

$L(2, 1)$ -LABELING OF THE ITERATED MYCIELSKI GRAPHS OF GRAPHS AND SOME PROBLEMS RELATED TO MATCHING PROBLEMS

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Abstract

In this paper, we study the $L(2, 1)$ -labeling of the Mycielski graph and the iterated Mycielski graph of graphs in general. For a graph G and all $t \geq 1$, we give sharp bounds for $\lambda(M^t(G))$ the $L(2, 1)$ -labeling number of the t -th iterated Mycielski graph in terms of the number of iterations t , the order n of G , the maximum degree Δ , and $\lambda(G)$ the $L(2, 1)$ -labeling number of G . For $t = 1$, we present necessary and sufficient conditions between the 4-star matching number of the complement graph and $\lambda(M(G))$ the $L(2, 1)$ -labeling number of the Mycielski graph of a graph, with some applications to special graphs. For all $t \geq 2$, we prove that for any graph G of order n , we have $2^{t-1}(n+2) - 2 \leq \lambda(M^t(G)) \leq 2^t(n+1) - 2$. Thereafter, we characterize the graphs achieving the upper bound $2^t(n+1) - 2$, then by using the Marriage Theorem and Tutte's characterization of graphs with a perfect 2-matching, we characterize all graphs without isolated vertices achieving the lower bound $2^{t-1}(n+2) - 2$. We determine the $L(2, 1)$ -labeling number for the Mycielski graph and the iterated Mycielski graph of some graph classes.

Keywords: frequency assignment, $L(2, 1)$ -labeling, Mycielski construction, matching.

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1. INTRODUCTION

The graphs considered in this paper are finite, simple, and undirected. For graph terminology, we refer to [23].

In 1992, Griggs and Yeh [11] studied a variation of the frequency assignment problem [12], where close transmitters must receive different channels and closer transmitters must receive different channels at least two apart. This problem is known as the $L(2, 1)$ -labeling problem, the main target is to come up with a frequency assignment with low-frequency bandwidth.

Formally, the $L(2, 1)$ -labeling of a graph $G = (V, E)$ is a function f from the vertex set V to the set of all nonnegative integers such that $|f(x) - f(y)| \geq 2$ if $d_G(x, y) = 1$ and $|f(x) - f(y)| \geq 1$ if $d_G(x, y) = 2$, where $d_G(x, y)$ is the distance between the vertices x and y in G . The *span* of an $L(2, 1)$ -labeling f is the difference between the largest and the smallest label used by f . We may always consider zero as the smallest label used, so that the span is the highest label assigned. A k - $L(2, 1)$ -labeling is an $L(2, 1)$ -labeling with no label greater than k , the minimum k so that G has a k - $L(2, 1)$ -labeling is called the $L(2, 1)$ -labeling number or λ -number of G , and denoted by $\lambda(G)$. An $L(2, 1)$ -labeling with span $\lambda(G)$ is called a λ -labeling.

The $L(2, 1)$ -labeling has been extensively studied (see surveys [3, 24]). The determination of the exact value of $\lambda(G)$ is an NP-Hard problem for graphs in general, it is NP-Complete to determine whether a graph admits an $L(2, 1)$ -labeling with span at most $\lambda \geq 4$ [7], the problem remains NP-Complete even restricted to some graph families (see NP-completeness results references in [3]). Therefore, the aim of the research was to bound the λ -number for graphs. By using the greedy algorithm, Griggs and Yeh [11] proved that $\lambda(G) \leq \Delta^2 + 2\Delta$ for any graph G , where Δ is the maximum degree of G . This upper bound was later improved by Gonçalves in [10] to $\Delta^2 + \Delta - 2$, and it is the best known upper bound for $\lambda(G)$ in terms of the maximum degree for graphs in general. Griggs and Yeh [11] conjectured that $\lambda(G) \leq \Delta^2$, for any graph G with $\Delta \geq 2$, it is called Δ^2 -conjecture and is one of the most captivating open problems about graph labeling with distance conditions. This conjecture was proven to be true by Havet *et al.* [13] for graphs with a large maximum degree. The $L(2, 1)$ -labeling number attracted attention not only for general graphs but also when considering specific graph classes. The decision version of the $L(2, 1)$ -labeling problem has been proven to be polynomial for complete graphs, paths, cycles, wheels, trees, complete k -partite graphs, among other few graph classes. For an overview on the subject of the $L(2, 1)$ -labeling (and its generalizations), we refer the reader to the surveys [3, 24].

In this paper, we investigate the $L(2, 1)$ -labeling of the Mycielski graph and the iterated Mycielski graph of graphs. In search of triangle-free graphs with a

large chromatic number, Mycielski [19] used the following transformation.

Definition 1.1. For a given graph $G = (V, E)$ of order n with $V = \{v_1, v_2, \dots, v_n\}$, the Mycielski graph of G , denoted $M(G)$, is the graph with vertex set $V \cup V' \cup \{u\}$, where $V' = \{v'_i : v_i \in V\}$ and edge set $E \cup \{v_i v'_j : v_i v_j \in E\} \cup \{v'_i u : v'_i \in V'\}$. The vertex v'_i is called the copy of the vertex v_i and u is called the root of $M(G)$.

The t -th iterated Mycielski graph of G , denoted $M^t(G)$, is defined recursively with $M^0(G) = G$ and for $t \geq 1$ $M^t(G) = M(M^{t-1}(G))$. If $t = 1$, $M^1(G)$ is the Mycielski graph of G and is denoted simply $M(G)$. It is known that $\chi(M(G)) = \chi(G) + 1$, and $\omega(M(G)) = \max\{2, \omega(G)\}$, for any graph G , where $\chi(G)$ and $\omega(G)$ are respectively the chromatic number and the clique number of G . Many aspects and invariants of the Mycielski graphs have been studied (see for example [2, 4, 5, 8, 16, 17, 20]), Mycielski graphs are known to be hard-to-color instances and are used for testing coloring algorithms [4]. The $L(2, 1)$ -labeling of the Mycielski graph of graphs has been previously investigated in [17] and [20]. A *4-star matching* H of a graph G is a subgraph such that H is a collection of vertex disjoint star graphs $K_{1,1}$, $K_{1,2}$, $K_{1,3}$ or $K_{1,4}$. The *4-star matching number* is the maximum order of a 4-star matching of G . In [17], Lin and Lam gave sufficient conditions on the 4-star matching number of the complement graph \bar{G} , so that $\lambda(M(G)) \leq 2n$ and $\lambda(M(G)) = 2n + k$, for any $k \geq 1$. This allows them to prove that $\lambda(M(G))$ can be computed in polynomial time for graphs with diameter at most 2, and then give the λ -number of the Mycielski graph of complete graph K_n , and the Mycielski graph of the graph join of complete graph and the empty graph. Shao and Solis-Oba in [20], also studied the $L(2, 1)$ -labeling number of the Mycielski and the iterated Mycielski graph of graphs. The authors as well gave the λ -number of the Mycielski graph of complete graph, and depending on the number of iterations determine the exact value or give bounds for $\lambda(M^t(K_n))$, then provided bounds for $\lambda(M^t(G))$ for any graph G .

In this paper, we continue the work started by Lin and Lam [17], and Shao and Solis-Oba [20]. In Section 2, we give some preliminary results about the Mycielski and iterated Mycielski graph of graphs, and some previous results on the $L(2, 1)$ -labeling number of graphs.

Section 3 is dedicated to the $L(2, 1)$ -labeling number of $M(G)$. First, we provide bounds involving the order n , the maximum degree Δ and the λ -number of G . Then we complete the equivalence relationship between the 4-star matching number and the $L(2, 1)$ -labeling number of the Mycielski graph of a graph. Afterward, we give applications of this result to the $L(2, 1)$ -labeling number of the Mycielski graph of some particular graphs, not mentioned in [17]. The end of Section 3 is dedicated to graphs with a lower bound $\lambda(M(G)) = n + 1$, we give a condition for a graph implying that $\lambda(M(G)) = n + 1$. Then we determine the $L(2, 1)$ -labeling number of $M(P_n)$ and $M(C_n)$ the Mycielski graph of path and

cycle respectively, which allow us to determine all the connected graphs realizing $\lambda(M(G))$ equal to 4, 6 and 7, respectively.

Section 4 is devoted to the t -th iterated Mycielski graph of graphs with $t \geq 2$. As in Section 3, we give bounds for $\lambda(M^t(G))$ in terms of the number of iterations t , the order, the maximum degree, and $\lambda(G)$. Then we show that for all $t \geq 2$, $\lambda(M^t(K_n)) = |M^t(K_n)| - 1 = 2^t(n + 1) - 2$, then we characterize all graphs having $\lambda(M^t(G)) = |M^t(G)| - 1 = 2^t(n + 1) - 2$. Later, we give a necessary and sufficient condition for any graph G without isolated vertices achieving a lower bound $2^{t-1}(n + 2) - 2$ for the λ -number of the iterated Mycielski graph of G , we apply that to get an upper bound that can be calculated in polynomial time for any graph G , then we determine $\lambda(M^t(P_n))$, and $\lambda(M^t(C_n))$. Finally, we propose a weak version of the Δ^2 -conjecture for the $L(2, 1)$ -labeling of the Mycielski and iterated Mycielski graph of graphs.

2. PRELIMINARIES AND PREVIOUS RESULTS

For a graph G , let Δ_{M^t} , $\deg_{M^t}(x)$, and $d_{M^t}(x, y)$ denote respectively, the maximum degree, the degree of a vertex x , and the distance between the vertices x and y in $M^t(G)$. If $t = 1$, we denote simply Δ_M , $\deg_M(x)$, and $d_M(x, y)$. As a consequence of Definition 1.1, we have the following.

Lemma 2.1. *If G is a graph of order n , then $|M^t(G)| = 2^t(n + 1) - 1$.*

Proof. From Definition 1.1, we have $|M(G)| = 2n + 1 = 2(n + 1) - 1$. By using induction on t , we can show that $|M^t(G)| = 2^t(n + 1) - 1$. ■

Observation 2.2. *If H is a subgraph of a graph G , then for any $t \geq 1$, $M^t(H)$ is a subgraph of $M^t(G)$.*

Lemma 2.3. *Let G be a graph of order n and maximum degree Δ . For any $t \geq 1$, we have $\Delta_{M^t} = \max\{2^{t-1}(n + 1) - 1, 2^t\Delta\}$.*

Proof. By Definition 1.1, we have $\deg_M(u) = n$, $\deg_M(x) = 2\deg_G(x)$, and $\deg_M(x') = \deg_G(x) + 1$ for all $x \in V$, where x' is the copy of the vertex x in $M(G)$. Then $\Delta_M = \max\{n, 2\Delta\}$. Suppose that for $k \geq 1$, we have $\Delta_{M^k} = \max\{2^{k-1}(n + 1) - 1, 2^k\Delta\}$.

For $k + 1$, if $2^{k-1}(n + 1) - 1 \geq 2^k\Delta$, then $\Delta_{M^k} = 2^{k-1}(n + 1) - 1$. Let v be a vertex of $M^k(G)$, such that $\deg_{M^k}(v) = \Delta_{M^k}$. From Definition 1.1 $\deg_{M^{k+1}}(v) = 2\deg_{M^k}(v) = 2^k(n + 1) - 2 \geq \deg_{M^{k+1}}(x)$, for all $x \in V_{M^k} \cup V'_{M^k}$. Also $\deg_{M^{k+1}}(u^{k+1}) = |M^k(G)| = 2^k(n + 1) - 1 > \deg_{M^{k+1}}(v)$, where u^{k+1} is the root of $M^{k+1}(G)$. So $\Delta_{M^{k+1}} = \deg_{M^{k+1}}(u^{k+1}) = 2^k(n + 1) - 1$.

Otherwise, if $2^k\Delta \geq 2^{k-1}(n + 1)$, then by the inductive hypothesis, we have $\Delta_{M^k} = \max\{2^{k-1}(n + 1) - 1, 2^k\Delta\} = 2^k\Delta$. We have $\deg_{M^{k+1}}(x) = 2\deg_{M^k}(x) \leq$

$2^{k+1}\Delta$, for all $x \in V_{M^k}$. For $x' \in V'_{M^k}$, $\deg_{M^{k+1}}(x') = \deg_{M^k}(x) + 1 \leq 2^k\Delta + 1 \leq 2^{k+1}\Delta$. Also $\deg_{M^{k+1}}(u^{k+1}) = 2^k(n+1) - 1 < 2^{k+1}\Delta$. Thus, $\Delta_{M^{k+1}} = 2^{k+1}\Delta$. It follows that $\Delta_{M^{k+1}} = \max\{2^k(n+1) - 1, 2^{k+1}\Delta\}$. ■

Notice that $M(G)$ is a connected graph if and only if G has no isolated vertices. The *diameter* of a graph $\text{diam}(G)$, is the greatest distance between any pair of vertices in G . If G is disconnected, then $\text{diam}(G)$ is considered to be infinite. In [8], Fisher *et al.* proved that $\text{diam}(M(G)) = \min\{\max\{2, \text{diam}(G)\}, 4\}$, for every graph G without isolated vertices. The following lemmas are a consequence of the proof of this result and the definition of $M(G)$.

