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PANCHROMATIC PATTERNS BY PATHS [1]

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Abstract

Let $H = (V_H, A_H)$ be a digraph, possibly with loops, and let $D = (V_D, A_D)$ be a loopless multidigraph with a colouring of its arcs $c : A_D \to V_H$. An H-path of D is a path $(v_0, \ldots, v_n)$ of D such that $(c(v_{i-1}, v_i), c(v_i, v_{i+1}))$ is an arc of H for every $1 \leq i \leq n-1$. For $u, v \in V_D$, we say that u reaches v by H-paths if there exists an H-path from u to v in D. A subset $S \subseteq V_D$ is absorbent by H-paths of D if every vertex in $V_D - S$ reaches some vertex in S by H-paths, and it is independent by H-paths if no vertex in S can reach another (different) vertex in S by H-paths. A kernel by H-paths is a subset of V_D which is independent by H-paths and absorbent by H-paths.

We define $\widetilde{\mathscr{B}}_1$ as the set of digraphs H such that any H-arc-coloured tournament has an absorbent by H-paths vertex; the set $\widetilde{\mathscr{B}}_2$ consists of the digraphs H such that any H-arc-coloured digraph D has an independent, absorbent by H-paths set; analogously, the set $\widetilde{\mathscr{B}}_3$ is the set of digraphs H such that every H-arc-coloured digraph D contains a kernel by H-paths.

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In this work, we point out similarities and differences between reachability by H-walks and reachability by H-paths. We present a characterization of the patterns H such that for every digraph D and every H-arc-colouring of D, every H-walk between two vertices in D contains an H-path with the same endpoints. We provide advances towards a characterization of the digraphs in $\widetilde{\mathscr{B}}_3$. In particular, we show that if a specific digraph is not in $\widetilde{\mathscr{B}}_3$, then the structure of the digraphs in the family is completely determined.

Keywords: kernel of a digraph, kernel by monochromatic paths, panchromatic pattern, kernel by H-walks, kernel by H-paths.

2020 Mathematics Subject Classification: 05C20, 05C69.

1. Introduction

A kernel in a digraph D is a subset K of V_D which is independent and absorbent[3]. Numerous generalizations of this concept have been studied in the literature, but one that has received particular attention is the notion of kernel by monochromatic walks. Given a digraph D whose arc set is coloured with a set of colours C, we say that a vertex u reaches a vertex v by monochromatic walks in D if there is a (directed) walk from u to v such that all its arcs have the same colour. With this generalized reachability concept, the notions of independence and absorbance by monochromatic walks come naturally; as one would expect, a subset S of V_D is independent by monochromatic walks if no vertex in S can reach another vertex in S by monochromatic walks, and it is absorbent by monochromatic walks if every vertex in $V_D - S$ reaches some vertex in S by monochromatic walks. Evidently, a kernel by monochromatic walks is a subset K of V_D which is independent and absorbent by monochromatic walks. Sands, Sauer and Woodrow proved in [13] that every digraph whose arc set is coloured with two colours has a kernel by monochromatic walks.

In [12], Linek and Sands further generalized kernels by monochromatic walks by considering a broader notion of reachability. Instead of colouring the arcs of the digraph D with an arbitrary set, they used the vertex set of another digraph H; with this setting, instead of only considering monochromatic walks, the arcs of H could be used to codify permitted colour changes in the walks of D to define reachability. Formally, a digraph H, possibly with loops, will be called a *pattern of colours*, or *pattern* for short; a digraph D is an *H-arc-coloured digraph* if it is an irreflexive multidigraph together with a colouring c of its arcs, $c\colon A_D \to V_H$. An *H-walk* W in D is a walk $W = (x_0, \dots, x_k)$ such that $(c(x_0, x_1), \dots, c(x_{k-1}, x_k))$ is a walk in H. For $u, v \in V_D$, we say that u *reaches* v by H-walks if there exists

[3] The set K is absorbent if for every vertex $u \in V_D - K$ there exists a vertex v in K such that (u, v) is an arc of D.

an H-walk from u to v in D. Naturally, this notion of reachability allows us to define independence by H-walks and absorbance by H-walks. A subset S of V_D is *independent by H-walks* if no vertex in S can reach by H-walks another vertex in S, and it is *absorbent by H-walks* if every vertex in $V_D - S$ can reach by H-walks some vertex in S; if S is both independent by H-walks and absorbent by H-walks, we say that S is a *kernel by H-walks.* Notice that if the only arcs of H are loops, then the only possible H-walks are precisely monochromatic walks, and hence, a kernel by H-walks is just a kernel by monochromatic walks. An *H-path* in D is an H-walk which is a path in D; the notions of independence by H-paths and absorbence by H-paths can be analogously defined. Notice that an H-path in D is a path whose sequence of colours induces a **walk** in H, not a path.

In this context, Arpin and Linek [1] introduced three interesting classes of patterns. The family $\mathscr{B}_1$ is the class of patterns H with the property that for every H-arc-coloured tournament T there exists a vertex which is absorbent by H-walks. Noticing that every independent set in a tournament has a single vertex, we may consider a special subclass of $\mathscr{B}_1$; the family $\mathscr{B}_2$ which consist of the patterns H with the property that for every H-arc-coloured digraph D there exists an absorbent by H-walks set which is an independent set. Further noticing that every independent by H-walks set is independent, the family $\mathscr{B}_3$ of the patterns H such that every H-arc-coloured digraph contains a kernel by H-walks, seems to be a nice subfamily of $\mathscr{B}_2$ to consider. The patterns in $\mathscr{B}_3$ are known as *panchromatic patterns* (by walks). In the same article, Arpin and Linek characterized the class $\mathscr{B}_2$; it consists of those patterns H with no odd directed cycle in their complement.

In [9] Galeana-Sánchez and Strausz presented a characterization of the class $\mathscr{B}_3$. Nonetheless, the authors of the present work recently found that the proof of a crucial lemma in the aforementioned work is incorrect, and thus, the characterization of this class remains as an open problem.

All the developments mentioned so far use a notion of reachability that relies on the concept of H-walks. It is easy to notice that every monochromatic walk contains a monochromatic path, so it comes to mind to consider reachability by H-paths instead of reachability by H-walks. Of course, this new notion of reachability yields new notions of independence, absorbance and kernels, but also, families analogous to the $\mathscr{B}_i$'s might be defined for $i \in \{1, 2, 3\}$ using reachability by H-paths. So, the aim of the present work is to study these concepts, similarities and differences with the reachability by H-walks case.

It is worth noticing that reachability problems in arc-coloured digraphs (and tournaments in particular) have received a lot of attention and many variants are being studied, consider, for instance, [3, 6, 7, 10].

