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DEGREE SUM CONDITION FOR VERTEX-DISJOINT 5-CYCLES

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Abstract

Let n and k be two integers and G a graph with $n = 5k$ vertices. Wang proved that if $\delta(G) \geq 3k$, then G contains k vertex disjoint cycles of length 5. In 2018, Chiba and Yamashita asked whether the degree condition can be replaced by degree sum condition. In this paper, we give a positive answer to this question.

Keywords: degree sum conditions; vertex disjoint 5-cycles.

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1. Introduction

Let G be a graph. We denote by $V(G)$ and $E(G)$ the vertex set and edge set of G, respectively, and call $|V(G)|$ the *order* of G. An edge joining two vertices x and y is denoted by xy. A class of subgraphs of G is *vertex disjoint*, or simply *disjoint*, if no two subgraphs in the class have a common vertex. The *length* of a cycle C (or a path P) is the number of edges on C (or P). For $X \subseteq V(G)$, the *neighbour set* of X, denoted by $N_G(X)$ or $N(X)$ with no confusion, is the set of the vertices not in X but adjacent to at least one vertex in X. In particular, if X consists of a single vertex x, then we call $|N_G(\{x\})|$ the *degree* of x and denote it by $d_G(x)$. We use $\delta(G)$ to denote the minimum degree of vertices in G. Define

$$\sigma_2(G) = \min\{d(x) + d(y) : x, y \in V(G), x \neq y, xy \notin E(G)\}.$$

The degree condition for the existence of cycle(s) with specified length(s) is one of the most elementary concerns in graph theory. A classic result should be

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the one given by Dirac [4] in 1952, which says that every graph with n vertices and minimum degree at least $n/2$ has a Hamilton cycle. Since then, this result has been generalized to various forms in terms of degree condition or degree sum condition. Corrádi and Hajnal [2] considered the maximum number of disjoint cycles in a graph and proved that if a graph G has at least $3k$ vertices and minimum degree at least $2k$, then G contains k disjoint cycles. In particular, when G has exactly $3k$ vertices, then G contains k disjoint triangles. Erdős and Faudree [6] conjectured that if G has $4k$ vertices and minimum degree $\delta(G) \geq 2k$, then G contains k vertex disjoint cycles of length 4. This conjecture was later confirmed by Wang [7]. In general, El-Zahar posed the following conjecture.

Conjecture 1 (El-Zahar, [5]). *Let G be a graph of order $n = n_1 + n_2 + \cdots + n_k$ with $n_i \geq 3$ for each $i \in \{1, 2, \ldots, k\}$. If $\delta(G) \geq \lceil \frac{n_1}{2} \rceil + \lceil \frac{n_2}{2} \rceil + \cdots + \lceil \frac{n_k}{2} \rceil$, then G has k disjoint cycles of length $n_1, n_2, \ldots, n_k$, where, for a real number r, $\lceil r \rceil$ is the least integer not less than r.*

In the same paper, El-Zahar also proved that the above conjecture is true for $k = 2$. In [1], Abbasi confirmed the conjecture for sufficiently large graphs by using the regularity lemma. Wang [8] proved the conjecture for the case when $n = 5k$ and $n_i = 5$ $(1 \leq i \leq k)$ as follows.

Theorem 2 (Wang, [8]). *Let k be a positive integer and G a graph of order $n = 5k$. If $\delta(G) \geq (n+k)/2$, then G contains k vertex disjoint cycles of length* 5.

In 2018, Chiba and Yamashita [3] posed the following question.

Question (Chiba and Yamashita, [3]). Let k be a positive integer and G a graph of order $n = 5k$. Is it true that if $\sigma_2(G) \geq n+k$, then G contains k vertex disjoint cycles of length 5?

In this paper, we give a positive answer to this question.

Theorem 3. *Let k be a positive integer and G a graph of order $n = 5k$. If $\sigma_2(G) \geq n + k$, then G contains k vertex disjoint cycles of length* 5.

2. Preliminaries

For $X, Y \subseteq V(G)$, we denote by $N(X) \,|\, Y$ the neighbour set of X restricted on Y, i.e., $N(X) \,|\, Y = N(X) \cap Y$. For simplicity, if X consists of a single vertex x, we simply write $N(\{x\}) \,|\, Y$ as $N(x) \,|\, Y$, and if H is a subgraph of G, then we simply write $N(X) \,|\, V(H)$ as $N(X) \,|\, H$. A *chord* of a cycle C is an edge not on C that joins two vertices of C, and the number of chords of C is denoted by $\tau(C)$. Further, for $x \in V(C)$, we use $\tau(x, C)$ to denote the number of chords of C that are incident with x. For a graph H, we say that

G *contains* H if G has a subgraph isomorphic to H and denote by $G \supseteq H$. For two graphs G and G', we denote by $G \uplus G'$ the vertex disjoint union of G and G'. For simplicity, we write the vertex disjoint union of k copies of a graph H as kH. For two disjoint vertex subsets, or subgraphs, A and B of G, we define $E(A, B)$ to be the set of all the edges of G between A and B and denote $e(A, B) = |E(A, B)|$. For a vertex subset X of G, we denote by $[X]$ the subgraph of G induced by X. Further, for subgraphs $G_1, G_2, \ldots, G_t$ of G, we write $[V(G_1) \cup V(G_2) \cup \cdots \cup V(G_t)] = [G_1, G_2, \ldots, G_t]$. For $t+1$ disjoint subgraphs $H, L_1, \ldots, L_t$ of G, we call $\{H, L_1, \ldots, L_t\}$ a *family* of G if $L_i \cong C_5$ for all $i \in \{1, 2, \ldots, t\}$. In particular, we call a family $\{H, L_1, \ldots, L_t\}$ *optimal* if $\sum_{i=1}^{t} \tau(L_i') \leq \sum_{i=1}^{t} \tau(L_i)$ for any family $\{H', L_1', \ldots, L_t'\}$ of $[H, L_1, \ldots, L_t]$ with $H' \cong H$ and $L_i' \cong C_5$ for all $i \in \{1, 2, \ldots, t\}$.

As usual, for $i \geq 3$, we denote by K_i, C_i and P_i the complete graph, cycle and path of order i, respectively. Following the notations in [8], let $B, F, F_1, F_2, F_3, F_4, F_5, K_4^+$ be the graphs as illustrated in Figure 1. Let L be a 5-cycle, $u \in V(L)$ and $x \in V(G) \setminus V(L)$. We write $x \to (L, u)$ if $[L - u + x] \supseteq C_5$. In particular, if $x \to (L, u)$ for all $u \in V(L)$, then we write $x \to L$. Further, for $\{v_1, v_2\} \subseteq V(G)$, we write $x \to (L, u; \{v_1, v_2\})$ if $x \to (L, u)$ and u is adjacent to both v_1 and v_2.

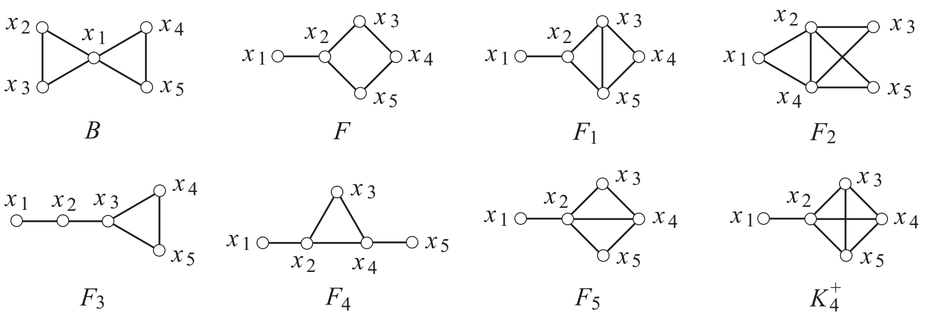

Figure 1. The subgraphs $B, F, F_1, F_2, F_3, F_4, F_5, K_4^+$.

To prove the main theorem, we introduce the following lemmas.

Lemma 4 [8]**.** *For a graph G, the following statements hold.*

(a) *Let B_1 and B_2 be two disjoint subgraphs of G with $B_1 \cong B_2 \cong B$ and R the set of the four vertices of degree 2 in B_1. If $e(R, B_2) \geq 13$, then $[B_1, B_2] \supseteq 2C_5$ or $[B_1, B_2] \supseteq B \uplus C_5$.*

(b) *Let D and L be two disjoint subgraphs of G with $D \cong B$ and $L \cong C_5$ and x the unique vertex of degree 4 in D. If $e(D - x, L) \geq 13$, then $[D, L] \supseteq 2C_5$.*

(c) *Let D and L be two disjoint subgraphs of G with $D \cong F_2$ and $L \cong C_5$ and R the set of the three vertices of degree 2 in D. If $e(R, L) \geq 10$, then*

$[D, L] \supseteq F_1 \uplus C_5$.

(d) *Let P and L be two disjoint subgraphs of G with $P \cong P_5$ and $L \cong C_5$. If $e(P, L) \geq 16$, $[P, L] \not\supseteq 2C_5$ and $\{P, L\}$ is optimal, then $[P, L] \supseteq F \uplus C_5$.*

(e) *Let $R \subseteq V(G)$ and L be a 5-cycle in $G - R$. If $|R| = 4$ and $e(R, L) \geq 13$, then $x \to (L, y; \{x_1, x_2\})$ for some $y \in V(L), x \in R$ and $\{x_1, x_2\} \subseteq R \setminus \{x\}$; or there are vertex labellings $R = \{x_1, x_2, x_3, x_4\}$ and $L = y_1y_2y_3y_4y_5y_1$ such that $N(x_1) \,|\, L = N(x_2) \,|\, L = \{y_1, y_2, y_3, y_4\}$, $N(x_3) \,|\, L = \{y_1, y_4, y_5\}$ and $N(x_4) \,|\, L = \{y_1, y_4\}$.*

Throughout the following, when we speak of a subgraph isomorphic to one of that in Figure 1, we always assume that its vertices are labelled as indicated in the figure. Further, for two vertex subsets $\{x_1, x_2, \ldots, x_k\}$ and $\{y_1, y_2, \ldots, y_h\}$, we write $e(\{x_1, x_2, \ldots, x_k\}, \{y_1, y_2, \ldots, y_h\})$ by $e(x_1x_2 \cdots x_k, y_1y_2 \cdots y_h)$ for simplicity.

Lemma 5 [8]**.** *For a graph G, the following statements hold.*

(a) *Let D and L be two disjoint subgraphs of G with $D \cong F$ and $L \cong C_5$. If $\{D, L\}$ is optimal and $e(D, L) \geq 16$, then $[D, L]$ contains one of $F_1 \uplus C_5$, $F_2 \uplus C_5$, $B \uplus C_5$ and $2C_5$; or the cycle L has a labelling $L = y_1y_2y_3y_4y_5y_1$ satisfying the following property*
$\mathbf{P}_1$: *$e(x_1, L) = 0$, $e(x_2x_4, L) = 10$, $N(x_3) \,|\, L = N(x_5) \,|\, L = \{y_1, y_2, y_4\}$, $\tau(L) = 4$ and $y_3y_5 \notin E(G)$.*

(b) *Let D, L and L' be three disjoint subgraphs of G with $D \cong F$, $L \cong L' \cong C_5$ and $L = y_1y_2y_3y_4y_5y_1$. If D and L satisfy the property $\mathbf{P}_1$ and $e(x_1x_3y_3y_5, L') \geq 13$, then $[D, L, L']$ contains either $F_1 \uplus 2C_5$ or $3C_5$.*

Lemma 6 [8]**.** *For a graph G, the following statements hold.*

(a) *Let D and L be two disjoint subgraphs of G with $D \cong F_1$ and $L \cong C_5$. If $\{D, L\}$ is optimal and $e(D, L) \geq 16$, then $[D, L]$ contains one of $K_4^+ \uplus C_5$, $K_4^+ \uplus B$, $B \uplus C_5$ and $2C_5$; or L has a labelling $L = y_1y_2y_3y_4y_5y_1$ satisfying the following property*
$\mathbf{P}_2$: *$e(x_1, L) = 0$, $e(y_1y_2y_4, D - x_1) = 12$, $N(y_3) \,|\, D = N(y_5) \,|\, D = \{x_3, x_5\}$, $\tau(L) = 4$ and $y_3y_5 \notin E$.*

(b) *Let D, L and L' be three disjoint subgraphs of G with $D \cong F_1$, $L \cong L' \cong C_5$ and $L = y_1y_2y_3y_4y_5y_1$. If D and L satisfy $\mathbf{P}_2$, $\{D, L, L'\}$ is optimal and $e(x_1x_4y_3y_5, L') \geq 13$, then $[D, L, L']$ contains either $K_4^+ \uplus 2C_5$ or $3C_5$.*

