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BOUNDS ON THE GLOBAL DOUBLE ROMAN DOMINATION NUMBER IN GRAPHS

Guoliang Hao [a], Shouliu Wei [b]

Seyed Mahmoud Sheikholeslami [c]

and

Xiaodan Chen [d,1]

[a] *College of Science*
East China University of Technology
Nanchang 330013, P.R. China

[b] *College of Mathematics and Data Science*
Minjiang University
Fuzhou 350108, P.R. China

[c] *Department of Mathematics*
Azarbaijan Shahid Madani University
Tabriz, I.R. Iran

[d] *College of Mathematics and Information Science*
Guangxi University
Nanning 530004, P.R. China

e-mail: guoliang-hao@163.com
wslwillow@126.com
s.m.sheikholeslami@azaruniv.ac.ir
x.d.chen@live.cn

Abstract

Let G be a simple graph of order n and let $\gamma_{gdR}(G)$ be the global double Roman domination number of G. In this paper, we give some upper bounds on the global double Roman domination number of G. In particular, we completely characterize the graph G with $\gamma_{gdR}(G) = 2n-2$ and $\gamma_{gdR}(G) = 2n-3$. Our results answer a question posed by Shao *et al.* (2019).

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[1]Corresponding author.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The *open neighborhood* of a vertex v in G is the set $N(v) = N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N[v] = N_G[v] = N(v) \cup \{v\}$. The *degree* of a vertex v in G is $d(v) = d_G(v) = |N(v)|$. The minimum degree and maximum degree among all vertices of G are denoted by $\delta = \delta(G)$ and $\Delta = \Delta(G)$, respectively. The *distance* $d(u, v)$ between two vertices u and v of a connected graph G is the length of a shortest (u, v)-path in G. The maximum distance among all pairs of vertices in G is the *diameter* of G, which is denoted by $\mathrm{diam}(G)$. A *diametral path* of G is a geodesic whose length equals the diameter of G. The *girth* $g(G)$ of G is the length of a shortest cycle in G.

As usual, let P_n, C_n and K_n denote the path, the cycle and the complete graph of order n, respectively, and let $K_{p,q}$ denote the complete bipartite graph with two partite sets having p and q vertices. The *complement* of a graph G is the graph $\overline{G}$, where $V(\overline{G}) = V(G)$ and $uv \in E(\overline{G})$ if and only if $uv \notin E(G)$. A complete subgraph of a graph G is called a *clique* of G. The *clique number* of a graph G, denoted by $\omega(G)$, is the maximum order among the complete subgraphs of G. A subset S of vertices of G is an *independent set* if no two vertices of S are adjacent in G. We denote the graph obtained from G by deleting one edge by $G - e$. For a subset X of vertices of G, the subgraph induced by X is denoted by $G[X]$. The *union* of graphs G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. If G is a disjoint union of k copies of a graph H, then we write $G = kH$.

A *double Roman dominating function* (abbreviated DRDF) on a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must be adjacent to at least two vertices assigned 2 or one vertex assigned 3 under f, whereas if $f(v) = 1$, then the vertex v must be adjacent to at least one vertex assigned 2 or 3. The *weight* of a DRDF f is the sum $\omega(f) = \sum_{v \in V(G)} f(v)$. The double Roman domination of graphs was introduced by Beeler *et al.* [5], and has been studied by several authors [1, 2, 6, 10, 11, 15, 16].

Global domination in graphs was introduced by Sampathkumar [13]. A subset S of vertices of a graph G is a *global dominating set* of G if S is a dominating set of both G and $\overline{G}$. The *global domination number* of G is the minimum cardinality of a global dominating set. The concept of global domination in graphs, with its many variations, is now well studied in, e.g., [3, 4, 7, 9, 12].

As a natural extension, the concept of global double Roman dominating function on a graph was introduced by Shao *et al.* [14] and further studied by Hao and Chen [8]. A DRDF $f : V(G) \to \{0, 1, 2, 3\}$ on G is called a *global double Roman dominating function* (GDRDF) on G if f is also a DRDF on the complement $\overline{G}$ of G. The *global double Roman domination number* of G, denoted by $\gamma_{gdR}(G)$, is the minimum weight of a GDRDF on G. Shao *et al.* [14] characterized the connected graph G of order n with $\gamma_{gdR}(G) = 2n$ and $\gamma_{gdR}(G) = 2n - 1$.

Theorem 1 [14]. *Let G be a connected graph of order $n \geq 3$. Then*

(i) $\gamma_{gdR}(G) = 2n$ *if and only if* $G = K_n$;

(ii) $\gamma_{gdR}(G) = 2n - 1$ *if and only if* $G = K_n - e$.

Shao *et al.* [14] further posed the problem of characterizing the graph G of order n with $\gamma_{gdR}(G) = 2n - 2$.

In this paper, we first establish some upper bounds on the global double Roman domination number of a graph and then we resolve the aforementioned problem by giving a complete characterization of the graph G with $\gamma_{gdR}(G) = 2n - 2$ and $\gamma_{gdR}(G) = 2n - 3$.

In particular, we will prove the following result.

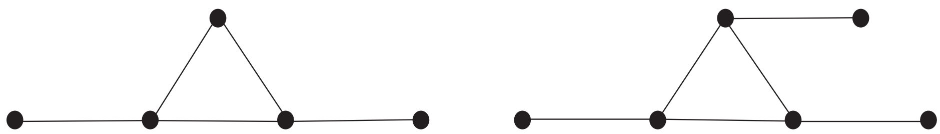
Theorem 2. *For any graph G of order n,*

(i) $\gamma_{gdR}(G) = 2n-2$ *if and only if G or $\overline{G}$ belongs to* $\{P_4, C_4, C_5, K_{1,3}, K_4^{--}, P_4 \cup K_1, 2K_2 \cup (n-4)K_1, G_1\}$,

(ii) $\gamma_{gdR}(G) = 2n - 3$ *if and only if G or $\overline{G}$ belongs to* $\{K_n^{--}, 3K_2 \cup (n-6)K_1, C_3 \cup (n-3)K_1, C_4 \cup K_1, P_4 \cup (n-4)K_1, G_2\}$,

where K_n^{--} denotes the graph obtained from K_n by deleting any two adjacent edges and, G_1 and G_2 are the graphs depicted in Figure 1(a) *and* (b), *respectively.*

Unless otherwise mentioned, we would hereafter require that $n \geq 5$ for the graphs $2K_2 \cup (n-4)K_1$, K_n^{--} and $C_3 \cup (n-3)K_1$, and that $n \geq 6$ for the graphs $3K_2 \cup (n-6)K_1$ and $P_4 \cup (n-4)K_1$.



