are at distance 3, then there is such an edge unless $\varphi \in R_3\cup R_4$. Moreover, if e_1 and e_2 are at distance 1 or 2, then there is such an edge unless $\varphi \in C_1$ or $\varphi \in C_2\cup R_2$, respectively. In conclusion, in all cases we can pick a required edge e contained in the path P and thus the restriction of φ to $T-(E_1\cup\{e\})$ does not satisfy (C4). We color e with the color 1, and are done by the induction hypothesis.

Case 1.2.2. *E_1 covers only one of u and v.* Suppose now that only one of the vertices u or v is incident with an edge from E_1, say u. Then we pick an uncolored edge e incident to v that is not adjacent to any edge from E_1; since $\varphi \notin C_1\cup C_2\cup C_3$, there is such an edge e. Now, if the restriction of φ to $T-(E_1\cup\{e\})$ does not satisfy (C4), then color e with the color 1, and by applying the induction hypothesis to the restriction of φ to $T-(E_1\cup\{e\})$, we obtain an extension of φ; otherwise, if the restriction of φ to $T-(E_1\cup\{e\})$ satisfies (C4), then as before, there are two precolored edges e_1 and e_2 joined by a path P. In this case, we need to consider two different subcases depending on whether v is contained in P, or not.

Subcase A. *v is contained in the path P from e_1 to e_2.* Let v be contained in P and let us first assume that u and v are adjacent. If v is not incident with e_1 or e_2, since all the edges in P are uncolored, then we can pick an uncolored edge e' incident to v that is not adjacent to any edge from E_1, but is contained in P; since $\varphi \notin C_1 \cup C_2 \cup C_3$, there is such an edge e'. In this case, the restriction of φ to $T - (E_1 \cup \{e'\})$ does not satisfy (C4), and we are done by applying the induction hypothesis to $T - (E_1 \cup \{e'\})$.

Suppose now that v is incident with e_1 or e_2. If $uv \in E(P)$, then we can pick an uncolored edge e'' that is not adjacent to any edge from $E_1 \cup \{e'\}$, but is contained in P; since $\varphi \notin C_2$, there is such an edge e''. Thus, in this case the restriction of φ to $T - (E_1 \cup \{e, e''\})$ does not satisfy (C4). Now, we may color the edges e and e'' by the color 1, and apply the induction hypothesis to $T - (E_1 \cup \{e, e''\})$ to obtain an extension of φ; otherwise, if $uv \notin E(P)$, then e must be contained in P, which is a contradiction; we conclude that this is not possible.

Finally, let us consider the case when u and v are nonadjacent. Since all the edges of P are uncolored, we can pick an uncolored edge e' incident to v that is not adjacent to any edge from E_1, but is contained in P; since $\varphi \notin C$, there is such an edge e'. In this case, the restriction of φ to $T - (E_1 \cup \{e'\})$ does not satisfy (C4). By coloring e' by the color 1, and applying the induction hypothesis to the restriction of φ to $T - (E_1 \cup \{e'\})$, we obtain an extension of φ.

Subcase B. *v is not contained in the path P from e_1 to e_2.* Let us now assume that v is not contained in P. Since all the edges of P are uncolored, then we pick uncolored edge e' not adjacent to any edge from $E_1 \cup \{e\}$, and which is contained in a path from e_1 to e_2; since v is not contained in P and by our assumption E_1 covers only u, there is such an edge e' provided that $\varphi \notin C$. Now, we may color the edges e and e' by the color 1, and apply the induction hypothesis to $T - (E_1 \cup \{e, e'\})$ to obtain an extension of φ.

Case 1.2.3. *no edge from E_1 covers u or v.* Finally, let us consider the case when no edge from E_1 is incident with a vertex of maximum degree. In this case, we consider two different subcases depending on whether u and v are adjacent or not.

Subcase A. *u and v are adjacent.* Assume u is adjacent to v and let us first assume that uv is not precolored. Then the edge $e = uv$ is not adjacent to any edge from E_1. If the restriction of φ to $T - (E_1 \cup \{e\})$ does not satisfy (C4), then color e by the color 1, and by induction hypothesis the restriction of φ to $T - (E_1 \cup \{e\})$ is extendable using the colors $2, 3$; otherwise, if the restriction of φ to $T - (E_1 \cup \{e\})$ satisfies (C4), then as before there are two precolored edges e_1 and e_2 joined by a path P. In this case, we need to consider different cases depending on whether u or v is contained in P, or not. Note that since the restriction of φ to $T - (E_1 \cup \{e\})$ satisfies (C4), at most one of u and v is contained in P.

Suppose first that exactly one of u or v is contained in P, by symmetry say u. If every neighbor of v distinct from u is incident with an edge from E_1, then we can pick an uncolored edge e' contained in P that is neither adjacent to any edge from $E_1 \cup \{e\}$ nor incident to u or v; since $\varphi \notin R_4$, there is such an edge e'. Now, we may color the edges e and e' by the color 1, and apply the induction hypothesis to $T - (E_1 \cup \{e, e'\})$ to obtain an extension of φ. Otherwise, if some neighbor $x \neq u$ of v is not incident with any edge from E_1, then we pick another uncolored edge $e'' \neq e$ incident to u that is not adjacent to any edge from $E_1 \cup \{xv\}$, and which is contained in P; since E_1 does not cover u, there is such an edge e''. Now, we may color the edges xv and e'' by the color 1, and apply the induction hypothesis to $T - (E_1 \cup \{xv, e''\})$ to obtain an extension of φ.

On the other hand, if none of u and v are contained in P, since u and v are the only vertices of degree 3, then we can pick an uncolored edge e' contained in P that is not adjacent to any edge from $E_1 \cup \{e\}$. By coloring e and e' by the color 1, the restriction of φ to $T - (E_1 \cup \{e, e'\})$ is extendable by the induction hypothesis using colors 2 and 3.

Suppose now that uv is precolored by some color distinct from 1. In this case, we can pick uncolored edges e and e' incident to u and v respectively that are not adjacent to any edge from E_1; since $\varphi \notin C_2$, there are such edges e and e'. Now, if the restriction of φ to $T - (E_1 \cup \{e, e'\})$ does not satisfy (C4), then we are done; otherwise, if the restriction of φ to $T - (E_1 \cup \{e, e'\})$ satisfies (C4), then as before there are two precolored edges e_1 and e_2 joined by a path P. Moreover, $e_1 = uv$ or $e_2 = uv$, so exactly one of u and v is contained in P, by symmetry say v. Then we can pick an uncolored edge e'' incident to v that is not adjacent to any edge from $E_1 \cup \{e\}$, and which is contained in P; since v is in P, and u and v are the only vertices of degree 3, there is such an edge e''. The restriction of φ to $E_1 \cup \{e, e''\}$ does not satisfy (C4), and we are done by the induction hypothesis.