Lemma 7. *Let D and L be two disjoint subgraphs of G with $D \cong F_3$ and $L \cong C_5$. If $e(D - x_3, L) \geq 13$, then $[D, L]$ contains either $2C_5$ or $F \uplus C_5$ or $B \uplus C_5$ or $F_4 \uplus C_5$.*

Proof. Let $L = y_1y_2y_3y_4y_5y_1$ and $R = V(D)\backslash\{x_3\}$. Suppose to the contrary that $[D, L]$ does not contain any of $2C_5$, $F \uplus C_5$, $B \uplus C_5$ and $F_4 \uplus C_5$. Suppose $e(x_4x_5, L) \geq 8$. Without loss of generality, assume that $e(x_4, L) \geq e(x_5, L)$. If $e(x_4, L) = 5$, then since $e(x_1x_2x_5, L) \geq 8$, L has a vertex y_j such that $x_ty_j, x_5y_j \in E(G)$ for some $t \in \{1, 2\}$ and so $x_4 \to (L, y_j; \{x_5, x_t\})$. Thus $[D, L]$ contains $2C_5$ or $F \uplus C_5$, a contradiction. Then $e(x_4x_5, L) = 8$ and $e(x_4, L) = e(x_5, L) = 4$, say $N(x_4)\,|\,L = \{y_1, y_2, y_3, y_4\}$. If $N(x_5)\,|\,L = \{y_1, y_2, y_3, y_5\}$ or $N(x_5)\,|\,L = \{y_1, y_2, y_4, y_5\}$ or $N(x_5)\,|\,L = \{y_1, y_3, y_4, y_5\}$ or $N(x_5)\,|\,L = \{y_2, y_3, y_4, y_5\}$, then since $e(x_1x_2, L) \geq 5$, L has a vertex y_j such that $x_r \to (L, y_j; \{x_s, x_t\})$ for some $t \in \{1, 2\}$ and $\{r, s\} = \{4, 5\}$. Hence $[D, L]$ contains $2C_5$ or $F \uplus C_5$, a contradiction. If $N(x_5)\,|\,L = \{y_1, y_2, y_3, y_4\}$, then $e(x_1x_2, y_2y_3) = 0$ for otherwise $[D, L]$ contains $2C_5$ or $F \uplus C_5$. Hence $e(x_1x_2, y_1y_4y_5) \geq 5$ and so $[x_1, x_2, y_1, y_4, y_5] \supseteq C_5$ and $[x_3, x_4, x_5, y_2, y_3] \supseteq C_5$, again a contradiction. Therefore $e(x_4x_5, L) \leq 7$ and hence $e(x_1x_2, L) \geq 6$. If $e(x_1, L) = 1$, say $x_1y_1 \in E(G)$, then $e(y_2y_3, x_4x_5) \leq 1$ (otherwise $[y_1, y_4, y_5, x_1, x_2] \supseteq C_5$ and $[x_3, x_4, x_5, y_2, y_3]$ contains C_5 or F or B). Similarly, $e(y_4y_5, x_4x_5) \leq 1$. Hence $e(R, L) \leq 10$, a contradiction. Now suppose $e(x_1, L) \geq 4$, say $N(x_1)\,|\,L \supseteq \{y_1, y_2, y_3, y_4\}$. If $x_1 \to (L, y_i)$ for some $i \in \{1, 2, 3, 4, 5\}$, then $e(y_i, x_4x_5) = 0$ (otherwise $[L - y_i + x_1] \supseteq C_5$ and $[D + y_i - x_1] \supseteq F_4$). This implies that $N(x_1)\,|\,L = \{y_1, y_2, y_3, y_4\}$, $e(x_4x_5, y_1y_4) = 4$, $e(x_2, L) = 5$. Therefore, $[y_2, y_3, x_1, x_2, x_3] \supseteq F$ and $[y_1, y_4, y_5, x_4, x_5] \supseteq C_5$, again a contradiction.

Case 1. $e(x_1, L) = 2$. In this case, $e(x_2, L) \geq 4$ as $e(x_1x_2, L) \geq 6$. First suppose that $N(x_1)\,|\,L = \{y_i, y_{i+1}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1)\,|\,L = \{y_1, y_2\}$, where and herein after, the subscript of a vertex y_i on C_5 means in i modulo 5 if $i > 5$. If $x_2y_4 \in E(G)$, then $y_1y_5y_4x_2x_1y_1 \cong y_2y_3y_4x_2x_1y_2 \cong C_5$. Hence, $e(y_1y_5, x_4x_5) \leq 1$ and $e(y_2y_3, x_4x_5) \leq 1$, for otherwise, $[y_2, y_3, x_3, x_4, x_5]$ and $[y_1, y_5, x_3, x_4, x_5]$ would contain C_5 or F or B. This means that $e(R, L) \leq 11$, a contradiction. We now assume $x_2y_4 \notin E(G)$ and, hence $N(x_2)\,|\,L = \{y_1, y_2, y_3, y_5\}$. We claim that $e(y_3y_4, x_4x_5) \leq 1$. If not, then $[y_1, y_2, y_5, x_1, x_2] \supseteq C_5$, $[y_3, y_4, x_3, x_4, x_5]$ contains C_5 or F or B, a contradiction. Similarly, $e(y_4y_5, x_4x_5) \leq 1$. Hence $e(R, L) \leq 12$, a contradiction. Next, suppose that $N(x_1)\,|\,L = \{y_i, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1)\,|\,L = \{y_1, y_3\}$. As $x_1 \to (L, y_2)$, $e(y_2, x_4x_5) = 0$. Recall that $e(x_2, L) \geq 4$. We have $[y_1, y_2, y_3, x_1, x_2] \supseteq C_5$. Therefore, $e(y_4y_5, x_4x_5) \leq 1$, for otherwise $[y_4, y_5, x_3, x_4, x_5]$ would contain C_5 or F or B. It follows that $e(R, L) \leq 12$, again a contradiction.

Case 2. $e(x_1, L) = 3$. First suppose that $N(x_1)\,|\,L = \{y_i, y_{i+1}, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1)\,|\,L = \{y_1, y_2, y_3\}$. As $x_1 \to (L, y_2)$, $e(y_2, x_4x_5) = 0$. Note that $[x_2, x_1, y_2, y_3, y_4] \supseteq F_4$ and $[x_2, x_1, y_1, y_2, y_5] \supseteq F_4$. Then $e(y_3y_4, x_4x_5) \leq 2$ and $e(y_5y_1, x_4x_5) \leq 2$ for otherwise $[x_3, x_4, x_5, y_3, y_4] \supseteq C_5$ and $[x_3, x_4, x_5, y_1, y_5]$ $\supseteq C_5$. Hence $e(R, L) \leq 12$, a contradiction. Next, suppose that $N(x_1)\,|\,L = \{y_i, y_{i+1}, y_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1)\,|\,L = \{y_1, y_2, y_4\}$. As $x_1 \to$

(L, y_i) for all $i \in \{3, 5\}$, $e(y_i, x_4x_5) = 0$. It follows that $e(x_2, y_3y_4y_5) \geq 2$ and so $[x_1, x_2, y_3, y_4, y_5] \supseteq F$. Then $e(y_1y_2, x_4x_5) \leq 2$ as $[x_3, x_4, x_5, y_1, y_2] \not\supseteq C_5$. Consequently, $e(R, L) \leq 12$, a contradiction. ■

Lemma 8. *Let D and L be two disjoint subgraphs of G such that $D \cong F_4$ and $L \cong C_5$. If $e(D - x_3, L) \geq 13$, then $[D, L]$ contains either $2C_5$ or $F \uplus C_5$ or $B \uplus C_5$.*

Proof. Let $L = y_1y_2y_3y_4y_5y_1$ and $R = V(D)\backslash\{x_3\}$. Suppose to the contrary that $[D, L]$ does not contain any of $2C_5$, $F \uplus C_5$ and $B \uplus C_5$. Without loss of generality assume that $e(x_1x_2, L) \geq e(x_4x_5, L)$. It is clear that $7 \leq e(x_1x_2, L) \leq 10$ and, hence $2 \leq e(x_1, L) \leq 5$. If $e(x_1, L) \geq 4$, say $N(x_1) \,|\, L \supseteq \{y_1, y_2, y_3, y_4\}$, then $e(y_i, x_2x_4x_5) \leq 1$ as $x_1 \to (L, y_i)$ for all $i \in \{2, 3, 5\}$. Further, we have $e(y_j, x_2x_4x_5) \leq 2$ for all $j \in \{1, 4\}$, for otherwise, $[D - x_1 + y_j]$ would contain C_5 and $[L - y_j + x_1]$ contain F. Hence $e(R, L) \leq 12$, a contradiction. Therefore, $e(x_1, L) \leq 3$.

Case 1. $e(x_1, L) = 2$. In this case, $e(x_2, L) = 5$ as $e(x_1x_2, L) \geq 7$. First suppose that $N(x_1) \,|\, L = \{y_i, y_{i+1}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L = \{y_1, y_2\}$. We claim that $e(y_iy_{i+1}, x_4x_5) \leq 2$ for all $i \in \{2, 3, 4, 5\}$, for otherwise, $[(V(L) \cup \{x_1, x_2\})\backslash\{y_i,\ y_{i+1}\}] \supseteq C_5$ and $[x_3, x_4, x_5, y_i, y_{i+1}] \supseteq F$. Then $e(y_1, x_4x_5) = e(y_4, x_4x_5) = 2$ as $e(x_4x_5, L) \geq 6$. It follows that $[y_2, y_3, x_1, x_2, x_3] \supseteq F$ and $[x_4, x_5, y_1, y_4, y_5] \supseteq C_5$, a contradiction. Next, suppose that $N(x_1) \,|\, L = \{y_i, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L = \{y_1, y_3\}$. We say that $e(y_1y_2, x_4x_5) \leq 2$ for otherwise $[x_1, x_2, y_5, y_4, y_3] \supseteq C_5$ and $[x_3, x_4, x_5, y_1, y_2] \supseteq F$. Similarly, $e(y_2y_3, x_4x_5) \leq 2$ and $e(y_4y_5, x_4x_5) \leq 2$. Then $e(y_1, x_4x_5) = e(y_3, x_4x_5) = 2$ as $e(x_4x_5, L) \geq 6$. Further, we have $e(y_2, x_4x_5) = 0$ as $x_1 \to (L, y_2)$. Then $e(y_4y_5, x_4x_5) = 2$ and so $[D, L] \supseteq F \uplus C_5$, a contradiction.

Case 2. $e(x_1, L) = 3$. In this case, $e(x_2, L) \geq 4$ as $e(x_1x_2, L) \geq 7$. First suppose that $N(x_1) \,|\, L = \{y_i, y_{i+1}, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L = \{y_1, y_2, y_3\}$. As $x_1 \to (L, y_2)$, $e(y_2, x_4x_5) \leq 1$. Further, if $y_2x_2 \in E(G)$, then $e(y_2, x_4x_5) = 0$. If $N(x_2) \,|\, L \supseteq \{y_4, y_5\}$, then $[y_1, y_2, y_5, x_1, x_2] \supseteq C_5$ and $[y_2, y_3, y_4, x_1, x_2] \supseteq C_5$. Further, $e(y_1y_5, x_4x_5) \leq 2$ and $e(y_3y_4, x_4x_5) \leq 2$ for otherwise $[y_1, y_5, x_3, x_4, x_5] \supseteq F$ and $[y_3, y_4, x_3, x_4, x_5] \supseteq F$. Hence $e(R, L) = e(x_1, L) + e(x_2, L) + e(y_2, x_4x_5) + e(y_1y_5, x_4x_5) + e(y_3y_4, x_4x_5) \leq 12$, no matter whether y_2x_2 is an edge in $E(G)$ or not, a contradiction. Then $N(x_2) \,|\, L \not\supseteq \{y_4, y_5\}$. Without loss of generality, we assume $N(x_2) \,|\, L = \{y_1, y_2, y_3, y_4\}$. Then $[y_4, y_5, y_1, x_1, x_2] \supseteq C_5$ and $[y_2, y_3, y_4, x_2, x_1] \supseteq C_5$. Hence $e(y_1y_5, x_4x_5) \leq 2$ and $e(y_2y_3, x_4x_5) \leq 2$. Then $e(y_4, x_4x_5) = 2$ as $e(x_4x_5, L) \geq 6$. This implies that $[L - y_4 + x_1] \supseteq F$ and $[D - x_1 + y_4] \supseteq C_5$, a contradiction. Next, suppose that $N(x_1) \,|\, L = \{y_i, y_{i+1}, y_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L = \{y_1, y_2, y_4\}$. As $x_1 \to (L, y_i)$ for all $i \in \{3, 5\}$, $e(y_i, x_2x_4x_5) \leq 1$. Then $e(y_j, x_2x_4x_5) = 3$ for some $j \in \{1, 2\}$. It follows that $[L - y_j + x_1] \supseteq F$ and $[D + y_j - x_1] \supseteq C_5$, a contradiction. ■

