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$L(2,1)$-LABELING OF THE ITERATED MYCIELSKI GRAPHS OF GRAPHS AND SOME PROBLEMS RELATED TO MATCHING PROBLEMS

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Abstract

In this paper, we study the $L(2,1)$-labeling of the Mycielski graph and the iterated Mycielski graph of graphs in general. For a graph G and all $t \geq 1$, we give sharp bounds for $\lambda(M^t(G))$ the $L(2,1)$-labeling number of the t-th iterated Mycielski graph in terms of the number of iterations t, the order n of G, the maximum degree $\triangle$, and $\lambda(G)$ the $L(2,1)$-labeling number of G. For $t = 1$, we present necessary and sufficient conditions between the 4-star matching number of the complement graph and $\lambda(M(G))$ the $L(2,1)$-labeling number of the Mycielski graph of a graph, with some applications to special graphs. For all $t \geq 2$, we prove that for any graph G of order n, we have $2^{t-1}(n+2)-2 \leq \lambda(M^t(G)) \leq 2^t(n+1)-2$. Thereafter, we characterize the graphs achieving the upper bound $2^t(n+1)-2$, then by using the Marriage Theorem and Tutte's characterization of graphs with a perfect 2-matching, we characterize all graphs without isolated vertices achieving the lower bound $2^{t-1}(n+2)-2$. We determine the $L(2,1)$-labeling number for the Mycielski graph and the iterated Mycielski graph of some graph classes.

Keywords: frequency assignment, $L(2,1)$-labeling, Mycielski construction, matching.

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1. Introduction

The graphs considered in this paper are finite, simple, and undirected. For graph terminology, we refer to [23].

In 1992, Griggs and Yeh [11] studied a variation of the frequency assignment problem [12], where close transmitters must receive different channels and closer transmitters must receive different channels at least two apart. This problem is known as the $L(2,1)$-labeling problem, the main target is to come up with a frequency assignment with low-frequency bandwidth.

Formally, the $L(2,1)$-*labeling* of a graph $G=(V,E)$ is a function f from the vertex set V to the set of all nonnegative integers such that $|f(x)-f(y)|\geq 2$ if $d_G(x,y)=1$ and $|f(x)-f(y)|\geq 1$ if $d_G(x,y)=2$, where $d_G(x,y)$ is the distance between the vertices x and y in G. The *span* of an $L(2,1)$-labeling f is the difference between the largest and the smallest label used by f. We may always consider zero as the smallest label used, so that the span is the highest label assigned. A k-$L(2,1)$-*labeling* is an $L(2,1)$-labeling with no label greater than k, the minimum k so that G has a k-$L(2,1)$-labeling is called the $L(2,1)$-*labeling number* or λ-*number* of G, and denoted by $\lambda(G)$. An $L(2,1)$-labeling with span $\lambda(G)$ is called a λ-*labeling*.

The $L(2,1)$-labeling has been extensively studied (see surveys [3, 24]). The determination of the exact value of $\lambda(G)$ is an NP-Hard problem for graphs in general, it is NP-Complete to determine whether a graph admits an $L(2,1)$-labeling with span at most $\lambda\geq 4$ [7], the problem remains NP-Complete even restricted to some graph families (see NP-completeness results references in [3]). Therefore, the aim of the research was to bound the λ-number for graphs. By using the greedy algorithm, Griggs and Yeh [11] proved that $\lambda(G)\leq \triangle^2+2\triangle$ for any graph G, where $\triangle$ is the maximum degree of G. This upper bound was later improved by Gonçalves in [10] to $\triangle^2+\triangle-2$, and it is the best known upper bound for $\lambda(G)$ in terms of the maximum degree for graphs in general. Griggs and Yeh [11] conjectured that $\lambda(G)\leq\triangle^2$, for any graph G with $\triangle\geq 2$, it is called $\triangle^2$-conjecture and is one of the most captivating open problems about graph labeling with distance conditions. This conjecture was proven to be true by Havet *et al.* [13] for graphs with a large maximum degree. The $L(2,1)$-labeling number attracted attention not only for general graphs but also when considering specific graph classes. The decision version of the $L(2,1)$-labeling problem has been proven to be polynomial for complete graphs, paths, cycles, wheels, trees, complete k-partite graphs, among other few graph classes. For an overview on the subject of the $L(2,1)$-labeling (and its generalizations), we refer the reader to the surveys [3, 24].

In this paper, we investigate the $L(2,1)$-labeling of the Mycielski graph and the iterated Mycielski graph of graphs. In search of triangle-free graphs with a large chromatic number, Mycielski [19] used the following transformation.

Definition 1.1. For a given graph $G = (V, E)$ of order n with $V = \{v_1, v_2, \ldots, v_n\}$, the Mycielski graph of G, denoted $M(G)$, is the graph with vertex set $V \cup V' \cup \{u\}$, where $V' = \{v'_i : v_i \in V\}$ and edge set $E \cup \{v_i v'_j : v_i v_j \in E\} \cup \{v'_i u : v'_i \in V'\}$. The vertex v'_i is called the copy of the vertex v_i and u is called the root of $M(G)$.

The t-th iterated Mycielski graph of G, denoted $M^t(G)$, is defined recursively with $M^0(G) = G$ and for $t \geq 1$ $M^t(G) = M(M^{t-1}(G))$. If $t = 1$, $M^1(G)$ is the Mycielski graph of G and is denoted simply $M(G)$. It is known that $\chi(M(G)) = \chi(G) + 1$, and $\omega(M(G)) = \max\{2, \omega(G)\}$, for any graph G, where $\chi(G)$ and $\omega(G)$ are respectively the chromatic number and the clique number of G. Many aspects and invariants of the Mycielski graphs have been studied (see for example [2, 4, 5, 8, 16, 17, 20]), Mycielski graphs are known to be hard-to-color instances and are used for testing coloring algorithms [4]. The $L(2,1)$-labeling of the Mycielski graph of graphs has been previously investigated in [17] and [20]. A 4-*star matching* H of a graph G is a subgraph such that H is a collection of vertex disjoint star graphs $K_{1,1}$, $K_{1,2}$, $K_{1,3}$ or $K_{1,4}$. The 4-*star matching number* is the maximum order of a 4-star matching of G. In [17], Lin and Lam gave sufficient conditions on the 4-star matching number of the complement graph $\overline{G}$, so that $\lambda(M(G)) \leq 2n$ and $\lambda(M(G)) = 2n + k$, for any $k \geq 1$. This allows them to prove that $\lambda(M(G))$ can be computed in polynomial time for graphs with diameter at most 2, and then give the λ-number of the Mycielski graph of complete graph K_n, and the Mycielski graph of the graph join of complete graph and the empty graph. Shao and Solis-Oba in [20], also studied the $L(2,1)$-labeling number of the Mycielski and the iterated Mycielski graph of graphs. The authors as well gave the λ-number of the Mycielski graph of complete graph, and depending on the number of iterations determine the exact value or give bounds for $\lambda(M^t(K_n))$, then provided bounds for $\lambda(M^t(G))$ for any graph G.

In this paper, we continue the work started by Lin and Lam [17], and Shao and Solis-Oba [20]. In Section 2, we give some preliminary results about the Mycielski and iterated Mycielski graph of graphs, and some previous results on the $L(2,1)$-labeling number of graphs.

Section 3 is dedicated to the $L(2,1)$-labeling number of $M(G)$. First, we provide bounds involving the order n, the maximum degree $\triangle$ and the λ-number of G. Then we complete the equivalence relationship between the 4-star matching number and the $L(2,1)$-labeling number of the Mycielski graph of a graph. Afterward, we give applications of this result to the $L(2,1)$-labeling number of the Mycielski graph of some particular graphs, not mentioned in [17]. The end of Section 3 is dedicated to graphs with a lower bound $\lambda(M(G)) = n+1$, we give a condition for a graph implying that $\lambda(M(G)) = n + 1$. Then we determine the $L(2,1)$-labeling number of $M(P_n)$ and $M(C_n)$ the Mycielski graph of path and cycle respectively, which allow us to determine all the connected graphs realizing $\lambda(M(G))$ equal to 4, 6 and 7, respectively.