Subcase B. *u and v are nonadjacent.* Let us now consider the case when u is not adjacent to v. We first pick an uncolored edge e incident to one of the vertices u or v, say u, that is not adjacent to any edge from E_1; since $\varphi \notin C_1 \cup C_2 \cup C_3$, there is such an edge e. Next we can pick another uncolored edge e' incident to v that is not adjacent to any edge from $E_1 \cup \{e\}$; otherwise, if such an edge e' does not exist, since $\varphi \notin C_1 \cup C_2 \cup C_3$, and u and v are nonadjacent, then this implies that u and v must be at distance 2 from each other. In this case, first we pick an uncolored edge e incident to u or v (say u) that is neither adjacent to any edge from E_1 nor contained in the path from u to v; since $\varphi \notin R_3$, there is such an edge e. Next we pick another uncolored edge e' incident to v that is not adjacent to any edge from $E_1 \cup \{e\}$; by our choice of e, there is such an edge e' provided that $\varphi \notin C$. In any case, if the restriction of φ to $T - (E_1 \cup \{e, e'\})$ does not satisfy (C4), then we color the edges e and e' with color 1, and apply the induction hypothesis to $E_1 \cup \{e, e'\}$.

Otherwise, if the restriction of φ to $T - (E_1 \cup \{e, e'\})$ satisfies (C4), then there are precolored edges e_1 and e_2 joined by a path P in $T - (E_1 \cup \{e, e'\})$. As before, either we pick new edge(s) incident to u and/or v contained in P, or choose an additional edge contained P (the latter applies if none of u and v is contained in P). Since the arguments are similar to the ones used above, the details are omitted.

Case 2. $\Delta(T) \geq 4$. Suppose now that $\Delta(T) \geq 4$; and let v be the unique vertex of maximum degree. We distinguish between two different cases based on the number of colors appearing on the edges of T under φ.

Case 2.1. *some color $c \neq 1$ appears on at least two edges under the precoloring φ.* If there is some color $c \neq 1$ that appears on at least two edges in T, then at most $\Delta(T)-1$ colors appear on the edges of T. If some edge of E_1 is incident with v and the restriction of φ to $T - E_1$ does not satisfy (C2), then by the induction hypothesis, the restriction of φ to $T - E_1$ is extendable to a proper edge coloring; thus φ is extendable. Otherwise, if the restriction of φ to $T - E_1$ satisfies (C2), then since color 1 appear on at least as many edges as any other color, there is a vertex u of degree $\Delta(T) - 1$ in $T - E_1$ that is incident with edges of $\Delta(T) - 3$ distinct colors $c_1, \ldots, c_{\Delta(T)-3}$, and two vertices v_1, v_2 such that uv_i is uncolored but v_i is incident with an edge colored by a fixed color $c \notin \{c_1, \ldots, c_{\Delta(T)-3}\}$. Since $\varphi \notin R_2$, there is an uncolored edge e incident to u that is not adjacent to any edge from E_1. By coloring e by the color 1, and applying the induction hypothesis to the restriction of φ to $T - (E_1 \cup \{e\})$, we obtain an extension of φ.

Assume now that no edge from E_1 is incident with v. Then we can pick an uncolored edge e incident to v that is not adjacent to any edge from E_1; since $\varphi \notin C_2 \cup C_3$, there is such an edge e. Now, if the restriction of φ to $T - (E_1 \cup \{e\})$ does not satisfy (C2), then color e with the color 1, and apply the induction hypothesis to the restriction of φ to $T - (E_1 \cup \{e\})$ to obtain an extension of φ.

On the other hand, if the restriction of φ to $T - (E_1 \cup \{e\})$ satisfies (C2), then as before there is a vertex u of degree $\Delta(T) - 1$ in $T - (E_1 \cup \{e\})$ that is incident with edges of $\Delta(T) - 3$ distinct colors $c_1, \ldots, c_{\Delta(T)-3}$, and two vertices v_1, v_2, such that uv_i is uncolored but v_i is incident with an edge colored by a fixed color $c \notin \{c_1, \ldots, c_{\Delta(T)-3}\}$. If e is incident to u, then since v is the unique vertex of degree $\Delta(T)$, this implies that $u = v$. Then instead of e, we pick another uncolored edge e' incident to v that is not adjacent to any edge from E_1; since $\varphi \notin R_2$, there is such an edge e'. By coloring e' by the color 1, and applying the induction hypothesis to the restriction of φ to $T - (E_1 \cup \{e'\})$, we obtain an extension of φ.

Otherwise, if e is not incident to u, then we can select an uncolored edge e' incident with u but not adjacent to any edge from E_1; since $\varphi \notin R_2$, there is such an edge e'. Now, if e is not adjacent to e', then we may color the edges e and e' by the color 1, and apply the induction hypothesis to $T - (E_1 \cup \{e, e'\})$ to obtain an extension of φ; otherwise, if e is adjacent to e', since v has maximum degree, there is another uncolored edge $e'' \neq e$ incident to v that is not adjacent to any edge from $E_1 \cup \{e'\}$. Finally, we may color the edges e' and e'' by the color 1, and apply the induction hypothesis to $T - (E_1 \cup \{e', e''\})$ to obtain an extension of φ.

Case 2.2. *all the colors* $1, 2, \ldots, \Delta(T)$ *appears on edges under the precoloring* φ. Suppose now that every color distinct from color 1 appears on exactly one edge in T. If some edge of E_1 is incident with v and the restriction of φ to $T - E_1$ does not satisfy (C1), then by the induction hypothesis, the restriction of φ to $T - E_1$ is extendable to a proper edge coloring using colors $2, 3, \ldots, \Delta(T)$; hence φ is extendable.

If the restriction of φ to $T - E_1$ satisfies (C1), then this implies that there is an uncolored edge e in $T - E_1$ such that e is adjacent to the edges precolored by the colors $2, 3, \ldots, \Delta(T)$. Since v is the unique vertex of degree $\Delta(T)$, it is not incident with e; otherwise e together with the edges precolored by the colors $2, 3, \ldots, \Delta(T)$ satisfy (C1), a contradiction to our assumption. By coloring e by the color 1, and applying the induction hypothesis to the restriction of φ to $T - (E_1 \cup \{e\})$, we obtain an extension of φ.