Lemma 9. *Let D and L be two disjoint subgraphs of G such that $D \cong F_5$ and $L \cong C_5$. If $e(D - x_2, L) \geq 13$, then $[D, L]$ contains either $2C_5$ or $F_1 \uplus C_5$ or $K_{2,3} \uplus C_5$.*

Proof. Let $L = y_1y_2y_3y_4y_5y_1$ and $R = V(D)\backslash\{x_2\}$. Suppose to the contrary that $[D, L]$ does not contain any of $2C_5$, $F_1 \uplus C_5$ and $K_{2,3} \uplus C_5$. If $e(x_1, L) \geq 4$, say $N(x_1)\,|\,L \supseteq \{y_1, y_2, y_3, y_4\}$, then $e(y_i, x_3x_4x_5) \leq 1$ for all $i \in \{2, 3, 5\}$ as $x_1 \to (L, y_i)$. Further, $e(y_4, x_3x_4x_5) \leq 1$ as $[x_1, y_1, y_2, y_3, y_5] \supseteq F_1$. Thus, $e(R, L) \leq 12$, a contradiction. Now suppose $e(x_1, L) = 1$, say $x_1y_1 \in E(G)$. Suppose that $x_3y_j \in E(G)$ for some $j \in \{2, 5\}$, then $[x_1, y_1, y_j, x_3, x_2] \supseteq C_5$. It follows that $e(x_4x_5, L-\{y_1, y_j\}) \leq 3$ for otherwise $[(L-\{y_1, y_j\}) \cup \{x_4, x_5\}]$ would contain C_5 or F_1 or $K_{2,3}$, a contradiction. Then $e(x_3, L) \geq e(R, L) - e(x_4x_5, L - \{y_1, y_j\}) - 4 - e(x_1, L) \geq 5$. Therefore, either $e(x_3, L) = 5$ or $e(x_3, L) \leq 3$. Similarly, either $e(x_5, L) = 5$ or $e(x_5, L) \leq 3$. If $e(x_3, L) \leq 3$ and $e(x_5, L) \leq 3$, then $e(R, L) \leq 12$, a contradiction. Without loss of generality assume that $e(x_3, L) = 5$. If $e(x_i, y_1) = 1$, $e(x_j, y_2y_5) = 2$ for some $i \in \{3, 4, 5\}$, $j \in \{3, 5\}$, $i \neq j$, then $[L - y_1 + x_j] \supseteq C_5$ and $[D + y_1 - x_j] \supseteq C_5$. This implies that $N(x_4)\,|\,L = \{y_2, y_3, y_4, y_5\}$ and $N(x_5)\,|\,L \supseteq \{y_3, y_4\}$ and so $[x_1, x_2, x_3, y_1, y_2] \supseteq C_5$ and $[x_4, x_5, y_3, y_4, y_5] \supseteq C_5$, a contradiction.

Case 1. $e(x_1, L) = 2$. First suppose that $N(x_1)\,|\,L = \{y_i, y_{i+1}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1)\,|\,L = \{y_1, y_2\}$. If $e(x_4, L) = 5$, then $e(x_3x_5, y_i) \leq 1$ for all $i \in \{1, 2\}$, for otherwise, $[L - y_i + x_4] \supseteq C_5$ and $[D + y_i - x_4] \supseteq K_{2,3}$. If $e(x_3x_5, y_3y_4y_5) \geq 5$, then $[x_1, x_2, x_4, y_1, y_2] \supseteq C_5$ and $[x_3, x_5, y_3, y_4, y_5] \supseteq C_5$, a contradiction. Thus, $e(x_3x_5, y_3y_4y_5) = 4$, $e(x_3x_5, y_i) = 1$ for all $i \in \{1, 2\}$. Further, either $[x_1, x_2, x_3, y_1, y_2] \supseteq C_5$ and $[x_4, x_5, y_3, y_4, y_5] \supseteq C_5$, or $[x_1, x_2, x_5, y_1, y_2] \supseteq C_5$ and $[x_3, x_4, y_3, y_4, y_5] \supseteq C_5$, a contradiction. Therefore $e(x_4, L) \leq 4$. If $e(x_4, y_1y_2) = 2$, then $e(y_{i+1}y_{i-1}, x_j) \leq 1$ for all $i \in \{1, 2\}$, $j \in \{3, 5\}$, for otherwise, $[L-y_i+x_j] \supseteq C_5$, $[D+y_i-x_j] \supseteq C_5$. Hence $e(R, L) \leq 12$, a contradiction. Therefore, $e(x_4, y_1y_2) \leq 1$. Without loss of generality assume that $e(x_3, L) \geq e(x_5, L)$. If $e(x_4, L) = 4$, say $N(x_4)\,|\,L = \{y_2, y_3, y_4, y_5\}$, then $[D + y_2 - x_3]$ would contain C_5 and $[L - y_2 + x_3]$ contain F_1 or C_5, a contradiction. Therefore $e(x_4, L) \leq 3$ and hence $e(x_3x_5, L) \geq 8$. If $e(x_3, L) = 5$, then $e(x_i, y_j) = 0$ for all $i \in \{4, 5\}$, $j \in \{1, 2\}$, for otherwise, $[L - y_j + x_3] \supseteq C_5$, $[D + y_j - x_3] \supseteq C_5$. Thus $e(x_4x_5, y_3y_4y_5) = 6$. Then $[x_1, x_2, x_3, y_1, y_2] \supseteq C_5$ and $[x_4, x_5, y_3, y_4, y_5] \supseteq C_5$, a contradiction. Hence $e(x_3, L) = e(x_5, L) = 4$ and $e(x_4, L) = 3$. If $x_4y_1 \in E(G)$, then $e(x_3x_5, y_2y_5) \leq 2$ and so $e(x_3x_5, y_1y_3y_4) = 6$. This implies that $[x_1, x_2, x_3, y_2, y_3] \supseteq C_5$ and $[x_4, x_5, y_1, y_4, y_5] \supseteq C_5$, a contradiction. Therefore, $x_4y_1 \notin E(G)$. Similarly, $x_4y_2 \notin E(G)$. Then $e(x_4, y_3y_4y_5) = 3$. We say that $e(x_3x_5, y_2y_3) \leq 3$ for otherwise $[x_1, y_1, y_2, x_3, x_2] \supseteq C_5$ and $[x_4, x_5, y_3, y_4, y_5] \supseteq C_5$. Then $e(x_3x_5, y_1y_4y_5) \geq 5$ and so $[x_1, x_2, x_4, y_2, y_3] \supseteq C_5$ and $[x_3, x_5, y_1, y_4, y_5] \supseteq C_5$, a contradiction.

Next, suppose that $N(x_1)\,|\,L = \{y_i, y_{i+2}\}$ for some $i \in \{1,2,3,4,5\}$, say $N(x_1)\,|\,L = \{y_1, y_3\}$. As $x_1 \to (L, y_2)$, $e(y_2, x_3x_5) = 0$. If $e(x_4, L) \leq 3$, then $e(x_3x_5, L - y_2) = 8$ and so $[L - y_1 + x_5] \supseteq F_1$ and $[D - x_5 + y_1] \supseteq C_5$, a contradiction. Without loss of generality assume that $e(x_3, L) \geq e(x_5, L)$. If $e(x_4, L) = 4$, then $N(x_3)\,|\,L = \{y_1, y_3, y_4, y_5\}$. We claim that $e(y_i, x_5) = 0$ for all $i \in \{1,3\}$, for otherwise, $[D - x_3 + y_i] \supseteq C_5$, $[L - y_i + x_3] \supseteq F_1$. Hence $e(R, L) \leq 12$, a contradiction. If $e(x_4, L) = 5$, then $e(x_3x_5, y_1y_5y_4) \leq 4$ for otherwise $[x_1, y_2, y_3, x_4, x_2] \supseteq C_5$ and $[x_3, x_5, y_1, y_5, y_4] \supseteq C_5$. It follows that $e(x_3x_5, y_3) = 2$ and so $[L - y_3 + x_4] \supseteq C_5$ and $[D + y_3 - x_4] \supseteq K_{2,3}$, again a contradiction.

Case 2. $e(x_1, L) = 3$. First suppose that $N(x_1)\,|\,L = \{y_i, y_{i+1}, y_{i+2}\}$ for some $i \in \{1,2,3,4,5\}$, say $N(x_1)\,|\,L = \{y_1, y_2, y_3\}$. As $x_1 \to (L, y_2)$, $e(y_2, x_3x_4x_5) \leq 1$. Further, $e(y_4, x_3x_4x_5) \leq 1$ as $[x_1, y_1, y_2, y_3, y_5] \supseteq F_1$. Similarly, $e(y_5, x_3x_4x_5) \leq 1$. Thus, $e(R, L) \leq 12$, a contradiction. Next, suppose that $N(x_1)\,|\,L = \{y_i, y_{i+1}, y_{i+3}\}$ for some $i \in \{1,2,3,4,5\}$, say $N(x_1)\,|\,L = \{y_1, y_2, y_4\}$. As $x_1 \to (L, y_i)$ for all $i \in \{3,5\}$, $e(y_i, x_3x_5) = 0$. Then $e(x_3x_5, L) \leq 6$ and so $e(x_4, L) \geq 4$. If $e(x_4, L) = 5$, then $e(y_i, x_3x_5) \leq 1$ for all $i \in \{1,2,4\}$, for otherwise, $[D - x_4 + y_i] \supseteq K_{2,3}$ and $[L - y_i + x_4] \supseteq C_5$. Hence $e(R, L) \leq 11$, a contradiction. If $e(x_4, L) = 4$, then $e(x_3x_5, y_1y_2y_4) = 6$. Further, if $e(x_4, y_3y_5) = 2$, then $[x_1, x_2, x_4, y_2, y_3] \supseteq C_5$ and $[x_3, x_5, y_1, y_4, y_5] \supseteq K_{2,3}$, a contradiction. Thus $e(x_4, y_3y_5) \leq 1$. Without loss of generality assume that $N(x_4)\,|\,L = \{y_1, y_2, y_3, y_4\}$. Then $[y_2, y_3, x_1, x_2, x_4] \supseteq C_5$ and $[y_1, y_4, y_5, x_3, x_5] \supseteq K_{2,3}$, again a contradiction. ■

3. Proof of Theorem 3

Let G be a graph of order $n = 5k$ with $\sigma_2(G) \geq n + k$. It is easy to see that G is Hamiltonian if $k = 1$. In the following, we always assume that $k \geq 2$. Suppose, for a contradiction, that $G \not\supseteq kC_5$. We may assume that G is maximal, i.e., $G + xy \supseteq kC_5$ for each pair of non-adjacent vertices x and y of G. Thus $G \supseteq P_5 \uplus (k-1)C_5$. Our proof will follow from the following lemmas.

Lemma 10. *For each $s \in \{1, 2, \ldots, k\}$, $G \not\supseteq sB \uplus (k-s)C_5$.*

Proof. To the contrary, suppose that $G \supseteq sB \uplus (k-s)C_5$ for some $s \in \{1, 2, \ldots, k\}$. Let s be the minimum number in $\{1, 2, \ldots, k\}$ for which $G \supseteq sB \uplus (k-s)C_5$ and let $B_1, \ldots, B_s, L_1, \ldots, L_{k-s}$ be k disjoint subgraphs of G with $B_i \cong B$ for $i \in \{1, 2, \ldots, s\}$ and $L_i \cong C_5$ for $i \in \{1, 2, \ldots, k-s\}$. Let R be the set of the four vertices of degree 2 in B_1. By Lemmas 4(a) and (b) and the minimality of s, we see that $e(R, B_i) \leq 12$ and $e(R, L_j) \leq 12$ for all $i \in \{2, 3, \ldots, s\}$ and $j \in \{1, 2, \ldots, k-s\}$. Therefore $\sum_{x \in R} d_G(x) \leq 12(k-1) + 8 = 12k - 4$. However, by the degree sum condition, we have $\sum_{x\in R} d_G(x) = (d_G(x_2)+d_G(x_4))+(d_G(x_3)+d_G(x_5)) \geq 12k$, a contradiction. ∎

Lemma 11. $G \supseteq F_1 \uplus (k-1)C_5$.