Lemma 2.4 [8]. *For v_i and v_j two non-isolated vertices in G , we have $d_M(u, v'_i) = 1$, $d_M(u, v_i) = 2$, $d_M(v'_i, v'_j) = 2$, $d_M(v_i, v'_i) = 2$, $d_M(v_i, v'_j) = \min\{3, d(v_i, v_j)\}$, and $d_M(v_i, v_j) = \min\{4, d(v_i, v_j)\}$.*

If v_i is an isolated vertex in G , then v_i is isolated in $M(G)$, and v'_i is adjacent to the root u .

Lemma 2.5. *If G is a graph without isolated vertices, then for $t \geq 1$, $\text{diam}(M^t(G)) = \min\{\max\{2, \text{diam}(G)\}, 4\}$.*

Proof. Based on [8], we have $\text{diam}(M(G)) = \min\{\max\{2, \text{diam}(G)\}, 4\}$. Suppose that for $k \geq 1$, we have $\text{diam}(M^k(G)) = \min\{\max\{2, \text{diam}(G)\}, 4\}$. We have $M^{k+1}(G) = M(M^k(G))$, so $\text{diam}(M^{k+1}(G)) = \min\{\max\{2, \text{diam}(M^k(G))\}, 4\}$. If $\text{diam}(G) = 1$ or 2 , then by the inductive hypothesis $\text{diam}(M^k(G)) = 2$, it follows that $\text{diam}(M^{k+1}(G)) = 2$. If $\text{diam}(G) = 3$, by the inductive hypothesis $\text{diam}(M^k(G)) = 3$ and so $\text{diam}(M^{k+1}(G)) = 3$. By using the same argument if $\text{diam}(G) \geq 4$, we get that $\text{diam}(M^{k+1}(G)) = 4$. ■

By Lemma 2.5, if the diameter of a graph G is 1 or 2, then the diameter of the t -th iterated Mycielski graph $M^t(G)$ is 2, for any $t \geq 1$. It is clear from the definition of the $L(2, 1)$ -labeling that any vertices at distance less or equal to 2 must be assigned distinct labels. So for any diameter two graph G , all the vertices must be assigned different labels $\lambda(G) \geq |G| - 1$. These arguments will also be used throughout the paper.

We recall some previous results on the $L(2, 1)$ -labeling of graphs.

Lemma 2.6 [11]. *If G is a graph of maximum degree $\Delta \geq 1$, then $\lambda(G) \geq \Delta + 1$. If $\lambda(G) = \Delta + 1$, then for every vertex v of degree Δ , $f(v) = 0$ or $f(v) = \Delta + 1$ for any λ -labeling f .*

For $t \geq 1$, from Lemma 2.6 and Lemma 2.3, an obvious lower bound for $\lambda(M^t(G))$ would be $\max\{2^{t-1}(n+1), 2^t\Delta + 1\}$.

Lemma 2.7 [6]. *If H is a subgraph of a graph G , then $\lambda(H) \leq \lambda(G)$.*

Theorem 2.8 [11]. *If G is a diameter 2 graph with maximum degree Δ , then $\lambda(G) \leq \Delta^2$.*

In the proof of Theorem 2.8, Griggs and Yeh proved that for a graph G of order n and maximum degree $\Delta \geq (n-1)/2 \geq 3$, we have $\lambda(G) < \Delta^2$. Since $\Delta_M = \max\{n, 2\Delta\}$ and $|M(G)| = 2n+1$, it means the Δ^2 -conjecture is true for the Mycielski graph of any graph G of order $n \geq 3$.

The *path covering number* $p_v(G)$ of a graph, is the smallest number of vertex-disjoint paths needed to cover all the vertices of a graph G . The *complement graph* \overline{G} of the graph G is the graph whose vertex set is V and where $xy \in E(\overline{G})$ if only if $xy \notin E(G)$. In [9], Georges *et al.* related the path covering number of the complement graph \overline{G} to the $L(2, 1)$ -labeling number of G .

Theorem 2.9 [9]. *For any graph G of order n , we have the following.*

- $\lambda(G) \leq n-1$ if and only if $p_v(\overline{G}) = 1$.
- $\lambda(G) = n+r-2$ if and only if $p_v(\overline{G}) = r \geq 2$.

3. THE MYCIELSKI GRAPH OF A GRAPH $M(G)$

3.1. Bounds for the $L(2, 1)$ -labeling number of $M(G)$

Theorem 3.1. *Let G be a graph of order $n \geq 1$ and maximum degree $\Delta \geq 0$. Then we have*

$$\max\{n+1, 2(\Delta+1)\} \leq \lambda(M(G)) \leq (n+1) + \lambda(G).$$

Proof. According to the definition of the Mycielski graph of a graph, the degree of the root $\deg_M(u) = n$, then $\lambda(M(G)) \geq n+1$. Otherwise, for $\Delta \geq 1$, we have the star graph $K_{1,\Delta}$ is a subgraph of G . Then by Observation 2.2 and Lemma 2.7, we have $\lambda(M(G)) \geq \lambda(M(K_{1,\Delta}))$. Since $\text{diam}(K_{1,\Delta}) = 2$ and $|K_{1,\Delta}| = \Delta+1$, it follows that $\text{diam}(M(K_{1,\Delta})) = 2$, and $\lambda(M(K_{1,\Delta})) \geq |M(K_{1,\Delta})| - 1 = 2(\Delta+1)$. Thus, $\lambda(M(G)) \geq 2(\Delta+1)$.

For the upper bound, let h be a λ -labeling of G . We denote $M(G)$ the Mycielski graph of G with vertex set $V(M(G)) = \{v_i, v'_i, u : 1 \leq i \leq n\}$, where v'_i is the copy of v_i in $M(G)$ and u is the root. Since every λ -labeling must assign the label 0 to a vertex of G , we consider without loss of generality that $h(v_n) = 0$. We define the following labeling f on $V(M(G))$.

$$f(x) = \begin{cases} i-1 & \text{if } x = v'_i, 1 \leq i \leq n, \\ n+h(v_i) & \text{if } x = v_i, 1 \leq i \leq n, \\ (n+1) + \lambda(G) & \text{if } x = u. \end{cases}$$

Now we will check that f is an $L(2, 1)$ -labeling of $M(G)$, we get five cases.

- We have $|f(v'_i) - f(v'_j)| = |i - j| \geq 1$ and $d_M(v'_i, v'_j) = 2$, for all $1 \leq i, j \leq n$ $i \neq j$.
- By Lemma 2.4, if $d_M(v_i, v_j) = 1$ (respectively, 2), then $d_G(v_i, v_j) = 1$ (respectively, 2). We have $|f(v_i) - f(v_j)| = |h(v_i) - h(v_j)|$. This means $|f(v_i) - f(v_j)| \geq 2$, if $d_M(v_i, v_j) = 1$ and $|f(v_i) - f(v_j)| \geq 1$, if $d_M(v_i, v_j) = 2$.
- For all $1 \leq i, j \leq n$, we have $|f(v_i) - f(v'_j)| = |n + h(v_i) - j + 1|$. The distance two conditions are respected for all the following cases.
 - (i) If $1 \leq j \leq n - 1$, then $|f(v_i) - f(v'_j)| \geq 2$.
 - (ii) If $j = n$ and $i = n$, we have $|f(v_n) - f(v'_n)| = 1$, and $d_M(v_n, v'_n) \geq 2$.
 - (iii) If $j = n$ and $d_G(v_i, v_n) = 1$, we have $|h(v_i) - h(v_n)| \geq 2$, so $h(v_i) \geq 2$. It follows that $|f(v_i) - f(v'_n)| \geq 2$.
 - (iv) If $j = n$ and $d_G(v_i, v_n) \geq 2$, by Lemma 2.4 we have $d_M(v_i, v'_n) \geq 2$, and $|f(v_i) - f(v'_n)| \geq 1$.
- For all $1 \leq i \leq n$, $|f(u) - f(v'_i)| = |(n + 1) + \lambda(G) - i + 1| \geq 2$.
- For all $1 \leq i \leq n$, $|f(u) - f(v_i)| = |(n + 1) + \lambda(G) - (n + h(v_i))| \geq 1$, and $d_M(u, v_i) \geq 2$.

So f is an $L(2, 1)$ -labeling of $M(G)$ with span $(n + 1) + \lambda(G)$. Hence $\lambda(M(G)) \leq (n + 1) + \lambda(G)$. ■

Corollary 3.2. *If G is a diameter 2 graph of maximum degree Δ , then $\lambda(M(G)) \leq 2(\Delta^2 + 1)$.*

Proof. By Theorem 2.8 for a diameter 2 graph, we have $\lambda(G) \leq \Delta^2$. Also, we have $|G| = n \leq \Delta^2 + 1$, known as the Moore bound due to Hoffman and Singleton [14]. By combining this with the upper bound of Theorem 3.1, we get that $\lambda(M(G)) \leq 2(\Delta^2 + 1)$. ■

The bound $2(\Delta^2 + 1)$ in Corollary 3.2 can only be attained by the Mycielski graph of diameter two Moore graphs [14], since the diameter of the Mycielski graph of these graphs is two, and these are the only diameter two graphs with order $\Delta^2 + 1$ and λ -number equal to Δ^2 [11]. The only known graphs achieving this bound are C_5 the cycle of order 5, the Petersen graph, and the Hoffman-Singleton graph.

3.2. $L(2, 1)$ -labeling number of the Mycielski graph of a graph and the star matching of the complement graph

By using the upper bound of Theorem 3.1 and Theorem 2.9, we can link the λ -number of $M(G)$ to the path covering of the complement graph \overline{G} . So if $p_v(\overline{G}) = 1$, i.e., \overline{G} has a Hamiltonian path, then $\lambda(M(G)) \leq 2n$, the equality holds

for diameter two graphs. Also if $p_v(\overline{G}) \geq 2$, then $\lambda(M(G)) \leq 2n + p_v(\overline{G}) - 1$. But for more relevant conditions, the study of the path covering of the complement of $M(G)$ is required.

We can see that for any graph G , $\overline{M}(G)$ the complement of the Mycielski graph of G is a connected graph. The neighborhood of u in $\overline{M}(G)$ is V . For all $1 \leq i \leq n$, $v_i v'_i \in E(\overline{M}(G))$. For $i \neq j$, $v'_i v'_j \in E(\overline{M}(G))$. Also $v_i v'_j, v_i v_j \in E(\overline{M}(G))$ if and only if $v_i v_j \notin E(G)$. The subgraph induced by the set V is \overline{G} . The subgraph induced by the set V' is the complete graph on n vertices.

Let m be an integer greater or equal to 2. An m -star matching H of G is a subgraph of G such that each component of H is isomorphic to a star graph $K_{1,i}$, with $1 \leq i \leq m$. The m -star matching number, denoted $s_m(G)$, is the maximum order of an m -star matching of G , an m -star matching of order $s_m(G)$ is said to be maximum. If $s_m(G) = |G|$, we say that G has a perfect m -star matching, a perfect m -star matching is known also as star-factor or $\{K_{1,1}, K_{1,2}, \dots, K_{1,m}\}$ -factor [1, 22]. The problem of finding whether or not a graph G admits a perfect m -star matching can be solved in polynomial time [15]. In [17], Lin and Lam studied the m -star matching and the m -star matching number $s_m(G)$. They delivered an algorithm to compute $s_m(G)$ running in $O(|V||E|)$. Then they related the 4-star matching number of \overline{G} to the path covering number of $\overline{M}(G)$. In the following we denote by $i_4(G)$ the number of vertices unmatched in a maximum 4-star matching of G , i.e. $i_4(G) = n - s_4(G)$.

Theorem 3.3 [17]. *For any graph G , we have the following.*

- (i) *If $i_4(\overline{G}) \leq 4$, then $p_v(\overline{M}(G)) = 1$.*
- (ii) *If $i_4(\overline{G}) \geq 5$, then $p_v(\overline{M}(G)) = \left\lceil \frac{i_4(\overline{G})}{2} \right\rceil - 1$.*

We show that the converse holds in both cases, similarly to Theorem 2.9 in [9].