Section 4 is devoted to the t-th iterated Mycielski graph of graphs with $t \geq 2$. As in Section 3, we give bounds for $\lambda(M^t(G))$ in terms of the number of iterations t, the order, the maximum degree, and $\lambda(G)$. Then we show that for all $t \geq 2$, $\lambda(M^t(K_n)) = |M^t(K_n)| - 1 = 2^t(n+1) - 2$, then we characterize all graphs having $\lambda(M^t(G)) = |M^t(G)| - 1 = 2^t(n+1) - 2$. Later, we give a necessary and sufficient condition for any graph G without isolated vertices achieving a lower bound $2^{t-1}(n+2) - 2$ for the λ-number of the iterated Mycielski graph of G, we apply that to get an upper bound that can be calculated in polynomial time for any graph G, then we determine $\lambda(M^t(P_n))$, and $\lambda(M^t(C_n))$. Finally, we propose a weak version of the $\triangle^2$-conjecture for the $L(2,1)$-labeling of the Mycielski and iterated Mycielski graph of graphs.

2. Preliminaries and Previous Results

For a graph G, let $\triangle_{M^t}$, $\deg_{M^t}(x)$, and $d_{M^t}(x,y)$ denote respectively, the maximum degree, the degree of a vertex x, and the distance between the vertices x and y in $M^t(G)$. If $t = 1$, we denote simply $\triangle_M$, $\deg_M(x)$, and $d_M(x,y)$. As a consequence of Definition 1.1, we have the following.

Lemma 2.1. *If G is a graph of order n, then $|M^t(G)| = 2^t(n+1) - 1$.*

Proof. From Definition 1.1, we have $|M(G)| = 2n+1 = 2(n+1) - 1$. By using induction on t, we can show that $|M^t(G)| = 2^t(n+1) - 1$. ■

Observation 2.2. *If H is a subgraph of a graph G, then for any $t \geq 1$, $M^t(H)$ is a subgraph of $M^t(G)$.*

Lemma 2.3. *Let G be a graph of order n and maximum degree $\triangle$. For any $t \geq 1$, we have $\triangle_{M^t} = \max\{2^{t-1}(n+1) - 1, 2^t\triangle\}$.*

Proof. By Definition 1.1, we have $\deg_M(u) = n$, $\deg_M(x) = 2\deg_G(x)$, and $\deg_M(x') = \deg_G(x) + 1$ for all $x \in V$, where x' is the copy of the vertex x in $M(G)$. Then $\triangle_M = \max\{n, 2\triangle\}$. Suppose that for $k \geq 1$, we have $\triangle_{M^k} = \max\{2^{k-1}(n+1) - 1, 2^k\triangle\}$.

For $k+1$, if $2^{k-1}(n+1) - 1 \geq 2^k\triangle$, then $\triangle_{M^k} = 2^{k-1}(n+1) - 1$. Let v be a vertex of $M^k(G)$, such that $\deg_{M^k}(v) = \triangle_{M^k}$. From Definition 1.1 $\deg_{M^{k+1}}(v) = 2\deg_{M^k}(v) = 2^k(n+1) - 2 \geq \deg_{M^{k+1}}(x)$, for all $x \in V_{M^k} \cup V'_{M^k}$. Also $\deg_{M^{k+1}}(u^{k+1}) = |M^k(G)| = 2^k(n+1) - 1 > \deg_{M^{k+1}}(v)$, where u^{k+1} is the root of $M^{k+1}(G)$. So $\triangle_{M^{k+1}} = \deg_{M^{k+1}}(u^{k+1}) = 2^k(n+1) - 1$.

Otherwise, if $2^k\triangle \geq 2^{k-1}(n+1)$, then by the inductive hypothesis, we have $\triangle_{M^k} = \max\{2^{k-1}(n+1) - 1, 2^k\triangle\} = 2^k\triangle$. We have $\deg_{M^{k+1}}(x) = 2\deg_{M^k}(x) \leq$

$2^{k+1}\triangle$, for all $x \in V_{M^k}$. For $x' \in V'_{M^k}$, $\deg_{M^{k+1}}(x') = \deg_{M^k}(x) + 1 \leq 2^k \triangle + 1 \leq 2^{k+1}\triangle$. Also $\deg_{M^{k+1}}(u^{k+1}) = 2^k(n+1) - 1 < 2^{k+1}\triangle$. Thus, $\triangle_{M^{k+1}} = 2^{k+1}\triangle$. It follows that $\triangle_{M^{k+1}} = \max\{2^k(n+1) - 1, 2^{k+1}\triangle\}$. ■

Notice that $M(G)$ is a connected graph if and only if G has no isolated vertices. The *diameter* of a graph $\mathrm{diam}(G)$, is the greatest distance between any pair of vertices in G. If G is disconnected, then $\mathrm{diam}(G)$ is considered to be infinite. In [8], Fisher *et al.* proved that $\mathrm{diam}(M(G)) = \min\{\max\{2, \mathrm{diam}(G)\}, 4\}$, for every graph G without isolated vertices. The following lemmas are a consequence of the proof of this result and the definition of $M(G)$.

Lemma 2.4 [8]**.** *For v_i and v_j two non-isolated vertices in G, we have $d_M(u, v'_i) = 1$, $d_M(u, v_i) = 2$, $d_M(v'_i, v'_j) = 2$, $d_M(v_i, v'_i) = 2$, $d_M(v_i, v'_j) = \min\{3, d(v_i, v_j)\}$, and $d_M(v_i, v_j) = \min\{4, d(v_i, v_j)\}$.*

If v_i is an isolated vertex in G, then v_i is isolated in $M(G)$, and v'_i is adjacent to the root u.

Lemma 2.5. *If G is a graph without isolated vertices, then for $t \geq 1$, $\mathrm{diam}(M^t(G)) = \min\{\max\{2, \mathrm{diam}(G)\}, 4\}$.*

Proof. Based on [8], we have $\mathrm{diam}(M(G)) = \min\{\max\{2, \mathrm{diam}(G)\}, 4\}$. Suppose that for $k \geq 1$, we have $\mathrm{diam}(M^k(G)) = \min\{\max\{2, \mathrm{diam}(G)\}, 4\}$. We have $M^{k+1}(G) = M(M^k(G))$, so $\mathrm{diam}(M^{k+1}(G)) = \min\{\max\{2, \mathrm{diam}(M^k(G)\}, 4\}$. If $\mathrm{diam}(G) = 1$ or 2, then by the inductive hypothesis $\mathrm{diam}(M^k(G)) = 2$, it follows that $\mathrm{diam}(M^{k+1}(G)) = 2$. If $\mathrm{diam}(G) = 3$, by the inductive hypothesis $\mathrm{diam}(M^k(G)) = 3$ and so $\mathrm{diam}(M^{k+1}(G)) = 3$. By using the same argument if $\mathrm{diam}(G) \geq 4$, we get that $\mathrm{diam}(M^{k+1}(G)) = 4$. ■

By Lemma 2.5, if the diameter of a graph G is 1 or 2, then the diameter of the t-th iterated Mycielski graph $M^t(G)$ is 2, for any $t \geq 1$. It is clear from the definition of the $L(2,1)$-labeling that any vertices at distance less or equal to 2 must be assigned distinct labels. So for any diameter two graph G, all the vertices must be assigned different labels $\lambda(G) \geq |G| - 1$. These arguments will also be used throughout the paper.

We recall some previous results on the $L(2,1)$-labeling of graphs.

Lemma 2.6 [11]**.** *If G is a graph of maximum degree $\triangle \geq 1$, then $\lambda(G) \geq \triangle + 1$. If $\lambda(G) = \triangle + 1$, then for every vertex v of degree $\triangle$, $f(v) = 0$ or $f(v) = \triangle + 1$ for any λ-labeling f.*

For $t \geq 1$, from Lemma 2.6 and Lemma 2.3, an obvious lower bound for $\lambda(M^t(G))$ would be $\max\{2^{t-1}(n+1), 2^t \triangle + 1\}$.