We refer to [2] and [5] for general concepts. If H is a spanning subdigraph of

H_1 we will say that H_1 is a spanning superdigraph of H. A digraph is *reflexive* if every vertex has a loop. As in [1], let H be a reflexive pattern such that $V_H = C_1 \cup \cdots \cup C_n$ where each C_i is nonempty and $C_i \cap C_j = \emptyset$ for $i \neq j$, the induced subdigraph by C_i is complete, and $C_i \times C_j \subset A_H$ whenever $i \neq j$ and $(C_i \times C_j) \cap A_H \neq \emptyset$. The digraph H' with $V_{H'} = \{C_1, \ldots, C_n\}$ and $(C_i, C_j) \in A_{H'}$ if and only if $(C_i \times C_j) \cap A_H \neq \emptyset$ is called a *contraction* of H and H is called an *expansion* of H'. Two different vertices x and y of H are *true twins* if $N^+(x) = N^+(y)$ and $N^-(x) = N^-(y)$. Notice that it follows from the previous definition that, if x and y are true twins, then we obtain isomorphic patterns if we delete either of them or we contract H where all contraction sets are singletons except for one which is the set $\{x, y\}$. We will denote this kind of contraction of H as H_{xy}. From the previous observation it is clear that H_{xy} is isomorphic to an induced subdigraph of H.

The rest of the paper is organized as follows. In Section 2, we introduce the $\widetilde{\mathscr{B}}_i$ classes, with $i \in \{1, 2, 3\}$, and present some properties of them that are a consequence of the results in [1]. In Section 3, we study the patterns H such that for every H-arc-coloured digraph, every H-walk between two vertices of D contains an H-path with the same endpoints. In contrast to the previous sections, in Section 4 some differences and similarities between reachability by H-walks and by H-paths are presented, in particular, for a pattern H we propose constructions to find infinite families of H-arc-coloured digraphs having kernel by H-paths but no kernel by H-walks, and vice versa. Section 5 is devoted to panchromatic patterns by paths; we analyze fifteen of the patterns on three vertices. In Section 6 we point out some properties of the pattern F_1, which is the only one we were unable to decide whether it is as a panchromatic pattern by paths. Section 7 closes this work by presenting conclusions and one open problem; we use the results developed in Section 6 to show that panchromatic patterns by paths have a nice structure, and could be completely characterized depending on whether $F_1 \in \widetilde{\mathscr{B}}_3$ or not.

2. $\widetilde{\mathscr{B}}_i$ Classes and Simple Consequences of Known Results

Let H be a pattern, and let D be an H-coloured multidgraph. Analogously to the concepts based on reachability by H-walks, we define a subset S of V_D to be *absorbent by H-paths* in D if every vertex in $V_D - S$ reaches by H-paths some vertex in S, and *independent by H-paths* if no vertex in S can reach another (different) vertex in S by H-paths. A *kernel by H-paths* is a subset of V_D which is absorbent by H-paths and independent by H-paths.

As we mentioned before, our main goal is to obtain a development parallel to that made in [1], but considering reachability by H-paths instead of reachability by H-walks. Hence, we start defining the following classes of digraphs.

- Let $\widetilde{\mathscr{B}_1}$ be the class of patterns H such that every H-arc-colourerd tournament contains a kernel by H-paths.
- Let $\widetilde{\mathscr{B}_2}$ be the class of patterns H such that every H-arc-coloured digraph D contains an independent set of V_D which is also absorbent by H-paths.
- Let $\widetilde{\mathscr{B}_3}$ be the class of patterns H such that for every H-arc-coloured digraph D contains a kernel by H-paths.

By simply considering the definitions it is easy to observe that $\widetilde{\mathscr{B}_3} \subseteq \widetilde{\mathscr{B}_2} \subseteq \widetilde{\mathscr{B}_1}$. Notice that if H is a complete reflexive digraph, then $H \in \widetilde{\mathscr{B}_3}$. Sands, Sauer and Woodrow proved in [13] that if H consists of two reflexive isolated vertices, then $H \in \widetilde{\mathscr{B}_3}$, hence $\widetilde{\mathscr{B}_i}$ is nonempty for $i \in \{1, 2, 3\}$. Naturally, we are interested in a characterization of these classes; our following result states some simple observations about them. The proof is the same as the one presented by Arpin and Linek for $\mathscr{B}_i$, for $i \in \{1, 2, 3\}$ [1], but we reproduce it for the sake of the reader.

Lemma 1. *Let H be a pattern.*

1. *If $H \in \widetilde{\mathscr{B}_i}$, then every vertex of H is reflexive, for $i \in \{1, 2, 3\}$.*
2. *If $H \in \widetilde{\mathscr{B}_i}$ and H_1 is an induced subdigraph of H, then $H_1 \in \widetilde{\mathscr{B}_i}$, for $i \in \{1, 2, 3\}$.*
3. *If H' is a contraction of H, then $H \in \widetilde{\mathscr{B}_i}$ if and only if $H' \in \widetilde{\mathscr{B}_i}$, for $i \in \{1, 2, 3\}$.*
4. *If $H \in \widetilde{\mathscr{B}_i}$ and H_1 is a spanning superdigraph of H, then $H_1 \in \widetilde{\mathscr{B}_i}$, for $i \in \{1, 2\}$.*

Proof. For part 1, consider a pattern H with a vertex x such that $(x, x) \notin A_H$. The H-arc-coloured digraph D obtained by colouring each arc of C_3 with x does not have a kernel by H-paths, and thus, $H \notin \widetilde{\mathscr{B}_i}$ for $i \in \{1, 2, 3\}$. For part 2, since every H_1-arc-colouring is also an H-arc-colouring, then any example showing $H_1 \notin \widetilde{\mathscr{B}_i}$ also shows that $H \notin \widetilde{\mathscr{B}_i}$, for $i \in \{1, 2, 3\}$. For part 4, we have that any H-path is also an H_1-path, thus $H_1 \in \widetilde{\mathscr{B}_i}$, for $i \in \{1, 2\}$.

The proof of part 3, although simple, is more elaborated, so we will only sketch an idea of the proof. First, since H' is isomorphic to an induced subdigraph of H, if $H \in \widetilde{\mathscr{B}_i}$, then $H' \in \widetilde{\mathscr{B}_i}$ as well. For the converse implication, let H' be obtained from H by contracting the sets $C_1, \ldots, C_r$. Suppose that D is an H-arc-coloured digraph with colouring c, and define an H'-arc-colouring c' of D by assigning color C_i to the arc (u, v) if and only if $c((u, v)) \in C_i$. With this colouring, D has a kernel by H'-paths K. It is not hard to verify that K is also a kernel by H-paths. ■

In [1], using order language, the digraphs in $\mathscr{B}_2$ are characterized as follows.