Theorem 3.4. *For any graph G , we have the following.*

- (a) *$i_4(\overline{G}) \leq 4$ if and only if $p_v(\overline{M}(G)) = 1$.*
- (b) *$\left\lceil \frac{i_4(\overline{G})}{2} \right\rceil = r \geq 3$ if and only if $p_v(\overline{M}(G)) = r - 1$.*

Proof. (a) Considering (i) and the contraposition of (ii) in Theorem 3.3, we get the necessity and sufficiency.

(b) Let $r \geq 3$. To verify (b) we proceed by induction on r , we prove first that (b) is true for $r = 3$.

Claim 3.5. *If $p_v(\overline{M}(G)) = 2$, then the root u is not an end-vertex of a path in a minimum path covering of $\overline{M}(G)$.*

Proof. If $p_v(\overline{M}(G)) = 2$, let P^1 and P^2 be the two paths of a minimum path covering of $\overline{M}(G)$. Suppose that u is an end-vertex of P^1 . Since u is adjacent in $\overline{M}(G)$ to every vertex in V , a vertex in V cannot be an end-vertex of P^2 , otherwise $\overline{M}(G)$ has a Hamiltonian path. So both ends of P^2 are from V' . Since the subgraph induced by V' is a complete graph, the other extremity of P^1 is in V . Let z be the other end of P^1 , x' and y' the ends of P^2 . Since u is adjacent to z , x' is adjacent to y' . If z' the copy of z belongs to P^1 , we have z' is adjacent to x' and y' , we can construct a Hamiltonian path of $\overline{M}(G)$. If z' belongs to P^2 , then since z is adjacent to z' , in this case also $\overline{M}(G)$ has a Hamiltonian path, a contradiction. \square

If $p_v(\overline{M}(G)) = 2$, let $x, y \in V$ and be such that x or its copy and y or its copy are end-vertices of the two different paths in a minimum path covering of $\overline{M}(G)$. We consider the graph H with vertex set V and edge set of its complement $E(\overline{H}) = E(\overline{G}) \cup \{xy\}$. It is clear that $p_v(\overline{M}(H)) = 1$, and $i_4(\overline{H}) \geq i_4(\overline{G}) - 2$. Since $p_v(\overline{M}(G)) = 2$, according to (a) we have $4 \geq i_4(\overline{H})$, and $i_4(\overline{G}) \geq 5$. It follows that $\left\lceil \frac{i_4(\overline{G})}{2} \right\rceil = 3$. So from Theorem 3.3(ii), we have Theorem 3.4(b) is true for $r = 3$.

Assume that (b) is true for $3 \leq r \leq k$, and let $r = k + 1$.

If $p_v(\overline{M}(G)) = k$, let $x, y \in V$ and be such that x or its copy and y or its copy are end-vertices of two different paths in a minimum path covering of $\overline{M}(G)$. We consider the graph H with vertex set V and edge set of its complement $E(\overline{H}) = E(\overline{G}) \cup \{xy\}$. We have $p_v(\overline{M}(H)) = k - 1$, and $i_4(\overline{H}) \geq i_4(\overline{G}) - 2$. So by the inductive hypothesis $\left\lceil \frac{i_4(\overline{H})}{2} \right\rceil = k$, hence $2k + 2 \geq i_4(\overline{G})$. Since $p_v(\overline{M}(G)) = k$, by the inductive hypothesis $i_4(\overline{G}) \geq 2k + 1$. It follows that $\left\lceil \frac{i_4(\overline{G})}{2} \right\rceil = k + 1$. Theorem 3.3(ii) completes the equivalence. \blacksquare

By combining Theorem 2.9 and Theorem 3.4, we get the following results.

Theorem 3.6. *Let G be any graph of order n . Then the following statements hold.*

- (a) $\lambda(M(G)) \leq 2n$ if and only if $i_4(\overline{G}) \leq 4$.
- (b) For any positive integer r , we have

$$\lambda(M(G)) = 2n + r \text{ if and only if } \left\lceil \frac{i_4(\overline{G})}{2} \right\rceil = r + 2.$$

Next, we give applications of the previous theorem to the λ -number of the Mycielski graph of certain graphs.

If the diameter of G is 1 or 2, then $\text{diam}(M(G)) = 2$, and we can conclude from Theorem 3.6 that $\lambda(M(G)) = 2n + \max \left\{ 2, \left\lceil \frac{i_4(\overline{G})}{2} \right\rceil \right\} - 2$.

Corollary 3.7. *Let G be a graph of order n . If the clique number $\omega(G) \leq 4$, then $\lambda(M(G)) \leq 2n$.*

Proof. By Theorem 3.6(a) if $\lambda(M(G)) > 2n$, then $i_4(\overline{G}) \geq 5$. This means that $\omega(G) \geq 5$. ■

The graphs with clique number less or equal to 4 in Corollary 3.7 include trees, planar graphs, and subcubic graphs.

If X is any subset of V , we denote $N_G(X)$ the set of all vertices in V adjacent to at least one vertex from X in G . In [17], a criterion for a graph to have a perfect m -star matching is given, this appeared also in [1, 15, 22].

Theorem 3.8 [1, 15, 17, 22]. *A graph G has a perfect m -star matching if and only if for any independent set S in G , $|N_G(S)| \geq |S|/m$.*

Corollary 3.9. *Let G be a graph of order n and maximum degree $\Delta \leq n - 2$. If $3(n - 1) + \delta \geq 4\Delta$, then $\lambda(M(G)) \leq 2n$.*

Proof. Let $\overline{\Delta}$ and $\overline{\delta}$ denote, respectively, the maximum and minimum degree of the complement graph \overline{G} . For any independent set S in \overline{G} , let $|E_{\overline{G}}(S)|$ denote the number of edges incident to the vertices of S in \overline{G} . We have

$$(1) \quad |N_{\overline{G}}(S)|\overline{\Delta} \geq |E_{\overline{G}}(S)| \geq \overline{\delta}|S|.$$

If $3(n - 1) + \delta \geq 4\Delta$, then since $\overline{\Delta} = (n - 1) - \delta$ and $\overline{\delta} = (n - 1) - \Delta$, we have $4\overline{\delta} \geq \overline{\Delta}$. Therefore from Inequality (1) we get that $|N_{\overline{G}}(S)| \geq |S|/4$, for any independent set S in \overline{G} . Then by Theorem 3.8, \overline{G} has a perfect 4-star matching. Hence from Theorem 3.6(a), we have $\lambda(M(G)) \leq 2n$. ■

From Corollary 3.9, any regular graph G of order n , except complete graphs, has $\lambda(M(G)) \leq 2n$. In [17], it is shown that for complete graph $\lambda(M(K_2)) = 4$ and $\lambda(M(K_n)) = 2n + \lceil \frac{n}{2} \rceil - 2$ for $n \geq 3$. Next, we determine the exact λ -number of the Mycielski graph of complete k -partite graphs.

Corollary 3.10. *Let G be a complete k -partite graph of order n , where the partite sets consist of p sets of order greater or equal 2 and q singletons.*

- If $q \leq 4$, then $\lambda(M(G)) = 2n$.
- If $q \geq 5$, then $\lambda(M(G)) = 2n + \lceil \frac{q}{2} \rceil - 2$.

Proof. We have \overline{G} is formed of p connected components that are complete graphs of order greater or equal to 2, and q isolated vertices. Therefore $i_4(\overline{G}) = q$. If $q \leq 4$, by Theorem 3.6(a), $\lambda(M(G)) \leq 2n$. Since $\text{diam}(M(G)) = 2$, it follows that $\lambda(M(G)) = 2n$. If $q \geq 5$, then by Theorem 3.6(b), $\lambda(M(G)) = 2n + \lceil \frac{q}{2} \rceil - 2$. ■

Let G_1, G_2 be two disjoint graphs. The disjoint union of G_1 and G_2 , denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The joint of G_1 and G_2 denoted $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2 .

Corollary 3.11. *Let G_1, G_2, \dots, G_k be a collection of disjoint graphs having, respectively, n_1, n_2, \dots, n_k vertices. Let $n = \sum_{i=1}^k n_i$. Then $\lambda(M(G_1 \vee G_2 \vee \dots \vee G_k)) = 2n + \max \left\{ 2, \left\lceil \frac{I}{2} \right\rceil \right\} - 2$, where $I = \sum_{i=1}^k i_4(\overline{G_i})$.*

Proof. Let $G = G_1 \vee G_2 \vee \dots \vee G_k$. We have $\overline{G} = \overline{G_1} \cup \overline{G_2} \cup \dots \cup \overline{G_k}$. It follows that $i_4(\overline{G}) = \sum_{i=1}^k i_4(\overline{G_i}) = I$. Thus, by Theorem 3.6(a), if $I \leq 4$, then $\lambda(M(G)) \leq 2n$. Since $\text{diam}(G) = 2$, it follows that $\lambda(M(G)) = 2n$. If $I \geq 5$, from Theorem 3.6(b), $\lambda(M(G)) = 2n + \left\lceil \frac{I}{2} \right\rceil - 2$. ■

3.3. Graphs with $\lambda(M(G)) = n + 1$

For $k \geq 1$, the k th power of a graph G is the graph G^k with vertex set V and edge set $E(G^k) = \{v_i v_j : 1 \leq d_G(v_i, v_j) \leq k\}$. Then the square of a graph G^2 has the edge set of its complement graph $E(\overline{G^2}) = \{v_i v_j : d_G(v_i, v_j) \geq 3\}$. Next we give a condition, so that $\lambda(M(G)) = n + 1$.

Lemma 3.12. *In a graph G of order n , if the vertex set V can be partitioned into $k \geq 1$ vertex-disjoint cliques in $\overline{G^2}$ such that at least $k - 1$ cliques are of order greater or equal 3, then $\lambda(M(G)) = n + 1$.*

Proof. Let $V = \bigcup_{r=1}^k S_r$ be such that S_r are vertex-disjoint cliques in $\overline{G^2}$ of order $|S_r| = n_r \geq 3$ for $1 \leq r \leq k - 1$, and $|S_k| = n_k \geq 1$, where $\sum_{r=1}^k n_r = n$. For $1 \leq r \leq k$, let us denote $S_r = \{v_{i,r} : 1 \leq i \leq n_r\}$, let $v'_{i,r}$ be the copy of the vertex $v_{i,r}$, and let u be the root of $M(G)$. We have $d_G(v_{i,r}, v_{j,r}) \geq 3$ for any two distinct vertices in S_r , so a vertex in S_{r+1} can be adjacent to at most one vertex in S_r . For $1 \leq r \leq k - 1$, the cliques S_r in $\overline{G^2}$ are symmetric of order greater or equal 3. We suppose without loss of generality that $d_G(v_{n_r,r}, v_{1,r+1}) \geq 2$, for $1 \leq r \leq k - 1$. Let $\psi_1 = 0$ and for $r \geq 2$, $\psi_r = \sum_{j=1}^{r-1} n_j$. With respect to the previous assumption, we label the vertices of $M(G)$ as following.

- For $1 \leq r \leq k - 1$, define $f(v_{1,r}) = \psi_r$. For $2 \leq i \leq n_r$, $f(v_{i,r}) = \psi_r + 1$. Also $f(v'_{1,r}) = \psi_r + 1$, and $f(v'_{2,r}) = \psi_r$. For $3 \leq i \leq n_r$, $f(v'_{i,r}) = \psi_r + i - 1$.
- If $|S_k| = 1$, then let $f(v_{1,k}) = n$, and $f(v'_{1,k}) = n - 1$.
- If $|S_k| = 2$, then let $f(v_{1,k}) = n - 2$, $f(v'_{1,k}) = n - 1$, $f(v_{2,k}) = n - 1$, and $f(v'_{2,k}) = n - 2$.
- If $|S_k| \geq 3$, then define $f(v_{1,k}) = \psi_k$. For $2 \leq i \leq n_k$, $f(v_{i,k}) = \psi_k + 1$. Also $f(v'_{1,k}) = \psi_k + 1$, and $f(v'_{2,k}) = \psi_k$. For $3 \leq i \leq n_k$, $f(v'_{i,k}) = \psi_k + i - 1$.

Finally, label the root u by $f(u) = n + 1$. We have $d_G(v_{i,r}, v_{j,r}) \geq 3$, and for $1 \leq r \leq k - 1$ we have $d_G(v_{n_r,r}, v_{1,r+1}) \geq 2$. This means by Lemma 2.4 that $d_M(v_{i,r}, v_{j,r}) \geq 3$, $d_M(v'_{i,r}, v_{j,r}) = 3$, and $d_M(v'_{n_r,r}, v_{1,r+1}) \geq 2$. The labeling f is an $L(2, 1)$ -labeling of $M(G)$ with span $n + 1$. Hence $\lambda(M(G)) = n + 1$. ■

In the case of the empty graph $\overline{K_n}$, we have $M(\overline{K_n}) \cong K_{1,n} \cup \overline{K_n}$. Since $\lambda(K_{1,n}) = n + 1$, we have $\lambda(M(\overline{K_n})) = n + 1$, we can get the same result using Lemma 3.12. We are now interested in some connected graphs, we consider the graph path P_n and cycle C_n .