Lemma 2.7 [6]**.** *If H is a subgraph of a graph G, then $\lambda(H) \leq \lambda(G)$.*

Theorem 2.8 [11]**.** *If G is a diameter 2 graph with maximum degree $\triangle$, then $\lambda(G) \leq \triangle^2$.*

In the proof of Theorem 2.8, Griggs and Yeh proved that for a graph G of order n and maximum degree $\triangle \geq (n-1)/2 \geq 3$, we have $\lambda(G) < \triangle^2$. Since $\triangle_M = \max\{n, 2\triangle\}$ and $|M(G)| = 2n+1$, it means the $\triangle^2$-conjecture is true for the Mycielski graph of any graph G of order $n \geq 3$.

The *path covering number* $p_v(G)$ of a graph, is the smallest number of vertex-disjoint paths needed to cover all the vertices of a graph G. The *complement graph* $\overline{G}$ of the graph G is the graph whose vertex set is V and where $xy \in E(\overline{G})$ if only if $xy \notin E(G)$. In [9], Georges *et al.* related the path covering number of the complement graph $\overline{G}$ to the $L(2,1)$-labeling number of G.

Theorem 2.9 [9]**.** *For any graph G of order n, we have the following.*

- $\lambda(G) \leq n-1$ *if and only if* $p_v(\overline{G}) = 1$.
- $\lambda(G) = n+r-2$ *if and only if* $p_v(\overline{G}) = r \geq 2$.

3. The Mycielski Graph of a Graph $M(G)$

3.1. Bounds for the $L(2,1)$-labeling number of $M(G)$

Theorem 3.1. *Let G be a graph of order $n \geq 1$ and maximum degree $\triangle \geq 0$. Then we have*

$$\max\{n+1, 2(\triangle+1)\} \leq \lambda(M(G)) \leq (n+1) + \lambda(G).$$

Proof. According to the definition of the Mycielski graph of a graph, the degree of the root $\deg_M(u) = n$, then $\lambda(M(G)) \geq n+1$. Otherwise, for $\triangle \geq 1$, we have the star graph $K_{1,\triangle}$ is a subgraph of G. Then by Observation 2.2 and Lemma 2.7, we have $\lambda(M(G)) \geq \lambda(M(K_{1,\triangle}))$. Since $\operatorname{diam}(K_{1,\triangle}) = 2$ and $|K_{1,\triangle}| = \triangle+1$, it follows that $\operatorname{diam}(M(K_{1,\triangle})) = 2$, and $\lambda(M(K_{1,\triangle})) \geq |M(K_{1,\triangle})| - 1 = 2(\triangle+1)$. Thus, $\lambda(M(G)) \geq 2(\triangle+1)$.

For the upper bound, let h be a λ-labeling of G. We denote $M(G)$ the Mycielski graph of G with vertex set $V(M(G)) = \{v_i, v_i', u : 1 \leq i \leq n\}$, where v_i' is the copy of v_i in $M(G)$ and u is the root. Since every λ-labeling must assign the label 0 to a vertex of G, we consider without loss of generality that $h(v_n) = 0$. We define the following labeling f on $V(M(G))$.

$$f(x) = \begin{cases} i-1 & \text{if } x = v_i',\ 1 \leq i \leq n, \\ n + h(v_i) & \text{if } x = v_i,\ 1 \leq i \leq n, \\ (n+1) + \lambda(G) & \text{if } x = u. \end{cases}$$

Now we will check that f is an $L(2,1)$-labeling of $M(G)$, we get five cases.

- We have $|f(v'_i) - f(v'_j)| = |i - j| \geq 1$ and $d_M(v'_i, v'_j) = 2$, for all $1 \leq i, j \leq n$ $i \neq j$.
- By Lemma 2.4, if $d_M(v_i, v_j) = 1$ (respectively, 2), then $d_G(v_i, v_j) = 1$ (respectively, 2). We have $|f(v_i) - f(v_j)| = |h(v_i) - h(v_j)|$. This means $|f(v_i) - f(v_j)| \geq 2$, if $d_M(v_i, v_j) = 1$ and $|f(v_i) - f(v_j)| \geq 1$, if $d_M(v_i, v_j) = 2$.
- For all $1 \leq i, j \leq n$, we have $|f(v_i) - f(v'_j)| = |n + h(v_i) - j + 1|$. The distance two conditions are respected for all the following cases.
  - (i) If $1 \leq j \leq n-1$, then $|f(v_i) - f(v'_j)| \geq 2$.
  - (ii) If $j = n$ and $i = n$, we have $|f(v_n) - f(v'_n)| = 1$, and $d_M(v_n, v'_n) \geq 2$.
  - (iii) If $j = n$ and $d_G(v_i, v_n) = 1$, we have $|h(v_i) - h(v_n)| \geq 2$, so $h(v_i) \geq 2$. It follows that $|f(v_i) - f(v'_n)| \geq 2$.
  - (iv) If $j = n$ and $d_G(v_i, v_n) \geq 2$, by Lemma 2.4 we have $d_M(v_i, v'_n) \geq 2$, and $|f(v_i) - f(v'_n)| \geq 1$.
- For all $1 \leq i \leq n$, $|f(u) - f(v'_i)| = |(n+1) + \lambda(G) - i + 1| \geq 2$.
- For all $1 \leq i \leq n$, $|f(u) - f(v_i)| = |(n+1) + \lambda(G) - (n + h(v_i))| \geq 1$, and $d_M(u, v_i) \geq 2$.

So f is an $L(2,1)$-labeling of $M(G)$ with span $(n+1) + \lambda(G)$. Hence $\lambda(M(G)) \leq (n+1) + \lambda(G)$. ∎

Corollary 3.2. *If G is a diameter 2 graph of maximum degree $\triangle$, then $\lambda(M(G)) \leq 2(\triangle^2 + 1)$.*

Proof. By Theorem 2.8 for a diameter 2 graph, we have $\lambda(G) \leq \triangle^2$. Also, we have $|G| = n \leq \triangle^2 + 1$, known as the Moore bound due to Hoffman and Singleton [14]. By combining this with the upper bound of Theorem 3.1, we get that $\lambda(M(G)) \leq 2(\triangle^2 + 1)$. ∎

The bound $2(\triangle^2 + 1)$ in Corollary 3.2 can only be attained by the Mycielski graph of diameter two Moore graphs [14], since the diameter of the Mycielski graph of these graphs is two, and these are the only diameter two graphs with order $\triangle^2 + 1$ and λ-number equal to $\triangle^2$ [11]. The only known graphs achieving this bound are C_5 the cycle of order 5, the Petersen graph, and the Hoffman-Singleton graph.

3.2. $L(2,1)$-labeling number of the Mycielski graph of a graph and the star matching of the complement graph

By using the upper bound of Theorem 3.1 and Theorem 2.9, we can link the λ-number of $M(G)$ to the path covering of the complement graph $\overline{G}$. So if $p_v(\overline{G}) = 1$, i.e., $\overline{G}$ has a Hamiltonian path, then $\lambda(M(G)) \leq 2n$, the equality holds for diameter two graphs. Also if $p_v(\overline{G}) \geq 2$, then $\lambda(M(G)) \leq 2n + p_v(\overline{G}) - 1$. But for more relevant conditions, the study of the path covering of the complement of $M(G)$ is required.

We can see that for any graph G, $\overline{M}(G)$ the complement of the Mycielski graph of G is a connected graph. The neighborhood of u in $\overline{M}(G)$ is V. For all $1 \leq i \leq n$, $v_i v_i' \in E(\overline{M}(G))$. For $i \neq j$, $v_i' v_j' \in E(\overline{M}(G))$. Also $v_i v_j', v_i v_j \in E(\overline{M}(G))$ if and only if $v_i v_j \notin E(G)$. The subgraph induced by the set V is $\overline{G}$. The subgraph induced by the set V' is the complete graph on n vertices.