(a) G_1 (b) G_2

Figure 1. The graphs G_1 and G_2.

The proof of Theorem 2 will be given in Section 3.

2. Upper Bounds

In this section we present some upper bounds for global double Roman domination number of graphs in terms of diameter, girth and clique number. We start with an upper bound on $\gamma_{gdR}(G)$ of a connected graph G with $\mathrm{diam}(G) \geq 4$.

Proposition 3. *For any connected graph G of order n with* $\mathrm{diam}(G) \geq 4$,

$$\gamma_{gdR}(G) \leq \begin{cases} 2n - \mathrm{diam}(G) - 1, & \text{if} \quad \mathrm{diam}(G) \equiv 2 \pmod 3, \\ 2n - \mathrm{diam}(G), & \text{if} \quad \mathrm{diam}(G) \equiv 0 \text{ or } 1 \pmod 3. \end{cases}$$

Proof. Let $v_1 v_2 \cdots v_k$ be a diametral path of G. If $\mathrm{diam}(G) \equiv 2 \pmod 3$, then one can check that the function f defined by $f(v_{3i+2}) = 3$, $f(v_{3i+1}) = f(v_{3i+3}) = 0$ for $0 \leq i \leq \frac{k-3}{3}$ and $f(x) = 2$ for each $x \in V(G) \backslash \{v_1, v_2, \ldots, v_k\}$, is a GDRDF on G and hence

$$\gamma_{gdR}(G) \leq \omega(f) = 2(n - \mathrm{diam}(G) - 1) + \mathrm{diam}(G) + 1 = 2n - \mathrm{diam}(G) - 1.$$

If $\mathrm{diam}(G) \equiv 0 \pmod 3$, then the function f defined by $f(v_k) = 2$, $f(v_{3i+2}) = 3$, $f(v_{3i+1}) = f(v_{3i+3}) = 0$ for $0 \leq i \leq \frac{k-4}{3}$ and $f(x) = 2$ for each $x \in V(G) \backslash \{v_1, v_2, \ldots, v_k\}$, is a GDRDF on G and hence

$$\gamma_{gdR}(G) \leq \omega(f) = 2(n - \mathrm{diam}(G) - 1) + \mathrm{diam}(G) + 2 = 2n - \mathrm{diam}(G).$$

If $\mathrm{diam}(G) \equiv 1 \pmod 3$, then the function f defined by $f(v_k) = 3, f(v_{k-1}) = 0$, $f(v_{3i+2}) = 3$, $f(v_{3i+1}) = f(v_{3i+3}) = 0$ for $0 \leq i \leq \frac{k-5}{3}$ and $f(x) = 2$ for each $x \in V(G) \backslash \{v_1, v_2, \ldots, v_k\}$, is a GDRDF on G and as above we have $\gamma_{gdR}(G) \leq 2n - \mathrm{diam}(G)$. This completes the proof. ∎

Proposition 4. *Let G be a connected graph of order n with* $\mathrm{diam}(G) \geq 3$ *and* $\omega(G) \geq 3$. *Then*

$$\gamma_{gdR}(G) \leq 2n - \omega(G) + 1.$$

Furthermore, if G has a clique H of size $\omega(G)$ such that each vertex of H but possibly one has at least two non-neighbors in $V(G) \setminus V(H)$, then

$$\gamma_{gdR}(G) \leq 2n - 2\omega(G) + 3.$$

Proof. Let H be a clique of G with size $\omega(G)$ and let $V(H) = \{v_1, v_2, \ldots, v_{\omega(G)}\}$. Define the function f on $V(G)$ by $f(x) = 1$ for each $x \in V(H) \backslash \{v_1\}$ and $f(x) = 2$ for each $x \in (V(G) \backslash V(H)) \cup \{v_1\}$. Since each vertex in $V(H) \backslash \{v_1\}$ is adjacent to v_1 in G, we have that f is a DRDF on G. On the other hand, since $\mathrm{diam}(G) \geq 3$, we have that each vertex in $V(H) \backslash \{v_1\}$ is adjacent to some vertex of $V(G) \backslash V(H)$ in $\overline{G}$ and hence f is a DRDF on $\overline{G}$. Thus f is a GDRDF on G. As a result,

$$\gamma_{gdR}(G) \leq \omega(f) = 2(n - \omega(G)) + \omega(G) + 1 = 2n - \omega(G) + 1.$$

Now, assume that each vertex of H but possibly v_1 has two non-neighbors in $V(G) \setminus V(H)$. Define the function f on $V(G)$ by $f(v_1) = 3$, $f(x) = 0$ for each

$x \in V(H)\backslash\{v_1\}$ and $f(x) = 2$ for each $x \in V(G) \setminus V(H)$. Since each vertex in $V(H) \setminus \{v_1\}$ is adjacent to v_1 in G, f is a DRDF on G. On the other hand, since each vertex in $V(H) \setminus \{v_1\}$ is adjacent to at least two vertices in $V(G) \setminus V(H)$ in $\overline{G}$, f is a DRDF on $\overline{G}$. Thus f is a GDRDF on G and hence

$$\gamma_{gdR}(G) \leq \omega(f) = 2(n - \omega(G)) + 3 = 2n - 2\omega(G) + 3,$$

as desired. ∎

Proposition 5. *Let G be a connected graph of order n with* $\mathrm{diam}(G) = 3$. *Then*

$$\gamma_{gdR}(G) \leq 2n - 2.$$

Furthermore, if there exists a diametral path $v_1v_2v_3v_4$ such that $d(v_1) \geq 2$ or $d(v_4) \geq 2$, then

$$\gamma_{gdR}(G) \leq 2n - 4.$$

Proof. Let $v_1v_2v_3v_4$ be a diametral path of G. It is easy to verify that the function f defined by $f(v_1) = f(v_4) = 3$, $f(v_2) = f(v_3) = 0$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2(n-4) + 6 = 2n - 2$.