Let P_n denote the graph path of order $n \geq 3$ with vertex set $V(P_n) = \{v_1, \dots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n - 1\}$. Denote $V(M(P_n)) = V(P_n) \cup \{v'_i : 1 \leq i \leq n\} \cup \{u\}$, where v'_i is the copy of the vertex v_i , and u is the root of $M(P_n)$.

Proposition 3.13.

$$\lambda(M(P_n)) = \begin{cases} 6 & \text{if } n = 3, 4, \\ 7 & \text{if } n = 5, \\ n + 1 & \text{if } n \geq 6. \end{cases}$$

Proof. • For $n = 3$, we have $\text{diam}(P_3) = 2$. So from Theorem 3.6, $\lambda(M(P_3)) = 6$.

• For $n = 4$, we have a 6- $L(2, 1)$ -labeling of $M(P_4)$ shown in Figure 1. Hence $\lambda(M(P_4)) \leq 6$. Also we have $M(P_3)$ is a subgraph of $M(P_4)$. By Lemma 2.7, it follows that $\lambda(M(P_4)) \geq \lambda(M(P_3)) = 6$. Thus, $\lambda(M(P_4)) = 6$.

• For $n = 5$, Figure 2 illustrates a 7- $L(2, 1)$ -labeling of $M(P_5)$. This implies also by Theorem 3.1 that $6 \leq \lambda(M(P_5)) \leq 7$.

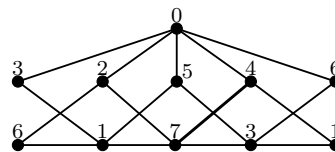
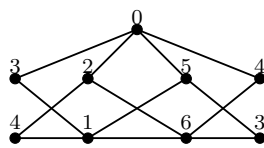


Figure 1. A 6- $L(2, 1)$ -labeling of $M(P_4)$. Figure 2. A 7- $L(2, 1)$ -labeling of $M(P_5)$.

Suppose that $\lambda(M(P_5)) = 6$. Then there is an $L(2, 1)$ -labeling f of $M(P_5)$ using labels in the set $L = \{0, 1, 2, 3, 4, 5, 6\}$. Since $\deg_M(u) = 5$, by Lemma 2.6, $f(u) = 0$ or $f(u) = 6$. Without loss of generality, we suppose that $f(u) = 0$. Since all the vertices are at distance less or equal to 2 from u , it is the only vertex with label 0. We denote by $N(v)$ the open neighborhood of a vertex v , and by $N^2(v)$ the set of all vertices at distance at most 2 from a vertex v in $M(P_5)$. We have $N(u) = \{v'_1, v'_2, v'_3, v'_4, v'_5\}$, and $d_M(v'_i, v'_j) = 2$, for $1 \leq i, j \leq 5$. So each vertex v'_i must receive a distinct label from the set $\{2, 3, 4, 5, 6\}$ different from

$f(u) = 0$. We have every vertex in $M(P_5)$ is at distance less or equal 2 from v_3 . It means that v_3 must receive a distinct label from $v'_1, v'_2, v'_3, v'_4, v'_5$, and u . Hence $f(v_3) = 1$, and v_3 is the only vertex with label 1. We have $\{u, v_3, v'_1, v'_2, v'_3, v'_4\} \subset N^2(v_2)$, and the vertices u, v_3, v'_1, v'_2, v'_3 and v'_4 receive distinct labels from the set L which leaves only the label assigned to v'_5 available for v_2 . Hence $f(v_2) = f(v'_5)$. Also, $\{u, v_3, v'_2, v'_3, v'_4, v'_5\} \subset N^2(v_4)$. By using the same arguments as before, we get that $f(v_4) = f(v'_1)$. $N^2(v_1) = \{u, v_2, v_3, v'_1, v'_2, v'_3\}$, and each vertex in $N^2(v_1)$ have a distinct label from L ($f(v_2) = f(v'_5)$). Then $f(v_1) = f(v'_4)$. Also $N^2(v_5) = \{u, v_3, v_4, v'_3, v'_4, v'_5\}$, with $f(v_4) = f(v'_1)$. Hence $f(v_5) = f(v'_2)$. We have $N(v_3) = \{v_2, v_4, v'_2, v'_4\}$, with $f(v_2) = f(v'_5)$, $f(v_4) = f(v'_1)$, and $f(v_3) = 1$. It follows that the labels assigned to v'_1, v'_2, v'_4 and v'_5 must be greater or equal to 3. Hence, the only remaining label for v'_3 is $f(v'_3) = 2$. We have v_2 and v_4 are adjacent to v'_3 , $f(v_2) = f(v'_5)$, $f(v_4) = f(v'_1)$, and $f(v'_3) = 2$. Then $f(v'_5)$ and $f(v'_1)$ must be greater than 3, hence $f(v'_5), f(v'_1) \in \{4, 5, 6\}$. Since v'_1 is adjacent to v_2 and $f(v_2) = f(v'_5)$, we have $|f(v'_5) - f(v'_1)| \geq 2$. Therefore $f(v'_1), f(v'_5) \in \{4, 6\}$, which means also that $f(v'_2), f(v'_4) \in \{3, 5\}$. Since $f(v_2) = f(v'_5)$, and $f(v_1) = f(v'_4)$, it follows that $|f(v'_5) - f(v'_4)| \geq 2$. Also, $f(v_4) = f(v'_1)$ and $f(v_5) = f(v'_2)$, hence $|f(v'_1) - f(v'_2)| \geq 2$. If $f(v'_1) = 4$, then since $|f(v'_1) - f(v'_2)| \geq 2$, $f(v'_2) \notin \{3, 5\}$, a contradiction. Now if $f(v'_1) = 6$, then $f(v'_5) = 4$. Since $|f(v'_5) - f(v'_4)| \geq 2$, $f(v'_4) \notin \{3, 5\}$, again a contradiction. Therefore $\lambda(M(P_5)) \geq 7$. Hence $\lambda(M(P_5)) = 7$.

• For $n \geq 6$, we define a labeling f on $V(M(P_n))$ as following.

$f(u) = 0$, $f(v'_1) = 6$, $f(v'_2) = 5$, $f(v'_3) = 4$, $f(v'_4) = 7$, $f(v'_5) = 2$, $f(v'_6) = 3$, and $f(v'_i) = i + 1$ if $i \geq 7$.

$f(v_1) = 7$, $f(v_2) = 1$, $f(v_3) = 3$, $f(v_4) = 6$, $f(v_5) = 1$, $f(v_6) = 4$, and for $i \geq 7$:

$f(v_i) = 6$ if $i \equiv 1 \pmod{3}$, $f(v_i) = 2$ if $i \equiv 2 \pmod{3}$, $f(v_i) = 4$ if $i \equiv 0 \pmod{3}$.

The idea is to come up with a 7- $L(2, 1)$ -labeling of the subgraph induced by $H = \{u, v_i, v'_i : 1 \leq i \leq 6\}$ isomorphic to $M(P_6)$. Then if $i \geq 7$, assign each vertex copy v'_i consecutive labels beginning with 8, and label the vertices v_i with labels 6, 2, 4 for $i \equiv 1 \pmod{3}$, $i \equiv 2 \pmod{3}$, and $i \equiv 0 \pmod{3}$, respectively. This is an $L(2, 1)$ -labeling of $M(P_n)$ with span $n + 1$. Hence $\lambda(M(P_n)) \leq n + 1$, for $n \geq 6$. It follows from Theorem 3.1 that $\lambda(M(P_n)) = n + 1$, for $n \geq 6$. ■

Let C_n be the graph cycle with vertex set $V(C_n) = \{v_0, v_1, \dots, v_{n-1}\}$ and edge set $E(C_n) = \{v_i v_{i+1 \pmod{n}} : 0 \leq i \leq n-1\}$, where the indices are taken modulo n . We denote $V(M(C_n)) = V(C_n) \cup \{v'_i : 1 \leq i \leq n\} \cup \{u\}$, we have $E(M(C_n)) = \{v_i v_{i+1 \pmod{n}}, v_i v'_{i+1 \pmod{n}}, v'_i v_{i+1 \pmod{n}} : 0 \leq i \leq n-1\} \cup \{v'_i u : 0 \leq i \leq n-1\}$.

Proposition 3.14.

$$\lambda(M(C_n)) = \begin{cases} 6 & \text{if } n = 3, \\ 8 & \text{if } n = 4, \\ 10 & \text{if } n = 5, \\ n + 1 & \text{if } n \geq 6. \end{cases}$$

Proof. • For $3 \leq n \leq 5$, since $\text{diam}(C_3) = 1$, $\text{diam}(C_4) = \text{diam}(C_5) = 2$, from Lemma 2.4, $\text{diam}(M(C_3)) = \text{diam}(M(C_4)) = \text{diam}(M(C_5)) = 2$. By applying Theorem 3.6, we get that $\lambda(M(C_3)) = 6$, $\lambda(M(C_4)) = 8$, and $\lambda(M(C_5)) = 10$.

• For $n \geq 6$, in Figure 3, Figure 4, and Figure 5, respectively, we present an $L(2,1)$ -labeling for $M(C_6)$, $M(C_7)$, and $M(C_8)$, respectively, with span 7, 8, and 9. It follows from the lower bound in Theorem 3.1 that $\lambda(M(C_6)) = 7$, $\lambda(M(C_7)) = 8$, and $\lambda(M(C_8)) = 9$.

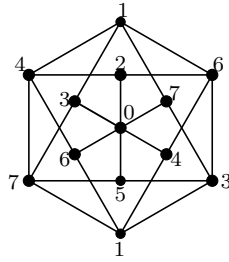


Figure 3. A 7- $L(2,1)$ -labeling of $M(C_6)$.

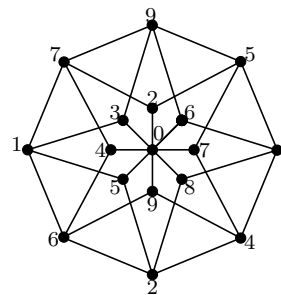
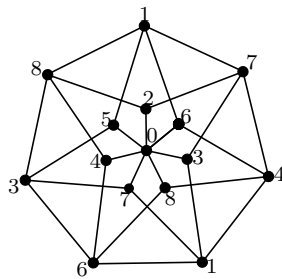


Figure 4. An 8- $L(2,1)$ -labeling of $M(C_7)$. Figure 5. A 9- $L(2,1)$ -labeling of $M(C_8)$.

For $n \geq 9$, we partition the vertex set $V(C_n)$ into cliques in $\overline{C_n^2}$ as following.

If $n \equiv 0 \pmod{3}$, for $0 \leq i \leq \frac{n}{3} - 1$, the sets $S_i = \{v_i, v_{i+\frac{n}{3}}, v_{i+2\frac{n}{3}}\}$ form disjoint cliques of order 3 in $\overline{C_n^2}$. We have $V(C_n) = \bigcup_{i=0}^{\frac{n}{3}-1} S_i$.

If $n \equiv 1 \pmod{3}$, for $0 \leq i \leq \lfloor \frac{n}{3} \rfloor - 1$, the sets $S_i = \{v_i, v_{i+\lfloor \frac{n}{3} \rfloor}, v_{i+2\lfloor \frac{n}{3} \rfloor}\}$ form disjoint cliques of order 3 in $\overline{C_n^2}$. We have $V(C_n) = \bigcup_{i=0}^{\lfloor \frac{n}{3} \rfloor - 1} S_i \cup \{v_{n-1}\}$.

If $n \equiv 2 \pmod{3}$, for $1 \leq i \leq \lceil \frac{n}{3} \rceil - 1$, the sets $S_i = \{v_i, v_{i+\lceil \frac{n}{3} \rceil}, v_{i+2\lceil \frac{n}{3} \rceil-1}\}$ form disjoint cliques of order 3 in $\overline{C_n^2}$, and $v_0 v_{\lceil \frac{n}{3} \rceil}$ is an edge in $\overline{C_n^2}$. We have $V(C_n) = \bigcup_{i=1}^{\lceil \frac{n}{3} \rceil - 1} S_i \cup \{v_0, v_{\lceil \frac{n}{3} \rceil}\}$.

The cycle C_n in the three cases verifies the condition in Lemma 3.12. Hence $\lambda(M(C_n)) = n + 1$, for $n \geq 6$. ■

For a connected graph G of order n in Theorem 3.1 we have $\lambda(M(G)) \geq n + 1$. It means that for any fixed positive integer k , there are finitely many connected graphs having $\lambda(M(G)) = k$. In the following, we determine all the connected graphs having $\lambda(M(G))$ equal to 4, 6 and 7. These are the smallest possible values for the λ -number of the Mycielski graph of any non-trivial connected graph.