Let m be an integer greater or equal to 2. An *m-star matching* H of G is a subgraph of G such that each component of H is isomorphic to a star graph $K_{1,i}$, with $1 \leq i \leq m$. The *m-star matching number*, denoted $s_m(G)$, is the maximum order of an m-star matching of G, an m-star matching of order $s_m(G)$ is said to be maximum. If $s_m(G) = |G|$, we say that G has a perfect m-star matching, a perfect m-star matching is known also as star-factor or $\{K_{1,1}, K_{1,2}, \ldots, K_{1,m}\}$-factor [1, 22]. The problem of finding whether or not a graph G admits a perfect m-star matching can be solved in polynomial time [15]. In [17], Lin and Lam studied the m-star matching and the m-star matching number $s_m(G)$. They delivered an algorithm to compute $s_m(G)$ running in $O(|V||E|)$. Then they related the 4-star matching number of $\overline{G}$ to the path covering number of $\overline{M}(G)$. In the following we denote by $i_4(G)$ the number of vertices unmatched in a maximum 4-star matching of G, i.e. $i_4(G) = n - s_4(G)$.

Theorem 3.3 [17]**.** *For any graph G, we have the following.*

(i) *If $i_4(\overline{G}) \leq 4$, then $p_v(\overline{M}(G)) = 1$.*

(ii) *If $i_4(\overline{G}) \geq 5$, then $p_v(\overline{M}(G)) = \left\lceil \frac{i_4(\overline{G})}{2} \right\rceil - 1$.*

We show that the converse holds in both cases, similarly to Theorem 2.9 in [9].

Theorem 3.4. *For any graph G, we have the following.*

(a) *$i_4(\overline{G}) \leq 4$ if and only if $p_v(\overline{M}(G)) = 1$.*

(b) *$\left\lceil \frac{i_4(\overline{G})}{2} \right\rceil = r \geq 3$ if and only if $p_v(\overline{M}(G)) = r - 1$.*

Proof. (a) Considering (i) and the contraposition of (ii) in Theorem 3.3, we get the necessity and sufficiency.

(b) Let $r \geq 3$. To verify (b) we proceed by induction on r, we prove first that (b) is true for $r = 3$.

Claim 3.5. *If $p_v(\overline{M}(G)) = 2$, then the root u is not an end-vertex of a path in a minimum path covering of $\overline{M}(G)$.*

Proof. If $p_v(\overline{M}(G)) = 2$, let P^1 and P^2 be the two paths of a minimum path covering of $\overline{M}(G)$. Suppose that u is an end-vertex of P^1. Since u is adjacent in $\overline{M}(G)$ to every vertex in V, a vertex in V cannot be an end-vertex of P^2, otherwise $\overline{M}(G)$ has a Hamiltonian path. So both ends of P^2 are from V'. Since the subgraph induced by V' is a complete graph, the other extremity of P^1 is in V. Let z be the other end of P^1, x' and y' the ends of P^2. Since u is adjacent to z, x' is adjacent to y'. If z' the copy of z belongs to P^1, we have z' is adjacent to x' and y', we can construct a Hamiltonian path of $\overline{M}(G)$. If z' belongs to P^2, then since z is adjacent to z', in this case also $\overline{M}(G)$ has a Hamiltonian path, a contradiction. □

If $p_v(\overline{M}(G)) = 2$, let $x, y \in V$ and be such that x or its copy and y or its copy are end-vertices of the two different paths in a minimum path covering of $\overline{M}(G)$. We consider the graph H with vertex set V and edge set of its complement $E(\overline{H}) = E(\overline{G}) \cup \{xy\}$. It is clear that $p_v(\overline{M}(H)) = 1$, and $i_4(\overline{H}) \geq i_4(\overline{G}) - 2$. Since $p_v(\overline{M}(G)) = 2$, according to (a) we have $4 \geq i_4(\overline{H})$, and $i_4(\overline{G}) \geq 5$. It follows that $\left\lceil \frac{i_4(\overline{G})}{2} \right\rceil = 3$. So from Theorem 3.3(ii), we have Theorem 3.4(b) is true for $r = 3$.

Assume that (b) is true for $3 \leq r \leq k$, and let $r = k + 1$.

If $p_v(\overline{M}(G)) = k$, let $x, y \in V$ and be such that x or its copy and y or its copy are end-vertices of two different paths in a minimum path covering of $\overline{M}(G)$. We consider the graph H with vertex set V and edge set of its complement $E(\overline{H}) = E(\overline{G}) \cup \{xy\}$. We have $p_v(\overline{M}(H)) = k - 1$, and $i_4(\overline{H}) \geq i_4(\overline{G}) - 2$. So by the inductive hypothesis $\left\lceil \frac{i_4(\overline{H})}{2} \right\rceil = k$, hence $2k+2 \geq i_4(\overline{G})$. Since $p_v(\overline{M}(G)) = k$, by the inductive hypothesis $i_4(\overline{G}) \geq 2k + 1$. It follows that $\left\lceil \frac{i_4(\overline{G})}{2} \right\rceil = k + 1$. Theorem 3.3(ii) completes the equivalence. ■

By combining Theorem 2.9 and Theorem 3.4, we get the following results.

Theorem 3.6. *Let G be any graph of order n. Then the following statements hold.*

(a) $\lambda(M(G)) \leq 2n$ *if and only if* $i_4(\overline{G}) \leq 4$.

(b) *For any positive integer r, we have*

$$\lambda(M(G)) = 2n + r \text{ if and only if } \left\lceil \frac{i_4(\overline{G})}{2} \right\rceil = r + 2.$$

Next, we give applications of the previous theorem to the λ-number of the Mycielski graph of certain graphs.

If the diameter of G is 1 or 2, then $\mathrm{diam}(M(G)) = 2$, and we can conclude from Theorem 3.6 that $\lambda(M(G)) = 2n + \max\left\{2, \left\lceil \frac{i_4(\overline{G})}{2} \right\rceil\right\} - 2$.

Corollary 3.7. *Let G be a graph of order n. If the clique number $\omega(G) \leq 4$, then $\lambda(M(G)) \leq 2n$.*

Proof. By Theorem 3.6(a) if $\lambda(M(G)) > 2n$, then $i_4(\overline{G}) \geq 5$. This means that $\omega(G) \geq 5$. ∎

The graphs with clique number less or equal to 4 in Corollary 3.7 include trees, planar graphs, and subcubic graphs.

If X is any subset of V, we denote $N_G(X)$ the set of all vertices in V adjacent to at least one vertex from X in G. In [17], a criterion for a graph to have a perfect m-star matching is given, this appeared also in [1, 15, 22].

Theorem 3.8 [1, 15, 17, 22]**.** *A graph G has a perfect m-star matching if and only if for any independent set S in G, $|N_G(S)| \geq |S|/m$.*

Corollary 3.9. *Let G be a graph of order n and maximum degree $\triangle \leq n-2$. If $3(n-1) + \delta \geq 4\triangle$, then $\lambda(M(G)) \leq 2n$.*

Proof. Let $\overline{\triangle}$ and $\overline{\delta}$ denote, respectively, the maximum and minimum degree of the complement graph $\overline{G}$. For any independent set S in $\overline{G}$, let $|E_{\overline{G}}(S)|$ denote the number of edges incident to the vertices of S in $\overline{G}$. We have

$$|N_{\overline{G}}(S)|\overline{\triangle} \geq |E_{\overline{G}}(S)| \geq \overline{\delta}|S|. \tag{1}$$

If $3(n-1) + \delta \geq 4\triangle$, then since $\overline{\triangle} = (n-1) - \delta$ and $\overline{\delta} = (n-1) - \triangle$, we have $4\overline{\delta} \geq \overline{\triangle}$. Therefore from Inequality (1) we get that $|N_{\overline{G}}(S)| \geq |S|/4$, for any independent set S in $\overline{G}$. Then by Theorem 3.8, $\overline{G}$ has a perfect 4-star matching. Hence from Theorem 3.6(a), we have $\lambda(M(G)) \leq 2n$. ∎

From Corollary 3.9, any regular graph G of order n, except complete graphs, has $\lambda(M(G)) \leq 2n$. In [17], it is shown that for complete graph $\lambda(M(K_2)) = 4$ and $\lambda(M(K_n)) = 2n + \lceil \frac{n}{2} \rceil - 2$ for $n \geq 3$. Next, we determine the exact λ-number of the Mycielski graph of complete k-partite graphs.