Theorem 2 [1]**.** *The following are equivalent.*

1. $H \in \mathscr{B}_2$.
2. H^c *has no odd cycle.*
3. H *is spanned by a quasiorder whose quotient partial order is a linear sum of* 1- *and* 2-*element antichains.*

We characterize the patterns that belong to $\widetilde{\mathscr{B}}_2$ as the reflexive digraphs such that their complement does not contain odd directed cycles, i.e., $\widetilde{\mathscr{B}}_2 = \mathscr{B}_2$. The analysis to obtain this characterization is analogous to that presented in [1], so some details will be omitted.

Theorem 3. *If H is a reflexive pattern, then $H \in \widetilde{\mathscr{B}}_2$ if and only if H^c does not contain any odd cycles.*

Proof. For the necessity, we prove the contrapositive. If H^c contains the odd cycle $(0, 1, 2, \ldots, 2k, 0)$ then let D be the odd cycle $D = (x_0, x_1, \ldots, x_{2k}, x_0)$ and define the H-colouring c such that $c((x_i, x_{i+1})) = i$, for $i \in \{0, 1, \ldots, 2k-1\}$, and $c((x_{2k}, x_0)) = 2k$. Every H-path in D is a single arc, hence D has no independent absorbent by H-paths set and $H \notin \widetilde{\mathscr{B}}_2$.

For the converse, suppose that H^c has no odd cycle. Let $H_1, \ldots, H_k$ be a topological ordering of the complements of the strong components of H^c. Since every H_i^c is strongly connected and has no odd cycles, then it is a bipartite digraph with bipartition (V_{i1}, V_{i2}), for $i \in \{1, \ldots, k\}$. It follows that in H every V_{ij} induces a complete reflexive digraph, for $i \in \{1, \ldots, k\}$ and $j \in \{1, 2\}$. Hence, $H[V_{i1}] \cup H[V_{i2}]$ is an expansion of $2K_1$ (or an expansion of K_1 in the case that V_{ij} is empty for some $j \in \{1, 2\}$). Then, by parts 3 and 4 of Lemma 1, we conclude $H_i \in \widetilde{\mathscr{B}}_2$.

By the choice of the ordering for the H_i's, the digraph H' obtained from $\bigcup_{i=1}^k H_i$ by adding all the arcs from vertices in H_j to vertices in H_i, for $i < j$, is a spanning subdigraph of H. By following a strategy similar to that of the proof of Lemma 4.1 in [1], it is not hard to conclude that $H' \in \widetilde{\mathscr{B}}_2$. Then, it follows from part 4 of Lemma 1 that $H \in \widetilde{\mathscr{B}}_2$. ■

3. Transitive Patterns

In this section we present a characterization of the patterns H such that for every digraph D and every H-arc-colouring of D, every H-walk between two vertices in D contains an H-path with the same endpoints.

Given a set of digraphs $\mathcal{S}$, we say that a digraph is $\mathcal{S}$-*free* if it does not contain any digraph in $\mathcal{S}$ as an induced subdigraph. Recall that if a property $\mathcal{P}$

is closed under taking induced subdigraphs, then there exists a family of forbidden digraphs, $\mathcal{F}$, not necessarily finite, that characterize the digraphs with that property. In other words, a digraph satisfies the property $\mathcal{P}$ if and only if it is $\mathcal{F}$-free. As an example, consider the following theorem.

Theorem 4. *If H is a reflexive digraph, then H is a transitive digraph if and only if it is $\mathcal{T}$-free, where $\mathcal{T}$ is the family of reflexive digraphs depicted in Figure* 1.

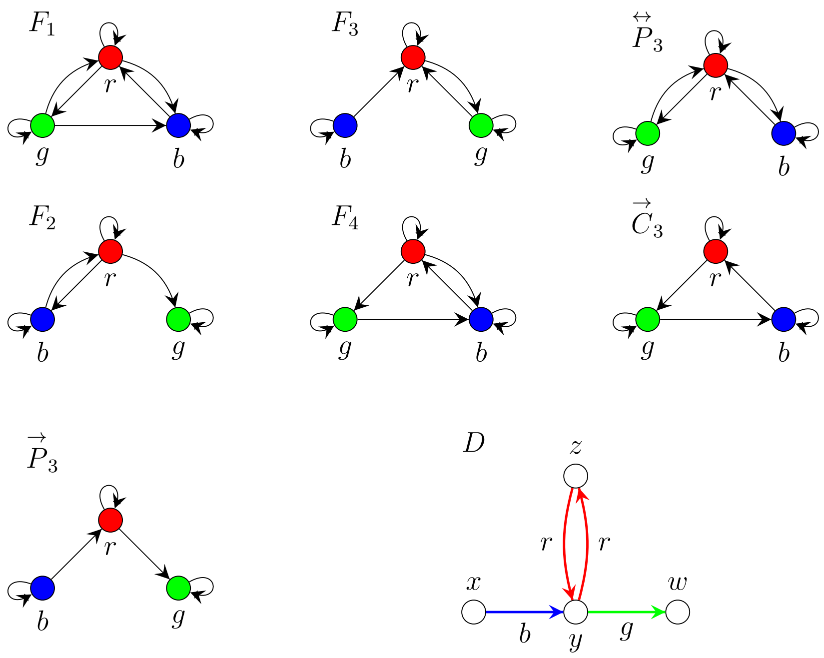

Figure 1. The family $\mathcal{T}$, and an example where (x, y, z, y, w) is an H-walk from x to w in D but there is no H-path from x to w in D, for every H in $\mathcal{T}$.

Proof. The necessity is trivial. For the sufficiency, notice that any non-transitive digraph has a subdigraph of order 3 where three arcs are fixed (two consecutive existing arcs and one absent arc which would form a transitive tournament with the two previous arcs). Then, only three arcs need to be chosen, leading to eight possibilities, two of which are isomorphic. Hence, the family $\mathcal{T}$ has seven digraphs. ■

If we consider the elements of $\mathcal{T}$ as colour patterns H, each of them is an example showing that not every H-walk contains an H-path with the same endpoints; see Figure 1. Our following result states that these are the only (minimal) patterns with this property.

Theorem 5. *Let H be a reflexive digraph. H is transitive if and only if for every H-arc-coloured digraph D, and every pair of different vertices x, y of D, every H-walk from x to y in D contains an H-path from x to y in D.*

Proof. We will prove the necessity by contrapositive. By Theorem 4, we suppose that H is a reflexive digraph containing one of the patterns in Figure 1 as an induced subdigraph. Hence, the digraph at the bottom right of the same figure is an H-arc-coloured digraph with vertices x and w such that an H-walk exists from x to w, but no H-path exists with the same endpoints.

For the sufficiency, suppose that H is a transitive reflexive digraph. Let D be an H-arc-coloured digraph, with colouring c, let x and y be different vertices in D and let $\gamma = (x = x_0, \ldots, x_n = y)$ be an H-walk from x to y in D. By definition, $c(\gamma) = (c_0, c_1, \ldots, c_{n-1})$ is a walk in H where $c_i = c((x_i, x_{i+1}))$ with $i \in \{0, 1, \ldots, n-1\}$. Moreover, since H is transitive it follows that $(c_i, c_j) \in A_H$ with $i < j$. Let P be a path from x to y contained in γ. Since $(c_i, c_j) \in A_H$ with $i < j$, then the colors of the arcs of P also form a walk in H. Therefore, P is an H-path from x to y in D. ■

The above result allows us to establish the following theorem.