Suppose that $d(v_1) \geq 2$ (the case when $d(v_4) \geq 2$ is similar). Let $w \in N(v_1)\backslash\{v_2\}$. Then the function f defined by $f(v_1) = f(v_4) = 3$, $f(v_2) = f(v_3) = f(w) = 0$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2(n-5) + 6 = 2n - 4$. ∎

Proposition 6. *Let G be a connected graph of order n with* $\mathrm{diam}(G) = 2$. *Then*

$$\gamma_{gdR}(G) \leq n + \delta(G) + 1.$$

Furthermore, if $\delta(G) \leq n - 4$, then

$$\gamma_{gdR}(G) \leq 2n - 4.$$

Proof. Let u be a vertex of G with minimum degree $\delta(G)$. Since $\mathrm{diam}(G) = 2$, we have that every vertex of $V(G)\backslash N[u]$ has a neighbor in $N(u)$ in G, and that every vertex of $V(G)\backslash N[u]$ is adjacent to u in $\overline{G}$. Thus the function f defined by $f(x) = 2$ for each $x \in N[u]$ and $f(x) = 1$ otherwise, is a GDRDF on G implying that $\gamma_{gdR}(G) \leq \omega(f) = (n - \delta(G) - 1) + 2\delta(G) + 2 = n + \delta(G) + 1$.

If $\delta(G) \leq n-5$, then it follows that $\gamma_{gdR}(G) \leq n + \delta(G) + 1 \leq 2n - 4$. So in the following we may assume that $\delta(G) = n - 4$. Let $N(u) = \{u_1, u_2, \ldots, u_{n-4}\}$ and $X = V(G) \setminus N[u] = \{v_1, v_2, v_3\}$. Since $\mathrm{diam}(G) = 2$, $N(v_i) \cap N(u) \neq \emptyset$ for each $v_i \in X$. We consider four cases.

Case 1. $n \geq 8$. Note that $\delta(G) = n - 4 \geq 4$. One can verify that the function g_1 defined by $g_1(u) = 3$, $g_1(x) = 2$ for each $x \in N(u)$ and $g_1(x) = 0$ otherwise, is a GDRDF on G implying that $\gamma_{gdR}(G) \leq \omega(g_1) = 2\delta(G) + 3 = 2n - 5$.

Case 2. $n = 7$. Note that $\delta(G) = n - 4 = 3$. If each $v_i \in X$ has at least two neighbors in $N(u)$, then the function g_1 defined above is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_1) = 2\delta(G) + 3 = 2n - 5$. As shown earlier, $N(v_i) \cap \{u_1, u_2, u_3\} \ne \emptyset$ for each $v_i \in X$. So in the following we may assume that some vertex, say v_1, in X has exactly one neighbor, say u_1, in $N(u)$. Since $\delta(G) = 3$, we obtain $v_1v_2, v_1v_3 \in E(G)$, and hence the function g_2 defined by $g_2(u) = g_2(v_1) = 3$, $g_2(u_1) = 2$ and $g_2(x) = 0$ otherwise, is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_2) = 8 < 2n - 4$.

Case 3. $n = 6$. Note that $\delta(G) = n - 4 = 2$. If each $v_i \in X$ is adjacent to both of u_1 and u_2, then the function g_1 defined above is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_1) = 7 < 2n - 4$. As shown earlier, $N(v_i) \cap \{u_1, u_2\} \ne \emptyset$ for each $v_i \in X$. Hence we may assume that there exists some vertex, say v_1, in X such that $N(v_1) \cap \{u_1, u_2\} = \{u_1\}$. Moreover, since $\delta(G) = 2$, we may assume that $v_1v_2 \in E(G)$.

If $v_1v_3 \in E(G)$, then the function g_2 defined above is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_2) = 8 = 2n - 4$. If $v_1v_3 \notin E(G)$ and $v_2v_3 \notin E(G)$, then $N(v_3) = \{u_1, u_2\}$ since $\delta(G) = 2$, and so the function g_3 defined by $g_3(u) = g_3(v_1) = 0$ and $g_3(x) = 2$ otherwise, is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_3) = 8 = 2n - 4$. So in the following we may assume that $v_1v_3 \notin E(G)$ and $v_2v_3 \in E(G)$.

Note that $N(v_2) \cap \{u_1, u_2\} \ne \emptyset$. If $N(v_2) \cap \{u_1, u_2\} = \{u_1\}$ (the case when $N(v_2) \cap \{u_1, u_2\} = \{u_2\}$ is similar), then the function g_4 defined by $g_4(u) = g_4(v_2) = 3$, $g_4(u_1) = 2$ and $g_4(x) = 0$ otherwise, is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_4) = 8 = 2n-4$. We next consider the case when $\{u_1, u_2\} \subseteq N(v_2)$.

Note that $N(v_3) \cap \{u_1, u_2\} \ne \emptyset$. If $u_1 \in N(v_3)$, then the function g_5 defined by $g_5(u) = g_5(u_1) = 3$, $g_5(u_2) = 2$ and $g_5(x) = 0$ otherwise, is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_5) = 8 = 2n - 4$. If $u_1 \notin N(v_3)$ and $u_2 \in N(v_3)$, then the function g_6 defined by $g_6(v_1) = g_6(v_3) = 0$ and $g_6(x) = 2$ otherwise, is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_6) = 8 = 2n - 4$.

Case 4. $n = 5$. Note that $\delta(G) = n - 4 = 1$. The function g_7 defined by $g_7(u) = g_7(u_1) = 3$ and $g_7(x) = 0$ otherwise, is a GDRDF on G implying that $\gamma_{gdR}(G) \le \omega(g_7) = 6 = 2n - 4$.

This completes the proof. ∎

Proposition 7. *For any connected graph G of order n with $g(G) \ge 6$,*

$$\gamma_{gdR}(G) \le \begin{cases} 2n - g(G), & \text{if} \quad g(G) \equiv 0, 2, 3, 4 \pmod 6, \\ 2n - g(G) + 1, & \text{if} \quad g(G) \equiv 1 \text{ or } 5 \pmod 6. \end{cases}$$

Proof. Let $C = v_1v_2 \cdots v_{g(G)}v_1$ be a cycle of G. If $g(G) \equiv 0, 3 \pmod 6$, then the function f defined by $f(v_{3i+2}) = 3$, $f(v_{3i+1}) = f(v_{3i+3}) = 0$ for $0 \le i \le \frac{g(G)-3}{3}$

and $f(x) = 2$ for each $x \in V(G)\backslash V(C)$, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2(n - g(G)) + g(G) = 2n - g(G)$.