Corollary 3.15. *For a connected graph G , we have the following.*

- (1) $\lambda(M(G)) = 4$ if and only if G is K_2 .
- (2) $\lambda(M(G)) = 6$ if and only if $G \in \{P_3, P_4, C_3\}$.
- (3) $\lambda(M(G)) = 7$ if and only if $G \in \{P_5, P_6, C_6\}$.

Proof. From Theorem 3.1, for a connected graph G of order n and maximum degree Δ , we have

$$(2) \quad \lambda(M(G)) \geq \max\{n + 1, 2(\Delta + 1)\}.$$

The only connected graph with $\Delta = 1$ is K_2 and we have $\lambda(M(K_2)) = 4$. If $\Delta \geq 2$, from the inequality (2), $\lambda(M(G)) \geq 6$. It follows that $\lambda(M(G)) = 4$ if and only if $G \cong K_2$. Also there is no connected graph with $\lambda(M(G)) = 5$.

The only connected graphs with $\Delta = 2$ are path graphs and cycles. Based on inequality (2), if $\Delta \geq 3$, then $\lambda(M(G)) \geq 8$. Then if $6 \leq \lambda(M(G)) \leq 7$, it means necessarily that G is a path or a cycle graph. In Proposition 3.13 and Proposition 3.14, the only connected graphs with $\lambda(M(G)) = 6$ are P_3, P_4 , and C_3 . Also the only connected graphs with $\lambda(M(G)) = 7$ are P_5, P_6 , and C_6 . ■

4. THE ITERATED MYCIELSKI GRAPH OF A GRAPH $M^t(G)$

4.1. Bounds for $\lambda(M^t(G))$

Theorem 4.1. *If G is a graph of order $n \geq 2$ and maximum degree $\Delta \geq 0$, then for $t \geq 2$ we have*

$$2^{t-1} \max\{n + 2, 2(\Delta + 2)\} - 2 \leq \lambda(M^t(G)) \leq (2^t - 1)(n + 1) + \lambda(G).$$

Proof. For a graph G of order $n \geq 2$ from Definition 1.1, we have $K_{1,n}$ is a subgraph of $M(G)$. Then by Observation 2.2, $M^{t-1}(K_{1,n})$ is a subgraph of $M^t(G)$. Since $\text{diam}(K_{1,n}) = 2$, it follows from Lemma 2.5 and Lemma 2.7 that $\lambda(M^t(G)) \geq \lambda(M^{t-1}(K_{1,n})) \geq |M^{t-1}(K_{1,n})| - 1$. By Lemma 2.1 $|M^{t-1}(K_{1,n})| = 2^{t-1}(n+2) - 1$, hence $\lambda(M^t(G)) \geq 2^{t-1}(n+2) - 2$, for $t \geq 2$. If $\Delta \geq 1$, we have $K_{1,\Delta}$ is a subgraph of G . By using the same arguments as before, we get that $\lambda(M^t(G)) \geq 2^t(\Delta + 2) - 2$.

On the other hand, for $t \geq 2$, we have $M^t(G) = M(M^{t-1}(G))$. So by the upper bound of Theorem 3.1, $\lambda(M^t(G)) \leq (|M^{t-1}(G)| + 1) + \lambda(M^{t-1}(G)) = 2^{t-1}(n+1) + \lambda(M^{t-1}(G))$. Recursively we get that $\lambda(M^t(G)) \leq \sum_{i=0}^{t-1} 2^i(n+1) + \lambda(G) = (2^t - 1)(n+1) + \lambda(G)$. ■

The lower bound $2^{t-1}(n+2) - 2$ and the upper bound of Theorem 4.1 are true also for $n = 1$. The upper bound coincides with the upper bound in Theorem 3.1 for $t = 1$. As a consequence we make the following observation.

Observation 4.2. *If a graph G of order n has $\lambda(G) \leq n - 1$, then for any $t \geq 1$, $\lambda(M^t(G)) \leq |M^t(G)| - 1 = 2^t(n+1) - 2$, and there is equality if G is of diameter two.*

Further, we denote $V^t = \{v_i^k : 1 \leq i \leq n \text{ and } 0 \leq k \leq 2^t - 1\}$, the set composed of the vertices of V and all their copies in $M^t(G)$, where v_i^1 is the copy of v_i^0 in $M(G)$. v_i^2 and v_i^3 are respectively the copies of v_i^0 and v_i^1 in $M^2(G)$. $v_i^4, v_i^5, v_i^6, v_i^7$ are respectively the copies of $v_i^0, v_i^1, v_i^2, v_i^3$ in $M^3(G)$ and so forth. In $M^t(G)$ for $0 \leq k \leq 2^{t-1} - 1$, we have $v_i^{2^{t-1}+k}$ is the exact copy of the vertex v_i^k from $M^{t-1}(G)$. For $t \geq 2$, let U_t be the set of all the roots (i.e. roots and their consecutive copies in all levels) in $M^t(G)$. Recursively $U_t = U_{t-1} \cup U'_{t-1} \cup \{u_{t,0}\}$ and $|U_t| = 2^t - 1$. We denote the set of roots $U_t = \{u_{i,j} : 1 \leq i \leq t \text{ and } 0 \leq j \leq 2^{t-i} - 1\}$ such that for example in $M^3(G)$, $u_{1,0}$ is the root of $M(G)$, $u_{1,1}$ the copy of $u_{1,0}$, and $u_{2,0}$ the root of $M^2(G)$. $u_{1,2}, u_{1,3}, u_{2,1}$ are respectively the copies of $u_{1,0}, u_{1,1}, u_{2,0}$, and $u_{3,0}$ is the root in $M^3(G)$, and so forth. Figure 6 illustrates an adjacency of a vertex and its copies v_i^k in $M^2(G)$, with respect to the above ordering.

Lemma 4.3. *If $d_G(v_i^0, v_j^0) \leq 2$, then for any $t \geq 1$ and all $0 \leq k, m \leq 2^t - 1$, we have $d_{M^t}(v_i^k, v_j^m) \leq 2$, and if v_i^0 is not an isolated vertex for $k \neq m$, we have $d_{M^t}(v_i^k, v_i^m) = 2$.*

Proof. By using Lemma 2.4 inductively, we get the results. ■

The *eccentricity* of a vertex v in a graph G , is the greatest distance between v and any other vertex in G . By Lemma 4.3, if a vertex has eccentricity 1 or 2 in G , then the vertex and all its copies are of eccentricity 2 in $M^t(G)$. In a graph G

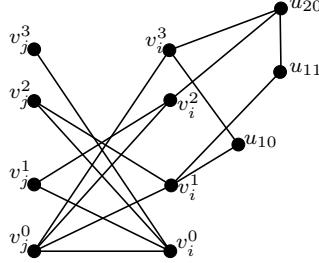


Figure 6. An example of an adjacency of the vertices v_i^k in $M^2(G)$.

without isolated vertices, we have from the definition of the Mycielski construction, the eccentricity of the root in $M(G)$ is 2, so from above the eccentricity of all the roots and their copies is 2 in $M^t(G)$, for any $t \geq 1$.

Proposition 4.4. *If G is a graph without isolated vertices of order n , with k vertices of eccentricity 2, then for $t \geq 1$, we have $\lambda(M^t(G)) \geq 2^{t-1}(n+k+2) - 2$.*

Proof. For $t \geq 1$, let $v_1^0, v_2^0, \dots, v_k^0$ be the vertices of eccentricity 2 in G . Let V_i^{t-1} be the set composed of a vertex v_i^0 and all its copies in $M^{t-1}(G)$. In $M^t(G)$, by Lemma 4.3 and Definition 1.1, the vertices in $\bigcup_{i=1}^k V_i^{t-1} \cup V'_{t-1} \cup U_{t-1} \cup \{u_{t,0}\}$ are all within distance two, where U_{t-1} is the set of roots and their copies in $M^{t-1}(G)$, V'_{t-1} is the set of copies of the vertices of $M^{t-1}(G)$ in $M^t(G)$, and $u_{t,0}$ is the root of $M^t(G)$. Hence $\lambda(M^t(G)) \geq \sum_{i=1}^k |V_i^{t-1}| + |V'_{t-1}| + |U_{t-1}| = k2^{t-1} + 2^{t-1}(n+1) - 1 + 2^{t-1} - 1 = 2^{t-1}(n+k+2) - 2$. ■

For a graph G of order n , by Proposition 4.4, if $\lambda(M(G)) = n+1$, then G has at most one vertex of eccentricity 2. Also for $t \geq 2$, if $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$, then no vertex in G has eccentricity 2. There exist graphs with one vertex of eccentricity 2 and $\lambda(M(G)) = n+1$. Figure 7 illustrates a tree graph T of order 9 with one vertex of eccentricity 2, having $\lambda(M(T)) = 10$. Based on Proposition 4.4, $\lambda(M^t(T)) \geq 2^{t-1}(n+3) - 2 > 2^{t-1}(n+2) - 2$. Therefore, if $\lambda(M(G)) = n+1$, then not necessarily $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$, for $t \geq 2$.

4.2. Graphs with $\lambda(M^t(G)) = 2^t(n+1) - 2$

Shao and Solis-Oba in [20], gave bounds for the λ -number of some iterated Mycielski graph of complete graph K_n . In the following, we give the exact value of the λ -number of $M^t(K_n)$, for any $t \geq 2$.

Theorem 4.5. *For any $t \geq 2$ and $n \geq 2$, we have $\lambda(M^t(K_n)) = 2^t(n+1) - 2$.*

Proof. For $n \geq 2$, we have $\text{diam}(K_n) = 1$, so by Lemma 2.5 for any $t \geq 2$, we have $\text{diam}(M^t(K_n)) = 2$. Let $V^2 = \{v_i^k : 0 \leq k \leq 3 \text{ and } 1 \leq i \leq n\}$ be the set

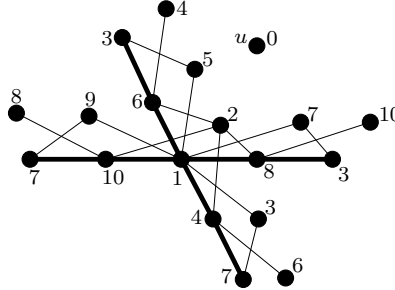


Figure 7. A 10- $L(2, 1)$ -labeling of the Mycielski graph of a tree T of order 9.

composed of the vertex of V and all their consecutive copies in $M^2(K_n)$. Let χ_i with $1 \leq i \leq n$ be a sequence of vertices in $M^2(K_n)$, where $\chi_i = v_i^2 v_i^0 v_i^1$ if i is odd and $\chi_i = v_i^1 v_i^0 v_i^2$ if i is even. We label the vertices of $M^2(K_n)$ using consecutive labels beginning with 0, in the following order $\chi_1 \chi_2 \cdots \chi_n v_n^3 v_{n-1}^3 \cdots v_1^3 u_{11} u_{10} u_{20}$.

This does not violate the distance two conditions, since two consecutive vertices are either a vertex and its copy or two vertices from the same level, which are successively at distance two. This leads to an $L(2, 1)$ -labeling of $M^2(K_n)$ with span $|M^2(K_n)| - 1$. Since the diameter is 2, then $\lambda(M^2(K_n)) = |M^2(K_n)| - 1$. From Observation 4.2 and Lemma 2.1, we get $\lambda(M^t(K_n)) = |M^t(K_n)| - 1 = 2^t(n + 1) - 2$, for any $t \geq 2$. ■

Since any graph G of order $n \geq 2$ is a subgraph of the complete graph K_n , we can conclude that for $t \geq 2$, we have $\lambda(M^t(G)) \leq |M^t(G)| - 1 = 2^t(n + 1) - 2$. This could also be proven using Theorem 3.6 by showing that for any graph G , the complement of the Mycielski graph $\overline{M}(G)$ has a perfect 4-star matching, which means by Theorem 3.6(a) that $\lambda(M^2(G)) \leq |M^2(G)| - 1$. Then the result follows from Observation 4.2 for any $t \geq 2$.

Corollary 4.6. *Let G_1 and G_2 be two graphs of the same order $|G_1| = |G_2| \geq 2$. Then for any $t \geq 2$, we have $\lambda(M^t(G_1)) + 2^t \leq \lambda(M^{t+1}(G_2))$.*

Proof. For $t \geq 2$, let G_1 and G_2 be two graphs such that $|G_1| = |G_2| = n \geq 2$. By Theorem 4.1 and Theorem 4.5, we have $\lambda(M^t(G_1)) \leq 2^t(n + 1) - 2$ and $\lambda(M^{t+1}(G_2)) \geq 2^t(n + 2) - 2$. Hence $\lambda(M^t(G_1)) + 2^t \leq \lambda(M^{t+1}(G_2))$. ■

Let us denote $\overline{M}^t(G)$ the complement graph of $M^t(G)$. The close relation between Hamiltonicity and the $L(2, 1)$ -labeling allow us to prove the following.