Proof. First, we claim that $G \supseteq F \uplus (k-1)C_5$. Let $\{H, L_1, L_2, \ldots, L_{k-1}\}$ be an optimal family of G with $H \cong P_5$. If $[H] \cong P_5 = x_1x_2x_3x_4x_5$, then $d_G(x_1) + d_G(x_5) \geq 6k$. Without loss of generality, we assume that $d_G(x_1) \geq 3k$. Since $\sigma_2(G) \geq 6k$, $\sum_{x\in V(H)} d_G(x) = d_G(x_1)+(d_G(x_2)+d_G(x_4))+(d_G(x_3)+d_G(x_5)) \geq 15k$. Then $e(H, G-V(H)) \geq 15k-8 = 15(k-1)+7$. Thus $e(H, L_i) \geq 16$ for some $i \in \{1, 2, \ldots, k-1\}$. By Lemma 4(d), $[H, L_i] \supseteq F \uplus C_5$ and, hence $G \supseteq F \uplus (k-1)C_5$. If $[H] \cong F_4$, then $e(x_1x_2x_4x_5, G-V(H)) \geq 12k-8 = 12(k-1)+4$. This implies that $e(x_1x_2x_4x_5, L_i) \geq 13$ for some $i \in \{1, 2, \ldots, k-1\}$. By Lemma 8 and Lemma 10, $[H, L_i] \supseteq F \uplus C_5$ and so $G \supseteq F \uplus (k-1)C_5$. If $[H] \cong F_3$, then $e(x_1x_2x_4x_5, G-V(H)) \geq 12k-7 = 12(k-1)+5$. Hence $e(x_1x_2x_4x_5, L_i) \geq 13$ for some $i \in \{1, 2, \ldots, k-1\}$. By Lemmas 7, 8 and 10, $[H, L_i] \supseteq F \uplus C_5$ and, therefore, $G \supseteq F \uplus (k-1)C_5$.

We first assume that $G \supseteq F_2 \uplus (k-1)C_5$ and let $\{H, L_1, L_2, \ldots, L_{k-1}\}$ be a family of G with $H \cong F_2$. Note that $d_G(x_1)+d_G(x_5) \geq 6k$, say $d_G(x_1) \geq 3k$. Then $e(x_1x_3x_5, G-V(H)) = d_G(x_1)+(d_G(x_3)+d_G(x_5))-6 \geq 9k-6 = 9(k-1)+3$ and, hence $e(x_1x_3x_5, L_i) \geq 10$ for some $i \in \{1, 2, \ldots, k-1\}$. So by Lemma 4(c), $[H, L_i] \supseteq F_1 \uplus C_5$ and, therefore, $G \supseteq F_1 \uplus (k-1)C_5$. We now assume that $G \not\supseteq F_2 \uplus (k-1)C_5$. Recalling that $G \supseteq F \uplus (k-1)C_5$, let $\{H, L_1, L_2, \ldots, L_{k-1}\}$ be an optimal family of G with $H \cong F$. If $[H] \cong F$ or $[H] \cong K_{2,3}$, say $\{x_1, x_2, x_3, x_4, x_5\} = V(H)$ with $x_1x_3, x_1x_5, x_3x_5 \notin E(G)$ and $x_2x_4 \notin E(G)$, then $d_G(x_1)+d_G(x_5) \geq 6k$, say $d_G(x_1) \geq 3k$. Further, $e(H, G-V(H)) \geq d_G(x_1)+(d_G(x_3)+d_G(x_5))+(d_G(x_2)+d_G(x_4))-12 \geq 15k-12 = 15(k-1)+3$. Then $e(H, L_i) \geq 16$ for some $i \in \{1, 2, \ldots, k-1\}$. By Lemma 5(a) and Lemma 10, the cycle L_i has a labelling $L_i = y_1y_2y_3y_4y_5y_1$ satisfying property $\mathbf{P}_1$. Therefore, $e(x_1x_3y_3y_5, G-V(H\cup L_i)) \geq 12k-17 = 12(k-2)+7$ and, hence, $e(x_1x_3y_3y_5, L_j) \geq 13$ for some $j \in \{1, \ldots, k-1\}\backslash\{i\}$. So by Lemma 5(b), we have $[H, L_i, L_j] \supseteq F_1 \uplus 2C_5$ and, hence, $G \supseteq F_1 \uplus (k-1)C_5$. We next assume that $G \not\supseteq K_{2,3} \uplus (k-1)C_5$. If $[H] \cong F_5$, then $e(x_1x_3x_4x_5, G-V(H)) = (d_G(x_1)+d_G(x_4))+(d_G(x_3)+d_G(x_5))-8 \geq 12k-8 = 12(k-1)+4$. Thus $e(x_1x_3x_4x_5, L_i) \geq 13$ for some $i \in \{1, 2, \ldots, k-1\}$. So by Lemma 9, $[H, L_i] \supseteq F_1 \uplus C_5$, i.e., $G \supseteq F_1 \uplus (k-1)C_5$. ∎

Lemma 12. *Let $\psi = \{H, L_1, L_2, \ldots, L_{k-1}\}$ be an optimal family of G with $H \cong F_1$ and let $T = \{x_2, x_4, x_5\}$. If $G \not\supseteq K_4^+ \uplus (k-1)C_5$, then for each $t \in \{1, 2, \ldots, k-1\}$, the following statements hold.*

(a) *If $e(x_1, L_t) = 5$, then $e(T, L_t) \leq 5$.*

(b) *If $e(x_1, L_t) = 4$, then $e(T, L_t) \leq 7$.*

(c) *If* $e(x_1, L_t) = 3$, *then* $e(T, L_t) \leq 9$.

(d) *If* $e(x_1, L_t) = 2$, *then* $e(T, L_t) \leq 11$.

(e) *If* $e(x_1, L_t) = 1$, *then* $e(T, L_t) \leq 12$.

(f) *If* $e(x_1, L_t) = 0$, *then* $e(T, L_t) \leq 15$.

Proof. Let $L_t = y_1y_2y_3y_4y_5y_1$ and $G_t = [H, L_t]$. If $e(x_1, L_t) = 5$, then $e(y_i, T) \leq 1$ for all $i \in \{1, 2, 3, 4, 5\}$ since $G_t \not\supseteq 2C_5$. Hence, (a) follows directly.

To prove (b), without loss of generality assume that $N(x_1) \,|\, L_t = \{y_1, y_2, y_3, y_4\}$. Suppose to the contrary that $e(T, L_t) \geq 8$. It is clear that $\tau(y_5, L_t) = 0$ for otherwise $x_1 \to L_t$ and so $G_t \supseteq 2C_5$. As $x_1 \to (L_t, y_i)$ for all $i \in \{2, 3, 5\}$, $e(y_i, T) \leq 1$. Hence, $e(y_1y_4, T) \geq 5$, say $e(y_4, T) = 3$ and $e(y_1, T) \geq 2$. If $e(y_5, x_2x_4) \geq 1$, then $[H - x_1 + y_5] \supseteq F_1$ and $\tau(L_t - y_5 + x_1) > \tau(L_t)$, contradicting the optimality of ψ. Thus, $e(y_5, x_2x_4) = 0$. If $y_5x_5 \in E(G)$, then $y_1x_2 \notin E(G)$ for otherwise $[y_1, y_2, y_3, x_1, x_2] \supseteq C_5$ and $[y_4, y_5, x_3, x_4, x_5] \supseteq C_5$. This means that $e(y_1, x_4x_5) = 2$ and so $[y_2, y_3, y_4, x_1, x_2] \supseteq C_5$ and $[y_1, y_5, x_3, x_4, x_5] \supseteq C_5$, a contradiction. Therefore, $y_5x_5 \notin E(G)$ and hence $e(y_1, T) = 3$, $e(y_2y_3, T) = 2$. If $e(y_2y_3, x_5) = 2$, then $[y_1, y_5, y_4, x_1, x_2] \supseteq C_5$ and $[y_2, y_3, x_3, x_4, x_5] \supseteq B$, contradicting Lemma 10. Hence, $e(y_2y_3, x_2x_4) \geq 1$, say $y_3x_2 \in E(G)$. We claim that $y_1y_3 \in E(G)$ for otherwise $[y_3, x_2, x_3, x_4, x_5] \supseteq F_1$ and $\tau(y_1y_2x_1y_4y_5y_1) > \tau(L_t)$. This implies that $[y_5, y_1, y_2, y_3, x_1] \supseteq K_4^+$ and $[y_4, x_2, x_3, x_4, x_5] \supseteq C_5$, a contradiction.

To prove (c), suppose to the contrary that $e(T, L_t) \geq 10$. Assume first $N(x_1) \,|\, L_t = \{y_i, y_{i+1}, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_2, y_3\}$. As $x_1 \to (L_t, y_2)$, $e(y_2, T) \leq 1$. For any $i \in \{1, 2, 3, 4, 5\}$, if $[(V(L_t) \cup \{x_1, x_2\}) \backslash \{y_i, y_{i+1}\}] \supseteq C_5$, then $e(y_iy_{i+1}, x_4x_5) \leq 2$ for otherwise $[x_3, x_4, x_5, y_i, y_{i+1}] \supseteq C_5$. If $e(x_2, y_4y_5) = 2$, then $e(T, L_t) = e(x_2, L_t) + e(x_4x_5, y_2) + e(x_4x_5, y_3y_4) + e(x_4x_5, y_1y_5) \leq 9$, no matter whether y_2x_2 is an edge of G or not, a contradiction. Hence $e(x_2, y_4y_5) \leq 1$, say $x_2y_4 \notin E(G)$. Further, if $e(x_2, y_3y_5) = 2$, then $e(x_4x_5, y_1y_2) \leq 2$ and $e(x_4x_5, y_4y_5) \leq 2$. Since $e(T, L_t) \geq 10$ and $x_2y_4 \notin E(G)$, $N(x_2) \,|\, L_t = \{y_1, y_2, y_3, y_5\}$ and $e(y_3, x_4x_5) = 2$. Therefore, $[x_3, x_2, x_1, y_1, y_2] \supseteq K_4^+$. Then $e(y_5, x_4x_5) = 0$ as $G_t \not\supseteq K_4^+ \uplus C_5$. Therefore, $e(y_4, x_4x_5) = 2$. Then $[x_1, x_2, y_1, y_2, y_5] \supseteq C_5$ and $[x_3, x_4, x_5, y_3, y_4] \supseteq C_5$, a contradiction. Therefore, $e(x_2, y_3y_5) \leq 1$. If $x_2y_5 \in E(G)$, then $e(x_4x_5, y_1y_2) \leq 2$, $e(x_4x_5, y_3y_4) \leq 2$ and so $e(T, L_t) \leq 9$, a contradiction. Thus $x_2y_5 \notin E(G)$. If $x_2y_1 \in E(G)$, then $e(x_4x_5, y_4y_5) \leq 2$ and so $e(T, L_t) = e(x_2, L_t) + e(x_4x_5, y_2) + e(x_4x_5, y_1y_3) + e(x_4x_5, y_4y_5) \leq 9$, no matter whether y_2x_2 is an edge of G or not, a contradiction. Hence $x_2y_1 \notin E(G)$. Similarly, $x_2y_3 \notin E(G)$. Then $e(T, L_t) \leq 9$, no matter whether y_2x_2 is an edge in $E(G)$ or not, again a contradiction.

Next, assume that $N(x_1) \,|\, L_t = \{y_i, y_{i+1}, y_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_2, y_4\}$. As $x_1 \to (L_t, y_i)$ for all $i \in \{3, 5\}$, $e(y_i, T) \leq 1$. Further, if $e(x_2, L_t) \leq 1$ or $e(x_2, L_t) = 2$, $N(x_2) \,|\, L_t \cap \{y_3, y_5\} \neq \emptyset$ or $e(x_2, L_t) = 3$,

$N(x_2) \mid L_t \supseteq \{y_3, y_5\}$, then $e(T, L_t) \leq 9$, a contradiction. Suppose $e(x_2, L_t) \leq 3$. If $x_2y_4 \in E(G)$, then $e(y_1y_5, x_4x_5) \leq 2$, $e(y_2y_3, x_4x_5) \leq 2$ and so $e(T, L_t) \leq 9$, a contradiction. Hence $x_2y_4 \notin E(G)$. Then $N(x_2) \mid L_t \supseteq \{y_1, y_2\}$. It follows that $e(y_1y_5, x_4x_5) \leq 2$, $e(y_2y_3, x_4x_5) \leq 2$ and so $e(T, L_t) \leq 9$, again a contradiction.