If $g(G) \equiv 2, 4 \pmod 6$, then the function f defined by $f(v_{2i}) = 2$, $f(v_{2i-1}) = 0$ for $1 \leq i \leq \frac{g(G)}{2}$ and $f(x) = 2$ for each $x \in V(G)\backslash V(C)$, is a GDRDF on G and as above we have $\gamma_{gdR}(G) \leq 2n - g(G)$.

If $g(G) \equiv 1 \pmod 6$, then the function f defined by $f(v_{g(G)}) = 2$, $f(v_{3i+2}) = 3$, $f(v_{3i+1}) = f(v_{3i+3}) = 0$ for $0 \leq i \leq \frac{g(G)-4}{3}$ and $f(x) = 2$ for each $x \in V(G)\backslash V(C)$, is a GDRDF on G and as above we have $\gamma_{gdR}(G) \leq 2n - g(G) + 1$.

If $g(G) \equiv 5 \pmod 6$, then the function f defined by $f(v_{g(G)}) = 3$, $f(v_{g(G)-1}) = 0$, $f(v_{3i+2}) = 3$, $f(v_{3i+1}) = f(v_{3i+3}) = 0$ for $0 \leq i \leq \frac{g(G)-5}{3}$ and $f(x) = 2$ for each $x \in V(G)\backslash V(C)$, is a GDRDF on G and as above we obtain $\gamma_{gdR}(G) \leq 2n - g(G) + 1$.

This completes the proof. ■

Proposition 8. *For any connected graph G of order $n \geq 6$ with $g(G) = 5$,*

$$\gamma_{gdR}(G) \leq 2n - 4.$$

Proof. Let $C = v_1v_2v_3v_4v_5v_1$ be a cycle of G. Since G is a connected graph of order $n \geq 6$, we may assume that v_1 has a neighbor $w \in V(G) \setminus V(C)$. One can check that the function f defined by $f(v_3) = f(v_5) = 0$ and $f(x) = 2$ for each $x \in V(G) \setminus \{v_3, v_5\}$, is a GDRDF on G and so $\gamma_{gdR}(G) \leq 2n - 4$, as desired. ■

3. Proof of Theorem 2

For any connected graph G, Proposition 3 indicates that to prove Theorem 2 it suffices to consider those graphs G with $\mathrm{diam}(G) \leq 3$. For the case of $\mathrm{diam}(G) = 3$, we have the following result.

Lemma 9. *Let G be a graph of order $n \geq 4$ with* $\mathrm{diam}(G) = 3$. *Then*

(i) $\gamma_{gdR}(G) = 2n - 2$ *if and only if* $G \in \{P_4, G_1\}$;

(ii) $\gamma_{gdR}(G) = 2n - 3$ *if and only if* $G = G_2$.

Proof. If $n = 4$, then $G = P_4$ and the result is trivial. So in the following we may assume that $n \geq 5$. Suppose now that $\gamma_{gdR}(G) \in \{2n - 2, 2n - 3\}$. Let $P = v_0v_1v_2v_3$ be a diametral path of G. By Proposition 5, we may assume that $d(v_0) = d(v_3) = 1$. We proceed with some claims.

Claim 1. $d(v_1) = d(v_2) = 3$.

Proof. Since $d(v_0) = d(v_3) = 1$ and $n \geq 5$ by our earlier assumption, we have $d(v_1) \geq 3$ or $d(v_2) \geq 3$. Suppose first that $d(v_1) = 2$ and $d(v_2) \geq 3$ (the case

$d(v_1) \geq 3$ and $d(v_2) = 2$ is similar). Then the function f defined by $f(v_1) = f(v_2) = 3$ and $f(x) = 0$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 6 \leq 2n-4$, which is a contradiction. Suppose second that $d(v_1) \geq 4$ (the case $d(v_2) \geq 4$ is similar). Note that $d(v_3) = 1$. Then the function f defined by $f(x) = 0$ for each $x \in N(v_1)\backslash\{v_2\}$, $f(v_1) = f(v_3) = 3$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2(n - |N(v_1)\backslash\{v_2\}| - 2) + 6 \leq 2n-4$, which is a contradiction. Consequently, we have $d(v_1) = d(v_2) = 3$. Thus Claim 1 holds. □

Claim 2. *v_1 and v_2 have a unique common neighbor, say v_4.*

Proof. Suppose that $u_1 \in N(v_1)\backslash\{v_0, v_2\}$ and $u_2 \in N(v_2)\backslash\{v_1, v_3\}$ are two distinct vertices. Note that $d(v_1) = d(v_2) = 3$ by Claim 1. Then the function f defined by $f(u_1) = f(u_2) = f(v_0) = f(v_3) = 0$, $f(v_1) = f(v_2) = 3$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n-6$, which is a contradiction. Therefore, by Claim 1, we have $N(v_1)\backslash\{v_0, v_2\} = N(v_2)\backslash\{v_1, v_3\}$, implying that Claim 2 holds. □

Claim 3. *$\gamma_{gdR}(G) = 2n-2$ if and only if $G = G_1$.*

Proof. Suppose that $\gamma_{gdR}(G) = 2n-2$. Now let $n \geq 6$. By Claims 1 and 2 and our earlier assumption, there must exist $v_5 \in V(G)\backslash\{v_0, v_1, v_2, v_3, v_4\}$ that is a neighbor of v_4. Then the function f defined by $f(v_0) = f(v_3) = f(v_5) = 1$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n-3$, which is a contradiction. Thus $n = 5$. Now, by combining Claims 1 and 2 and our earlier assumption, we have $G = G_1$. On the other hand, if $G = G_1$, then it is easy to verify that $\gamma_{gdR}(G) = 2n-2$. Thus Claim 3 holds. □

Claim 4. *$\gamma_{gdR}(G) = 2n-3$ if and only if $G = G_2$.*

Proof. Suppose that $\gamma_{gdR}(G) = 2n-3$. Let $n \geq 7$. By our earlier assumption and Claims 1 and 2, we have $V(G)\backslash\{v_0, v_1, v_2, v_3, v_4\} \subseteq N(v_4)$ since $\operatorname{diam}(G) = 3$. Then the function f defined by $f(v_0) = f(v_2) = 2$, $f(v_3) = f(v_4) = 3$ and $f(x) = 0$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 10 \leq 2n-4$, which is a contradiction. This implies that $n \in \{5, 6\}$. Then by our earlier assumption and Claims $1, 2$ and 3, we have $G = G_2$. On the other hand, if $G = G_2$, then it is easy to verify that $\gamma_{gdR}(G) = 2n-3$. Thus Claim 4 holds. □

The proof is completed. ■

We next discuss the case of $\operatorname{diam}(G) = 2$. We first consider the graph G with $\operatorname{diam}(G) = \omega(G) = 2$.