Corollary 4.7. *For any graph G and any $t \geq 2$, $\overline{M}^t(G)$ is a Hamiltonian graph.*

Proof. Let G be a graph of order n . First we show that $\overline{M}^2(G)$ is Hamiltonian.

Let χ_i with $2 \leq i \leq n$ be a sequence of vertices in $\overline{M^2}(G)$, where $\chi_i = v_i^2 v_i^0 v_i^1$ if i is odd, and $\chi_i = v_i^1 v_i^0 v_i^2$ if i is even. Take the vertices of $\overline{M^2}(G)$ in the following order, $v_1^0 v_1^1 \chi_2 \chi_3 \cdots \chi_n v_n^3 v_{n-1}^3 \cdots v_1^3 v_1^2 u_{11} u_{10} u_{20} v_1^0$.

Notice that this is similar to the order proposed in Theorem 4.5 for labeling $M^2(K_n)$. Since every two consecutive vertices are non-adjacent in $M^2(G)$, then the vertices of $\overline{M^2}(G)$ taken in the above order form a Hamiltonian cycle. Thus, for any graph G we have $\overline{M^2}(G)$ is Hamiltonian. For $t \geq 2$, since $M^t(G) \cong M^2(M^{t-2}(G))$, $\overline{M^t}(G)$ is a Hamiltonian graph for any $t \geq 2$. ■

Next we characterize the graphs with $\lambda(M^t(G)) = 2^t(n+1) - 2$, for $t \geq 2$.

Theorem 4.8. *Let G be a graph of order $n \geq 2$. Then for $t \geq 2$, we have $\lambda(M^t(G)) = 2^t(n+1) - 2$ if and only if $G \cong K_n$ or $\text{diam}(G) = 2$.*

Proof. For $t \geq 2$, if $G \cong K_n$, then by Theorem 4.5 we have $\lambda(M^t(G)) = 2^t(n+1) - 2$. If $\text{diam}(G) = 2$, from Theorem 4.5 we have $\lambda(M^t(G)) \leq 2^t(n+1) - 2$. By Lemma 2.5, $\text{diam}(M^t(G)) = 2$, the vertices must be assigned distinct labels, hence $\lambda(M^t(G)) = 2^t(n+1) - 2$.

Conversely, suppose that G is a graph of order $n \geq 2$, with $\text{diam}(G) \geq 3$. So there are at least two vertices at distance greater or equal to 3, one from another. Without loss of generality, we suppose that $d_G(v_1^0, v_n^0) \geq 3$. For $t = 2$, let χ_i with $2 \leq i \leq n-1$ be a sequence of vertices in $M^2(G)$, where $\chi_i = v_i^2 v_i^0 v_i^1$ if i is odd, and $\chi_i = v_i^1 v_i^0 v_i^2$ if i is even. The labeling f assigns consecutive labels to the vertices beginning with 0 in the following order, $v_1^0 v_1^1 \chi_2 \chi_3 \cdots \chi_{n-1} v_{n-1}^3 v_{n-2}^3 \cdots v_1^3 v_1^2$.

This is similar to the order in Theorem 4.5. The maximum label assigned is $f(v_1^2) = 4n - 5$. We have $d_G(v_1^0, v_n^0) \geq 3$, so by Lemma 2.4 we have $d_{M^2}(v_1^2, v_n^0) \geq 3$, and $d_{M^2}(v_1^2, v_1^1) = 3$. We label $f(v_n^0) = f(v_1^2) = 4n - 5$, $f(v_1^1) = 4n - 4$, $f(v_n^2) = 4n - 3$, $f(v_n^3) = 4n - 2$, $f(u_{11}) = 4n - 1$, $f(u_{10}) = 4n$, $f(u_{20}) = 4n + 1$. This is a valid $L(2, 1)$ -labeling of $M^2(G)$ with span $4n + 1$. Hence $\lambda(M^2(G)) \leq 4n + 1 = 4(n+1) - 3$. From the upper bound of Theorem 3.1 and Theorem 4.1, for all $t \geq 3$, we have $\lambda(M^t(G)) \leq (2^{t-2} - 1)(|M^2(G)| + 1) + \lambda(M^2(G))$. Since $|M^2(G)| = 4(n+1) - 1$, it follows that for all $t \geq 2$, $\lambda(M^t(G)) \leq 2^t(n+1) - 3$. ■

4.3. Graphs with $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$

Lemma 4.9. *Let $t \geq 2$ and $1 \leq i, j \leq n$. Then for $1 \leq k \leq 2^{t-1} - 1$, we have $d_{M^t}(v_i^k, v_j^{2^{t-1}+k}) = 2$, and for $2^{t-1} + 1 \leq k \leq 2^t - 1$, we have $d_{M^t}(v_i^k, v_j^{2^{t-1}-1}) = 2$.*

Proof. For $1 \leq k \leq 2^{t-1} - 1$, we have $v_j^{2^{t-1}+k}$ is the copy of v_j^k in $M^t(G)$. Since $d_{M^{t-1}}(v_i^k, v_j^k) = 2$, by Lemma 2.4 we have $d_{M^t}(v_i^k, v_j^{2^{t-1}+k}) = 2$.

For $t \geq 2$, v_i^3 is the copy of v_i^1 . So by Lemma 2.4 $d_{M^2}(v_i^3, v_j^1) = 2$. Since $d_{M^2}(v_i^3, v_j^2) = 2$, by using Lemma 2.4 inductively, we can show that for $2^{t-1} + 1 \leq$

$k \leq 2^t - 1$, we have $d_{M^t}(v_i^k, v_j^{2^{t-1}-1}) = 2$. ■

Lemma 4.10. *If v_i^0 and v_j^0 are not isolated vertices, then for $0 \leq k \leq 2^{t-1} - 1$, we have $d_{M^t}(v_i^k, v_j^{2^t-k-1}) = \min \{3, d_G(v_i^0, v_j^0)\}$.*

Proof. We have $v_i^{2^t-k-1}$ is the copy of $v_i^{2^{t-1}-k-1}$ in $M^t(G)$. Based on Lemma 2.4, we have $d_{M^t}(v_i^k, v_j^{2^t-k-1}) = \min \{3, d_{M^{t-1}}(v_i^k, v_j^{2^{t-1}-k-1})\}$. If $0 \leq k \leq 2^{t-2} - 1$, we have $d_{M^{t-1}}(v_i^k, v_j^{2^{t-1}-k-1}) = \min \{3, d_{M^{t-2}}(v_i^k, v_j^{2^{t-2}-k-1})\}$. Otherwise, if $2^{t-2} \leq k \leq 2^{t-1} - 1$, by symmetry $k = 2^{t-1} - m - 1$ where $0 \leq m \leq 2^{t-2} - 1$, so $d_{M^{t-1}}(v_i^k, v_j^{2^{t-1}-k-1}) = d_{M^{t-1}}(v_i^{2^{t-1}-m-1}, v_j^m) = \min \{3, d_{M^{t-2}}(v_i^{2^{t-2}-m-1}, v_j^m)\}$. By recursively using Lemma 2.4, we get $d_{M^t}(v_i^k, v_j^{2^t-k-1}) = \min \{3, d_G(v_i^0, v_j^0)\}$. ■

In the case where v_i^0 or v_j^0 are isolated vertices, for $1 \leq k \leq 2^{t-1} - 1$, we have $d_{M^t}(v_i^k, v_j^{2^t-k-1}) = 3$.

The direct product $G \times K_2$, called the *canonical double cover* (or *Kronecker double cover*) is a bipartite graph with two partition sets $X = V \times \{x\}$ and $Y = V \times \{y\}$, where $(v_i, x)(v_j, y) \in E(G \times K_2)$ if and only if $v_i v_j \in E(G)$.

From Lemma 4.10, $v_i^{2^{t-1}-1} v_j^{2^{t-1}} \in E(M^t(G))$ if and only if $v_i^0 v_j^0 \in E(G)$. Since two copies of the same vertex or copies from the same level are non-adjacent, we have the following result.

Observation 4.11. *For $t \geq 2$, let $S = \{v_i^{2^{t-1}-1}, v_i^{2^{t-1}} : 1 \leq i \leq n\}$. In $M^t(G)$, the subgraph induced by the vertices in S is isomorphic to $G \times K_2$.*

A *matching* in a graph G is a collection of vertex-disjoint edges in G , a *perfect matching* is a matching that covers all the vertices of G . The following theorem known as the *Marriage Theorem*, gives a criterion for any bipartite graph $G = (X, Y)$ to have a perfect matching.

Theorem 4.12 (The Marriage Theorem). *Let $G = (X, Y)$ be a bipartite graph. Then G has a perfect matching if and only if $|X| = |Y|$ and for any $S \subseteq X$, $|N_G(S)| \geq |S|$.*

A *2-matching* of a graph G is an assignment of weights 0, 1, or 2 to the edges of G such that the sum of weights of edges incident to any vertex in G is less or equal to 2 (see Chapter 6 in [18]). A 2-matching of a graph G can be seen as components with degree vertex at most 2. The sum of weights in a 2-matching is called the *size*. The maximum size of a 2-matching is denoted by $\nu_2(G)$, which can be computed in polynomial time [21]. A *perfect 2-matching* is a 2-matching where the sum of weights incident to any vertex in G is exactly 2. Tutte in [21], provides a characterization for the existence of perfect 2-matching of a graph.

Theorem 4.13 [21]. *A graph G has a perfect 2-matching if and only if for any independent set $S \subseteq V$, $|N_G(S)| \geq |S|$.*

A perfect 2-matching can be seen as a spanning subgraph in which each component is a single edge K_2 or a cycle. Since every even cycle has a perfect matching, a graph with a perfect 2-matching has a spanning subgraph in which each component is a single edge or an odd cycle. It is easy to see from the two preceding Theorem 4.12 and Theorem 4.13, that the existence of perfect 2-matching in a graph G is equivalent to that $G \times K_2$ admits a perfect matching.

Theorem 4.14. *Let G be a graph without isolated vertices of order $n \geq 2$. Then for $t \geq 2$, $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$ if and only if for any $S \subseteq V$, $|D_2(S)| \geq |S|$, where $D_2(S) = \{x \in V : \exists v \in S, d_G(x, v) > 2\}$.*

Proof. Let G be a graph without isolated vertices of order $n \geq 2$ such that for $t \geq 2$, $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$. Let f be a λ -labeling of $M^t(G)$, using labels from the set $L = \{0, \dots, 2^{t-1}(n+2) - 2\}$. From Lemma 4.3, we have $d_{M^t}(v_i^k, u) \leq 2$ and $d_{M^t}(u, u') \leq 2$, for all $v_i^k \in V^t$ and all $u, u' \in U_t$. The roots are assigned distinct labels, different from the labels assigned to the vertices in V^t . So for $2^{t-1} \leq k \leq 2^t - 1$, we have $f(v_i^k) \in L \setminus f(U_t)$ and $|L \setminus f(U_t)| = 2^{t-1}n$. For $1 \leq i, j \leq n$, we have $d_{M^t}(v_i^k, v_j^m) = 2$, where $2^{t-1} \leq k, m \leq 2^t - 1$. It follows that the $2^{t-1}n$ vertices v_i^k where $2^{t-1} \leq k \leq 2^t - 1$, and $1 \leq i \leq n$, have distinct labels and use all the labels in $L \setminus f(U_t)$. By Lemma 4.9, we have $d_{M^t}(v_i^k, v_j^{2^{t-1}-1}) = 2$, for $2^{t-1} + 1 \leq k \leq 2^t - 1$. The only labels remaining in $L \setminus f(U_t)$, for the vertices $v_j^{2^{t-1}-1}$, are those assigned to the vertices $v_i^{2^{t-1}}$. Since $d_{M^t}(v_i^{2^{t-1}-1}, v_j^{2^{t-1}-1}) = 2$ and $d_{M^t}(v_i^{2^{t-1}}, v_j^{2^{t-1}}) = 2$, we have $f(v_i^{2^{t-1}-1}) \neq f(v_j^{2^{t-1}-1})$ and $f(v_i^{2^{t-1}}) \neq f(v_j^{2^{t-1}})$. It follows that for any vertex $v_j^{2^{t-1}}$, there is one and only one vertex $v_i^{2^{t-1}-1}$ such that $f(v_i^{2^{t-1}-1}) = f(v_j^{2^{t-1}})$. Let (v_i, x) and (v_j, y) , $1 \leq i, j \leq n$, denote the vertices of $G \times K_2$, where $(v_i, x)(v_j, y) \in E(G \times K_2)$ if and only if $v_i^0 v_j^0 \in E(G)$. Let $M = \{(v_i, x)(v_j, y) : f(v_i^{2^{t-1}-1}) = f(v_j^{2^{t-1}})\}$. Since $f(v_i^{2^{t-1}-1}) = f(v_j^{2^{t-1}})$, we have by Lemma 4.10 that $d_G(v_i^0, v_j^0) \geq 3$. From Observation 4.11, M is a perfect matching of the graph $\overline{G^2} \times K_2$, then by Theorem 4.12 we get the necessity.