Corollary 3.10. *Let G be a complete k-partite graph of order n, where the partite sets consist of p sets of order greater or equal 2 and q singletons.*

- *If $q \leq 4$, then $\lambda(M(G)) = 2n$.*
- *If $q \geq 5$, then $\lambda(M(G)) = 2n + \lceil \frac{q}{2} \rceil - 2$.*

Proof. We have $\overline{G}$ is formed of p connected components that are complete graphs of order greater or equal to 2, and q isolated vertices. Therefore $i_4(\overline{G}) = q$. If $q \leq 4$, by Theorem 3.6(a), $\lambda(M(G)) \leq 2n$. Since $\mathrm{diam}(M(G)) = 2$, it follows that $\lambda(M(G)) = 2n$. If $q \geq 5$, then by Theorem 3.6(b), $\lambda(M(G)) = 2n + \lceil \frac{q}{2} \rceil - 2$. ∎

Let G_1, G_2 be two disjoint graphs. The disjoint union of G_1 and G_2, denoted by $G_1 \cup G_2$, is the graph with vertex set $V(G_1) \cup V(G_2)$ and edge set $E(G_1) \cup E(G_2)$. The joint of G_1 and G_2 denoted $G_1 \vee G_2$ is the graph obtained from $G_1 \cup G_2$ by joining each vertex of G_1 to each vertex of G_2.

Corollary 3.11. *Let $G_1, G_2, \ldots, G_k$ be a collection of disjoint graphs having, respectively, $n_1, n_2, \ldots, n_k$ vertices. Let $n = \sum_{i=1}^{k} n_i$. Then $\lambda(M(G_1 \vee G_2 \vee \cdots \vee G_k)) = 2n + \max\left\{2, \left\lceil \frac{I}{2} \right\rceil\right\} - 2$, where $I = \sum_{i=1}^{k} i_4(\overline{G_i})$.*

Proof. Let $G = G_1 \vee G_2 \vee \cdots \vee G_k$. We have $\overline{G} = \overline{G_1} \cup \overline{G_2} \cup \cdots \cup \overline{G_k}$. It follows that $i_4(\overline{G}) = \sum_{i=1}^{k} i_4(\overline{G_i}) = I$. Thus, by Theorem 3.6(a), if $I \leq 4$, then $\lambda(M(G)) \leq 2n$. Since $\operatorname{diam}(G) = 2$, it follows that $\lambda(M(G)) = 2n$. If $I \geq 5$, from Theorem 3.6(b), $\lambda(M(G)) = 2n + \left\lceil \frac{I}{2} \right\rceil - 2$. ■

3.3. Graphs with $\boldsymbol{\lambda(M(G)) = n + 1}$

For $k \geq 1$, the *kth power* of a graph G is the graph G^k with vertex set V and edge set $E(G^k) = \{v_i v_j : 1 \leq d_G(v_i, v_j) \leq k\}$. Then the square of a graph G^2 has the edge set of its complement graph $E(\overline{G^2}) = \{v_i v_j : d_G(v_i, v_j) \geq 3\}$. Next we give a condition, so that $\lambda(M(G)) = n + 1$.

Lemma 3.12. *In a graph G of order n, if the vertex set V can be partitioned into $k \geq 1$ vertex-disjoint cliques in $\overline{G^2}$ such that at least $k - 1$ cliques are of order greater or equal 3, then $\lambda(M(G)) = n + 1$.*

Proof. Let $V = \bigcup_{r=1}^{k} S_r$ be such that S_r are vertex-disjoint cliques in $\overline{G^2}$ of order $|S_r| = n_r \geq 3$ for $1 \leq r \leq k - 1$, and $|S_k| = n_k \geq 1$, where $\sum_{r=1}^{k} n_r = n$. For $1 \leq r \leq k$, let us denote $S_r = \{v_{i,r} : 1 \leq i \leq n_r\}$, let $v'_{i,r}$ be the copy of the vertex $v_{i,r}$, and let u be the root of $M(G)$. We have $d_G(v_{i,r}, v_{j,r}) \geq 3$ for any two distinct vertices in S_r, so a vertex in S_{r+1} can be adjacent to at most one vertex in S_r. For $1 \leq r \leq k - 1$, the cliques S_r in $\overline{G^2}$ are symmetric of order greater or equal 3. We suppose without loss of generality that $d_G(v_{n_r,r}, v_{1,r+1})) \geq 2$, for $1 \leq r \leq k - 1$. Let $\psi_1 = 0$ and for $r \geq 2$, $\psi_r = \sum_{j=1}^{r-1} n_j$. With respect to the previous assumption, we label the vertices of $M(G)$ as following.

- For $1 \leq r \leq k - 1$, define $f(v_{1,r}) = \psi_r$. For $2 \leq i \leq n_r$, $f(v_{i,r}) = \psi_r + 1$. Also $f(v'_{1,r}) = \psi_r + 1$, and $f(v'_{2,r}) = \psi_r$. For $3 \leq i \leq n_r$, $f(v'_{i,r}) = \psi_r + i - 1$.
- If $|S_k| = 1$, then let $f(v_{1,k}) = n$, and $f(v'_{1,k}) = n - 1$.
- If $|S_k| = 2$, then let $f(v_{1,k}) = n - 2$, $f(v'_{1,k}) = n - 1$, $f(v_{2,k}) = n - 1$, and $f(v'_{2,k}) = n - 2$.
- If $|S_k| \geq 3$, then define $f(v_{1,k}) = \psi_k$. For $2 \leq i \leq n_k$, $f(v_{i,k}) = \psi_k + 1$. Also $f(v'_{1,k}) = \psi_k + 1$, and $f(v'_{2,k}) = \psi_k$. For $3 \leq i \leq n_k$, $f(v'_{i,k}) = \psi_k + i - 1$.

Finally, label the root u by $f(u) = n+1$. We have $d_G(v_{i,r}, v_{j,r})) \geq 3$, and for $1 \leq r \leq k-1$ we have $d_G(v_{n_r,r}, v_{1,r+1})) \geq 2$. This means by Lemma 2.4 that $d_M(v_{i,r}, v_{j,r})) \geq 3$, $d_M(v'_{i,r}, v_{j,r})) = 3$, and $d_M(v'_{n_r,r}, v_{1,r+1})) \geq 2$. The labeling f is an $L(2,1)$-labeling of $M(G)$ with span $n+1$. Hence $\lambda(M(G)) = n+1$. ■

In the case of the empty graph $\overline{K_n}$, we have $M(\overline{K_n}) \cong K_{1,n} \cup \overline{K_n}$. Since $\lambda(K_{1,n}) = n+1$, we have $\lambda(M(\overline{K_n})) = n+1$, we can get the same result using Lemma 3.12. We are now interested in some connected graphs, we consider the graph path P_n and cycle C_n.

Let P_n denote the graph path of order $n \geq 3$ with vertex set $V(P_n) = \{v_1, \ldots, v_n\}$ and edge set $E(P_n) = \{v_i v_{i+1} : 1 \leq i \leq n-1\}$. Denote $V(M(P_n)) = V(P_n) \cup \{v'_i : 1 \leq i \leq n\} \cup \{u\}$, where v'_i is the copy of the vertex v_i, and u is the root of $M(P_n)$.

Proposition 3.13.

$$\lambda(M(P_n)) = \begin{cases} 6 & \textit{if } n = 3, 4, \\ 7 & \textit{if } n = 5, \\ n+1 & \textit{if } n \geq 6. \end{cases}$$

Proof. • For $n = 3$, we have $\operatorname{diam}(P_3) = 2$. So from Theorem 3.6, $\lambda(M(P_3)) = 6$.