Assume now that no edge from E_1 is incident with v. Then we can pick an uncolored edge e incident to v that is not adjacent to any edge from E_1; since $\varphi \notin C_1 \cup C_2$, there is such an edge e. Now, if the restriction of φ to $T - (E_1 \cup \{e\})$ does not satisfy (C1), then color e with the color 1, and we are done; otherwise, if the restriction of φ to $T - (E_1 \cup \{e\})$ satisfies (C1), then this implies that there is an uncolored edge e' in $T - (E_1 \cup \{e\})$ such that e' is adjacent to the edges precolored by the colors $2, 3, \ldots, \Delta(T)$. Now, if e' is incident to v, then we pick e' instead of e, since $\varphi \notin C_1$, e' is not adjacent to any edge from E_1, and we can apply the induction hypothesis to $T - (E_1 \cup \{e'\})$.

On the other hand, if e' is not incident to v, then since $\varphi \notin C_1$, e' is not adjacent to any edge from E_1. Now, if e' is not adjacent to e, then we may color the edges e and e' by the color 1, and apply the induction hypothesis to $T - (E_1 \cup \{e, e'\})$ to obtain an extension of φ; otherwise, if e' is adjacent to e, since v has maximum degree, there is another uncolored edge $e'' \neq e$ incident to v that is not adjacent to any edge from $E_1 \cup \{e'\}$. Now, we may color the edges e' and e'' by the color 1, and apply the induction hypothesis to $T - (E_1 \cup \{e', e''\})$ to obtain an extension of φ. This completes the proof of Theorem 2.3. ■

3. Avoiding and Extending Edge Precolorings Simultaneously

Let φ be a proper precoloring of $\Delta(T)+1-r$ edges of T, where $1 \leq r \leq \Delta(T)+1$, and ψ be an arbitrary (i.e., not necessarily proper) coloring of r edges in T. We shall prove that there is a proper $\Delta(T)$-edge coloring of T that extends φ and which avoids ψ unless the colorings φ and ψ satisfy any of the following conditions.

(T1) exactly one edge is ψ-colored, and φ satisfies any of the conditions (C1)–(C4);

(T2) there is a vertex v of degree $\Delta(T)$ such that every edge incident with v is either ψ-colored c, φ-colored by a color distinct from c, or not φ-colored or ψ-colored, but adjacent to an edge φ-colored c;

(T3) there is an edge uv that is ψ-colored c and for every $i \in \{1, 2, \ldots, \Delta(T)\} \setminus \{c\}$ there is an edge incident with u or v that is φ-colored i;

(T4) there is a vertex u incident with $\Delta(T) - 3$ edges φ-colored by $c_1, c_2, \ldots, c_{\Delta(T)-3}$, respectively, and there are two colors $c, c' \notin \{c_1, c_2, \ldots, c_{\Delta(T)-3}\}$ and edges uv, uw such that either

- both uv and uw are ψ-colored c and adjacent to an edge φ-colored c', or
- uv is ψ-colored c and adjacent to an edge φ-colored c', and uw is ψ-colored c' and adjacent to an edge φ-colored c, or
- uv is ψ-colored c and adjacent to an edge φ-colored c', and uw is uncolored but w is incident with two edges φ-colored by colors c and c';

(T5) $\Delta(T) = 3$ and there are two uncolored adjacent edges uv and uw with $d(v) = d(w) = 3$ in T and a color c such that every edge incident with v or w except uv and uw, is either ψ-colored c, φ-colored by some color distinct from c, or uncolored but adjacent to an edge φ-colored c;

(T6) $\Delta(T) = 3$ and there is an uncolored edge uv with $d(u) = d(v) = 3$ in T and two colors c and c' such that every edge incident with u except uv is either ψ-colored c, φ-colored by some color distinct from c, or uncolored but adjacent to an edge φ-colored c, and every edge incident with v except uv is either ψ-colored c', φ-colored by some color distinct from c', or uncolored but adjacent to an edge φ-colored c';

(T7) $\Delta(T) = 3$ and there is a vertex v of degree $d(v) = 3$ that is adjacent to a vertex u in T such that every edge incident with v except uv is either ψ-colored c, φ-colored by a color distinct from c, or uncolored but adjacent to an edge φ-colored c, and uv is uncolored but u is incident with an edge ux that is ψ-colored $c' \neq c$ and x is incident with another edge φ-colored $c'' \notin \{c, c'\}$;

(T8) $\Delta(T) = 2$ and there are two precolored edges e_1 and e_2 in T such that

(i) they are at odd distance and either both are colored under φ or ψ by different colors, or one edge is colored under φ and the other edge is under ψ, and they are colored by the same color;

(ii) they are at even distance and either both are colored by the same color under φ or ψ, or one edge is colored under φ and the other edge is colored under ψ, and they are colored differently.

For $i = 1, \dots, 8$, we denote by T_i the set of all ordered pairs (φ, ψ) of partial colorings of a tree T satisfying the corresponding condition above, and we set $\mathbb{T} = \bigcup T_i$. It is readily verified that if (φ, ψ) satisifes one of the conditions above, then there is no extension of φ that avoids ψ.

We shall use the following result from [6] on extending an edge precoloring where the precolored edges induce several connected subtrees.

Theorem 3.1. *Let T be a tree with maximum degree $\Delta(T) = k \geq 3$. If at most $k-1$ connected subgraphs of T are properly edge-colored using k colors, and the distance between any two vertices in two different precolored subgraphs is at least 3, then this partial edge coloring is extendable to a proper k-edge coloring of T.*

We shall also need the following lemma.

Lemma 3.2. *Let φ' be a proper $\Delta(T)$-edge precoloring of T with $\Delta(T) \geq 3$. Suppose that the precolored edges induce two connected subgraphs T_{H_1} and T_{H_2} of T, where $|E(T_{H_1})| = 1$; then φ' is extendable unless there is exactly one uncolored edge uv between T_{H_1} and T_{H_2} with $u \in V(T_{H_1})$ and $v \in V(T_{H_2})$ such that u is incident to an edge colored c, and v is incident to $\Delta(T) - 1$ edges with distinct colors from $\{1, 2, \dots, \Delta(T)\} \setminus \{c\}$.*

Proof. Without loss of generality we assume that $E(T_{H_1})$ is colored with 1. If $|E(T_{H_1})| + |E(T_{H_2})| \leq \Delta(T) - 1$, then since at most $\Delta(T) - 1$ edges are colored, by Corollary 2.2, φ' is extendable.

So we now assume that at least $\Delta(T)$ edges are colored under φ'. Suppose first exactly one uncolored edge uv with $u \in V(T_{H_1})$ and $v \in V(T_{H_2})$ is contained in a path from T_{H_1} to T_{H_2}. If v is incident to edges colored by every color from $\{2, \dots, \Delta(T)\}$, then these edges together with the edge in T_{H_1} form C_1, which is not extendable; otherwise, if v is incident with edges of at most $\Delta(T) - 2$ distinct colors from $\{2, \dots, \Delta(T)\}$, then there is some color $c \in \{2, \dots, \Delta(T)\}$ that does not appear at v. Thus, we may color uv by the color c, to obtain a connected colored subgraph, and we are done by Theorem 3.1.