Theorem 6. *Let H be a transitive reflexive digraph and D an H-arc-coloured digraph. The subset K of V_D is a kernel by H-walks of D if and only if it is a kernel by H-paths of D.*

Proof. Let $K \subseteq V_D$ be a kernel by H-walks of D. Since K is independent by H-walks in D and every H-path is an H-walk in D then, in particular, K is an independent by H-paths set in D. To show that K is an absorbent by H-paths set in D, consider a vertex x in $V_D - K$. Since K is absorbent by H-walks in D, then there is a w in K such that there is an H-walk from x to w in D, say γ. By Theorem 5, γ contains an H-path from x to w in D. Therefore K is a kernel by H-paths of D.

Conversely, let $K \subseteq V_D$ be a kernel by H-paths of D. Every H-path is an H-walk in D, so we have that K is absorbent by H-walks. It follows from Theorem 5 that every H-walk contains an H-path; since there are no H-paths between vertices in K, it follows that there are neither H-walks between vertices of K, and thus, K is independent by H-walks. ■

4. Kernels by H-Paths and Kernels by H-Walks

As we said before, this section is devoted, despite the similarities, to exhibit the differences between reachability by H-paths and reachability by H-walks. In particular, we will show that adding a new vertex to a digraph D, and making it adjacent to and from every vertex in D, preserves the existence and non-existence of kernels by H-walks and kernels by H-paths. Thus, for some patterns H_1 and H_2, it is possible to construct infinitely many digraphs having a kernel by H_1-paths but no kernel by H_1-walks and also infinitely many digraphs having a kernel by H_2-walks but no kernel by H_2-paths.

Since the existence of an H-walk between two vertices does not guarantee the existence of an H-path between those vertices, and the concatenation of two H-paths is not always an H-walk, we claim that if D has a kernel by H-walks, then D not necessarily has a kernel by H-paths, as the example in Figure 2 shows. In Figure 2, we have that $\{w\}$ is a kernel by H_1-walks of D_1, because (x, y, z, z, y, w) is a spanning H_1-walk in D_1 that finishes in w. It is easy to check that D_1 has no kernel by H_1-paths (notice that every independent by H_1-paths set of D_1 has cardinality one).

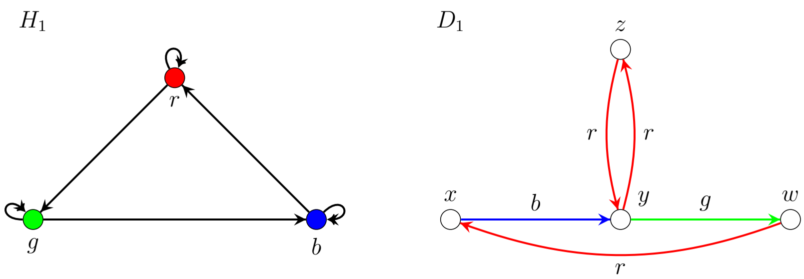

Figure 2. The set $\{w\}$ is a kernel by H_1-walks of D_1 and D_1 has no kernel by H_1-paths.

We also claim that if D has a kernel by H-paths, then D not necessarily has a kernel by H-walks as the example in Figure 3 shows. In Figure 3, we have that $\{u, v\}$ is an H_2-kernel of D_2. It is easy to see that D_2 has no kernel by H_2-walks (notice that every independent by H_2-walks set in D_2 has cardinality one because (u, y, z, y, v) is an H_2-walk in D_2).

To convince the reader that these are not isolated cases, we now exhibit infinite families with the same behaviour as the previous two examples.

Given two multidigraphs D_1 and D_2, we define the multidigraph $D_1 \bullet D_2$, which we call the *linear sum* of D_1 and D_2, to have vertex set $V_{D_1 \bullet D_2} = V_{D_1} \cup V_{D_2}$ (disjoint union) and arc set $A_{D_1 \bullet D_2} = A_{D_1} \cup A_{D_2} \cup (V_{D_1} \times V_{D_2})$.

Let H be a pattern, and let D_1 and D_2 be two H-arc-coloured digraphs with colourings c_1 and c_2, respectively. Consider the linear sum of D_1 and D_2. We define an H-arc-colouring c of $D_1 \bullet D_2$ as follows: $c : A_{D_1 \bullet D_2} \to V_H$ is such that

$$c(a) = \begin{cases} c_1(a) & \text{if } a \in A_{D_1}, \\ c_2(a) & \text{if } a \in A_{D_2}, \\ c_0 & \text{if } a = (u, v) \text{ with } u \in V_{D_1} \text{ and } v \in V_{D_2}, \text{ where } c_0 \text{ is in } V_H. \end{cases}$$

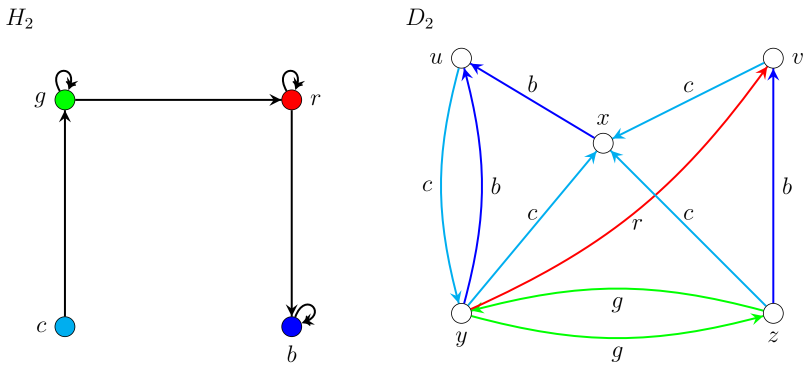

Figure 3. $\{u, v\}$ is a kernel by H-paths of D_2 and D_2 has no kernel by H_2-walks.

Based on the previous H-arc-colouring we obtain the following lemmas.

Lemma 7. *Let H be a pattern. If D_1 and D_2 are H-arc-coloured digraphs, then $D_1 \bullet D_2$ has a kernel by H-paths if and only if D_2 has kernel by H-paths.*

Proof. Suppose first that $D_1 \bullet D_2$ has a kernel by H-paths. We will show that D_2 has a kernel by H-paths. Let K be a kernel by H-paths of $D_1 \bullet D_2$. We will prove that K is a kernel by H-paths of D_2.