• For $n = 4$, we have a 6-$L(2,1)$-labeling of $M(P_4)$ shown in Figure 1. Hence $\lambda(M(P_4)) \leq 6$. Also we have $M(P_3)$ is a subgraph of $M(P_4)$. By Lemma 2.7, it follows that $\lambda(M(P_4)) \geq \lambda(M(P_3)) = 6$. Thus, $\lambda(M(P_4)) = 6$.

• For $n = 5$, Figure 2 illustrates a 7-$L(2,1)$-labeling of $M(P_5)$. This implies also by Theorem 3.1 that $6 \leq \lambda(M(P_5)) \leq 7$.

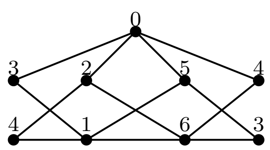

Figure 1. A 6-$L(2,1)$-labeling of $M(P_4)$.

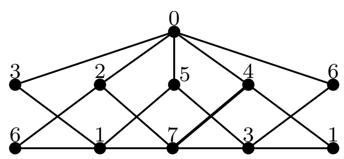

Figure 2. A 7-$L(2,1)$-labeling of $M(P_5)$.

Suppose that $\lambda(M(P_5)) = 6$. Then there is an $L(2,1)$-labeling f of $M(P_5)$ using labels in the set $L = \{0,1,2,3,4,5,6\}$. Since $\deg_M(u) = 5$, by Lemma 2.6, $f(u) = 0$ or $f(u) = 6$. Without loss of generality, we suppose that $f(u) = 0$. Since all the vertices are at distance less or equal to 2 from u, it is the only vertex with label 0. We denote by $N(v)$ the open neighborhood of a vertex v, and by $N^2(v)$ the set of all vertices at distance at most 2 from a vertex v in $M(P_5)$. We have $N(u) = \{v'_1, v'_2, v'_3, v'_4, v'_5\}$, and $d_M(v'_i, v'_j) = 2$, for $1 \leq i, j \leq 5$. So each vertex v'_i must receive a distinct label from the set $\{2,3,4,5,6\}$ different from

$f(u) = 0$. We have every vertex in $M(P_5)$ is at distance less or equal 2 form v_3. It means that v_3 must receive a distinct label from $v'_1, v'_2, v'_3, v'_4, v'_5$, and u. Hence $f(v_3) = 1$, and v_3 is the only vertex with label 1. We have $\{u, v_3, v'_1, v'_2, v'_3, v'_4\} \subset N^2(v_2)$, and the vertices u, v_3, v'_1, v'_2, v'_3 and v'_4 receive distinct labels from the set L which leaves only the label assigned to v'_5 available for v_2. Hence $f(v_2) = f(v'_5)$. Also, $\{u, v_3, v'_2, v'_3, v'_4, v'_5\} \subset N^2(v_4)$. By using the same arguments as before, we get that $f(v_4) = f(v'_1)$. $N^2(v_1) = \{u, v_2, v_3, v'_1, v'_2, v'_3\}$, and each vertex in $N^2(v_1)$ have a distinct label from L ($f(v_2)$=$f(v'_5)$). Then $f(v_1) = f(v'_4)$. Also $N^2(v_5) = \{u, v_3, v_4, v'_3, v'_4, v'_5\}$, with $f(v_4) = f(v'_1)$. Hence $f(v_5) = f(v'_2)$. We have $N(v_3) = \{v_2, v_4, v'_2, v'_4\}$, with $f(v_2) = f(v'_5)$, $f(v_4) = f(v'_1)$, and $f(v_3) = 1$. It follows that the labels assigned to v'_1, v'_2, v'_4 and v'_5 must be greater or equal to 3. Hence, the only remaining label for v'_3 is $f(v'_3) = 2$. We have v_2 and v_4 are adjacent to v'_3, $f(v_2) = f(v'_5)$, $f(v_4) = f(v'_1)$, and $f(v'_3) = 2$. Then $f(v'_5)$ and $f(v'_1)$ must be greater than 3, hence $f(v'_5), f(v'_1) \in \{4, 5, 6\}$. Since v'_1 is adjacent to v_2 and $f(v_2) = f(v'_5)$, we have $|f(v'_5) - f(v'_1)| \geq 2$. Therefore $f(v'_1), f(v'_5) \in \{4, 6\}$, which means also that $f(v'_2), f(v'_4) \in \{3, 5\}$. Since $f(v_2) = f(v'_5)$, and $f(v_1) = f(v'_4)$, it follows that $|f(v'_5) - f(v'_4)| \geq 2$. Also, $f(v_4) = f(v'_1)$ and $f(v_5) = f(v'_2)$, hence $|f(v'_1) - f(v'_2)| \geq 2$. If $f(v'_1) = 4$, then since $|f(v'_1) - f(v'_2)| \geq 2$, $f(v'_2) \notin \{3, 5\}$, a contradiction. Now if $f(v'_1) = 6$, then $f(v'_5) = 4$. Since $|f(v'_5) - f(v'_4)| \geq 2$, $f(v'_4) \notin \{3, 5\}$, again a contradiction. Therefore $\lambda(M(P_5)) \geq 7$. Hence $\lambda(M(P_5)) = 7$.

- For $n \geq 6$, we define a labeling f on $V(M(P_n))$ as following.

$f(u) = 0$, $f(v'_1) = 6$, $f(v'_2) = 5$, $f(v'_3) = 4$, $f(v'_4) = 7$, $f(v'_5) = 2$, $f(v'_6) = 3$, and $f(v'_i) = i + 1$ if $i \geq 7$.

$f(v_1) = 7$, $f(v_2) = 1$, $f(v_3) = 3$, $f(v_4) = 6$, $f(v_5) = 1$, $f(v_6) = 4$, and for $i \geq 7$:

$f(v_i) = 6$ if $i \equiv 1 \pmod 3$, $f(v_i) = 2$ if $i \equiv 2 \pmod 3$, $f(v_i) = 4$ if $i \equiv 0 \pmod 3$.

The idea is to come up with a 7-$L(2,1)$-labeling of the subgraph induced by $H = \{u, v_i, v'_i : 1 \leq i \leq 6\}$ isomorphic to $M(P_6)$. Then if $i \geq 7$, assign each vertex copy v'_i consecutive labels beginning with 8, and label the vertices v_i with labels $6, 2, 4$ for $i \equiv 1 \pmod 3$, $i \equiv 2 \pmod 3$, and $i \equiv 0 \pmod 3$, respectively. This is an $L(2,1)$-labeling of $M(P_n)$ with span $n + 1$. Hence $\lambda(M(P_n)) \leq n + 1$, for $n \geq 6$. It follows from Theorem 3.1 that $\lambda(M(P_n)) = n + 1$, for $n \geq 6$. ■

Let C_n be the graph cycle with vertex set $V(C_n) = \{v_0, v_1, \ldots, v_{n-1}\}$ and edge set $E(C_n) = \{v_i v_{i+1 (\mathrm{mod}\ n)} : 0 \leq i \leq n-1\}$, where the indices are taken modulo n. We denote $V(M(C_n)) = V(C_n) \cup \{v'_i : 1 \leq i \leq n\} \cup \{u\}$, we have $E(M(C_n)) = \left\{v_i v_{i+1 (\mathrm{mod}\ n)}, v_i v'_{i+1 (\mathrm{mod}\ n)}, v'_i v_{i+1 (\mathrm{mod}\ n)} : 0 \leq i \leq n-1\right\} \cup \{v'_i u : 0 \leq i \leq n-1\}$.