Suppose now at least two uncolored edges are contained in a shortest path from a vertex of T_{H_1} to a vertex of T_{H_2}. Since $\Delta(T) \geq 3$, we can color the edges in this path greedily; so again we obtain a connected colored subgraph, and we are done by Theorem 3.1. ∎

Theorem 3.3. *Let φ be a proper $\Delta(T)$-edge coloring of $\Delta(T)+1-r$ edges of the tree T and ψ be a partial coloring of r edges in T such that no edge has the same color under φ and ψ, where $1 \leq r \leq \Delta(T)+1$. If $(\varphi, \psi) \notin \mathbb{T}$, then there is an extension of φ that avoids ψ.*

Proof. Let $E_{\varphi,\psi}$ be the set of edges in $E(T)$ that are colored under φ or ψ. Without loss of generality we assume that no edge is colored under both φ and ψ. First we shall show that if $(\varphi, \psi) \notin \mathbb{T}$, then we can define a new proper coloring φ' of $E_{\varphi,\psi}$ from φ by coloring every ψ-colored edge that is not colored under φ in such a way that the resulting precoloring avoids ψ.

If all ψ-colored edges are pairwise non-adjacent, then we can trivially define the coloring φ' by assigning some color from $\{1, 2, \dots, \Delta(T)\} \setminus \{\psi(e)\}$ to any edge e that is colored under ψ so that φ' is proper and avoids ψ unless (T3) holds. Moreover, if every ψ-colored edge is adjacent to at most $\Delta(T)-2$ edges from $E_{\varphi,\psi}$, then we can define the coloring φ' of $E_{\varphi,\psi}$ from φ by greedily assigning some color from $\{1, 2, \dots, \Delta(T)\} \setminus \{\psi(e)\}$ to any edge e that is colored under ψ so that φ' is proper and avoids ψ. Thus we may assume that at least one ψ-colored edge e_1 is adjacent to at least $\Delta(T)-1$ edges from $E_{\varphi,\psi}$. If e_1 is the only edge with this property, then since $(\varphi, \psi) \notin \mathbb{T}$, at most $\Delta(T)-2$ colors distinct from $\psi(e_1)$ appear on φ-colored edges that are adjacent to e_1; hence, we can define the required coloring φ' by first coloring e_1, and then greedily coloring all other ψ-colored edges in T.

Suppose now that two ψ-colored edges e and e' are both adjacent to $\Delta(T)-1$ edges from $E_{\varphi,\psi}$. Then e and e' have a common vertex u that is incident with

(a) $\Delta(T)-1$ edges from $E_{\varphi,\psi}$, or

(b) $\Delta(T)$ edges from $E_{\varphi,\psi}$.

If (a) holds, then e and e' are both adjacent to one additional edge each from $E_{\varphi,\psi}$ that is not incident with u. It is straightforward to check that unless (φ, ψ) satisfies (T4), we can define the required coloring φ'.

Suppose now that (b) holds, and let us first consider the case when all ψ-colored edges incident with u have the same color. Then (T2) holds unless there is some φ-colored edge e'' incident with u such that $\varphi(e'') = \psi(e)$. Since by assumption (T2) does not hold, this means that we can greedily color the ψ-colored edges incident with u, starting with the edge that is adjacent to most edges from $E_{\varphi,\psi}$.

Suppose now that at least two colors appear on the ψ-colored edges incident with u. Assume $\psi(e) = c$, $\psi(e') = c'$ and that e is adjacent to the largest number of edges from $E_{\varphi,\psi}$. If e is adjacent to $\Delta(T)-1$ edges from $E_{\varphi,\psi}$, then since $\psi(e) \neq \psi(e')$, we can define the coloring φ' by first coloring e by the color c' if no edge incident with u is φ-colored c'; alternatively, if there is such an edge incident with u, then we first color e by the color in $\{1, \dots, \Delta(T)\}$ not appearing on an edge incident with u under φ. On the other hand, if e is adjacent to $\Delta(T)$ edges from $E_{\varphi,\psi}$, then we can similarly define the required coloring φ' unless (T2) holds.

In conclusion, since $(\varphi, \psi) \notin \mathbb{T}$, we can define the coloring φ' from φ by recoloring every ψ-colored edge so that φ' is proper and avoids ψ. In the following we shall, using Theorem 2.3, establish Theorem 3.3 by proving that φ' is extendable to a proper $\Delta(T)$-edge coloring. The case when $\Delta(T) \leq 2$ is trivial, so assume that $\Delta(T) \geq 3$. We first consider the case when $\Delta(T) = 3$.

Case 1. $\Delta(T) = 3$. Since at most $\Delta + 1$ edges are colored under φ', by Theorem 2.3 there is an extension of φ that avoids ψ unless $\varphi' \in C_1 \cup C_2 \cup C_3 \cup R_3 \cup R_4$ because when $\Delta(T) = 3$, R_2 is a special case of R_3. Our strategy is to define a new proper edge coloring from φ' that avoids ψ and is extendable. We shall prove that this is possible unless $(\varphi, \psi) \in \mathbb{T}$. Throughout Case 1, we shall denote three arbitrary colors from $\{1, 2, 3\}$ by c, c' and c'', respectively.

Case 1.1. $\varphi' \in C_1$. If $\varphi' \in C_1$, then there is an uncolored edge uv in T such that u is incident with edges of $k < \Delta(T) = 3$ distinct colors and v is incident to $3 - k$ edges colored with $3 - k$ other distinct colors under φ'. In this case we distinguish between two different cases, whether uv is adjacent to three, or four φ'-colored edges.

Let us first assume that three φ'-colored edges are adjacent to uv. Suppose without loss of generality that one edge e φ'-colored c is incident with u and the remaining two edges φ'-colored c' and c'' are incident with v. Now, if we can define a new coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring the edge e incident with u by one of the colors c' or c'', say c', then color uv by the color c, and we are done by Lemma 3.2; otherwise, if this is not possible then either

(a) e is φ-colored c, or

(b) e is ψ-colored and adjacent to an edge φ-colored by the color in $\{1, 2, 3\} \setminus \{\varphi'(e), \psi(e)\}$.

If (a) holds, then since $\varphi \notin C_1$, one or both φ'-colored edges incident with v are colored under ψ. If both are ψ-colored c, or if one edge is ψ-colored c and the other edge is ψ-colored by a color distinct from c and adjacent to an edge φ-colored c, then (T2) holds. Moreover, if one edge is φ-colored and the other edge is ψ-colored by a color distinct from c and adjacent to an edge φ-colored c, then (C2) holds. Since $(\varphi, \psi) \notin \mathbb{T}$, we can recolor one of the edges incident with v by the color c to obtain a new proper edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ; then color uv by some color not appearing at u or v, and we are done by Lemma 3.2.