Now suppose $e(x_2, L_t) \geq 4$. If $y_3y_5 \in E(G)$, then $x_1 \to L_t$. Since $e(T, L_t) \geq 10$, there is a vertex y_i for some $i \in \{1, 2, 3, 4, 5\}$ such that $e(y_i, T) \geq 2$. Then $[H - x_1 + y_i] \supseteq C_5$ and $[L_t - y_i + x_1] \supseteq C_5$, a contradiction. Thus $y_3y_5 \notin E(G)$. If $x_2y_3 \in E(G)$ and $y_1y_3 \notin E(G)$, then $[y_3, x_2, x_3, x_4, x_5] \cong F_1$ and $\tau(L_t) < \tau(L_t - y_3 + x_1)$, a contradiction. Therefore, if $x_2y_3 \in E(G)$, then $y_1y_3 \in E(G)$. Similarly, if $x_2y_5 \in E(G)$, then $y_2y_5 \in E(G)$. We claim that $e(y_1y_2y_4, x_4x_5) = 6$ if $|\,N(x_2) \mid L_t\,| = 4$. If $x_2y_3 \in E(G)$ and $x_2y_5 \in E(G)$, then $e(y_1y_2y_4, x_4x_5) = 6$ since $e(y_i, T) \leq 1$ for all $i \in \{3, 5\}$. Without loss of generality, assume that $x_2y_3 \in E(G)$ and $x_2y_5 \notin E(G)$. If $e(y_5, x_4x_5) = 0$, then $e(y_1y_2y_4, x_4x_5) = 6$ as $e(y_3, T) \leq 1$. If $e(y_5, x_4x_5) = 2$, then $[x_1, y_1, y_2, y_3, y_4] \supseteq C_5$ and $[y_5, x_2, x_3, x_4, x_5] \supseteq C_5$, a contradiction. If $e(y_5, x_4x_5) = 1$, then $e(y_4, x_4x_5) = 0$ for otherwise $[y_1, y_2, y_3, x_1, x_2] \supseteq C_5$ and $[y_4, y_5, x_3, x_4, x_5]$ would contain C_5 or B. Therefore, $e(T, L_t) \leq 9$, a contradiction. If $N(x_2) \mid L_t = \{y_1, y_2, y_3, y_4\}$, then $[y_4, y_5, x_3, x_4, x_5] \supseteq F_1$. Further, $\tau(L_t) = 4$ as $\tau(L_t) \geq \tau(y_1y_2y_3x_2x_1y_1)$. Similarly, if $N(x_2) \mid L_t = \{y_1, y_2, y_3, y_5\}$ or $N(x_2) \mid L_t = \{y_1, y_2, y_4, y_5\}$, then we also have $\tau(L_t) = 4$. If $N(x_2) \mid L_t = \{y_2, y_3, y_4, y_5\}$, then $[x_1, y_1, x_3, x_4, x_5] \supseteq F_1$. Further, $\{y_1y_3, y_1y_4, y_2y_5\} \subseteq E(G)$ as $\tau(L_t) \geq \tau(y_2y_3y_4y_5x_2y_2)$. If $N(x_2) \mid L_t = \{y_1, y_3, y_4, y_5\}$, then $[x_1, y_2, x_3, x_4, x_5] \supseteq F_1$. Further, $\{y_1y_3, y_2y_4, y_2y_5\} \subseteq E(G)$ as $\tau(L_t) \geq \tau(y_1x_2y_3y_4y_5y_1)$. If $N(x_2) \mid L_t = \{y_1, y_2, y_3, y_4, y_5\}$, then $e(y_1y_2y_4, x_4x_5) \geq 5$ since $e(y_i, T) \leq 1$ for all $i \in \{3, 5\}$. We claim that $\tau(L_t) = 4$ if $N(x_2) \mid L_t = \{y_1, y_2, y_3, y_4, y_5\}$. If $e(y_4, x_4x_5) = 2$, then $[x_1, y_4, x_3, x_4, x_5] \supseteq F_1$. Further, $\tau(L_t) = 4$ as $\tau(L_t) \geq \tau(y_1y_2y_3x_2y_5y_1)$. If $e(y_4, x_4x_5) \leq 1$, then $e(y_1y_2, x_4x_5) = 4$ and so $[x_1, y_1, x_3, x_4, x_5] \supseteq F_1$ and $[x_1, y_2, x_3, x_4, x_5] \supseteq F_1$. Further, $\tau(L_t) = 4$ as $\tau(L_t) \geq \tau(x_2y_2y_3y_4y_5x_2)$ and $\tau(L_t) \geq \tau(y_1x_2y_3y_4y_5y_1)$.

Let $R = \{x_1, x_3, y_3, y_5\}$. If $x_3y_3 \in E(G)$, then $x_2y_3 \notin E(G)$ and so $N(x_2) \mid L_t = \{y_1, y_2, y_4, y_5\}$. Then $[y_1, y_4, y_5, x_1, x_2] \supseteq C_5$ and $[y_2, y_3, x_3, x_4, x_5] \supseteq C_5$, a contradiction. Thus $x_3y_3 \notin E(G)$. Similarly, $x_3y_5 \notin E(G)$. Hence R is an independent set. As $e(R, G - V(G_t)) \geq 12k - 18 = 12(k-2) + 6$, $e(R, L_i) \geq 13$ for some $i \in \{1, 2, \ldots, k-1\} \backslash \{t\}$.

Claim 1. *If* $u \to (L_i, z; \{v, w\})$ *for some* $z \in V(L_i), u \in R$ *and* $\{v, w\} \subseteq R \setminus \{u\}$, *then* $[G_t, L_i] \supseteq 3C_5$.

Proof. We separate the proof into two cases.

Case 1. $e(x_2, y_3y_5) = 2$. In this case, $e(x_4x_5, y_3y_5) = 0$. Further, $e(x_4x_5, y_1y_2y_4) = 6$ if $|N(x_2)|L_t| = 4$ and $e(x_4x_5, y_1y_2y_4) \geq 5$ if $|N(x_2)|L_t| = 5$. Recall that $\tau(L_t) = 4$ if $|N(x_2)|L_t| = 5$. Hence, y_1, y_2 and y_4 are symmetric in $[L_t]$ if $|N(x_2)|L_t| = 5$. Without loss of generality, assume that $e(x_4x_5, y_1y_4) = 4$.

Further, by the symmetry of x_1, y_3 and y_5 in $[L_t + x_1]$, we need only to consider the following cases. If $x_1 \to (L_i, z; \{y_3, y_5\})$ for some $z \in V(L_i)$, then $[L_i - z + x_1] \supseteq C_5$, $[y_1, y_2, y_3, y_5, z] \supseteq C_5$ and $[x_2, x_3, x_4, x_5, y_4] \supseteq C_5$. If $x_1 \to (L_i, z; \{y_3, x_3\})$ for some $z \in V(L_i)$, then $[L_i - z + x_1] \supseteq C_5$, $[y_3, x_2, x_5, x_3, z] \supseteq C_5$ and $[y_1, y_2, y_4, y_5, x_4] \supseteq C_5$. If $x_3 \to (L_i, z; \{y_3, y_5\})$ for some $z \in V(L_i)$, then $[L_i - z + x_3] \supseteq C_5$, $[y_1, y_2, y_3, y_5, z] \supseteq C_5$ and $[x_1, x_2, x_4, x_5, y_4] \supseteq C_5$.

Case 2. $e(x_2, y_3y_5) = 1$, say $N(x_2) \,|\, L_t = \{y_1, y_2, y_3, y_4\}$. In this case, $e(x_4x_5, y_1y_2y_4) = 6$ and $e(x_4x_5, y_3) = 0$. Further, $e(x_4x_5, y_5) = 0$ for otherwise $[y_1, y_2, y_3, x_1, x_2] \supseteq C_5$ and $[y_4, y_5, x_3, x_4, x_5]$ would contain C_5 or B. By the symmetry of x_1 and y_3 in $[L_t + x_1]$, we need only to consider the following cases. If $x_1 \to (L_i, z; \{y_3, y_5\})$ for some $z \in V(L_i)$, then $[L_i - z + x_1] \supseteq C_5$, $[y_1, y_2, y_3, y_5, z] \supseteq C_5$ and $[x_2, x_3, x_4, x_5, y_4] \supseteq C_5$. If $x_1 \to (L_i, z; \{y_3, x_3\})$ for some $z \in V(L_i)$, then $[L_i - z + x_1] \supseteq C_5$, $[y_3, x_2, x_5, x_3, z] \supseteq C_5$ and $[y_1, y_2, y_4, y_5, x_4] \supseteq C_5$. If $x_1 \to (L_i, z; \{y_5, x_3\})$ for some $z \in V(L_i)$, then $[L_i - z + x_1] \supseteq C_5$, $[y_4, y_5, z, x_2, x_3] \supseteq C_5$ and $[y_1, y_2, y_3, x_4, x_5] \supseteq C_5$.

If $x_3 \to (L_i, z; \{y_3, y_5\})$ for some $z \in V(L_i)$, then $[L_i - z + x_3] \supseteq C_5$, $[y_1, y_2, y_3, z, y_5] \supseteq C_5$ and $[x_1, x_2, x_4, x_5, y_4] \supseteq C_5$. If $x_3 \to (L_i, z; \{y_3, x_1\})$ for some $z \in V(L_i)$, then $[L_i - z + x_3] \supseteq C_5$, $[x_1, y_2, x_2, y_3, z] \supseteq C_5$ and $[y_1, y_5, y_4, x_4, x_5] \supseteq C_5$. If $y_5 \to (L_i, z; \{x_3, x_1\})$ for some $z \in V(L_i)$, then $[L_i - z + y_5] \supseteq C_5$, $[x_1, x_2, x_3, x_5, z] \supseteq C_5$ and $[y_1, y_2, y_3, y_4, x_4] \supseteq C_5$. If $y_5 \to (L_i, z; \{y_3, x_1\})$ for some $z \in V(L_i)$, then $[L_i - z + y_5] \supseteq C_5$, $[y_1, y_2, y_3, x_1, z] \supseteq C_5$ and $[x_2, x_3, x_4, x_5, y_4] \supseteq C_5$. □

By Claim 1 and Lemma 4(e), there are vertex labellings $R = \{a_1, a_2, a_3, a_4\}$ and $L_i = b_1b_2b_3b_4b_5b_1$ such that $e(a_1a_2, b_1b_2b_3b_4) = 8$, $e(a_3, b_1b_5b_4) = 3$ and $e(a_4, b_1b_4) = 2$. Recall that $e(x_4x_5, y_1y_2y_4) = 6$ if $|N(x_2)|L_t| = 4$, and $e(x_4x_5, y_1y_2y_4) \geq 5$ and $\tau(L_t) = 4$ if $|N(x_2)|L_t| = 5$. Assume first that $|N(x_2)|L_t| = 5$. Then x_1, y_3 and y_5 are symmetric in $[L_t+x_1]$. Further, y_1, y_2 and y_4 are symmetric in $[L_t]$. Without loss of generality, assume that $e(x_4x_5, y_1y_4) = 4$. If $x_3 \in \{a_1, a_2\}$, say $\{x_3, y_3\} = \{a_1, a_2\}$, then $[x_1, b_1, y_5, b_4, b_5] \supseteq C_5$, $[x_2, x_3, x_4, x_5, y_1] \supseteq C_5$ and $[b_2, b_3, y_2, y_3, y_4] \supseteq B$, a contradiction. If $x_3 \notin \{a_1, a_2\}$, say $\{x_1, y_3\} = \{a_1, a_2\}$, then $[x_3, y_5, b_1, b_5, b_4] \supseteq C_5$, $[x_1, x_2, x_4, x_5, y_1] \supseteq C_5$ and $[b_2, b_3, y_2, y_3, y_4] \supseteq B$, again a contradiction.

Next, assume that $|N(x_2)|L_t| = 4$. If $e(x_2, y_3y_5) = 2$, then x_1, y_3 and y_5 are symmetric in $[L_t + x_1]$. If $x_3 \in \{a_1, a_2\}$, say $\{x_3, y_3\} = \{a_1, a_2\}$, then $[x_1, b_1, y_5, b_4, b_5] \supseteq C_5$, $[x_2, x_3, x_4, x_5, y_1] \supseteq C_5$ and $[b_2, b_3, y_2, y_3, y_4] \supseteq B$, or $[x_1, b_1, y_5, b_4, b_5] \supseteq C_5$, $[x_2, x_3, x_4, x_5, y_2] \supseteq C_5$ and $[b_2, b_3, y_1, y_3, y_4] \supseteq B$, a contradiction. If $x_3 \notin \{a_1, a_2\}$, say $\{x_1, y_3\} = \{a_1, a_2\}$, then $[x_3, y_5, b_1, b_5, b_4] \supseteq C_5$, $[y_3, x_2, x_5, x_4, y_4] \supseteq C_5$ and $[b_2, b_3, x_1, y_1, y_2] \supseteq B$, again a contradiction. If $e(x_2, y_3y_5) = 1$, then $N(x_2) \,|\, L_t = \{y_1, y_2, y_3, y_4\}$ or $N(x_2) \,|\, L_t = \{y_1, y_2, y_4, y_5\}$. Without loss of generality, assume that $N(x_2) \,|\, L_t = \{y_1, y_2, y_3, y_4\}$. By the symmetry of x_1 and y_3 in $[L_t + x_1]$, we need only to consider the following cases. If $\{x_3, y_3\} = \{a_1, a_2\}$, then $[x_1, b_1, y_5, b_4, b_5] \supseteq C_5$, $[x_2, x_3, x_4, x_5, y_1] \supseteq C_5$ and $[b_2, b_3, y_2, y_3, y_4] \supseteq B$, a contradiction. If $\{x_3, y_5\} = \{a_1, a_2\}$, then $[x_1, b_1, y_3, b_4, b_5] \supseteq C_5$, $[x_2, x_3, x_4, x_5, y_1] \supseteq C_5$ and $[b_2, b_3, y_2, y_4, y_5] \supseteq B$, a contradiction. If $\{x_1, y_5\} = \{a_1, a_2\}$, then $[x_3, b_1, y_3, b_4, b_5] \supseteq C_5$, $[x_1, x_2, x_4, x_5, y_2] \supseteq C_5$ and $[b_2, b_3, y_1, y_4, y_5] \supseteq B$, a contradiction. If $\{x_1, y_3\} = \{a_1, a_2\}$, then $[x_3, b_1, y_5, b_4, b_5] \supseteq C_5$, $[x_1, x_2, x_4, x_5, y_4] \supseteq C_5$ and $[b_2, b_3, y_1, y_2, y_3] \supseteq B$, again a contradiction.