Proposition 3.14.

$$\lambda(M(C_n)) = \begin{cases} 6 & \textit{if } n = 3, \\ 8 & \textit{if } n = 4, \\ 10 & \textit{if } n = 5, \\ n+1 & \textit{if } n \geq 6. \end{cases}$$

Proof. • For $3 \leq n \leq 5$, since $\mathrm{diam}(C_3) = 1$, $\mathrm{diam}(C_4) = \mathrm{diam}(C_5) = 2$, from Lemma 2.4, $\mathrm{diam}(M(C_3)) = \mathrm{diam}(M(C_4)) = \mathrm{diam}(M(C_5)) = 2$. By applying Theorem 3.6, we get that $\lambda(M(C_3)) = 6$, $\lambda(M(C_4)) = 8$, and $\lambda(M(C_5)) = 10$.

• For $n \geq 6$, in Figure 3, Figure 4, and Figure 5, respectively, we present an $L(2,1)$-labeling for $M(C_6)$, $M(C_7)$, and $M(C_8)$, respectively, with span 7, 8, and 9. It follows from the lower bound in Theorem 3.1 that $\lambda(M(C_6)) = 7$, $\lambda(M(C_7)) = 8$, and $\lambda(M(C_8)) = 9$.

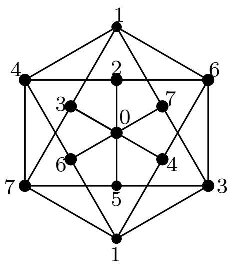

Figure 3. A 7-$L(2,1)$-labeling of $M(C_6)$.

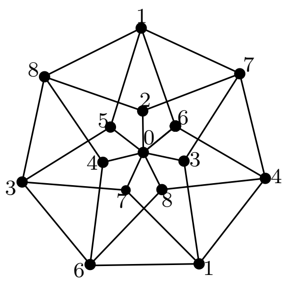

Figure 4. An 8-$L(2,1)$-labeling of $M(C_7)$.

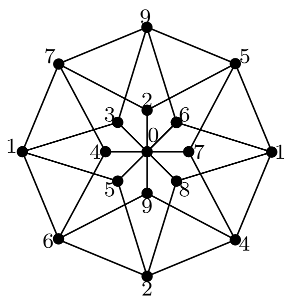

Figure 5. A 9-$L(2,1)$-labeling of $M(C_8)$.

For $n \geq 9$, we partition the vertex set $V(C_n)$ into cliques in $\overline{C_n^2}$ as following.

If $n \equiv 0 \pmod 3$, for $0 \leq i \leq \frac{n}{3} - 1$, the sets $S_i = \{v_i, v_{i+\frac{n}{3}}, v_{i+2\frac{n}{3}}\}$ form disjoint cliques of order 3 in $\overline{C_n^2}$. We have $V(C_n) = \bigcup_{i=0} S_i$.

If $n \equiv 1 \pmod 3$, for $0 \le i \le \lfloor \frac{n}{3} \rfloor - 1$, the sets $S_i = \left\{ v_i, v_{i+\lfloor \frac{n}{3} \rfloor}, v_{i+2\lfloor \frac{n}{3} \rfloor} \right\}$ form disjoint cliques of order 3 in $\overline{C_n^2}$. We have $V(C_n) = \bigcup_{i=0} S_i \cup \{v_{n-1}\}$.

If $n \equiv 2 \pmod 3$, for $1 \le i \le \lceil \frac{n}{3} \rceil - 1$, the sets $S_i = \left\{ v_i, v_{i+\lceil \frac{n}{3} \rceil}, v_{i+2\lceil \frac{n}{3} \rceil - 1} \right\}$ form disjoint cliques of order 3 in $\overline{C_n^2}$, and $v_0 v_{\lceil \frac{n}{3} \rceil}$ is an edge in $\overline{C_n^2}$. We have $V(C_n) = \bigcup_{i=1} S_i \cup \left\{ v_0, v_{\lceil \frac{n}{3} \rceil} \right\}$.

The cycle C_n in the three cases verifies the condition in Lemma 3.12. Hence $\lambda(M(C_n)) = n+1$, for $n \ge 6$. ■

For a connected graph G of order n in Theorem 3.1 we have $\lambda(M(G)) \ge n+1$. It means that for any fixed positive integer k, there are finitely many connected graphs having $\lambda(M(G)) = k$. In the following, we determine all the connected graphs having $\lambda(M(G))$ equal to $4, 6$ and 7. These are the smallest possible values for the λ-number of the Mycielski graph of any non-trivial connected graph.

Corollary 3.15. *For a connected graph G, we have the following.*

(1) $\lambda(M(G)) = 4$ *if and only if G is K_2.*

(2) $\lambda(M(G)) = 6$ *if and only if $G \in \{P_3, P_4, C_3\}$.*

(3) $\lambda(M(G)) = 7$ *if and only if $G \in \{P_5, P_6, C_6\}$.*

Proof. From Theorem 3.1, for a connected graph G of order n and maximum degree $\triangle$, we have

$$\lambda(M(G)) \ge \max\{n+1, 2(\triangle + 1)\}. \tag{2}$$

The only connected graph with $\triangle = 1$ is K_2 and we have $\lambda(M(K_2)) = 4$. If $\triangle \ge 2$, from the inequality (2), $\lambda(M(G)) \ge 6$. It follows that $\lambda(M(G)) = 4$ if and only if $G \cong K_2$. Also there is no connected graph with $\lambda(M(G)) = 5$.

The only connected graphs with $\triangle = 2$ are path graphs and cycles. Based on inequality (2), if $\triangle \ge 3$, then $\lambda(M(G)) \ge 8$. Then if $6 \le \lambda(M(G)) \le 7$, it means necessarily that G is a path or a cycle graph. In Proposition 3.13 and Proposition 3.14, the only connected graphs with $\lambda(M(G)) = 6$ are $P_3, P_4,$ and C_3. Also the only connected graphs with $\lambda(M(G)) = 7$ are $P_5, P_6,$ and C_6. ■

4. The Iterated Mycielski Graph of a Graph $M^t(G)$

4.1. Bounds for $\lambda(M^t(G))$

Theorem 4.1. *If G is a graph of order $n \ge 2$ and maximum degree $\triangle \ge 0$, then for $t \ge 2$ we have*

$$2^{t-1} \max\{n+2, 2(\triangle+2)\} - 2 \le \lambda(M^t(G)) \le (2^t - 1)(n+1) + \lambda(G).$$

Proof. For a graph G of order $n \geq 2$ from Definition 1.1, we have $K_{1,n}$ is a subgraph of $M(G)$. Then by Observation 2.2, $M^{t-1}(K_{1,n})$ is a subgraph of $M^t(G)$. Since $\mathrm{diam}(K_{1,n}) = 2$, it follows from Lemma 2.5 and Lemma 2.7 that $\lambda(M^t(G)) \geq \lambda(M^{t-1}(K_{1,n})) \geq |M^{t-1}(K_{1,n})| - 1$. By Lemma 2.1 $|M^{t-1}(K_{1,n})| = 2^{t-1}(n+2) - 1$, hence $\lambda(M^t(G)) \geq 2^{t-1}(n+2) - 2$, for $t \geq 2$. If $\triangle \geq 1$, we have $K_{1,\triangle}$ is a subgraph of G. By using the same arguments as before, we get that $\lambda(M^t(G)) \geq 2^t(\triangle + 2) - 2$.

On the other hand, for $t \geq 2$, we have $M^t(G) = M(M^{t-1}(G))$. So by the upper bound of Theorem 3.1, $\lambda(M^t(G)) \leq (|M^{t-1}(G)| + 1) + \lambda(M^{t-1}(G)) = 2^{t-1}(n+1) + \lambda(M^{t-1}(G))$. Recursively we get that $\lambda(M^t(G)) \leq \sum_{i=0}^{t-1} 2^i(n+1) + \lambda(G) = (2^t - 1)(n+1) + \lambda(G)$. ∎

The lower bound $2^{t-1}(n+2)-2$ and the upper bound of Theorem 4.1 are true also for $n = 1$. The upper bound coincides with the upper bound in Theorem 3.1 for $t = 1$. As a consequence we make the following observation.