Suppose now that (b) holds. If every edge incident with v is ψ-colored c, or φ-colored by a color distinct from c, then (T7) holds. Since again $(\varphi, \psi) \notin \mathbb{T}$, we may assume that at least one edge incident with v is ψ-colored by a color distinct from c. We can recolor this edge by the color c, then color uv by the color not appearing at u or v, and thereafter apply Lemma 3.2.

Suppose now that four φ'-colored edges are adjacent to uv. Without loss of generality we assume that there are two φ'-colored edges e_1 and e_2 that are incident with u and v, respectively, such that $\varphi'(e_1) = \varphi'(e_2) = c$. Then there are two φ'-colored edges e and e' that are incident with u and v, respectively, such that $\varphi'(e) = c'$ and $\varphi'(e') = c''$. Note that if $\psi(e) = c$, then we can recolor e by the color c'', and then uv by the color c', or if $\psi(e') = c$, then we recolor e' by the color c', and then uv by the color c'' to obtain an extendable coloring that avoids ψ and extends φ. Thus it suffices to consider the following three different cases.

(a) e and e' are both φ-colored c' and c'' respectively;

(b) one of them, say e', is φ-colored c'' and e is ψ-colored c'';

(c) e is ψ-colored c'' and e' is ψ-colored c'.

If (a) holds, then since $\varphi \notin C_1$, no edge φ'-colored c is colored under φ. Moreover, if e_1 is ψ-colored c'', or e_2 is ψ-colored c', then (T2) holds. Thus we may recolor e_1 and e_2 by colors c'' and c', respectively, and then uv by color c to obtain a properly colored connected subgraph. The resulting coloring avoids ψ and agrees with φ, and we are done by Theorem 3.1.

Suppose now that (b) holds. If $\varphi(e_1) = c$ or $\psi(e_1) = c''$, then (T2) holds; since by assumption $(\varphi, \psi) \notin \mathbb{T}$, we may assume that $\psi(e_1) = c'$. Thus we can recolor e_1 by c'' and e by color c, and then uv by c' to obtain a proper coloring that avoids ψ and agrees with φ. Again we may now invoke Theorem 3.1 to deduce that there an extension of φ that avoids ψ.

Let us now consider the case when (c) holds. If $\varphi(e_1) = \varphi(e_2) = c$, then (T6) holds. Moreover, if $\psi(e_1) = c''$ and $\varphi(e_2) = c$, or $\varphi(e_1) = c$ and $\psi(e_2) = c'$, then again (T6) holds. By symmetry, we may assume that $\psi(e_1) = c'$. We can define a new coloring from φ' by recoloring e_1 by c'' and e by color c, and finally coloring uv by color c' to obtain a properly colored connected subgraph that agrees with φ and avoids ψ.

Case 1.2. *$\varphi' \in C_2$ and $\varphi' \notin C_1$.* The condition implies that there is a vertex x of degree 3 that is incident with an edge φ'-colored $c \in \{1, 2, 3\}$, and two vertices v_1, v_2, such that for $i = 1, 2$, xv_i is uncolored but v_i is incident with an edge colored by a fixed color $c' \neq c$.

Let e, e_1, e_2 be the edges incident with x, v_1, v_2, respectively, with $\varphi'(e) = c$ and $\varphi'(e_1) = \varphi'(e_2) = c'$. If we can define a new coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring e by the color c', then we may color xv_1 and xv_2 greedily, and we are done by Lemma 3.2; otherwise, if we cannot recolor e in such a way, then either

(a) e is φ-colored c, or

(b) e is ψ-colored c', or

(c) e is ψ-colored c'' and adjacent to an edge φ'-colored c'.

If (a) holds, then since $\varphi \notin C_2$, one or both of e_1 and e_2 are colored under ψ. If both e_1 and e_2 are colored under ψ, then at least one of them is not adjacent to any other edge from $E_{\varphi,\psi}$. Thus we can define a new edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring this edge by some color from $\{1,2,3\}$. Thereafter, color xv_1 and xv_2 greedily, and by Lemma 3.2, φ'' is extendable to a proper 3-coloring. Hence, there is an extension of φ that avoids ψ.

Suppose now that one of them, say e_2, is colored under φ and e_1 is colored under ψ. If e_1 is not adjacent to any other edge from $E_{\varphi,\psi}$, then recolor e_1 by some color from $\{1,2,3\}$ and proceed as above; otherwise, if $\psi(e_1)=c$ and e_1 is adjacent to an edge φ-colored c'' (or if $\psi(e_1)=c''$ and e_1 is adjacent to an edge φ-colored c), then (T6) or (T7) holds. Moreover, if $\psi(e_1)=c$ and v_1 is incident with another edge ψ-colored c, then (T6) holds. Since by assumption $(\varphi,\psi)\notin \mathbb{T}$, we can recolor e_1 (and possibly one additional ψ-colored edge adjacent to e_1 if any) by some color from $\{1,2,3\}$ to obtain a new proper edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ, color xv_1 and xv_2 greedily, and finally we apply Lemma 3.2 to complete the coloring.

Suppose now that (b) holds. If $\varphi(e_1)=\varphi(e_2)=c'$, then (T2) holds. Moreover, if one edge, say e_2, is φ-colored c', $\psi(e_1)=c$ and e_1 is adjacent to an edge φ-colored c'' (or if $\varphi(e_2)=c'$, $\psi(e_1)=c''$ and e_1 is adjacent to an edge φ-colored c), then (T6) or (T7) holds. Moreover, if $\varphi(e_2)=c'$, $\psi(e_1)=c$ and v_1 is incident with another edge ψ-colored c, then (T6) holds. Since by assumption $(\varphi,\psi)\notin\mathbb{T}$, we can define a new edge coloring from φ' that avoids ψ by recoloring e_1 (and possibly also one additional ψ-colored edge adjacent to e_1) by some color from $\{1,2,3\}$. Thereafter, color xv_1 and xv_2 greedily, and finally we apply Lemma 3.2 to complete the coloring.

Let us now consider the case when (c) holds. If $\varphi(e_1)=\varphi(e_2)=c'$, then (T7) holds. Since again $(\varphi,\psi)\notin\mathbb{T}$, we may assume that e_1 or e_2 is colored under ψ. Then we can define a new edge coloring of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring e_1 or e_2 by some color from $\{1,2,3\}$. The obtained coloring is extendable and avoids ψ.