Since every vertex in V_{D_1} dominates every vertex in V_{D_2}, it is not hard to notice that $K \subseteq V_{D_2}$. On the other hand, by the definition of linear sum and its associated colouring c, it follows that any H-path in $D_1 \bullet D_2$ between two vertices x and y in V_{D_2}, is also an H-path in D_2. Hence, since K is a kernel by H-paths of $D_1 \bullet D_2$, then K is a kernel by H-paths of D_2.

Conversely, we will prove that a kernel by H-paths of D_2 is also a kernel by H-paths of $D_1 \bullet D_2$. Again, the definition of c guarantees that an H-path in D_2 is also an H-path in $D_1 \bullet D_2$. Also, by the definition of $D_1 \bullet D_2$, every vertex in V_{D_1} dominates every vertex in V_{D_2} and no vertex in V_{D_2} dominates a vertex in V_{D_1}. It follows that a kernel by H-paths in D_2 is also a kernel by H-paths in $D_1 \bullet D_2$. ■

The proof of our following lemma is analogous to the proof of Lemma 7, so we omit it.

Lemma 8. *Let H be a pattern. If D_1 and D_2 are H-arc-coloured digraphs, then $D_1 \bullet D_2$ has a kernel by H-walks if and only if D_2 has a kernel by H-walks.*

As noted above, using the example in Figure 3, Lemma 7 and the contrapositive of Lemma 8 we can recursively construct an infinite family of digraphs having a kernel by H-paths and no kernel by H-walks as follows. Let H_2 and D_2 be digraphs as in Figure 3. Define

- $D^1 = D_2$;
- $D^{j+1} = K_1 \bullet D^j$.

The resulting family of digraphs clearly has the desired properties.

Similarly, using the example in Figure 2, Lemma 8 and the contrapositive of Lemma 7, an infinite family of digraphs having a kernel by H-walks and no kernel by H-paths can be recursively defined as follows. Let H_1 and D_1 be digraphs as in Figure 2.

- $E^1 = D_1$;
- $E^{j+1} = K_1 \bullet E^j$.

Hence, the problem of determining the existence of a kernel by H-paths and the problem of determining the existence of a kernel by H-walks are indeed different problems.

5. Panchromatic Patterns by Paths

We now turn our attention to $\widetilde{\mathscr{B}}_3$. It follows from part 2 of Lemma 1 that $\widetilde{\mathscr{B}}_3$ is a hereditary class of digraphs, and hence, it can be characterized through a set of forbidden induced subdigraphs. In [8], such a characterization was given for panchromatic patterns (by walks) and, although it is based on the characterization of $\mathscr{B}_3$ found in [9] (which we mentioned earlier is flawed), a similar approach can be used to describe the structure of the panchromatic patterns by paths. The idea is to classify all the patterns on three vertices; knowing which of them are panchromatic patterns by paths gives us enough information to describe the general structure of patterns in this family. As a simple example, let us consider the reflexive digraphs $3K_1$ and $\overrightarrow{P_2} + K_1$, which are not in $\widetilde{\mathscr{B}}_3$. From the former, we deduce that the independence number of any digraph in $\widetilde{\mathscr{B}}_3$ is at most two, and thus, digraphs in $\widetilde{\mathscr{B}}_3$ have at most two connected components. From the latter, we obtain that if a digraph in $\widetilde{\mathscr{B}}_3$ has two connected components, both of them must be symmetric digraphs. So, each each pattern on three vertices not belonging to $\widetilde{\mathscr{B}}_3$ impose additional restrictions on the structure of digraphs in $\widetilde{\mathscr{B}}_3$. For a summary of the knowledge obtained by analyzing all the patterns on three vertices the impatient reader might jump directly to Section 7.

We begin our analysis with a simple observation relating kernels by H-paths with kernels by H-walks. It is a direct consequence of Theorem 6, so its proof will be omitted.

Proposition 9. *If H is a transitive reflexive digraph, then H is a panchromatic pattern by paths if and only if H is a panchromatic pattern* (*by walks*).

Arpin and Linek showed in [1] that every reflexive digraph with order one or two is a panchromatic pattern. These patterns are trivially transitive, so by

Proposition 9 we have that the panchromatic patterns with order one or two are also panchromatic patterns by paths.

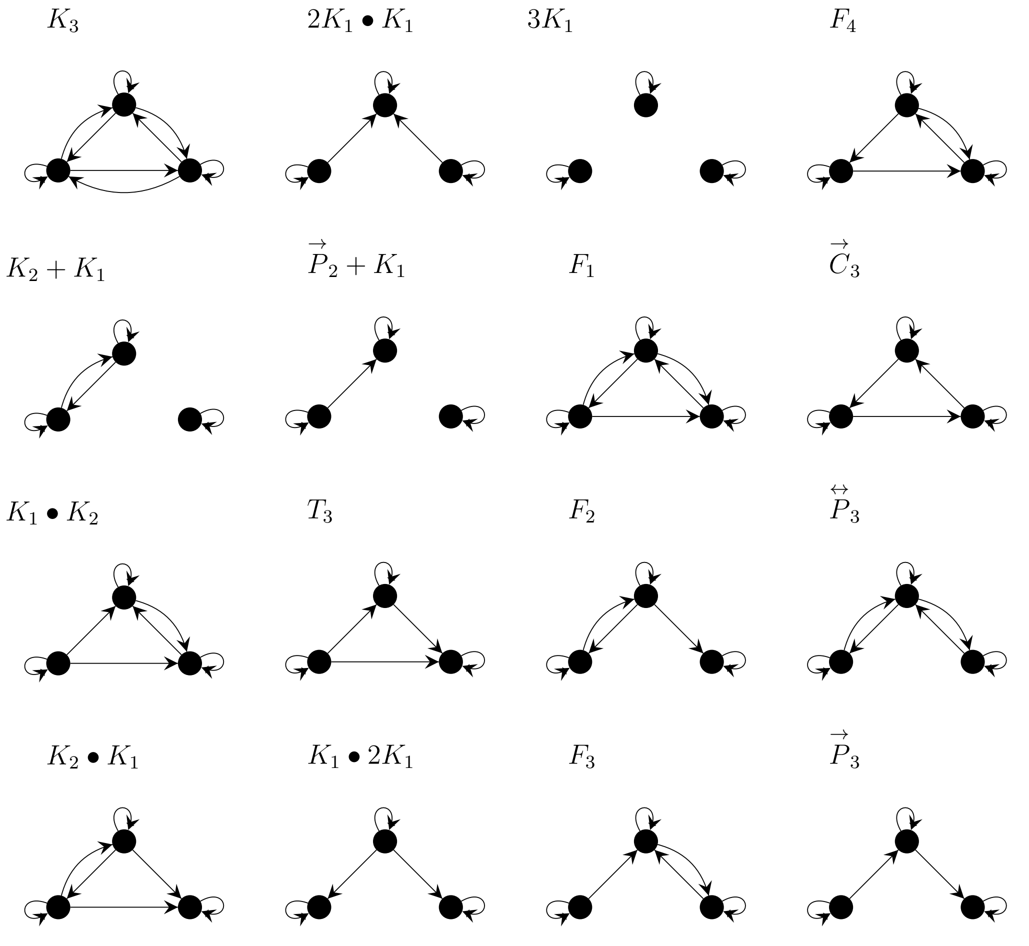

Figure 4. All non-isomorphic reflexive digraphs on 3 vertices.