To prove (d), suppose to the contrary that $e(T, L_t) \geq 12$. Assume first $N(x_1) \,|\, L_t = \{y_i, y_{i+1}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_2\}$. For any $i \in \{1, 2, 3, 4, 5\}$, if $[(V(L_t) \cup \{x_1, x_2\}) \backslash \{y_i, y_{i+1}\}] \supseteq C_5$, then $e(y_i y_{i+1}, x_4 x_5) \leq 2$ for otherwise $[x_3, x_4, x_5, y_i, y_{i+1}] \supseteq C_5$. It follows that if $x_2 y_4 \in E(G)$, then $e(y_1 y_5, x_4 x_5) \leq 2$, $e(y_2 y_3, x_4 x_5) \leq 2$ and so $e(T, L_t) \leq 11$, a contradiction. Hence $x_2 y_4 \notin E(G)$. If $e(x_2, y_3 y_5) = 2$, then $e(y_3 y_4, x_4 x_5) \leq 2$, $e(y_4 y_5, x_4 x_5) \leq 2$ and so $e(y_1 y_2, x_4 x_5) = 4$ and $e(x_2, y_1 y_2 y_3 y_5) = 4$. Further, $y_3 x_5 \notin E(G)$ and $y_5 x_4 \notin E(G)$ as $G_t \not\supseteq 2C_5$. Moreover, $x_4 y_4, x_5 y_5 \in E(G)$ as $e(T, L_t) \geq 12$. Clearly, $G_t \supseteq 2C_5$, a contradiction. Therefore, $e(x_2, y_3 y_5) \leq 1$ and hence $e(x_2, y_1 y_2 y_3 y_5) \leq 3$. If $e(x_2, y_1 y_2 y_3 y_5) = 3$, say $x_2 y_3 \in E(G)$, then $e(x_4 x_5, y_4 y_5) \leq 2$. It follows that $e(T, L_t) \leq 11$, a contradiction. Hence $e(x_2, y_1 y_2 y_3 y_5) \leq 2$. Then $e(x_4 x_5, L_t) = 10$ and so $[x_1, y_1, x_4, y_3, y_2] \supseteq C_5$ and $[y_4, y_5, x_2, x_3, x_5] \supseteq B$, a contradiction. Next, assume that $N(x_1) \,|\, L_t = \{y_i, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_3\}$. As $x_1 \to (L_t, y_2)$, $e(y_2, T) \leq 1$. If $e(x_2, y_1 y_3) \geq 1$, then $[x_1, x_2, y_1, y_2, y_3] \supseteq C_5$ and so $e(x_4 x_5, y_4 y_5) \leq 2$. Then $e(T, L_t) = e(x_2, L_t) + e(x_4 x_5, y_4 y_5) + e(x_4 x_5, y_1 y_3) + e(x_4 x_5, y_2) \leq 11$, no matter whether $x_2 y_2 \in E(G)$ or $x_2 y_2 \notin E(G)$, a contradiction. Therefore, $e(x_2, y_1 y_3) = 0$ and hence $e(T, L_t) \leq 11$, again a contradiction.

To prove (e), suppose to the contrary that $e(T, L_t) \geq 13$. Without loss of generality assume that $x_1 y_1 \in E(G)$. If $e(x_2, y_3 y_4) \geq 1$, say $x_2 y_3 \in E(G)$, then $e(y_4 y_5, x_4 x_5) \leq 2$ for otherwise $[y_1, y_2, y_3, x_1, x_2] \supseteq C_5$ and $[x_3, x_4, x_5, y_4, y_5] \supseteq C_5$. Therefore, $6 \geq e(x_4 x_5, y_1 y_2 y_3) = e(T, L_t) - e(x_2, L_t) - e(x_4 x_5, y_4 y_5) \geq 6$. Then $e(x_4 x_5, y_1 y_2 y_3) = 6$ and $e(x_2, L_t) = 5$. Further, $[x_1, y_1, y_5, y_4, x_2] \supseteq C_5$ and $[y_2, y_3, x_3, x_4, x_5] \supseteq C_5$, a contradiction. Therefore, $e(x_2, y_3 y_4) = 0$ and hence $e(x_2, L_t) = 3$, $e(x_4 x_5, L_t) = 10$. It follows that $[x_1, y_1, x_5, x_3, x_2] \supseteq C_5$ and $[y_2, y_3, y_4, y_5, x_4] \supseteq C_5$, again a contradiction. ∎

Lemma 13. $G \supseteq K_4^+ \uplus (k-1) C_5$.

Proof. Suppose to the contrary that $G \not\supseteq K_4^+ \uplus (k-1) C_5$. By Lemma 11, let $\psi = \{H, L_1, L_2, \ldots, L_{k-1}\}$ be an optimal family of G with $H \cong F_1$. Suppose that $e(H, L_i) \geq 16$ for some $i \in \{1, 2, \ldots, k-1\}$, say $e(H, L_1) \geq 16$. By Lemma 6(a) and Lemma 10, we may first assume that there exists a labelling $L_1 = y_1 y_2 y_3 y_4 y_5 y_1$ with property $\mathbf{P}_2$. Let $R = \{x_1, x_4, y_3, y_5\}$ and $G_1 = [H, L_1]$. Then $e(R, V(G_1) - R) \leq 16$ and so $e(R, G - V(G_1)) \geq (d_{G_1}(x_1) + d_{G_1}(x_4)) + (d_{G_1}(y_3) +$

$d_{G_1}(y_5)) - 16 \geq 12k - 16 = 12(k-2) + 8$. Hence, $e(R, L_i) \geq 13$ for some $i \in \{2, 3, \dots, k-1\}$. So by Lemma 6(b), $[G_1, L_i] \supseteq K_4^+ \uplus 2C_5$ and, therefore $G \supseteq K_4^+ \uplus (k-1)C_5$, a contradiction. Next, we assume that $G \supseteq K_4^+ \uplus B \uplus (k-2)C_5$ and let $K_4^+, B, L_1, L_2, \dots, L_{k-2}$ be k disjoint subgraphs of G with $L_i \cong C_5$ for $i \in \{1, 2, \dots, k-2\}$. Let $B = y_1y_2y_3y_1y_4y_5y_1$ and $R' = \{y_2, y_3, y_4, y_5\}$. We claim that $e(R', K_4^+) \leq 15$. Suppose to the contrary that $e(R', K_4^+) \geq 16$. If $e(x_1, R') \leq 1$, then $e(R', x_2x_3x_4x_5) \geq 15$. It is easy to see that $[K_4^+, B] \supseteq K_4^+ \uplus C_5$, a contradiction. If $e(x_1, R') \geq 3$, say $x_1y_2, x_1y_3, x_1y_4 \in E(G)$, then $[B - y_i + x_1] \supseteq C_5$ for all $i \in \{2, 3, 5\}$. However, $e(y_j, x_2x_3x_4x_5) \geq 2$ for some $j \in \{2, 3, 5\}$ and so $[K_4^+ - x_1 + y_j] \supseteq C_5$, a contradiction. If $e(x_1, R') = 2$, then we just need to consider $x_1y_2, x_1y_4 \in E(G)$ and $x_1y_2, x_1y_3 \in E(G)$. If $x_1y_2, x_1y_4 \in E(G)$, then $[B - y_i + x_1] \supseteq C_5$ for all $i \in \{3, 5\}$. However, $e(y_j, x_2x_3x_4x_5) \geq 2$ for some $j \in \{3, 5\}$ and so $[K_4^+ - x_1 + y_j] \supseteq C_5$, a contradiction. If $x_1y_2, x_1y_3 \in E(G)$, then $e(x_2, y_2y_3) = 0$ for otherwise $[x_1, x_2, y_1, y_2, y_3] \supseteq C_5$ and $[x_3, x_4, x_5, y_4, y_5] \supseteq C_5$, a contradiction. Then $e(y_2y_3, x_3x_4x_5) = 6$ and $e(y_4y_5, x_2x_3x_4x_5) = 8$ and so $[K_4^+, B] \supseteq 2C_5$, again a contradiction. Further, $e(R', G - V(K_4^+ \cup B)) \geq 12k - 15 - 8 = 12(k-2) + 1$. Hence $e(R', L_j) \geq 13$ for some $j \in \{1, 2, \dots, k-2\}$. So by Lemma 4(b), $[B, L_j] \supseteq 2C_5$, i.e., $G \supseteq K_4^+ \uplus (k-1)C_5$, a contradiction. Therefore, $e(H, L_i) \leq 15$ for each $i \in \{1, 2, \dots, k-1\}$. It follows that $d_G(x_1) > 3k$, for otherwise, we obtain $e(H, G - V(H)) = (d_G(x_1) + d_G(x_3)) + (d_G(x_2) + d_G(x_4)) + (d_G(x_1) + d_G(x_5)) - d_G(x_1) - 12 \geq 18k - d_G(x_1) - 12 \geq 15k - 12 > 15(k-1)$, then there exists $i \in \{1, 2, \dots, k-1\}$ such that $e(H, L_i) \geq 16$, a contradiction.

For r with $0 \leq r \leq 5$, let $\mathcal{A}_r = \{L_t \mid e(x_1, L_t) = r,\ 1 \leq t \leq k-1\}$ and $a_r = |\mathcal{A}_r|$. It is clear that $a_0 + a_1 + a_2 + a_3 + a_4 + a_5 = k - 1$. Further, it can be seen that

$$(1)\quad d_G(x_1) = d_H(x_1) + \sum_{r=0}^{5} \sum_{L_t \in \mathcal{A}_r} e(x_1, L_t) = 1 + a_1 + 2a_2 + 3a_3 + 4a_4 + 5a_5.$$

Let $R_1 = \{x_1, x_2, x_4, x_5\}$. By Lemma 12, we obtain

$$(2)\quad \begin{aligned} \sum_{x \in R_1} d_G(x) &= \sum_{x \in R_1} d_H(x) + \sum_{r=0}^{5} \sum_{L_t \in \mathcal{A}_r} e(R_1, L_t) \\ &\leq 9 + 15a_0 + 13a_1 + 13a_2 + 12a_3 + 11a_4 + 10a_5. \end{aligned}$$

By (1) and (2), we obtain $d_G(x_1) + \sum_{x \in R_1} d_G(x) \leq 10 + 15a_0 + 14a_1 + 15a_2 + 15a_3 + 15a_4 + 15a_5 = 15k - 5 - a_1$. But by the degree sum condition, we have $d_G(x_1) + \sum_{x \in R_1} d_G(x) \geq 15k$, which is a contradiction. ■

Lemma 14. *For any family $\{H, L_1, \dots, L_{k-1}\}$ of G with $H \cong K_4^+$, $d_G(x_2) < 3k$.*

Proof. Suppose to the contrary that $d_G(x_2) \geq 3k$ for some family $\psi = \{H, L_1, \ldots, L_{k-1}\}$ with $H \cong K_4^+$. Further, we assume that $\sum_{i=1}^{k-1} \tau(L'_i) \leq \sum_{i=1}^{k-1} \tau(L_i)$ for any family $\{H', L'_1, \ldots, L'_{k-1}\}$ with $H' \cong K_4^+$ and $d_G(x_2) \geq 3k$. Let $Q = [x_2, x_3, x_4, x_5]$ and $T = [x_3, x_4, x_5]$. Then $Q \cong K_4$ and $T \cong K_3$. For r with $0 \leq r \leq 5$, let $\mathcal{B}_r = \{L_t \mid e(x_1, L_t) = r, 1 \leq t \leq k-1\}$ and $b_r = |\mathcal{B}_r|$. It is clear that $b_0 + b_1 + b_2 + b_3 + b_4 + b_5 = k - 1$.