Lemma 10. *Let G be a graph of order $n \geq 4$ with* $\operatorname{diam}(G) = \omega(G) = 2$. *Then*

(i) $\gamma_{gdR}(G) = 2n-2$ *if and only if* $G \in \{C_4, C_5, K_{1,3}\}$;

(ii) $\gamma_{gdR}(G) \neq 2n-3$.

Proof. By Proposition 6 we may assume that $\delta(G) \geq n-3$. If $\delta(G) = n-1$, then $G = K_n$, a contradiction to our assumption that $\operatorname{diam}(G) = 2$. Therefore, $\delta(G) \in \{n-2, n-3\}$. Let u be a vertex of G with degree $\delta(G)$. Note that G is triangle-free since $\omega(G) = 2$. We have the following two claims.

Claim 1. *If $\delta(G) = n-2$, or $\delta(G) = n-3$ and $n \in \{4, 5\}$, then*

(i) $\gamma_{gdR}(G) = 2n-2$ *if and only if* $G \in \{C_4, C_5, K_{1,3}\}$;

(ii) $\gamma_{gdR}(G) \neq 2n-3$.

Proof. Recall that $\operatorname{diam}(G) = 2$ and G is triangle-free. First, suppose that $\delta(G) = n-2$. If $n = 4$, then clearly $G = C_4$ and $\gamma_{gdR}(G) = 2n-2$. If $n \geq 5$, then every vertex of $N(u)$ has degree at most 2, a contradiction to our assumption that $\delta(G) = n-2 \geq 3$.

Second, suppose that $\delta(G) = n-3$ and $n \in \{4, 5\}$. If $n = 4$, then $\delta(G) = n-3 = 1$ and hence $G = K_{1,3}$, implying that $\gamma_{gdR}(G) = 2n-2$. Assume next that $n = 5$. Clearly $\delta(G) = n-3 = 2$. Moreover, since G is triangle-free, this forces $\Delta(G) \in \{2, 3\}$. If $\Delta(G) = 2$, then $G = C_5$ and we have $\gamma_{gdR}(G) = 2n-2$. If $\Delta(G) = 3$, then it is not difficult to check that $G = K_{2,3}$ and clearly $\gamma_{gdR}(G) = 2n-4$. Claim 1 holds. □

Claim 2. *If $\delta(G) = n-3$ and $n \geq 6$, then $\gamma_{gdR}(G) \leq 2n-4$.*

Proof. Let $N(u) = \{u_1, u_2, \ldots, u_{n-3}\}$ and let $V(G) \setminus N[u] = \{v_1, v_2\}$. Assume that $v_1v_2 \in E(G)$. Since $\operatorname{diam}(G) = 2$ and G is triangle-free, we have that each $u_i \in N(u)$ is adjacent to exactly one of v_1 and v_2 and so $d(u_i) = 2 < n-3 = \delta(G)$, a contradiction. Therefore, we obtain $v_1v_2 \notin E(G)$. Since $\delta(G) = n-3 \geq 3$, one can observe that the function f defined by $f(u_1) = f(u_2) = f(v_1) = f(v_2) = 1$ and $f(x) = 2$ otherwise, is a GDRDF of G implying that $\gamma_{gdR}(G) \leq \omega(f) = 2n-4$. Thus Claim 2 holds. □

The proof is completed. ■

For the case when $\operatorname{diam}(G) = 2$ and $\omega(G) \geq 3$, we need more detailed analysis. For this purpose, we shall adopt the following notations. For a graph G with $\operatorname{diam}(G) = 2$, let U_1 be a subset of vertices of G that induces a maximum clique in G, and let $U_2 = V(G) \setminus U_1$ and $|U_i| = n_i$ for $i \in \{1, 2\}$.

Lemma 11. *Let G be a graph of order $n \geq 4$ with $\operatorname{diam}(G) = 2$ and $\omega(G) \geq 3$ and let U_2 induce a clique. Then*

(i) $\gamma_{gdR}(G) = 2n-2$ *if and only if* $G = K_4^{--}$, $\overline{G} = P_4 \cup K_1$ *or* $\overline{G} = 2K_2 \cup (n-4)K_1$ $(n \geq 5)$;

(ii) $\gamma_{gdR}(G) = 2n-3$ *if and only if* $G = K_n^{--}$ $(n \geq 5)$, $\overline{G} = C_4 \cup K_1$, $\overline{G} = 3K_2 \cup (n-6)K_1$ $(n \geq 6)$, *or* $\overline{G} = P_4 \cup (n-4)K_1$ $(n \geq 6)$.

Proof. We have the following claims.

Claim 1. *If* $n_2 = 1$, *then*

(i) $\gamma_{gdR}(G) = 2n-2$ *if and only if* $G = K_4^{--}$;

(ii) $\gamma_{gdR}(G) = 2n-3$ *if and only if* $G = K_n^{--}$ $(n \geq 5)$.

Proof. The sufficiency is trivial. We proceed to show the necessity. Suppose that $\gamma_{gdR}(G) \in \{2n-2, 2n-3\}$. Let $U_2 = \{u\}$. By Proposition 6, we have $\delta(G) \geq n-3$ and hence u is not adjacent to at most two vertices of U_1. This implies that $G \in \{K_n - e, K_n^{--}\}$. If $G = K_n - e$, then by Theorem 1(ii), $\gamma_{gdR}(G) = 2n-1$, a contradiction. Therefore, we have $G = K_n^{--}$ $(n \geq 4)$. Claim 1 holds. □

Claim 2. *If* $n_2 = 2$, *then*

(i) $\gamma_{gdR}(G) = 2n-2$ *if and only if* $\overline{G} = P_4 \cup K_1$ *or* $\overline{G} = 2K_2 \cup (n-4)K_1$ $(n \geq 5)$;

(ii) $\gamma_{gdR}(G) = 2n-3$ *if and only if* $\overline{G} = C_4 \cup K_1$ *or* $\overline{G} = P_4 \cup (n-4)K_1$ $(n \geq 6)$.