Using Proposition 9 we can analyze nine of the patterns on three vertices depicted in Figure 4. In [1], Arpin and Linek proved that the patterns K_3, $K_1 \bullet K_2$, $K_2 \bullet K_1$ and $K_2 + K_1$ belong to $\mathscr{B}_3$, moreover, all of them are transitive patterns, hence, they also belong to $\widetilde{\mathscr{B}}_3$. Analogously, the patterns $2K_1 \bullet K_1$, $\vec{P}_2 + K_1$, T_3, $K_1 \bullet 2K_1$ and $3K_1$ do not belong to $\widetilde{\mathscr{B}}_3$.

The following result was proved by Arpin and Linek in [1]; although they stated it for panchromatic patterns, it is easy to notice that the same proof works for panchromatic patterns by paths.

Lemma 10. *Let H be a pattern. If H contains a walk W such that $W = (x_0, x_1, \ldots, x_k)$ and*

1. *for all x_j, $0 \leq j \leq k-1$, there is a colour $c_j \in V_H$ such that $(x_j, c_j) \notin A_H$,*
2. *$(x_k, x_0) \notin A_H$,*

then $H \notin \widetilde{\mathscr{B}_3}$.

Note that every pattern of $\overrightarrow{P}_3$, $\overrightarrow{C}_3$, F_3 and F_4 has an asymmetric arc (x_0, x_1) with third vertex c_0 such that (x_0, c_0) is not an arc. By Lemma 10, $\overrightarrow{P}_3$, $\overrightarrow{C}_3$, F_3 and F_4 are not panchromatic patterns.

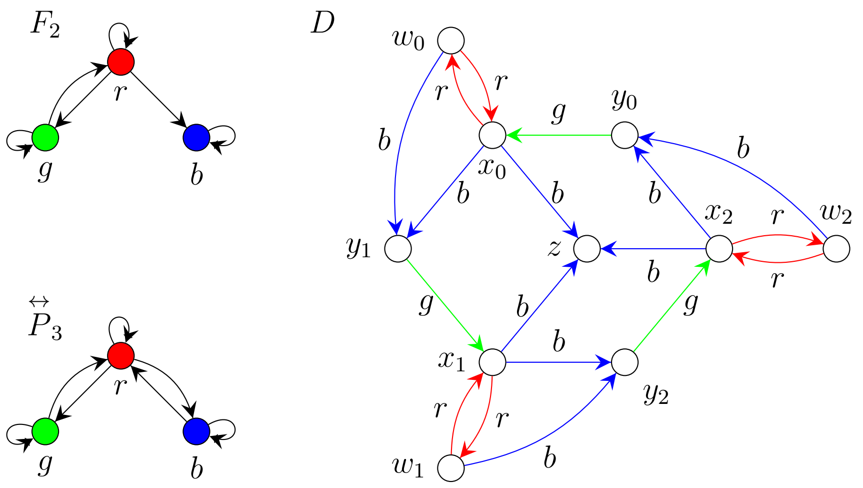

Figure 5. Patterns $\overleftrightarrow{P}_3$ and F_2 with a digraph D with no kernel by $\overleftrightarrow{P}_3$-paths and no kernel by F_2-paths.

We will show that $\overleftrightarrow{P}_3$ and F_2 are not panchromatic patterns by paths. Consider $\overleftrightarrow{P}_3$, F_2 and D as in the Figure 5. First, we will show that the F_2-arc-coloured digraph D has no kernel by F_2-paths. Proceeding by contradiction, suppose that D has a kernel by F_2-paths, K. Since z is a sink in D, then z is an element of K. Note that y_i does not reach z by F_2-paths, with $i \in \{0,1,2\}$, thus $K \neq \{z\}$. Moreover, the paths (w_i, x_i, z), with $i \in \{0,1,2\}$, are F_2-paths in D, thus w_i and x_i are not in K. On the other hand, since (y_i, x_i, w_i, y_{i+1}) are F_2-paths, only one element of $\{y_0, y_1, y_2\}$ is in K. Nevertheless, if y_i is in K, that is $K = \{z, y_i\}$, then y_{i+1} cannot reach any vertex of K, which is a contradiction. Therefore, D has no kernel by F_2 paths.

Analogously, if we consider D as a $\overleftrightarrow{P}_3$-arc-coloured digraph and exchanging F_2 by $\overleftrightarrow{P}_3$ we obtain that D has no kernel by $\overleftrightarrow{P}_3$-paths. We can conclude that $\overleftrightarrow{P}_3$ and F_2 are not panchromatic patterns by paths.

It is important to note that, in Figure 5, $\{z\}$ is a kernel by F_2-walks and a kernel by $\overset{\leftrightarrow}{P}_3$-walks in D. Hence, determine whether F_2 and $\overset{\leftrightarrow}{P}_3$ are panchromatic patterns (by walks) remains an open problem.

So far, we have dealt with fifteen of the sixteen reflexive patterns on three vertices. The remaining three-vertex pattern is harder to deal with. The following section is devoted to pattern F_1.

6. The Pattern F_1

So far, for every pattern on three vertices we know whether it is a panchromatic pattern by paths, except for the pattern F_1. Arpin and Linek proved in [1] that F_1 is a panchromatic pattern (by walks) using the concept of closure of a coloured digraph. Unfortunately, their technique is neither directly applicable in the path case, nor seems to be modifiable to suit it. One interesting fact that should be pointed out from Arpin and Linek's technique is that they construct a multidigraph (parallel arcs are present) having the same reachability of a given F_1-arc-coloured digraph, and where some substructures always exist. From here, one may ask whether there might be an F_1-arc-coloured multidigraph without a kernel by F_1-paths, while every F_1-arc-coloured digraph (without parallel arcs) has a kernel by F_1-paths. If so, then looking for a counterexample would imply looking at multidigraphs, which represents a broader search space than just considering digraphs. Fortunately, it is not the case; the main result of this section implies that we can only consider digraphs.

Theorem 11. *If every F_1-arc-coloured digraph has a kernel by F_1-paths, then every F_1-arc-coloured multidigraph has a kernel by F_1-paths.*

Proof. We will prove that for every F_1-arc-coloured multidigraph D there is an F_1-arc-coloured digraph $\widehat{D}$ such that D has a kernel by F_1-paths if and only if $\widehat{D}$ has a kernel by F_1-paths. Let be r, g and b the vertices of F_1, and assume that the only missing arc in F_1 is (b, g).