Claim 2. *For each $t \in \{1, 2, \ldots, k-1\}$, the following statements hold.*

(a) *If $e(x_1, L_t) = 5$, then $e(Q, L_t) \leq 5$.*

(b) *If $e(x_1, L_t) = 4$, then $e(Q, L_t) \leq 9$ except possible one L_t with $e(Q, L_t) = 10$.*

(c) *If $e(x_1, L_t) = 3$, then $e(Q, L_t) \leq 12$.*

(d) *If $e(x_1, L_t) = 2$, then $e(Q, L_t) \leq 15$.*

(e) *If $e(x_1, L_t) = 1$, then $e(Q, L_t) \leq 16$.*

(f) *If $e(x_1, L_t) = 0$, then $e(Q, L_t) \leq 20$.*

Proof. Let $L_t = y_1y_2y_3y_4y_5y_1$ and $G_t = [H, L_t]$. If $e(x_1, L_t) = 5$, then $e(y_i, Q) \leq 1$ for all $i \in \{1, 2, 3, 4, 5\}$ since $G_t \not\supseteq 2C_5$. Hence, (a) follows directly.

To prove (b), say $N(x_1) \mid L_t = \{y_1, y_2, y_3, y_4\}$. First, we claim that $\{y_1y_3, y_2y_4, y_1y_4\} \subseteq E(L_t)$, $N(x_2) \mid L_t = \{y_1, y_2, y_3, y_4\}$ and $N(x_3) \mid L_t = N(x_4) \mid L_t = N(x_5) \mid L_t = \{y_1, y_4\}$ if $e(Q, L_t) \geq 10$. It is clear that $\tau(y_5, L_t) = 0$ for otherwise $x_1 \to L_t$ and so $G_t \supseteq 2C_5$. As $x_1 \to (L_t, y_i)$ for $i \in \{2, 3, 5\}$, $e(y_i, Q) \leq 1$. Hence, $e(y_1y_4, Q) \geq 7$, say $e(y_1, Q) \geq 3$ and $e(y_4, Q) = 4$. If $x_2y_5 \in E(G)$, then $[Q + y_5] \supseteq K_4^+$ and $\tau(x_1y_1y_2y_3y_4x_1) > \tau(L_t)$, contradicting the definition of ψ. Hence $x_2y_5 \notin E(G)$. We say that $e(x_i, y_j) = 0$ for all $i \in \{3, 4, 5\}$ and $j \in \{2, 3\}$. If not, then $[x_1, x_2, x_i, y_2, y_3] \supseteq C_5$ and $[(V(T) \cup \{y_1, y_4, y_5\}) \backslash \{x_i\}] \supseteq C_5$. If $e(y_5, T) \geq 1$, then $x_2y_1 \notin E(G)$ for otherwise $[x_1, x_2, y_1, y_2, y_3] \supseteq C_5$ and $[x_3, x_4, x_5, y_4, y_5] \supseteq C_5$. Hence $e(y_1, T) = 3$. Then $[x_1, x_2, y_2, y_3, y_4] \supseteq C_5$ and $[x_3, x_4, x_5, y_1, y_5] \supseteq C_5$, a contradiction. Therefore, $e(y_5, T) = 0$. Then $e(y_1, Q) = e(y_4, Q) = 4$ and $e(y_2y_3, x_2) = 2$. If $y_1y_3 \notin E(G)$ or $y_2y_4 \notin E(G)$, then $[x_3, x_2, x_1, y_3, y_2] \supseteq K_4^+$ and $\tau(y_1x_4x_5y_4y_5y_1) > \tau(L_t)$, a contradiction. Hence $y_1y_3, y_2y_4 \in E(G)$. If $y_1y_4 \notin E(G)$, then either $d_G(y_1) \geq 3k$ or $d_G(y_4) \geq 3k$, say $d_G(y_1) \geq 3k$. Then $[y_5, y_1, y_2, y_3, x_1] \supseteq K_4^+$ and $\tau(y_4x_2x_3x_4x_5y_4) > \tau(L_t)$, contradicting the definition of ψ. Hence, $y_1y_4 \in E(G)$. Next, we claim that at most one L_t with $e(x_1, L_t) = 4$ and $e(Q, L_t) = 10$. If not, then there is another one $L'_t = z_1z_2z_3z_4z_5z_1$ with $e(x_1, L'_t) = 4$ and $e(Q, L'_t) = 10$. Without loss of generality, assume that $N(x_1) \mid L'_t = \{z_1, z_2, z_3, z_4\}$. Then $\{z_1z_3, z_2z_4, z_1z_4\} \subseteq E(L'_t)$, $N(x_2) \mid L'_t = \{z_1, z_2, z_3, z_4\}$ and $N(x_3) \mid L'_t = N(x_4) \mid L'_t = N(x_5) \mid L'_t = \{z_1, z_4\}$. Therefore, $[y_1, y_5, y_4, x_4, x_5] \supseteq C_5$, $[x_2, z_4, z_5, z_1, x_3] \supseteq C_5$ and $[y_2, y_3, x_1, z_2, z_3] \supseteq B$, contradicting Lemma 10.

To prove (c), suppose to the contrary that $e(Q, L_t) \geq 13$. Assume first $N(x_1) \mid L_t = \{y_i, y_{i+1}, y_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \mid L_t = \{y_1, y_2, y_4\}$.

As $x_1 \to (L_t, y_i)$ for all $i \in \{3,5\}$, $e(y_i, Q) \leq 1$. Further, $y_3y_5 \notin E(G)$ as $x_1 \not\to L_t$. It follows that $e(y_1y_2y_4, Q) \geq 11$ and so $e(x_2, y_1y_4) \geq 1$, $e(x_2, y_2y_4) \geq 1$. Then $[x_1, x_2, y_1, y_5, y_4] \supseteq C_5$ and $[x_1, x_2, y_2, y_3, y_4] \supseteq C_5$. Further, as $e(y_i, T) \geq 2$ for all $i \in \{1,2\}$, $e(y_3y_5, T) = 0$ for otherwise $[x_3, x_4, x_5, y_2, y_3] \supseteq C_5$ or $[x_3, x_4, x_5, y_1, y_5] \supseteq C_5$, a contradiction. Then $N(y_3)|Q \cup N(y_5)|Q \subseteq \{x_2\}$.

Let $R = \{x_1, x_4, y_3, y_5\}$. Then $e(R, V(G_t) - R) \leq 18$ and so $e(R, G - V(G_t)) \geq 12k - 18 = 12(k-2) + 6$ and hence $e(R, L_i) \geq 13$ for some $i \in \{1, 2, 3, \ldots, k-1\} \backslash \{t\}$. Note that $e(Q, L_t) \geq 13$ and $N(y_3)|Q \cup N(y_5)|Q \subseteq \{x_2\}$. If $u \to (L_i, z; \{v, w\})$ for some $z \in V(L_i), u \in R$ and $\{v, w\} \subseteq R \setminus \{u\}$, then $[G_t, L_i] \supseteq 3C_5$, a contradiction. By Lemma 4(e), there are vertex labellings $L_i = z_1z_2z_3z_4z_5z_1$ and $R = \{a_1, a_2, a_3, a_4\}$ such that $e(a_1a_2, z_1z_2z_3z_4) = 8$, $e(a_3, z_1z_5z_4) = 3$ and $e(a_4, z_1z_4) = 2$. If $\{a_1, a_2\} = \{x_1, x_4\}$, then set $\{r, s\} = \{1, 2\}$ with $y_r \in N(x_1) \,|\, L_t \cap N(x_4) \,|\, L_t$. We can see that $[x_1, y_r, x_4, z_2, z_3] \supseteq C_5$, $[y_3, y_5, z_1, z_5, z_4] \supseteq C_5$ and $[x_2, x_3, x_5, y_s, y_4] \supseteq C_5$, a contradiction. If $\{a_1, a_2\} = \{x_1, y_i\}$ for some $i \in \{3, 5\}$, say $\{a_1, a_2\} = \{x_1, y_5\}$, then $[x_1, y_1, y_5, z_2, z_3] \supseteq C_5$, $[y_3, x_4, z_1, z_5, z_4] \supseteq C_5$ and $[y_2, y_4, x_2, x_3, x_5] \supseteq C_5$, a contradiction. If $\{a_1, a_2\} = \{x_4, y_i\}$ for some $i \in \{3, 5\}$, say $\{a_1, a_2\} = \{x_4, y_5\}$, then we set $\{r, s\} = \{1, 4\}$ with $x_4y_r \in E(G)$. It is clear that $[x_4, y_r, y_5, z_2, z_3] \supseteq C_5$, $[x_1, y_3, z_1, z_5, z_4] \supseteq C_5$ and $[x_2, x_3, x_5, y_2, y_s] \supseteq C_5$, a contradiction. Hence $\{a_1, a_2\} = \{y_3, y_5\}$. Therefore, $[y_3, y_4, y_5, z_2, z_3] \supseteq C_5$, $[x_1, x_4, z_1, z_5, z_4] \supseteq C_5$ and $[x_2, x_3, x_5, y_1, y_2] \supseteq C_5$, again a contradiction.

Next, assume that $N(x_1) \,|\, L_t = \{y_i, y_{i+1}, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_2, y_3\}$. As $x_1 \to (L_t, y_2)$, $e(y_2, Q) \leq 1$. Suppose $e(x_2, y_4y_5) \geq 1$, say $x_2y_5 \in E(G)$. Then $e(y_3y_4, T) \leq 3$ for otherwise $[x_1, x_2, y_5, y_1, y_2] \supseteq C_5$ and $[x_3, x_4, x_5, y_3, y_4] \supseteq C_5$. Further, if $x_2y_4 \in E(G)$, then similarly, $e(y_1y_5, T) \leq 3$ and so $e(Q, L_t) \leq 12$, a contradiction. Hence $x_2y_4 \notin E(G)$. Since $e(Q, L_t) \geq 13$, then $e(y_1y_5, Q) = 8$, $e(y_3y_4, T) = 3$, $e(y_2, Q) = 1$ and $x_2y_3 \in E(G)$. If $e(y_4, T) \geq 1$, then $[x_3, x_4, x_5, y_4, y_5] \supseteq C_5$ and $[y_1, y_2, y_3, x_1, x_2] \supseteq C_5$, a contradiction. Thus $e(y_4, T) = 0$ and so $e(y_3, T) = 3$. Then $[x_1, y_2, y_1, x_3, x_2] \supseteq C_5$ and $[y_3, y_4, y_5, x_4, x_5] \supseteq C_5$, a contradiction. Therefore, $e(x_2, y_4y_5) = 0$. If $e(x_2, y_1y_3) \geq 1$, then $[x_1, x_2, y_1, y_2, y_3] \supseteq C_5$ and so $e(y_4y_5, T) \leq 3$. It follows that $e(Q, L_t) = e(y_4y_5, T) + e(y_1y_3, T) + e(y_2, T) + e(x_2, y_1y_2y_3) \leq 12$, no matter whether y_2x_2 is an edge of $E(G)$ or not, a contradiction. Therefore, $e(x_2, y_1y_3y_4y_5) = 0$ and hence $e(T, y_1y_3y_4y_5) = 12$. Then $[x_1, y_3, x_3, y_1, y_2] \supseteq C_5$ and $[x_2, x_4, y_4, y_5, x_5] \supseteq C_5$, again a contradiction.