Conversely, suppose that for any $S \subseteq V$, we have $|D_2(S)| \geq |S|$. This means by Theorem 4.13 that the graph $\overline{G^2}$ has a perfect 2-matching, which means that $\overline{G^2}$ has a spanning subgraph H , whose connected components are vertex-disjoint edges or odd cycles. Let E^1, E^2, \dots, E^r be the K_2 components, and let C^1, C^2, \dots, C^s be the odd cycle components of H .

Further, we denote $x_i^0 y_i^0$ is the edge E^i and $c_{1,i}^0 c_{2,i}^0 \dots c_{n_i,i}^0$ is the odd cycle

C^i , where $n_i = |C^i|$. We define an $L(2, 1)$ -labeling f to the vertices of $M^t(G)$ as follows.

Suppose that $r \geq 2$. First we label the vertices x_1^k, y_1^k with $0 \leq k \leq 2^t - 1$, where x_1^k and y_1^k are the vertices x_1^0 and y_1^0 and their consecutive copies. The labeling f assigns in descending order the labels $2^{t-1} - 1, 2^{t-1} - 2, \dots, 0$, respectively, to $x_1^0, x_1^1, \dots, x_1^{2^{t-1}-1}$ and the labels $2^t - 1, 2^t - 2, \dots, 2^{t-1}$, respectively, to $x_1^{2^{t-1}}, x_1^{2^{t-1}+1}, \dots, x_1^{2^t-1}$. Then assign the same list of consecutive labels, now in ascending order $0, 1, \dots, 2^{t-1} - 1$, respectively, to the vertices $y_1^{2^{t-1}}, y_1^{2^{t-1}+1}, \dots, y_1^{2^t-1}$ and the labels $2^{t-1}, 2^{t-1} + 1, \dots, 2^t - 1$, respectively, to $y_1^0, y_1^1, \dots, y_1^{2^{t-1}-1}$.

- For $0 \leq k \leq 2^{t-1} - 1$, $f(x_1^k) = 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(x_1^k) = 3 \times 2^{t-1} - k - 1$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(y_1^k) = k + 2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(y_1^k) = k - 2^{t-1}$.

We have $f(x_1^k) = f(y_1^m)$ if $m = 2^t - k - 1$. Since $x_1^0 y_1^0 \in E(\overline{G^2})$, we have $d_G(x_1^0, y_1^0) \geq 3$, so by Lemma 4.10 $d_{M^t}(x_1^k, y_1^{2^t-k-1}) = 3$. Otherwise $f(x_1^k) \neq f(y_1^m)$. Since x_1^0 and y_1^0 are not adjacent in G , we have $d_{M^t}(x_1^k, y_1^m) \geq 2$, for all $0 \leq k, m \leq 2^t - 1$. Also $d_{M^t}(x_1^k, x_1^m) = d_{M^t}(y_1^k, y_1^m) = 2$, $f(x_1^k) \neq f(x_1^m)$ and $f(y_1^k) \neq f(y_1^m)$. The smallest label is $f(x_1^{2^{t-1}-1}) = f(y_1^{2^{t-1}}) = 0$, the maximum label is $f(x_1^{2^{t-1}}) = f(y_1^{2^{t-1}-1}) = 2^t - 1$.

For $2 \leq i \leq r$, we have $d_G(x_i^0, y_i^0) \geq 3$, so a vertex in E^{i-1} cannot be adjacent in G to both x_i^0 and y_i^0 . Since in every E^i the vertices x_i^0 and y_i^0 are symmetric, we rearrange the vertices of each E^i depending on the cases.

(i) If x_{i-1}^0 is adjacent in G to a vertex in E^i , we consider without loss of generality that x_{i-1}^0 is adjacent to y_i^0 .

(ii) If x_{i-1}^0 is not adjacent to E^i and y_{i-1}^0 is adjacent, we let $d_G(y_{i-1}^0, x_i^0) = 1$. Otherwise the vertices in E^{i-1} and E^i are mutually non-adjacent. This means that $d_G(x_{i-1}^0, x_i^0) \geq 2$, and $d_G(y_{i-1}^0, y_i^0) \geq 2$, for all $2 \leq i \leq r$.

With respect to the above assumptions, we label the vertices x_i^k and y_i^k with $2 \leq i \leq r$, as following.

- For $2 \leq i \leq r - 1$, and $0 \leq k \leq 2^t - 1$, $f(x_i^k) = (i - 1)2^t + f(x_1^k)$, and $f(y_i^k) = (i - 1)2^t + f(y_1^k)$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(x_r^k) = (r - 1)2^t + f(x_1^k)$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(x_r^k) = (r - 1)2^t + k$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(y_r^k) = r2^t - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(y_r^k) = (r - 1)2^t + f(y_1^k)$.

The labeling f uses distinct labels from $(i-1)2^t, \dots, i2^t - 1$, for every pair of x_i^k, y_i^m , where $m = 2^t - k - 1$, by using the same pattern for x_1^k, y_1^m (except for x_r^k, y_r^k). In the case where $r = 1$, let for $0 \leq k \leq 2^{t-1} - 1$, $f(x_1^k) = 2^{t-1} - k - 1$, for $2^{t-1} \leq k \leq 2^t - 1$, $f(x_1^k) = k$, for $0 \leq k \leq 2^{t-1} - 1$, $f(y_1^k) = 2^t - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(y_1^k) = k - 2^{t-1}$. The only vertices from two different components, with the difference between the labels equal to 1, are for $x_{i-1}^{2^{t-1}}$ and $y_{i-1}^{2^{t-1}-1}$, with both $x_i^{2^{t-1}-1}$ and $y_i^{2^{t-1}}$. This does not violate the distance two conditions, since $d_G(x_{i-1}^0, x_i^0) \geq 2$, and $d_G(y_{i-1}^0, y_i^0) \geq 2$, for all $2 \leq i \leq r$. The maximum label assigned is $f(x_r^{2^t-1}) = f(y_r^0) = r2^t - 1$.

If $s \geq 1$, next we label the vertices of the odd cycle components C^i . We make the following claim.

Claim 4.15. *For a vertex v in G not in the odd cycle component $C^i = c_{1,i}^0 c_{2,i}^0 \dots c_{n_i,i}^0$, there is at least one edge $c_{p,i}^0 c_{q,i}^0 \in C^i$ such that v is not adjacent in G to both $c_{p,i}^0$ and $c_{q,i}^0$.*

Proof. We prove this by using contradiction. We suppose that v is adjacent to at least one endpoint of any $c_{p,i}^0 c_{q,i}^0 \in C^i$. We may assume that v is adjacent to $c_{1,i}^0$. Since $d_G(c_{1,i}^0, c_{2,i}^0) \geq 3$, v is not adjacent to $c_{2,i}^0$, so v is adjacent to $c_{3,i}^0$, and so forth. Hence, if j is odd, then v is adjacent to $c_{j,i}^0$, and if j is even, then v is not adjacent to $c_{j,i}^0$. Since v is adjacent to $c_{1,i}^0$, v is not adjacent to $c_{n_i,i}^0$. It follows that n_i is even, a contradiction. \square

Since the cycles C^i are symmetric, we may consider that $d_G(y_r, c_{1,1}^0) \geq 2$, and $d_G(y_r, c_{n_1,1}^0) \geq 2$, and for $1 \leq i \leq s-1$, $d_G(c_{n_i,i}^0, c_{1,i+1}^0) \geq 2$, and $d_G(c_{n_i,i}^0, c_{n_{i+1},i+1}^0) \geq 2$. We label the vertices $c_{j,i}^k$ where $1 \leq j \leq n_i$, $1 \leq i \leq s$ and $0 \leq k \leq 2^t - 1$, with respect to the above assumptions.

- For $0 \leq k \leq 2^{t-1} - 1$, $f(c_{1,1}^k) = r2^t + 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(c_{1,1}^k) = r2^t + k$.
- For $2 \leq j \leq n_1 - 1$ and all $0 \leq k \leq 2^t - 1$, $f(c_{j,1}^k) = f(c_{1,1}^k) + (j-1)2^{t-1}$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(c_{n_1,1}^k) = f(c_{1,1}^k) + (n_1 - 1)2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(c_{n_1,1}^k) = f(c_{1,1}^{2^t-k-1})$.

The smallest label for the vertices $c_{i,1}^k$ is $f(c_{1,1}^{2^{t-1}-1}) = f(c_{n_1,1}^{2^t-1}) = r2^t$, and the maximum is $f(c_{n_1,1}^0) = f(c_{n_1-1,1}^{2^t-1}) = r2^t + n_1 2^{t-1} - 1$. Now let $\varphi_i =$

$r2^t + \sum_{j=1}^{i-1} n_j 2^{t-1}$. For $2 \leq i \leq s$, we label $f(c_{1,i}^{2^{t-1}-1}) = f(c_{n_i,i}^{2^{t-1}}) = \varphi_i$, then we label vertices in the following way.

- For $0 \leq k \leq 2^{t-1} - 1$, $f(c_{1,i}^k) = \varphi_i + 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(c_{1,i}^k) = \varphi_i + k$.
- For $2 \leq j \leq n_i - 1$ and all $0 \leq k \leq 2^t - 1$, $f(c_{j,i}^k) = f(c_{1,i}^k) + (j-1)2^{t-1}$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(c_{n_i,i}^k) = f(c_{1,i}^k) + (n_i - 1)2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(c_{n_i,i}^k) = f(c_{1,i}^{2^t-k-1})$.

The labeling f uses $n_i 2^{t-1}$ distinct labels for the $n_i 2^t$ vertices of each component C^i and their copies. For $0 \leq k \leq 2^{t-1} - 1$, we have $f(c_{1,i}^k) = f(c_{n_i,i}^{2^t-k-1})$, and for $2 \leq j \leq n_i$ $f(c_{j,i}^k) = f(c_{j-1,i}^{2^t-k-1})$. It is possible, since $d_G(c_{j,i}^0, c_{j-1,i}^0) \geq 3$, which means by Lemma 4.10 that $d_{M^t}(c_{j,i}^k, c_{j-1,i}^{2^t-k-1}) = 3$. For two vertices $c_{j,i}^k, c_{l,i}^m$ from the same component, the difference between the labels is equal to 1 in the following cases.

(i) The vertices are copies of the same vertex, or if $2^{t-1} \leq k, m \leq 2^t - 1$, in those two cases $d_{M^t}(c_{j,i}^k, c_{l,i}^m) = 2$.

(ii) For $l = j + 1$, we have $d_G(c_{j,i}^0, c_{j+1,i}^0) \geq 3$, then $d_{M^t}(c_{j,i}^k, c_{j+1,i}^m) \geq 2$.

(iii) If $l = j + 2$, $k = 2^t - 1$ and $m = 2^{t-1} - 1$, we have from Lemma 4.9 $d_{M^t}(c_{j,i}^{2^t-1}, c_{l,i}^{2^{t-1}-1}) = 2$. For two vertices from different odd cycle components, we have the difference between the labels assigned is equal to 1, it happens only for $c_{n_i,i}^0$ and $c_{n_{i-1},i}^{2^{t-1}-1}$ with $c_{1,i+1}^{2^{t-1}-1}$ and $c_{n_{i+1},i+1}^{2^{t-1}-1}$. For $1 \leq i \leq s-1$, we have $d_G(c_{n_i,i}^0, c_{n_{i+1},i+1}^0) \geq 2$ and $d_G(c_{n_i,i}^0, c_{1,i+1}^0) \geq 2$. Also from Lemma 4.9 the vertices are at distance greater or equal 2 in $M^t(G)$.

The maximum label assigned is $f(c_{n_s,s}^0) = f(c_{n_s-1,s}^{2^t-1}) = r2^t + \sum_{j=1}^s n_j 2^{t-1} - 1 = n2^{t-1} - 1$.