Proof. The sufficiency is trivial. We proceed to show the necessity. Since $n_1 = \omega(G) \geq 3$, we have $n \geq 5$. Suppose that $\gamma_{gdR}(G) \in \{2n-2, 2n-3\}$. Let $U_2 = \{u_1, u_2\}$. Since $\delta(G) \geq n-3$, by Proposition 6, and U_1 induces a maximum clique, we must have $|U_1 \backslash N(u_i)| \in \{1, 2\}$ for each $i \in \{1, 2\}$.

First, suppose that $|U_1 \backslash N(u_1)| = |U_1 \backslash N(u_2)| = 1$. If $U_1 \backslash N(u_1) = U_1 \backslash N(u_2)$, then $N[u_1]$ induces a clique of order $n_1 + 1$, which contradicts the maximality of U_1. Thus $U_1 \backslash N(u_1) \neq U_1 \backslash N(u_2)$. Let $U_1 \backslash N(u_1) = \{v_1\}$ and $U_1 \backslash N(u_2) = \{v_2\}$, where $v_1 \neq v_2$. It is easy to check that $\{u_1, u_2, v_1, v_2\}$ induces $2K_2$ and $U_1 \backslash \{v_1, v_2\}$ induces $(n-4)K_1$ in $\overline{G}$. As a result, we have $\overline{G} = 2K_2 \cup (n-4)K_1$ $(n \geq 5)$.

Second, suppose that $|U_1 \backslash N(u_1)| = |U_1 \backslash N(u_2)| = 2$. Without loss of generality, assume that $U_1 \backslash N(u_1) = \{v_1, v_2\}$ and $U_1 \backslash N(u_2) = \{v_3, v_4\}$. If v_1, v_2 and v_3 are distinct, then the function f defined by $f(u_2) = f(v_1) = f(v_2) = 0$, $f(u_1) = f(v_3) = 3$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n-4$, a contradiction. Thus $v_3 \in \{v_1, v_2\}$. Similarly, we have $v_4 \in \{v_1, v_2\}$. This implies that $U_1 \backslash N(u_1) = U_1 \backslash N(u_2) = \{v_1, v_2\}$. Then it is easy to check that $\{u_1, u_2, v_1, v_2\}$ induces C_4 and $U_1 \backslash \{v_1, v_2\}$ induces $(n-4)K_1$ in $\overline{G}$. As a result, we have $\overline{G} = C_4 \cup (n-4)K_1$. Moreover, if $n \geq 6$, then let $C_4 = x_1x_2x_3x_4x_1$ and define the function f on $V(G)$ by $f(x_1) = f(x_3) = 0$ and $f(x) = 2$ for the remaining vertices. Clearly, f is a GDRDF on G and so $\gamma_{gdR}(G) \leq 2n-4$, a contradiction. Therefore, we have $n = 5$ and so $\overline{G} = C_4 \cup K_1$.

Finally, suppose that one of $|U_1 \backslash N(u_1)|$ and $|U_1 \backslash N(u_2)|$ is equal to one and the other is equal to two. Without loss of generality, assume that $U_1 \backslash N(u_1) = \{v_1\}$ and $U_1 \backslash N(u_2) = \{v_2, v_3\}$. If v_1, v_2 and v_3 are distinct, then the function f defined by $f(u_1) = f(v_2) = f(v_3) = 0$, $f(u_2) = f(v_1) = 3$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n-4$, a contradiction. Thus $v_1 \in \{v_2, v_3\}$ and hence we may assume that $U_1 \backslash N(u_1) = \{v_2\}$ and $U_1 \backslash N(u_2) =$

$\{v_2, v_3\}$. It is easy to check that $\{u_1, u_2, v_2, v_3\}$ induces P_4 and $U_1\backslash\{v_2, v_3\}$ induces $(n-4)K_1$ in $\overline{G}$. As a result, we have $\overline{G} = P_4 \cup (n-4)K_1$ $(n \geq 5)$. Claim 2 holds. $\square$

Claim 3. *If $n_2 \geq 3$, then $\gamma_{gdR}(G) \leq 2n-3$ with equality if and only if $\overline{G} = 3K_2 \cup (n-6)K_1$ $(n \geq 6)$.*

Proof. Since $n_1 = \omega(G) \geq 3$, we have $n \geq 6$. By Proposition 6, it suffices to consider the case that $\delta(G) \geq n-3$. Let $U_2 = \{u_1, u_2, \ldots, u_{n_2}\}$. Assume first that there exists some vertex, say u_1, in U_2 such that $|U_1\backslash N(u_1)| \geq 2$. Let v_1 and v_2 be two distinct vertices in $U_1\backslash N(u_1)$. Moreover, since $\delta(G) \geq n-3$, u_1 is adjacent to all vertices in $U_1\backslash\{v_1, v_2\}$ and so $\{v_1, v_2\} = U_1\backslash N(u_1)$. Then there must exist two vertices $u \in U_2$ and $v_3 \in U_1\backslash\{v_1, v_2\}$ such that $uv_3 \notin E(G)$ (for otherwise, $V(G)\backslash\{v_1, v_2\}$ induces a clique of order $n_1 + n_2 - 2 \geq n_1 + 1$, which is a contradiction to the maximality of U_1). Note that $u \neq u_1$ since $\{v_1, v_2\} = U_1\backslash N(u_1)$. One can verify that the function f defined by $f(u) = f(v_1) = f(v_2) = 0$, $f(u_1) = f(v_3) = 3$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n-4$. So in the following we may assume that for each $i \in \{1, 2, \ldots, n_2\}$, $|U_1\backslash N(u_i)| \leq 1$. Moreover, since U_1 induces a maximum clique, we have that for each $i \in \{1, 2, \ldots, n_2\}$, $|U_1\backslash N(u_i)| \geq 1$. This forces $|U_1\backslash N(u_i)| = 1$ for each $i \in \{1, 2, \ldots, n_2\}$.

Suppose now that there exist two vertices, say u_1 and u_2, in U_2 such that $(U_1\backslash N(u_1)) \cap (U_1\backslash N(u_2)) \neq \emptyset$. Without loss of generality, assume that $v_1 \in (U_1\backslash N(u_1)) \cap (U_1\backslash N(u_2))$. By our earlier assumption, we have $U_1\backslash N(u_1) = U_1\backslash N(u_2) = \{v_1\}$. This implies that $(U_1\backslash\{v_1\}) \cup \{u_1, u_2\}$ induces a clique of order n_1+1, a contradiction to the maximality of U_1. Thus $(U_1\backslash N(u_i)) \cap (U_1\backslash N(u_j)) = \emptyset$ for $1 \leq i < j \leq n_2$.