Let $A_D[x, y] := \{f \in A_H : \ f$ has head x and tail $y\}$. We construct $\widehat{D}$, an F_1-arc-coloured digraph obtained from D through the following modifications. Whenever there are vertices u and v in V_D with $|A_D[u, v]| \geq 2$, then

1. If there is $f \in A_D[u, v]$ with colour r, we replace $A_D[u, v]$ by only one arc from u to v, e_{uv}, coloured r.
2. If $A_D[u, v] = \{f, f'\}$ is such that f is coloured g and f' is coloured b, then we create new vertices $s_{u,v}$ and $t_{u,v}$, add them to V_D, and create new arcs (u, v), $(u, s_{u,v})$, $(s_{u,v}, t_{u,v})$ and $(s_{u,v}, v)$, with colours g, b, g and b, respectively, add them to A_D and delete f and f' from A_D.

Let S be the set of all new vertices $s_{u,v}$ in $\widehat{D}$ and T the set of all new vertices $t_{u,v}$ in $\widehat{D}$.

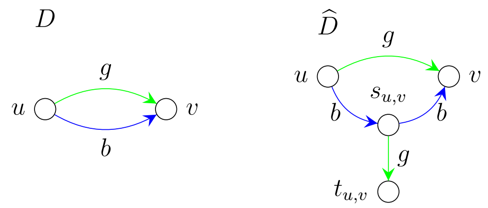

Figure 6. The construction of $\widehat{D}$ in Theorem 11.

Claim 1. *Let $x, y \in V_D$. There is an F_1-path from x to y in D if and only if there is an F_1-path from x to y in $\widehat{D}$.*

Proof. Let W be the F_1-path W defined by $W = (x = x_0, \ldots, x_n = y)$ in D. Construct $\widehat{W}$, an F_1-path from x to y in $\widehat{D}$, by performing the following modifications on W. For each $i \in \{0, 1, \ldots, n-1\}$ such that $|A_D[x_i, x_{i+1}]| \geq 2$, we replace the arc from x_i to x_{i+1} used in W by

1. $e_{x_i x_{i+1}}$ coloured r, if there is $f \in A_D[x_i, x_{i+1}]$ coloured r,
2. (x_i, x_{i+1}) coloured g, if there is no $f \in A_D[x_i, x_{i+1}]$ coloured r and the arc from x_i to x_{i+1} in W is coloured g,
3. $(x_i, s_{x_i, x_{i+1}}, x_{i+1})$ the monochromatic path coloured b, if there is no $f \in A_D[x_i, x_{i+1}]$ coloured r and the arc from x_i to x_{i+1} in W is coloured b.

Since W is an F_1-path in D, it is enough to check that for each arc substitution in W, the sequence of colours of the new arcs are still a walk in F_1. For each substitution of type 1, $(x_{i-1}, x_i, x_{i+1}, x_{i+2})$ is an F_1-path in $\widehat{D}$ because $N^+(r) = N^-(r) = V(F_1)$ and (x_i, x_{i+1}) is coloured r in $\widehat{W}$. For each substitution of type 2, $(x_{i-1}, x_i, x_{i+1}, x_{i+2})$ is an F_1-path in $\widehat{D}$ because the arc from x_i to x_{i+1} in W is coloured g and the new arc (x_i, x_{i+1}) in $\widehat{W}$ is coloured g. For each substitution of type 3, $(x_{i-1}, x_i, s_{x_i, x_{i+1}}, x_{i+1}, x_{i+2})$ is an F_1-path because the arc from x_i to x_{i+1} in W is coloured b and the path $(x_i, s_{x_i, x_{i+1}}, x_{i+1})$ is monochromatic with colour b in $\widehat{D}$. Hence, $\widehat{W}$ is an F_1-path from x to y in $\widehat{D}$.

Conversely, let $\widehat{W} = (x = x_0, x_1, \ldots, x_n = y)$ be an F_1-path from x to y in $\widehat{D}$ and construct an F_1-path W from x to y in D by modifying $\widehat{W}$ in the following way. For each $i \in \{1, \ldots, n-1\}$ such that there is $x_i \in S$, we replace the monochromatic path $(x_{i-1}, s_{x_{i-1}, x_{i+1}} = x_i, x_{i+1})$ coloured b by the arc from x_{i-1} to x_{i+1} coloured b in D, and for each $i \in \{0, 1, \ldots, n-1\}$ such that $|A_D[x_i, x_{i+1}]| \geq 2$, different from those obtained by the first modifications, we replace the arc from x_i to x_{i+1} in $\widehat{W}$ by

1. $f \in A_D[x_i, x_{i+1}]$ coloured r if the arc x_i to x_{i+1} in $\widehat{W}$ is $e_{x_i x_{i+1}}$ coloured r,

2. the arc from x_i to x_{i+1} coloured g if there is no $f \in A_D[x_i, x_{i+1}]$ coloured r and the arc fromx_i to x_{i+1} in $\widehat{W}$ is coloured g.

Note that the above modifications change a monochromatic path of colour b into an arc coloured b and replace arcs by other arcs with the same colour. Therefore, it follows that W is an F_1-path from x to y in D. This ends the proof of Claim 1. □

Let K be a kernel by F_1-paths in D. We consider $\widehat{K} = K \cup T$, and we will prove that $\widehat{K}$ is a kernel by F_1-paths in $\widehat{D}$. Notice that $\widehat{K} \cap S = \emptyset$ and T is reached by F_1-paths only by the vertices in S. Also, each vertex in T is a sink in $\widehat{D}$, so there are no F_1-paths that start in T. By the independence by F_1-paths of K and Claim 1, there is no F_1-path in $\widehat{D}$ between vertices of K. Since $S \cap K = \emptyset$ and the observations at the beginning of this paragraph, there are neither F_1-paths in $\widehat{D}$ between vertices of K and vertices of T, nor between vertices of T. Hence $\widehat{K}$ is independent by F_1-paths in $\widehat{D}$.

Let x be a vertex in $V_{\widehat{D}} - \widehat{K}$. If $x \in S$, then there is $t \in T$ such that $(x, t) \in A_{\widehat{D}}$. If $x \in V_D - K$, then there is $y \in K$ such that there exist an F_1-path from x to y in D, by Claim 1 there is an F_1-path from x to y in $\widehat{D}$. Hence, $\widehat{K}$ is absorbent by F_1-paths in $\widehat{D}$, and therefore $\widehat{K}$ is a kernel by F_1-paths in $\widehat{D}$.

Let $\widetilde{K}$ be a kernel by F_1-paths in $\widehat{D}$. Consider $K = \widetilde{K} \cap V_D$. We will prove that K is a kernel by F_1-paths in D. Since every vertex in T is a sink in $\widehat{D}$, $T \subset \widetilde{K}$ and consequently $S \cap \widetilde{K} = \emptyset$. Moreover, the only vertices that reach T by F_1-paths in $\widehat{D}$ are the vertices in S. Since $\widetilde{K}$ is independent by F_1-paths in $\widehat{D}$, then by Claim 1, $K = \widetilde{K} \cap V_D$ is an independent by F_1-paths set in D. Hence, if $x \in V_D - K$, then by the absorbency by F_1-paths of $\widetilde{K}$ in $\widehat{D}$, we have that K is an absorbent by F_1-paths set in D. Therefore K is a kernel by F_1-paths in D.