Case 1.3. $\varphi'\in C_3$. Then there is a vertex y of degree 3 and three vertices v_1,v_2,v_3, such that for $i=1,2,3$, yv_i is uncolored but v_i is incident with an edge colored by a fixed color c under φ'. For $i=1,2,3$, let e_i be the edges incident with v_i that are colored c under φ'. Note that since $\varphi\notin C_3$, at least one of the e_i is colored under ψ. If there is some e_i from $\{e_1,e_2,e_3\}$ that is ψ-colored and not adjacent to any other edge from $E_{\varphi,\psi}$, then we can define a new edge coloring of $E_{\varphi,\psi}$ from φ' that avoids ψ by coloring this edge by the color in $\{1,2,3\}\setminus\{c,\psi(e_i)\}$, and then by coloring the edge yv_1 by the color c; otherwise, we may assume that exactly one e_i, say e_1, is ψ-colored and adjacent to an edge from $E_{\varphi,\psi}$. If $\psi(e_1) = c'$ and e_1 is adjacent to an edge φ-colored c'', then (T6) or (T7) holds. Moreover, if $\psi(e_1) = c'$ and v_1 is incident with another edge ψ-colored c', then (T6) holds. Since by assumption $(\varphi, \psi) \notin \mathbb{T}$, we can define a new edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring e_1 (and possibly one additional ψ-colored edge adjacent to e_1 if any) by some color from $\{1, 2, 3\}$, and thereafter color yv_1 by the color c. Next, we color the edges yv_2 and yv_3 greedily, and apply Lemma 3.2 to find an extension of φ that avoids ψ.

Case 1.4. $\varphi' \in R_3$. If $\varphi' \in R_3$, then there are two uncolored adjacent edges uv and uw with $d(v) = d(w) = 3$ in T such that every edge incident with v or w except uv and uw, is either uncolored but adjacent to an edge colored c, or colored by a color distinct from c under φ'. Now, if there is some ψ-colored edge that is not incident with v or w, then we can define a new proper edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring this edge by some color from $\{1, 2, 3\}$; since the resulting coloring is not in R_3, it is extendable.

Let us now assume that every ψ-colored edge is incident with v or w. If every φ'-colored edge incident with v or w is ψ-colored c, or φ-colored by a color distinct from c, then (T6) holds. Since by assumption $(\varphi, \psi) \notin \mathbb{T}$ and at least one edge is ψ-colored, we may assume that there is some edge ψ-colored $c' \neq c$ that is incident with v or w. We can recolor this edge by the color c, and thereafter color every uncolored edge incident with v and w greedily; Theorem 3.1 then yields that there is an extension of this precoloring, and, consequently, an extension of φ that avoids ψ.

Case 1.5. $\varphi' \in R_4$. Finally, let us consider the case when $\varphi' \in R_4$ and $\varphi' \notin C_1 \cup C_2$. Then there is an uncolored edge pq with $d(p) = d(q) = 3$, and there are two colors c and c' such that every edge incident with p except pq is either uncolored but adjacent to an edge colored c under φ' or colored by some color distinct from c, and every edge incident with q except pq is either uncolored but adjacent to an edge colored c' or colored by some color distinct from c' under φ'. Now, if there is some ψ-colored edge that is not incident with p or q, then we can recolor this edge by some color from $\{1, 2, 3\}$ to obtain a new partial edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ; since this coloring is not in R_4, it is extendable.

Now assume that every ψ-colored edge is incident with p or q. If every φ'-colored edge incident with p is ψ-colored c, or φ-colored by a color distinct from c, and if every φ'-colored edge incident with q is ψ-colored c', or φ-colored by a color distinct from c', then (T6) holds. Since again $(\varphi, \psi) \notin \mathbb{T}$ and at least one edge is ψ-colored, we may assume that there is some edge ψ-colored $c'' \neq c$ incident with p, and we can recolor this edge by the color c. Thereafter we color every uncolored edge incident with p and q greedily and invoke Theorem 3.1 to complete the argument.

Case 2. $\Delta(T) \geq 4$. Let us now consider the case when $\Delta(T) \geq 4$. Since altogether at most $\Delta(T)+1$ edges are colored under φ', it follows from Theorem 2.3 that there is an extension of φ that avoids ψ unless $\varphi' \in C_1 \cup C_2 \cup C_3 \cup R_2$.

Case 2.1. $\varphi' \in C_1$. If $\varphi' \in C_1$, then there is an uncolored edge uv in T such that u is incident with edges of $k < \Delta(T)$ distinct colors and v is incident to $\Delta(T) - k$ edges colored with $\Delta(T) - k$ other distinct colors under φ'. We distinguish between two subcases: whether uv is adjacent to $\Delta(T)$ or $\Delta(T)+1$ φ'-colored edges.

Subcase A. *uv is adjacent to $\Delta(T)$ φ'-colored edges.* Note that since $\varphi \notin C_1$, at least one ψ-colored edge e is incident with u or v, say u. We shall consider some different cases.

Suppose first that at least three φ'-colored edges are incident with v. Thus at least three colors c_1, c_2, c_3 do not appear on edges incident with u. Since uv is adjacent to $\Delta(T)$ φ'-colored edges, there exists some color $c \in \{c_1, c_2, c_3\}$ such that neither is e ψ-colored c nor is an edge adjacent to e φ'-colored c. Then we can define a new proper edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring e by the color c. This implies that the color $\varphi'(e)$ does not appear at u or v; color uv by this color to obtain a colored connected subgraph, and we are done by Lemma 3.2.

Suppose now that exactly two φ'-colored edges e_1 and e_2 are incident with v, where $\varphi'(e_i) = c_i$. Then we can similarly define a coloring φ'' from φ' that avoids ψ by recoloring some ψ-colored edge incident with u by one of the colors c_1 or c_2 unless e is the only ψ-colored edge incident with u, $\psi(e) = c_1$, and e is adjacent to an edge e' φ'-colored c_2 (or $\psi(e) = c_2$ and $\varphi'(e') = c_1$). Now, if e' is ψ-colored and $\varphi'(e') = c_2$, then we first recolor e' by some color distinct from c_2 and $\psi(e')$, and then recolor e by the color c_2; the obtained coloring φ'' avoids ψ. Next, we color uv by the color $\varphi'(e)$ and deduce that there is an extension from Lemma 3.2. Otherwise, if e' is φ-colored, then (T4) holds unless e_1 or e_2 is ψ-colored, say e_1. Since $(\varphi, \psi) \notin \mathbb{T}$ and $\Delta(T) \geq 4$, we can recolor e_1 by some color that appears at u to obtain a coloring that avoids ψ; then color uv by the color c_1, and we are done by Lemma 3.2.