Let $U_1 = \{v_1, v_2, \ldots, v_{n_1}\}$. By our earlier assumptions, we may assume that $U_1\backslash N(u_i) = \{v_i\}$ for each $1 \leq i \leq n_2$. It is easy to check that $\{u_i, v_i : 1 \leq i \leq n_2\}$ induces n_2K_2 and $U_1\backslash\{v_i : 1 \leq i \leq n_2\}$ induces $(n-2n_2)K_1$ in $\overline{G}$. As a result, we have $\overline{G} = n_2K_2 \cup (n-2n_2)K_1$. If $n_2 \geq 4$, then clearly $\gamma_{gdR}(G) \leq 2n-4$. If $n_2 = 3$, that is, if $\overline{G} = 3K_2 \cup (n-6)K_1$ $(n \geq 6)$, then clearly $\gamma_{gdR}(G) = 2n-3$. Claim 3 holds. $\square$

This completes the proof. ■

Lemma 12. *Let G be a graph of order $n \geq 5$ with* $\mathrm{diam}(G) = 2$ *and $\omega(G) \geq 3$ and let U_2 be an independent set with $n_2 \geq 2$. Then*

(i) $\gamma_{gdR}(G) = 2n-2$ *if and only if* $\overline{G} = P_4 \cup K_1$;

(ii) $\gamma_{gdR}(G) = 2n-3$ *if and only if* $\overline{G} \in \{C_3 \cup (n-3)K_1$ $(n \geq 5), P_4 \cup (n-4)K_1$ $(n \geq 6)\}$.

Proof. Suppose that $\gamma_{gdR}(G) \in \{2n-2, 2n-3\}$. By Proposition 6, we must have $\delta(G) \geq n-3$. Since U_2 is independent and U_1 induces a maximum clique, we must have $n_2 = 2$. Let $U_2 = \{u_1, u_2\}$. Since U_1 induces a maximum clique, we have $|U_1 \backslash N(u_1)| \geq 1$ and $|U_1 \backslash N(u_2)| \geq 1$. On the other hand, it follows from $\delta(G) \geq n-3$ that $|U_1 \backslash N(u_1)| = |U_1 \backslash N(u_2)| = 1$. Assume that $U_1 \backslash N(u_1) = \{v_1\}$ and $U_1 \backslash N(u_2) = \{v_2\}$.

If $v_2 \neq v_1$, then it is easy to check that $\{u_1, u_2, v_1, v_2\}$ induces P_4 and $U_1 \backslash \{v_1, v_2\}$ induces $(n-4)K_1$ in $\overline{G}$, implying that $\overline{G} = P_4 \cup (n-4)K_1$ $(n \geq 5)$. If $v_2 = v_1$, then it is easy to check that $\{u_1, u_2, v_1\}$ induces C_3 and $U_1 \backslash \{v_1\}$ induces $(n-3)K_1$ in $\overline{G}$, implying that $\overline{G} = C_3 \cup (n-3)K_1$ $(n \geq 5)$.

The converse is clear, which completes the proof. ∎

Lemma 13. *Let G be a graph of order $n \geq 4$ with* $\operatorname{diam}(G) = 2$ *and* $\omega(G) \geq 3$ *and let U_2 neither induce a clique nor be an independent set. Then*

$$\gamma_{gdR}(G) \leq 2n - 4.$$

Proof. According to Proposition 6, we can assume that $\delta(G) \geq n-3$. Since U_2 neither induces a clique nor is an independent set, we have that $n_2 \geq 3$. Since $\omega(G) \geq 3$ and U_1 is a maximum clique of G, we have $n_1 \geq 3$ and $U_1 \setminus N(u) \neq \emptyset$ for each $u \in U_2$. Moreover, since $\delta(G) \geq n-3$, each vertex u in U_2 is adjacent to all vertices in $U_2 \backslash \{u\}$ except at most one. It follows that the induced subgraph $G[U_2]$ is connected. Since U_2 does not induce a clique, $G[U_2]$ contains $P_3 = u_1u_2u_3$ as an induced subgraph. In particular, $u_1u_3 \notin E(G)$. Moreover, since $\delta(G) \geq n-3$ and $U_1 \setminus N(u_i) \neq \emptyset$ for each $i \in \{1,3\}$, we obtain $|U_1 \setminus N(u_i)| = 1$ for $i \in \{1,3\}$. We proceed with some claims.

Claim 1. *If $N(u_i) \cap N(u_2) \cap U_1 = \emptyset$ for some $i \in \{1,3\}$, then $\gamma_{gdR}(G) \leq 2n-4$.*

Proof. Without loss of generality, assume that $N(u_1) \cap N(u_2) \cap U_1 = \emptyset$. As shown earlier, $|U_1 \backslash N(u_1)| = 1$. Hence we may assume that $U_1 \backslash N(u_1) = \{v_1\}$. Moreover, since $N(u_1) \cap N(u_2) \cap U_1 = \emptyset$, we have $(U_1 \backslash \{v_1\}) \cap N(u_2) = N(u_1) \cap N(u_2) \cap U_1 = \emptyset$. If $v_1 \notin N(u_2)$, then $U_1 \cap N(u_2) = \emptyset$ and hence $d(u_2) \leq n - |U_1| - 1 = n - n_1 - 1 = n - \omega(G) - 1 \leq n-4 < \delta(G)$, a contradiction. Thus we have $v_1 \in N(u_2)$. Moreover, since $(U_1 \backslash \{v_1\}) \cap N(u_2) = \emptyset$ as shown earlier, we have $U_1 \cap N(u_2) = \{v_1\}$. One can verify that the function f defined by $f(u_1) = f(u_2) = 0$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n-4$. Thus Claim 1 holds. □

Claim 2. *If $N(u_1) \cap N(u_2) \cap N(u_3) \cap U_1 = \emptyset$, then $\gamma_{gdR}(G) \leq 2n-4$.*

Proof. If $N(u_i) \cap N(u_2) \cap U_1 = \emptyset$ for some $i \in \{1,3\}$, then by Claim 1, $\gamma_{gdR}(G) \leq 2n-4$. So in the following we may assume that $N(u_i) \cap N(u_2) \cap U_1 \neq \emptyset$ for each $i \in \{1,3\}$. Without loss of generality, assume that $v_1 \in N(u_1) \cap N(u_2) \cap U_1$ and $v_2 \in N(u_3) \cap N(u_2) \cap U_1$. Since $N(u_1) \cap N(u_2) \cap N(u_3) \cap U_1 = \emptyset$, we have that $v_1 \neq v_2$, $v_1 \notin N(u_3)$ and $v_2 \notin N(u_1)$. Moreover, since U_1 induces a maximum clique, u_2 is not adjacent to at least one vertex, say v_3, in U_1. As shown earlier, $|U_1 \setminus N(u_3)| = 1$ and $v_1 \notin N(u_3)$. Thus we have $v_3 \in N(u_3)$. Then the function

f defined by $f(u_2) = f(u_3) = f(v_2) = 0$, $f(u_1) = f(v_3) = 3$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n - 4$. Thus Claim 2 holds. $\square$

By Claim 2, we can assume that $v_1 \in N(u_1) \cap N(u_2) \cap N(u_3) \cap U_1$ in the following.