We finally label the remaining $2^t - 1$ roots with consecutive labels beginning with the label $n2^{t-1}$ in the following order

$$u_{1,2^{t-1}-1} u_{1,2^{t-1}-2} \cdots u_{1,0} u_{2,2^{t-2}-1} u_{2,2^{t-2}-2} \cdots u_{2,0} u_{3,2^{t-3}-1} \cdots u_{t,0}.$$

Since $d_{M^t}(u_{1,2^{t-1}-1}, c_{n_s,s}^0) = 2$, $d_{M^t}(u_{1,2^{t-1}-1}, c_{n_s-1,s}^{2^t-1}) = 2$, $d_{M^t}(u_{i,j}, u_{i,j-1}) = 2$, and $d_{M^t}(u_{i,0}, u_{i+1,2^{t-(i+1)}-1}) = 2$, this produces an $L(2, 1)$ -labeling with span $2^{t-1}(n+2) - 2$. In Figure 8, we show an $L(2, 1)$ -labeling with the same schema for $M^2(G)$, where $\overline{G^2}$ has a perfect 2-matching consisting of two K_2 components and two cycles of order 3 and 5, respectively. Hence from the lower bound of Theorem 4.1 for $t \geq 2$, we have $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$. ■

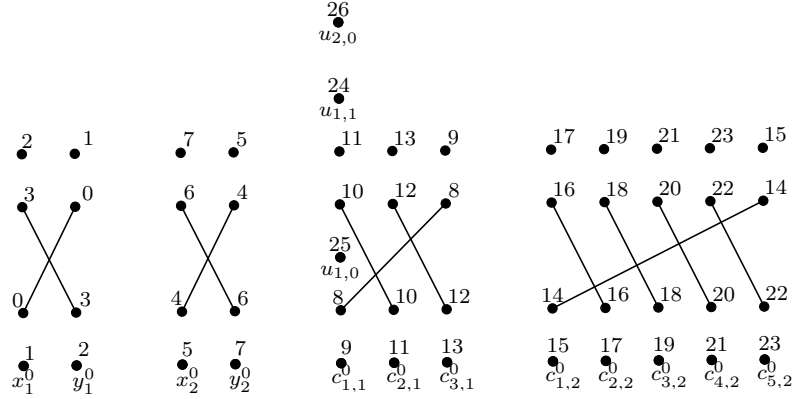


Figure 8. An $L(2,1)$ -labeling of $M^2(G)$ as in Theorem 4.6, where $\overline{G^2}$ has a perfect 2-matching with two K_2 components and two cycles of order 3 and 5, here the edges represent a perfect matching of $\overline{G^2} \times K_2$.

The labeling defined in Theorem 4.14 is a valid $L(2,1)$ -labeling for any graph G of order $n \geq 2$. If $\overline{G^2}$ has a perfect 2-matching, then we can label the vertices of $M^t(G)$ with a labeling having span $2^{t-1}(n+2) - 2$. Next, we give an upper bound for $\lambda(M^t(G))$ in terms of the maximum size of a 2-matching of $\overline{G^2}$.

Theorem 4.16. *Let G be a graph of order $n \geq 2$, with $\nu_2(\overline{G^2}) = p$. Then for $t \geq 2$, we have $\lambda(M^t(G)) \leq 2^{t-1}(2n - p + 2) - 2$.*

Proof. Let G be a graph with $\nu_2(\overline{G^2}) = p$. So there is an induced subgraph H of $\overline{G^2}$ of order p such that H has a perfect 2-matching. Let V_H be the set of vertices of H . From Theorem 4.14, we can label the vertices of $M^t(G[V_H])$ with an $L(2,1)$ -labeling f with span $2^{t-1}(p+2) - 2$, where $f(u_{t,0}) = 2^{t-1}(p+2) - 2$.

Now in $M^t(G)$, if $p < n$, then the vertices remaining unlabeled by f are the vertices in $V \setminus V_H$ and their copies. Let us denote v_i^k , where $1 \leq i \leq q$, and $0 \leq k \leq 2^t - 1$, such that $p + q = n$, the vertices of $V \setminus V_H$ and their consecutive copies. Let χ_i with $2 \leq i \leq q$ be a sequence of vertices in $M^t(G)$, where $\chi_i = v_i^2 v_i^0 v_i^1$ if i is odd, and $\chi_i = v_i^1 v_i^0 v_i^2$ if i is even. The only vertex labeled $2^{t-1}(p+2) - 2$ by f is $u_{t,0}$. Using consecutive labels we label the vertices v_i^k , with $1 \leq i \leq q$ beginning with the label $2^{t-1}(p+2) - 1$, in the following order $v_1^0 v_1^2 v_1^1 \chi_2 \cdots \chi_q v_q^3 v_q^3 v_{q-1}^3 \cdots v_1^3 v_1^4 \cdots v_q^4 v_q^5 \cdots v_1^{2^t-1}$.

This produces an $L(2,1)$ -labeling with span $2^{t-1}(p+2) - 2 + 2^t(n-p) = 2^{t-1}(2n - p + 2) - 2$. ■

Similarly to Subsection 3.3, we put interest in connected graphs, the path P_n and cycle C_n , which we use to determine some connected graphs with the smallest $\lambda(M^t(G))$.

Corollary 4.17. *For $t \geq 2$,*

$$\lambda(M^t(P_n)) = \begin{cases} 4 \times 2^t - 2 & \text{if } n = 3, 4, 5, \\ 2^{t-1}(n+2) - 2 & \text{if } n \geq 6. \end{cases}$$

Proof. For $n = 3$, we have $\text{diam}(P_3) = 2$. By Theorem 4.8 for $t \geq 2$ we have $\lambda(M^t(P_3)) = 4 \times 2^t - 2$.

For $n = 4$, $\overline{P_4^2}$ consists of a single edge and 2 isolated vertices. So $\nu_2(\overline{P_4^2}) = 2$, it follows from Theorem 4.16 that $\lambda(M^t(P_4)) \leq 4 \times 2^t - 2$. Since $M^t(P_3)$ is a subgraph of $M^t(P_4)$, from above $\lambda(M^t(P_4)) = 4 \times 2^t - 2$.

For $n = 5$, $\overline{P_5^2}$ consists of 2 independent edges and one isolated vertex. Hence $\nu_2(\overline{P_5^2}) = 4$, so from Theorem 4.16, $\lambda(M^t(P_5)) \leq 4 \times 2^t - 2$. Also $M^t(P_3)$ is a subgraph of $M^t(P_5)$, then $\lambda(M^t(P_5)) = 4 \times 2^t - 2$.

For $n \geq 6$, it is easy to see that the path P_n verifies the condition of Theorem 4.14, thus $\lambda(M^t(P_n)) = 2^{t-1}(n+2) - 2$. ■

Corollary 4.18. *For $t \geq 2$,*

$$\lambda(M^t(C_n)) = \begin{cases} 4 \times 2^t - 2 & \text{if } n = 3, \\ 5 \times 2^t - 2 & \text{if } n = 4, \\ 6 \times 2^t - 2 & \text{if } n = 5, \\ 2^{t-1}(n+2) - 2 & \text{if } n \geq 6. \end{cases}$$

Proof. We have $\text{diam}(C_3) = 1$, and $\text{diam}(C_4) = \text{diam}(C_5) = 2$. So by Theorem 4.8, for $t \geq 2$, we have $\lambda(M^t(C_3)) = 4 \times 2^t - 2$, $\lambda(M^t(C_4)) = 5 \times 2^t - 2$, and $\lambda(M^t(C_5)) = 6 \times 2^t - 2$. If $n \geq 6$, then the cycle C_n satisfies the condition of Theorem 4.14, thus $\lambda(M^t(C_n)) = 2^{t-1}(n+2) - 2$. ■

Corollary 4.19. *Let G be a connected graph, for $t \geq 2$ we have the following.*

- (1) $\lambda(M^t(G)) = 3 \times 2^t - 2$ if and only if G is K_2 .
- (2) $\lambda(M^t(G)) = 4 \times 2^t - 2$ if and only if $G \in \{P_3, P_4, P_5, P_6, C_3, C_6\}$.
- (3) $\lambda(M^t(G)) = 9 \times 2^{t-1} - 2$ if and only if $G \in \{P_7, C_7\}$.

Proof. From the lower bound of Theorem 4.1, for $t \geq 2$, we have

$$(3) \quad \lambda(M^t(G)) \geq 2^{t-1} \max\{n+2, 2(\Delta+2)\} - 2.$$

We have K_2 is the only connected graph with $\Delta = 1$, by Theorem 4.5 $\lambda(M^t(K_2)) = 3 \times 2^t - 2$. Based on inequality (3), if $\Delta \geq 2$, then $\lambda(M^t(G)) \geq 4 \times 2^t - 2$. Therefore, $\lambda(M^t(G)) = 3 \times 2^t - 2$ if and only if $G \cong K_2$.

If $\Delta = 2$, then G is either a path graph or a cycle. Then the graphs in Corollary 4.17 and Corollary 4.18 are the only connected graphs with $\Delta = 2$.

From inequality (3), if $\Delta \geq 3$, then $\lambda(M^t(G)) \geq 5 \times 2^t - 2$. Hence, based on Corollary 4.17 and Corollary 4.18, we can conclude that $\lambda(M^t(G)) = 4 \times 2^t - 2$ if and only if $G \in \{P_3, P_4, P_5, P_6, C_3, C_6\}$. Also, $\lambda(M^t(G)) = 9 \times 2^{t-1} - 2$ if and only if $G \in \{P_7, C_7\}$. ■

For any other non-trivial connected graph G not mentioned in Corollary 4.19 for $t \geq 2$, we have $\lambda(M^t(G)) \geq 5 \times 2^t - 2$.

5. OPEN PROBLEMS

From the statement of the Δ^2 -conjecture, and the upper bound of Theorem 3.1 and Theorem 4.1, we propose a weaker conjecture for the $L(2, 1)$ -labeling number of the Mycielski graph and the iterated Mycielski graph of graphs.

Conjecture 5.1. *For any graph G of order $n \geq 1$, with maximum degree Δ , and for all $t \geq 1$, we have $\lambda(M^t(G)) \leq (2^t - 1)(n + 1) + \Delta^2$.*

It is clear from Theorem 3.1 and Theorem 4.1 that if $\lambda(G) \leq \Delta^2$, then for any $t \geq 1$, $\lambda(M^t(G)) \leq (2^t - 1)(n + 1) + \Delta^2$.

Remark 5.2. For any positive integers t, t' such that $t' > t \geq 1$, if $\lambda(M^t(G)) \leq (2^t - 1)(n + 1) + \Delta^2$, then $\lambda(M^{t'}(G)) \leq (2^{t'} - 1)(n + 1) + \Delta^2$.

Proof. From the definition of the iterated Mycielski graph of a graph G , for $t' > t \geq 1$, we have $M^{t'}(G) = M^{t'-t}(M^t(G))$. From the upper bound of Theorem 3.1 and Theorem 4.1, we get that $\lambda(M^{t'}(G)) \leq (2^{t'-t} - 1)(n + 1) + \lambda(M^t(G))$. Therefore if $\lambda(M^t(G)) \leq (2^t - 1)(n + 1) + \Delta^2$, then

$$\begin{aligned} \lambda(M^{t'}(G)) &\leq (2^{t'-t} - 1)(n + 1) + \lambda(M^t(G)) \\ &\leq (2^{t'-t} - 1)(n + 1) + (2^t - 1)(n + 1) + \Delta^2 = (2^{t'-t} + 2^t - 2)(n + 1) + \Delta^2. \end{aligned}$$

For $t' > t \geq 1$, we have

$$\begin{aligned} (2^{t'} - 1) - (2^{t'-t} + 2^t - 2) &= 2^{t'} - 2^{t'-t} - 2^t + 1 \\ &= 2^t(2^{t'-t}) - 2^{t'-t} - (2^t - 1) = 2^{t'-t}(2^t - 1) - (2^t - 1) = (2^t - 1)(2^{t'-t} - 1) > 0. \end{aligned}$$

It means that $\lambda(M^{t'}(G)) \leq (2^{t'} - 1)(n + 1) + \Delta^2$. ■

Remark 5.2 shows that if Conjecture 5.1 is true for an iteration $t \geq 1$, then it is true for any iteration greater than t .

From our study, for any $t \geq 1$, the only graphs with at least one edge that we know having $\lambda(M^t(G)) = (2^t - 1)(n + 1) + \Delta^2$, are the graph K_2 , and the graphs achieving the bound in Corollary 3.2, which are the cycle C_5 , the Petersen

graph, the Hoffman-Singleton graph, and possibly a diameter two Moore graph of maximum degree 57, and order $57^2 + 1$ if such graph exists.

The complexity of the $L(2, 1)$ -labeling problem should be investigated more, whether for the Mycielski graph of graphs in general or the Mycielski graph of graphs not studied yet. For instance, trees, since the $L(2, 1)$ -labeling number can be determined in polynomial time for trees [6], we may ask if it is also the case for the Mycielski graphs generated from trees?

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