It remains to consider the case when exactly one φ'-colored edge, colored c, is incident with v. If we can define a coloring φ'' from φ' that avoids ψ by recoloring a ψ-colored edge incident to u by the color c, then, as before, the desired result follows. Otherwise, every edge incident with u is either ψ-colored c, φ-colored by a color distinct from c, or ψ-colored and adjacent to an edge φ'-colored c. If there is some ψ-colored edge $e' = ux, x \neq v$, such that $\psi(ux) \neq c$ and x is incident to another ψ-colored edge e'' such that $\varphi'(e'') = c$, then we can first recolor e'' by some color distinct from c and $\psi(e'')$, and then recolor e' by the color c; otherwise, if there are no such edges e' and e'', then (T2) or (C2) holds unless the edge at v is ψ-colored. Since $(\varphi, \psi) \notin \mathbb{T}$, we may recolor this edge by some color that appears at u, and then recolor uv by the color c to obtain an extendable precoloring that avoids ψ and agrees with φ.

Subcase B. *uv is adjacent to $\Delta(T)+1$ φ'-colored edges.* Suppose now that uv is adjacent to $\Delta(T)+1$ edges that are colored under φ'. Since $(\varphi,\psi) \notin \mathbb{T}$, at least one ψ-colored edge e is incident with u or v, say u.

Suppose first that at least three φ'-colored edges are incident with v. Thus at least three colors c_1, c_2, c_3 appear on some edges incident with v. Suppose first that there is a ψ-colored edge e' incident with u such that $\varphi'(e')$ appears once under φ'. Then there is some color $c' \in \{c_1, c_2, c_3\}$ such that e' is not ψ-colored c' and no other edge incident with u is φ'-colored c'. Hence, we can define a new proper edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring e' by the color c'; since the color $\varphi'(e')$ now does not appear at u or v, we can color uv by this color and invoke Theorem 3.1 to complete the argument.

On the other hand, if there is no such ψ-colored edge e', then e is the only ψ-colored edge incident with u and the color $\varphi'(e) = c$ appears at v on some edge vw under φ'. We now consider the two different cases that vw is ψ-colored and that vw is φ-colored c.

Suppose first that vw is ψ-colored; if three φ'-colored edges are incident with u, then we can define a new proper edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring vw and e by some color distinct from c that appears at u and v, respectively. Since the color c does not appear at u or v, we can recolor uv by c, and we are done as before. Otherwise, if it is not possible to recolor vw by some color distinct from c that appears at u, then vw is ψ-colored by some color c'' and only two φ'-colored edges colored c and c'' are incident with u; that is, the only φ'-colored edges incident with u are e and one additional edge φ-colored c''. Now, if every edge incident with v except vw is ψ-colored c'', or φ-colored, then (T2) holds. Since $(\varphi,\psi) \notin \mathbb{T}$, we may assume that there exists some ψ-colored edge $f \neq vw$ incident with v such that $\psi(f) \neq c''$. We can recolor f by the color c'', and then color uv by the color $\varphi'(f)$ to obtain an extendable coloring that avoids ψ and agrees with ϕ.

Suppose now that vw is φ-colored c and assume that three φ'-colored edges are incident with u. Since $\varphi \notin C_1$, there must exist some ψ-colored edge f incident with v. Then we can define a coloring φ'' from φ that avoids ψ by recoloring f by some color distinct from c that appears at u; otherwise, if only two φ'-colored edges colored c and c'' are incident with u (e and only one φ-colored edge colored c'' are incident with u), then since $(\varphi,\psi) \notin C_1 \cup T_2$, there must exist some ψ-colored edge f' incident with v such that $\psi(f') \neq c''$. We can recolor f' by the color c'', and then recolor uv by the color $\varphi'(f')$, and, again, we are done by Theorem 3.1.

Suppose now that exactly two φ'-colored edges e_1 and e_2, colored $\varphi'(e_i) = c_i$, are incident with v. Since the number of φ'-colored edges adjacent to uv exceeds

$\Delta(T)$ and $\varphi' \in C_1$, exactly one of c_1 or c_2, say c_1, appears at u on some edge ux under φ'. If e_2 is ψ-colored, then recolor e_2 by some color that appears at u. This implies that c_2 does not appear at u or v; color uv by this color, and we are done by Theorem 3.1.

Suppose now that e_2 is φ-colored c_2; then

(a) ux is φ-colored c_1, or

(b) ux is ψ-colored.

We first consider the case when (a) holds; if every edge incident with u is ψ-colored c_2, or φ-colored, then (T2) or (C1) holds. Since $(\varphi, \psi) \notin \mathbb{T}$, there must exist some ψ-colored edge f incident with u such that $\psi(f) \neq c_2$. We can recolor f by the color c_2, and then recolor uv by the color $\varphi'(f)$; the resulting precoloring is extendable.

Suppose now that (b) holds. If $\psi(ux) = c_2$, then we proceed as in the preceding paragraph; so we assume that ux is ψ-colored by a color distinct from c_2. Then every other edge incident with u is either φ-colored, or ψ-colored c_2, otherwise we can again proceed as in the preceding paragraph. We can now recolor ux by the color c_2; if there exists some additional ψ-colored edge f incident with u, then recolor f by the color c_1. This implies that the color $\varphi'(f)$ does not appear at u or v; recolor uv by this color, and we are by Theorem 3.1. Otherwise, if ux is the only ψ-colored edge incident with u, then since $\varphi \notin C_1$ and e_2 is colored under φ, e_1 must be colored under ψ. Thus we can recolor e_1 by some color that appears at u from $\{1, \dots, \Delta(T)\} \setminus \{\psi(e_1), c_1, c_2\}$ since $\Delta(T) \geq 4$. Finally, color uv by the color c_1 and the desired result follows.

Case 2.2. $\varphi' \in C_2$ *and* $\varphi' \notin C_1$. The condition implies that there is a vertex x of degree $\Delta(T)$ that is incident with edges of $\Delta(T) - k$ distinct colors $c_1, \dots, c_{\Delta(T)-k}$, and k vertices $v_1, \dots, v_k$, where $2 \leq k < \Delta(T)$, such that for $i = 1, \dots, k$, xv_i is uncolored but v_i is incident with an edge colored by a fixed color $c \notin \{c_1, \dots, c_{\Delta(T)-k}\}$. Since φ does not satisfy (C2), at least one of these edges is colored under ψ. We shall distinguish between the two different cases that every such ψ-colored edge is incident with x and that this does not hold. Suppose first that the latter holds, i.e., some ψ-colored edge e is incident with some v_i, say v_1, and $\varphi'(e) = c$. Note that e can be adjacent to at most one other edge from $E_{\varphi,\psi}$. Thus we can define a new proper edge coloring φ'' of $E_{\varphi,\psi}$ from φ' that avoids ψ by recoloring e by some color from $\{1, 2, \dots, \Delta(T)\}$ so that the resulting coloring is proper, avoids ψ and does not satisfy (C2). Thereafter, color the edge xv_1 by the color c, and then color the remaining xv_i greedily to obtain a precolored connected subgraph, and we are done by invoking Lemma 3.2.