Claim 3. *If* $(U_1 \backslash N(u_1)) \cap (U_1 \backslash N(u_3)) \neq \emptyset$, *then* $\gamma_{gdR}(G) \leq 2n - 4$.

Proof. Without loss of generality, assume that $v_2 \in (U_1 \backslash N(u_1)) \cap (U_1 \backslash N(u_3))$. Since $|U_1 \setminus N(u_i)| = 1$ for each $i \in \{1, 3\}$, we obtain that u_1 and u_3 must be adjacent to all vertices in $U_1 \backslash \{v_2\}$. Moreover, since U_1 induces a maximum clique, $U_1 \backslash N(u_2) \neq \emptyset$. One can check that the function f defined by $f(v_2) = 3, f(u_2) = 1$, $f(u_1) = f(u_3) = 0$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n - 4$. Claim 3 is true. $\square$

As shown earlier, $|U_1 \backslash N(u_i)| = 1$ for each $i \in \{1, 3\}$. Further, by Claim 3, we may assume that $(U_1 \backslash N(u_1)) \cap (U_1 \backslash N(u_3)) = \emptyset$. Then there must exist two distinct vertices, say v_2 and v_3, such that $U_1 \backslash N(u_1) = \{v_2\}$ and $U_1 \backslash N(u_3) = \{v_3\}$.

Claim 4. *If* $v_i \in N(u_2)$ *for some* $i \in \{2, 3\}$, *then* $\gamma_{gdR}(G) \leq 2n - 4$.

Proof. First, suppose that $v_2 \in N(u_2)$. Since U_1 induces a maximum clique, there exists some vertex, say x_i, in $U_1 \backslash \{v_1, v_2\}$ such that $x_i \notin N(u_2)$. It is easy to verify that the function f defined by $f(u_3) = f(v_2) = f(x_i) = 0$, $f(u_1) = f(u_2) = 3$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n - 4$.

Second, suppose that $v_2 \notin N(u_2)$ and $v_3 \in N(u_2)$. Then the function f defined by $f(u_1) = f(v_2) = f(v_3) = 0$, $f(u_2) = f(u_3) = 3$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n - 4$. Thus Claim 4 holds. $\square$

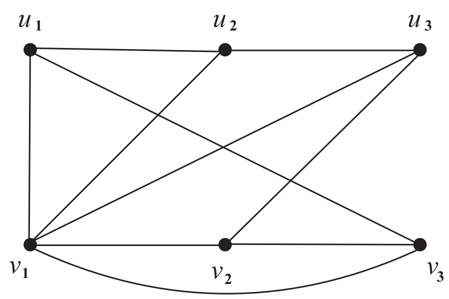


Figure 2. The graph G_3.

By Claim 4, we may assume that $v_2 \notin N(u_2)$ and $v_3 \notin N(u_2)$. By our earlier assumptions, we note that $v_1 \in N(u_1) \cap N(u_2) \cap N(u_3) \cap U_1$, $U_1 \backslash N(u_1) = \{v_2\}$ and $U_1 \backslash N(u_3) = \{v_3\}$. Thus G contains G_3 as an induced subgraph, where G_3

is depicted in Figure 2. Then the function f defined by $f(u_2) = f(u_3) = 0$ and $f(x) = 2$ otherwise, is a GDRDF on G and hence $\gamma_{gdR}(G) \leq \omega(f) = 2n - 4$, which completes the proof. ■

Now, we are ready to give the proof of Theorem 2.

***Proof of Theorem* 2.** For any connected graph G of order $n \geq 4$, by combining Theorem 1, Proposition 3 and Lemmas 9, 10, 11, 12 and 13, we can conclude that

- $\gamma_{gdR}(G) = 2n - 2$ if and only if $G \in \{P_4, C_4, C_5, K_{1,3}, K_4^{--}, G_1\}$ or $\overline{G} \in \{2K_2 \cup (n-4)K_1 (n \geq 5), P_4 \cup K_1\}$;
- $\gamma_{gdR}(G) = 2n - 3$ if and only if $G \in \{K_n^{--} (n \geq 5), G_2\}$ or $\overline{G} \in \{3K_2 \cup (n-6)K_1 (n \geq 6), C_3 \cup (n-3)K_1 (n \geq 5), C_4 \cup K_1, P_4 \cup (n-4)K_1 (n \geq 6)\}$.

Recall that the complement $\overline{G}$ of a disconnected graph G is connected and $\gamma_{gdR}(G) = \gamma_{gdR}(\overline{G})$. Moreover, note that the complements of P_4, C_5, G_1 and G_2 are connected graphs. Therefore, for any disconnected graph G of order $n \geq 4$, we have

- $\gamma_{gdR}(G) = 2n - 2$ if and only if $\overline{G} \in \{C_4, K_{1,3}, K_4^{--}\}$ or $G \in \{2K_2 \cup (n-4)K_1 (n \geq 5), P_4 \cup K_1\}$;
- $\gamma_{gdR}(G) = 2n - 3$ if and only if $\overline{G} = K_n^{--}$ $(n \geq 5)$ or $G \in \{3K_2 \cup (n-6)K_1 (n \geq 6), C_3 \cup (n-3)K_1 (n \geq 5), C_4 \cup K_1, P_4 \cup (n-4)K_1 (n \geq 6)\}$.

Note that for any graph G of order $n \leq 3$, it is not difficult to check that $\gamma_{gdR}(G) \in \{2n, 2n-1\}$. Now, by combining the above arguments, we can obtain the desired result. This completes the proof of Theorem 2. ■

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