To prove (d), suppose to the contrary that $e(Q, L_t) \geq 16$. Assume first $N(x_1) \,|\, L_t = \{y_i, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_3\}$. As $x_1 \to (L_t, y_2)$, $e(y_2, Q) \leq 1$. Therefore, $e(Q, y_1y_3y_4y_5) \geq 15$ and hence $e(x_2, y_1y_3) \geq 1$. Then $[x_1, x_2, y_1, y_2, y_3] \supseteq C_5$ and so $e(y_4y_5, T) \leq 3$. This implies that $e(Q, y_1y_3y_4y_5) \leq 13$, a contradiction. Next, assume that $N(x_1) \,|\, L_t = \{y_i, y_{i+1}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_2\}$. If $x_2y_4 \in E(G)$, then

$e(y_2y_3, T) \le 3$ for otherwise $[x_1, x_2, y_4, y_5, y_1] \supseteq C_5$ and $[x_3, x_4, x_5, y_2, y_3] \supseteq C_5$, a contradiction. Similarly, if $x_2y_4 \in E(G)$, then $e(y_1y_5, T) \le 3$. This means that $e(Q, L_t) \le 14$, a contradiction. Hence $x_2y_4 \notin E(G)$. If $e(x_2, y_3y_5) \ge 1$, say $x_2y_5 \in E(G)$, then $[x_1, x_2, y_5, y_1, y_2] \supseteq C_5$ and so $e(y_3y_4, T) \le 3$. Further, $e(y_1y_2y_5, Q) = 12$ as $e(Q, L_t) \ge 16$. This implies that $[x_1, y_1, x_5, x_4, x_2] \supseteq C_5$ and $[y_2, y_3, y_4, y_5, x_3] \supseteq C_5$, a contradiction. Hence $e(x_2, y_3y_5) = 0$. If $y_3x_3 \in E(G)$, then $e(y_1y_4y_5, x_4x_5) \le 4$ for otherwise $[x_1, y_2, y_3, x_3, x_2] \supseteq C_5$ and $[x_4, x_5, y_1, y_4, y_5] \supseteq C_5$. Thus $e(Q, L_t) = e(x_2, y_1y_2)+e(x_3, L_t)+e(x_4x_5, y_1y_4y_5)+e(x_4x_5, y_2y_3) \le 15$, a contradiction. Then $y_3x_3 \notin E(G)$. Similarly, $y_3x_4, y_3x_5 \notin E(G)$. Then $e(Q, L_t) = e(x_2, y_1y_2) + e(x_3x_4x_5, y_1y_2y_4y_5) \le 14$, again a contradiction.

To prove (e), suppose to the contrary that $e(Q, L_t) \ge 17$. Without loss of generality assume that $x_1y_1 \in E(G)$. Suppose $e(x_2, y_3y_4) \ge 1$, say $x_2y_3 \in E(G)$. Then $[x_1, x_2, y_1, y_2, y_3] \supseteq C_5$ and so $e(y_4y_5, T) \le 3$. Since $e(Q, L_t) \ge 17$, $e(y_1y_2y_3, Q) = 12$, $e(y_4y_5, T) = 3$ and $e(x_2, y_4y_5) = 2$. Then $[x_1, x_2, y_4, y_5, y_1] \supseteq C_5$ and $[x_3, x_4, x_5, y_2, y_3] \supseteq C_5$, a contradiction. Hence $e(x_2, y_3y_4) = 0$ and so $e(T, L_t) \ge 14$. This implies that $e(x_i, y_2y_5) = 2$ and $y_1x_j \in E(G)$ for some $\{i, j\} \subseteq \{3, 4, 5\}$ with $i \ne j$. Then $[L_t - y_1 + x_i] \supseteq C_5$ and $[H + y_1 - x_i] \supseteq C_5$, a contradiction. $\square$

Recall that $b_0 + b_1 + b_2 + b_3 + b_4 + b_5 = k - 1$. Further, we can see that

$$d_G(x_1) = d_H(x_1) + \sum_{r=0}^{5} \sum_{L_t \in \mathcal{B}_r} e(x_1, L_t) = 1 + b_1 + 2b_2 + 3b_3 + 4b_4 + 5b_5. \tag{3}$$

By Claim 2, there may exist one $L'_t \in \mathcal{B}_4$ with $e(H, L'_t) = 14$. Further, we obtain

$$\begin{aligned} \sum_{x \in V(H)} d_G(x) &= \sum_{x \in V(H)} d_H(x) + \sum_{r=0}^{3} \sum_{L_t \in \mathcal{B}_r} e(H, L_t) + \sum_{L_t \in \mathcal{B}_5} e(H, L_t) \\ &\quad + \sum_{L_t \in \mathcal{B}_4 \setminus \{L'_t\}} e(H, L_t) + e(H, L'_t) \\ &\le 14 + 20b_0 + 17b_1 + 17b_2 + 15b_3 + 10b_5 + (13b_4 + 1) \\ &= 15 + 20b_0 + 17b_1 + 17b_2 + 15b_3 + 13b_4 + 10b_5. \end{aligned} \tag{4}$$

Combining (4) with (3), we obtain $2d_G(x_1)+\sum_{x \in V(H)} d_G(x) \le 17+20b_0+19b_1+21b_2 + 21b_3 + 21b_4 + 20b_5 = 21k - b_0 - 2b_1 - b_5 - 4$. Since $d_G(x_2) \ge 3k$, we have $2d_G(x_1) + \sum_{x \in V(H)} d_G(x) = d_G(x_2) + (d_G(x_1) + d_G(x_3)) + (d_G(x_1) + d_G(x_4)) + (d_G(x_1) + d_G(x_5)) \ge 21k$, which is a contradiction. ■

Let $\psi = \{H, L_1, L_2, \dots, L_{k-1}\}$ be an optimal family of G with $H \cong K_4^+$, and let $\mathcal{B}_r$ and b_r be defined as in the proof of Lemma 14. Let $Q = [x_2, x_3, x_4, x_5]$ and $T = [x_3, x_4, x_5]$.

Lemma 15. *For each $t \in \{1, 2, \ldots, k-1\}$, the following statements hold.*

(a) *If $e(x_1, L_t) = 5$, then $e(T, L_t) \le 3$.*

(b) *If $e(x_1, L_t) = 4$, then $e(T, L_t) \le 6$.*

(c) *If $e(x_1, L_t) = 3$, then $e(T, L_t) \le 9$.*

(d) *If $1 \le e(x_1, L_t) \le 2$, then $e(T, L_t) \le 12$.*

(e) *If $e(x_1, L_t) = 0$, then $e(T, L_t) \le 15$.*

Proof. Let $L_t = y_1y_2y_3y_4y_5y_1$ and $G_t = [H, L_t]$. To prove (a), suppose to the contrary that $e(T, L_t) \ge 4$. As $G_t \not\supseteq 2C_5$, $e(y_i, Q) \le 1$ for all $i \in \{1, 2, 3, 4, 5\}$. Further, by the optimality of ψ, we obtain $[L_t] \cong K_5$. Without loss of generality assume that $y_1x_3 \in E(G)$. Then $y_1x_2 \notin E(G)$ and so $d_G(y_1) \ge 3k$, $[Q+y_1] \supseteq K_4^+$, $[L_t - y_1 + x_1] \cong K_5$. Therefore, we may assume that $d_G(x_1) \ge 3k$. Since $e(T, L_t) \ge 4$, $e(x_l, L_t) \ge 2$ for some $l \in \{3, 4, 5\}$, say $x_3y_1, x_3y_2 \in E(G)$. Then $[x_2, x_1, y_3, y_4, y_5] \supseteq K_4^+$, $[x_3, x_4, x_5, y_1, y_2] \supseteq B$. Let $L' = [x_3, x_4, x_5, y_1, y_2]$ and $R' = \{y_1, y_2, x_4, x_5\}$. Note that $e(R', x_2x_1y_3y_4y_5) \le 11$ and $\sum_{x \in R'} d_{L'}(x) = 8$. Then $e(R', G - V(G_t)) \ge 12k - 19 = 12(k-2) + 5$ and so $e(R', L_i) \ge 13$ for some $i \in \{1, \ldots, k-1\} \backslash \{t\}$. By Lemma 4(b), $[L', L_i] \supseteq 2C_5$. This contradicts Lemma 14 as $[x_2, x_1, y_3, y_4, y_5] \supseteq K_4^+$ and $d_G(x_1) \ge 3k$.

To prove (b), suppose to the contrary that $e(T, L_t) \ge 7$. Without loss of generality assume that $N(x_1) \,|\, L_t = \{y_1, y_2, y_3, y_4\}$. It is clear that $\tau(y_5, L_t) = 0$ for otherwise $x_1 \to L_t$ and so $G_t \supseteq 2C_5$. As $x_1 \to (L_t, y_i)$ for $i \in \{2, 3, 5\}$, $e(y_i, T) \le 1$. If $e(y_5, T) = 1$, then $[Q + y_5] \supseteq K_4^+$ and $\tau(L_t - y_5 + x_1) > \tau(L_t)$, contradicting the optimality of ψ. Therefore, $e(y_5, T) = 0$ and hence $e(y_1y_4, T) \ge 5$. Without loss of generality, assume that $e(y_1, T) = 3$ and $y_4x_3, y_4x_4 \in E(G)$. Further, since $e(T, L_t) \ge 7$, $e(y_2y_3, T) \ge 1$. If $x_3 \in N(y_2)|T \cup N(y_3)|T$, then $[x_1, y_2, y_3, x_3, x_2] \supseteq C_5$ and $[y_1, y_4, y_5, x_4, x_5] \supseteq C_5$, a contradiction. If $x_4 \in N(y_2)|T \cup N(y_3)|T$, then $[x_1, y_2, y_3, x_4, x_2] \supseteq C_5$ and $[y_1, y_4, y_5, x_3, x_5] \supseteq C_5$, a contradiction. If $x_5 \in N(y_2)|T \cup N(y_3)|T$, then $[x_1, y_2, y_3, x_5, x_2] \supseteq C_5$ and $[y_1, y_4, y_5, x_3, x_4] \supseteq C_5$, again a contradiction.

To prove (c), suppose to the contrary that $e(T, L_t) \ge 10$. Assume first $N(x_1) \,|\, L_t = \{y_i, y_{i+1}, y_{i+2}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_2, y_3\}$. As $x_1 \to (L_t, y_2)$, $e(y_2, T) \le 1$. Without loss of generality assume that $e(y_1y_4, T) \ge e(y_3y_5, T)$. Then $[x_1, y_2, y_3, x_i, x_2] \supseteq C_5$ and $[(V(T) \cup \{y_1, y_4, y_5\}) \backslash \{x_i\}] \supseteq C_5$, where $x_i \in N(y_2)|T \cup N(y_3)|T$, a contradiction. Next, assume that $N(x_1) \,|\, L_t = \{y_i, y_{i+1}, y_{i+3}\}$ for some $i \in \{1, 2, 3, 4, 5\}$, say $N(x_1) \,|\, L_t = \{y_1, y_2, y_4\}$. As $x_1 \to (L_t, y_i)$ for $i \in \{3, 5\}$, $e(y_i, T) \le 1$. Since $e(T, L_t) \ge 10$, $e(y_1y_4, x_4x_5) \ge 3$ and $e(y_3y_5, T) \ge 1$, say $x_3y_3 \in E(G)$. Then $[x_1, y_2, y_3, x_3, x_2] \supseteq C_5$ and $[y_1, y_5, y_4, x_4, x_5] \supseteq C_5$, a contradiction.

To prove (d), suppose to the contrary that $e(T, L_t) \ge 13$. Without loss of generality assume that $x_1y_1 \in E(G)$. Since $e(T, L_t) \ge 13$, $e(y_2y_5, T) \ge 4$. Assume first that $e(y_2y_5, T) = 4$, say $x_3 \in N(y_2)|T \cap N(y_5)|T$. Further, since $e(T, L_t) \geq 13$, $e(y_1, T) = 3$. Then $[x_1, y_1, x_2, x_4, x_5] \supseteq C_5$ and $[x_3, y_2, y_3, y_4, y_5] \supseteq C_5$, a contradiction. Next, assume that $e(y_2y_5, T) \geq 5$, say $\{x_3, x_4\} \subseteq N(y_2)|T \cap N(y_5)|T$. Since $e(T, L_t) \geq 13$, $e(y_1, T) \geq 1$. If $y_1x_3 \in E(G)$, then $[x_1, y_1, x_2, x_3, x_5] \supseteq C_5$ and $[x_4, y_2, y_3, y_4, y_5] \supseteq C_5$, a contradiction. If $y_1x_4 \in E(G)$, then $[x_1, y_1, x_2, x_4, x_5] \supseteq C_5$ and $[x_3, y_2, y_3, y_4, y_5] \supseteq C_5$, a contradiction. If $y_1x_5 \in E(G)$, then $[x_1, y_1, x_2, x_4, x_5] \supseteq C_5$ and $[x_3, y_2, y_3, y_4, y_5] \supseteq C_5$, again a contradiction. ■

Let $R = \{x_1, x_3, x_4, x_5\}$. By Lemma 15, we obtain

$$\begin{aligned}\sum_{x\in R} d_G(x) &= \sum_{x\in R} d_H(x) + \sum_{r=0}^{5}\sum_{L_t\in\mathcal{B}_r} e(R, L_t) \\ &\leq 10 + 15b_0 + 13b_1 + 14b_2 + 12b_3 + 10b_4 + 8b_5.\end{aligned} \tag{5}$$

Combining (5) with (3), we have $2d_G(x_1) + \sum_{x\in R} d_G(x) \leq 12 + 15b_0 + 15b_1 + 18b_2 + 18b_3 + 18b_4 + 18b_5 = 18k - 3b_0 - 3b_1 - 6$. But by the degree sum condition, we have $2d_G(x_1) + \sum_{x\in R} d_G(x) \geq 18k$, a contradiction. This proves Theorem 3.

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