Suppose now that every ψ-colored edge is incident with x. Note that since $\varphi' \notin C_1$ and $\Delta(T) + 1$ edges are precolored in T, such a ψ-colored edge e' is adjacent to at most $\Delta(T) - 2$ other edges from $E_{\varphi,\psi}$. Thus we can define a coloring φ'' that avoids ψ by recoloring a ψ-colored edge incident to x by the color c, unless every edge incident with x is either ψ-colored c, φ-colored by a color distinct from c, or adjacent to an edge φ-colored c, that is, (T2) holds. Thereafter, color the edges xv_i greedily as in the preceding paragraph.

Case 2.3. $\varphi' \in C_3$. The condition implies that there is a vertex y of degree $\Delta(T)$ and vertices $v_1, v_2, \dots, v_{\Delta(T)}$ such that, for $i = 1, 2, \dots, \Delta(T)$, yv_i is uncolored but v_i is incident with an edge colored by a fixed color c under φ'. For $i = 1, 2, \dots, \Delta(T)$, let e_i be the edge incident with v_i that is colored c under φ'. Now, since $\varphi \notin C_3$, at least one of the e_i, say e_1, is colored under ψ. Then e_1 is adjacent to at most one edge from $E_{\varphi,\psi}$, so we can recolor e_1 by some color from $\{1, 2, \dots, \Delta(T)\}$ to obtain a proper coloring φ'' of $E_{\varphi,\psi}$ that avoids ψ. Thereafter, color yv_1 by the color c, color the other yv_i greedily, and then apply Lemma 3.2.

Case 2.4. $\varphi' \in R_2$. The condition implies that there are two uncolored adjacent edges uv and uw in T such that u is incident with edges of $\Delta(T) - 3$ distinct colors $c_1, \dots, c_{\Delta(T)-3}$, and both v and w are incident to 2 edges colored with 2 other distinct colors $c_{\Delta(T)-2}, c_{\Delta(T)-1}$. If there is some ψ-colored edge e that is incident with u, then e is adjacent to exactly $\Delta(T) - 4$ edges from $E_{\varphi,\psi}$. Thus we can define a new edge coloring φ'' of $E_{\varphi,\psi}$ from φ', which avoids ψ, by recoloring e by some color that appears at v. The resulting coloring is not in R_2, so we can color every uncolored edge incident with u greedily to obtain an edge-colored connected subgraph; the result now follows by invoking Theorem 3.1.

Suppose now that no ψ-colored edge is incident with u. Without loss of generality, we assume that some ψ-colored edge e' is incident with v. Since e' is adjacent to exactly one edge from $E_{\varphi,\psi}$, we can define a coloring φ'' by recoloring e' by some color from $\{1, 2, \dots, \Delta(T)\}$ so that the resulting coloring is proper and avoids ψ. Thereafter, color uv by the color $\varphi'(e')$, and proceed as in the preceding paragraph.

This completes the proof of Theorem 3.3. ■

4. Restricted Extension of Precolored Matchings

In [6], we considered the problem of extending an edge coloring of a matching in a tree, as well as the problem of avoiding a not necessarily proper partial edge coloring and proved the following two theorems.

Theorem 4.1. *Let T be a tree with maximum degree $\Delta(T) \geq 3$, and M a precolored distance-2 matching of T. If no vertex v satisfies that $\Delta(T) - 1$ uncolored edges incident with v are all adjacent to edges precolored by a fixed color c, then the precoloring can be extended to a proper $\Delta(T)$-edge coloring of T.*

The condition that at most $\Delta(T) - 2$ edges incident to a given vertex can be adjacent to edges that are precolored by a fixed color c is necessary, that is, Theorem 4.1 becomes false if we replace $\Delta(T) - 2$ by $\Delta(T) - 1$. Furthermore, Theorem 4.1 directly implies the following.

Corollary 4.2. *Let T be a tree with maximum degree $\Delta(T) \geq 3$. Every precoloring of a distance-3 matching in T can be extended to a proper $\Delta(T)$-edge coloring of T.*

Theorem 4.3. *Let T be a tree with maximum degree $\Delta(T) \geq 3$. If φ is a partial $\Delta(T)$-edge coloring of T where every vertex is incident with at most $\Delta(T) - 2$ edges colored by the same color, then φ is avoidable.*

Here, we state a result generalizing these two theorems; we omit the proof since one may proceed exactly as in the proofs of the preceeding theorems [6].

Theorem 4.4. *Let T be a tree with maximim degree $\Delta(T) \geq 3$, φ a precoloring of a distance-2 matching of T, and ψ a partial edge coloring of T. If no vertex v satisfies that*

- *m edges incident with v are ψ-colored by a fixed color c, $0 \leq m \leq \Delta(T) - 1$, and $\Delta(T) - 1 - m$ edges incident with v are uncolored but adjacent to edges φ-colored c, or*
- *$\Delta(T) - 2$ edges incident with v are ψ-colored c and one edge incident with v is φ-colored by a color distinct from c,*

then φ can be extended to a proper $\Delta(T)$-edge coloring of T that avoids ψ.

The graph in Figure 1 shows that the conditions in the preceding theorem are sharp.

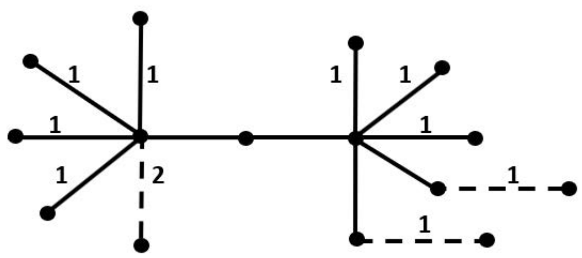

Figure 1. A representative T of a class of trees with a precoloring φ of a distance-2 matching and a precoloring ψ, such that φ is not extendable to an edge coloring avoiding ψ. Dashed edges indicate edges precolored under φ.

Acknowledgements

The first author was supported by a grant from the Swedish Research Council (2017-05077).

The second author thanks Addis Ababa University, Ethiopia, and Linköping University, Sweden, for being accepted as student in the international PhD program between these universities. We also thank International Science Program in Uppsala, Sweden, for financial and material support, which supports PhD training by establishing and strengthening collaborative research in mathematics.

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Received 13 September 2021
Revised 12 June 2022
Accepted 13 June 2022
Available online 27 June 2022