The contrapositive of the statement of the theorem now follows directly using the previous construction. ■

The following corollary is immediate.

Corollary 12. *If every F_1-arc-coloured digraph (without parallel arcs) has a kernel by F_1-paths, then F_1 is a panchromatic pattern by paths.*

7. Conclusions

To round up our comparison between reachability by H-paths and reachability by H-walks, let us recall that Arpin and Linek proved in [1] that there is a pattern in $\mathscr{B}_2$ which is not in $\mathscr{B}_3$. As we showed, F_2 is not a panchromatic pattern by paths, and it is easy to see that F_2^c has no odd cycles, then $F_2 \in \widetilde{\mathscr{B}}_2 - \widetilde{\mathscr{B}}_3$. Therefore $\widetilde{\mathscr{B}}_3 \subset \widetilde{\mathscr{B}}_2$.

To reach a conclusion regarding $\widetilde{\mathscr{B}}_3$, let $\mathcal{F}$ be the family of digraphs having $3K_1$, $\overrightarrow{P}_2 + K_1$, $\overrightarrow{P}_3$, $2K_1 \bullet K_1$, $K_1 \bullet 2K_1$, F_2, F_3, F_4, $\overleftrightarrow{P}_3$, $\overrightarrow{C}_3$, and T_3 as elements (see Figure 4). Also, let $\mathcal{F}'$ be defined as $\mathcal{F}' = \mathcal{F} \cup \{F_1\}$. By a *strong clique* in a digraph D we mean a subset S of V_D such that every pair of distinct vertices is adjacent by a symmetric arc.

The following results describe the structure of $\mathcal{F}$-free and $\mathcal{F}'$-free digraphs, respectively.

Lemma 13 [8]. *A digraph D is $\mathcal{F}$-free if and only if V_D admits a partition (V_1, V_2) such that V_i is a strong clique for $i \in \{1, 2\}$, and either there are no arcs between vertices in V_1 and vertices in V_2, or every vertex in V_1 dominates every vertex in V_2 and vertices in V_2 may dominate vertices in V_1.*

Although we provide a complete proof for our following lemma, the interested reader might refer to [11] for a deeper insight of this type of results.

Lemma 14. *A digraph D is $\mathcal{F}'$-free if and only if V_D admits a partition (V_1, V_2) such that V_i is a strong clique for $i \in \{1, 2\}$, and either there are no arcs between vertices in V_1 and vertices in V_2, or every vertex in V_1 dominates every vertex in V_2 and no vertex in V_2 dominates a vertex in V_1.*

Proof. Let D be an $\mathcal{F}'$-free digraph. Notice that $\mathcal{F}'$ contains every possible orientation of P_3 and also an independent set on three vertices, so the underlying graph of D is either a complete graph or a disjoint union of two complete graphs. In the latter case, since $\overrightarrow{P}_2 + K_1 \in \mathcal{F}'$, it must be the case that V_D admits a partition in two strong cliques, without further arcs. In the former case, since $F_1, \overrightarrow{C}_3, T_3 \in \mathcal{F}'$, we have that V_D admits a partition into two strong cliques V_1 and V_2 (the subdigraph of D induced by its symmetric arcs has independence number 2 and is P_3-free). Also, since $F_1, F_4 \in \mathcal{F}'$, all the asymmetric arcs of D have, without loss of generality, tail in V_1 and head in V_2. Thus, the desired result follows.

The remaining implication is immediate. ∎

By now, except for F_1, we have a complete classification of digraphs of order 3; those digraphs not in $\mathcal{F}$ are panchromatic patterns by paths, and digraphs in $\mathcal{F}$ are not. Although we do not know whether F_1 is a panchromatic pattern by paths, we have enough information to completely describe what would happen if $F_1 \notin \widetilde{\mathscr{B}}_3$, and provide some useful properties of panchromatic patterns by paths if the contrary occurs.

Theorem 15. *Let H be a pattern. If $F_1 \notin \widetilde{\mathscr{B}}_3$, then $H \in \widetilde{\mathscr{B}}_3$ if and only if V_H admits a partition (V_1, V_2) such that V_i is a strong clique for $i \in \{1, 2\}$, and either there are no arcs between vertices in V_1 and vertices in V_2, or every vertex in V_1 dominates every vertex in V_2 and no vertex in V_2 dominates a vertex in V_1.*

Proof. If H is a panchromatic pattern by paths, then it is $\mathcal{F}'$-free, and the result follows from Lemma 14. If H has a partition with the stated properties, then it can be contracted to either $2K_1$ or $\overrightarrow{P}_2$. Since both are panchromatic patterns by paths, it follows from Lemma 1 that H is a panchromatic pattern by paths. ■

Theorem 16. *Let H be a pattern and suppose that $F_1 \in \widetilde{\mathscr{B}}_3$. If $H \in \widetilde{\mathscr{B}}_3$ then V_H admits a partition (V_1, V_2) such that V_i is a strong clique for $i \in \{1, 2\}$, and either there are no arcs between vertices in V_1 and vertices in V_2, or every vertex in V_1 dominates every vertex in V_2 and vertices in V_2 may dominate vertices in V_1.*

Proof. It is immediate from the fact that a panchromatic pattern by paths is $\mathcal{F}$-free, and thus, has the required structure by Lemma 13. ■

We would like to point one open problem regarding panchromatic patterns by paths.

Problem 17. Determine whether F_1 is a panchromatic pattern by paths.

If the answer to Problem 17 is negative, this would settle the characterization of panchromatic patterns by paths. A positive answer would lead to ask whether there are other minimal obstructions for a digraph to be a panchromatic pattern by paths. Any such minimal obstruction should be $\mathcal{F}$-free, and thus would have the structure described in Theorem 16; if no such obstruction exists, the converse of Theorem 16 would be true, resulting in a characterization.

In [4], dynamic H-walks and dynamic H-paths were introduced as generalization of H-walks and H-paths, respectively. Naturally, generalizations of $\mathscr{B}_i$ and $\widetilde{\mathscr{B}_i}$ were defined. In the same work, find $2K_1$-arc-coloured multidigraph with no kernel by dynamic $2K_1$-paths, it is indicated as an open problem. Consider the $\overleftrightarrow{P}_3$-arc-coloured digraph D of Figure 5; if we replace every arc with colour r of D by two parallel arcs, one with colour g and the other with colour b, then we obtain $2K_1$-arc-coloured multidigraph D' with no kernel by dynamic $2K_1$-paths.

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