Observation 4.2. *If a graph G of order n has $\lambda(G) \leq n-1$, then for any $t \geq 1$, $\lambda(M^t(G)) \leq |M^t(G)| - 1 = 2^t(n+1) - 2$, and there is equality if G is of diameter two.*

Further, we denote $V^t = \{v_i^k : 1 \leq i \leq n \text{ and } 0 \leq k \leq 2^t - 1\}$, the set composed of the vertices of V and all their copies in $M^t(G)$, where v_i^1 is the copy of v_i^0 in $M(G)$. v_i^2 and v_i^3 are respectively the copies of v_i^0 and v_i^1 in $M^2(G)$. $v_i^4, v_i^5, v_i^6, v_i^7$ are respectively the copies of $v_i^0, v_i^1, v_i^2, v_i^3$ in $M^3(G)$ and so forth. In $M^t(G)$ for $0 \leq k \leq 2^{t-1} - 1$, we have $v_i^{2^{t-1}+k}$ is the exact copy of the vertex v_i^k from $M^{t-1}(G)$. For $t \geq 2$, let U_t be the set of all the roots (i.e. roots and their consecutive copies in all levels) in $M^t(G)$. Recursively $U_t = U_{t-1} \cup U'_{t-1} \cup \{u_{t,0}\}$ and $|U_t| = 2^t - 1$. We denote the set of roots $U_t = \{u_{i,j} : 1 \leq i \leq t \text{ and } 0 \leq j \leq 2^{t-i} - 1\}$ such that for example in $M^3(G)$, $u_{1,0}$ is the root of $M(G)$, $u_{1,1}$ the copy of $u_{1,0}$, and $u_{2,0}$ the root of $M^2(G)$. $u_{1,2}$, $u_{1,3}$, $u_{2,1}$ are respectively the copies of $u_{1,0}$, $u_{1,1}$, $u_{2,0}$, and $u_{3,0}$ is the root in $M^3(G)$, and so forth. Figure 6 illustrates an adjacency of a vertex and its copies v_i^k in $M^2(G)$, with respect to the above ordering.

Lemma 4.3. *If $d_G(v_i^0, v_j^0) \leq 2$, then for any $t \geq 1$ and all $0 \leq k, m \leq 2^t - 1$, we have $d_{M^t}(v_i^k, v_j^m) \leq 2$, and if v_i^0 is not an isolated vertex for $k \neq m$, we have $d_{M^t}(v_i^k, v_i^m) = 2$.*

Proof. By using Lemma 2.4 inductively, we get the results. ∎

The *eccentricity* of a vertex v in a graph G, is the greatest distance between v and any other vertex in G. By Lemma 4.3, if a vertex has eccentricity 1 or 2 in G, then the vertex and all its copies are of eccentricity 2 in $M^t(G)$. In a graph G

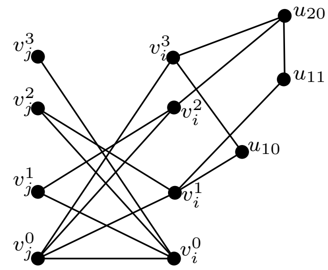

Figure 6. An example of an adjacency of the vertices v_i^k in $M^2(G)$.

without isolated vertices, we have from the definition of the Mycielski construction, the eccentricity of the root in $M(G)$ is 2, so from above the eccentricity of all the roots and their copies is 2 in $M^t(G)$, for any $t \geq 1$.

Proposition 4.4. *If G is a graph without isolated vertices of order n, with k vertices of eccentricity 2, then for $t \geq 1$, we have $\lambda(M^t(G)) \geq 2^{t-1}(n+k+2)-2$.*

Proof. For $t \geq 1$, let $v_1^0, v_2^0, \ldots, v_k^0$ be the vertices of eccentricity 2 in G. Let V_i^{t-1} be the set composed of a vertex v_i^0 and all its copies in $M^{t-1}(G)$. In $M^t(G)$, by Lemma 4.3 and Definition 1.1, the vertices in $\bigcup_{i=1}^k V_i^{t-1} \cup V'_{t-1} \cup U_{t-1} \cup \{u_{t,0}\}$ are all within distance two, where U_{t-1} is the set of roots and their copies in $M^{t-1}(G)$, V'_{t-1} is the set of copies of the vertices of $M^{t-1}(G)$ in $M^t(G)$, and $u_{t,0}$ is the root of $M^t(G)$. Hence $\lambda(M^t(G)) \geq \sum_{i=1}^k |V_i^{t-1}| + |V'_{t-1}| + |U_{t-1}| = k2^{t-1} + 2^{t-1}(n+1) - 1 + 2^{t-1} - 1 = 2^{t-1}(n+k+2) - 2$. ■

For a graph G of order n, by Proposition 4.4, if $\lambda(M(G)) = n+1$, then G has at most one vertex of eccentricity 2. Also for $t \geq 2$, if $\lambda(M^t(G)) = 2^{t-1}(n+2)-2$, then no vertex in G has eccentricity 2. There exist graphs with one vertex of eccentricity 2 and $\lambda(M(G)) = n+1$. Figure 7 illustrates a tree graph T of order 9 with one vertex of eccentricity 2, having $\lambda(M(T)) = 10$. Based on Proposition 4.4, $\lambda(M^t(T)) \geq 2^{t-1}(n+3) - 2 > 2^{t-1}(n+2) - 2$. Therefore, if $\lambda(M(G)) = n+1$, then not necessarily $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$, for $t \geq 2$.

4.2. Graphs with $\lambda(M^t(G)) = 2^t(n+1) - 2$

Shao and Solis-Oba in [20], gave bounds for the λ-number of some iterated Mycielski graph of complete graph K_n. In the following, we give the exact value of the λ-number of $M^t(K_n)$, for any $t \geq 2$.

Theorem 4.5. *For any $t \geq 2$ and $n \geq 2$, we have $\lambda(M^t(K_n)) = 2^t(n+1) - 2$.*

Proof. For $n \geq 2$, we have $\mathrm{diam}(K_n) = 1$, so by Lemma 2.5 for any $t \geq 2$, we have $\mathrm{diam}(M^t(K_n)) = 2$. Let $V^2 = \{v_i^k : 0 \leq k \leq 3 \text{ and } 1 \leq i \leq n\}$ be the set

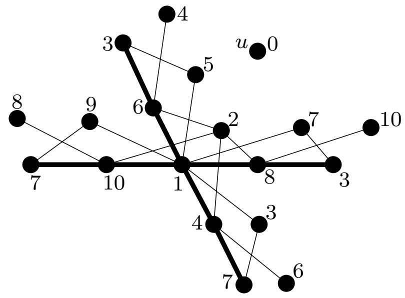

Figure 7. A 10-$L(2,1)$-labeling of the Mycielski graph of a tree T of order 9.

composed of the vertice of V and all their consecutive copies in $M^2(K_n)$. Let χ_i with $1 \leq i \leq n$ be a sequence of vertices in $M^2(K_n)$, where $\chi_i = v_i^2 v_i^0 v_i^1$ if i is odd and $\chi_i = v_i^1 v_i^0 v_i^2$ if i is even. We label the vertices of $M^2(K_n)$ using consecutive labels beginning with 0, in the following order $\chi_1 \chi_2 \cdots \chi_n v_n^3 v_{n-1}^3 \cdots v_1^3 u_{11} u_{10} u_{20}$.

This does not violate the distance two conditions, since two consecutive vertices are either a vertex and its copy or two vertices from the same level, which are successively at distance two. This leads to an $L(2,1)$-labeling of $M^2(K_n)$ with span $|M^2(K_n)| - 1$. Since the diameter is 2, then $\lambda(M^2(K_n)) = |M^2(K_n)| - 1$. From Observation 4.2 and Lemma 2.1, we get $\lambda(M^t(K_n)) = |M^t(K_n)| - 1 = 2^t(n+1) - 2$, for any $t \geq 2$. ∎

Since any graph G of order $n \geq 2$ is a subgraph of the complete graph K_n, we can conclude that for $t \geq 2$, we have $\lambda(M^t(G)) \leq |M^t(G)| - 1 = 2^t(n+1) - 2$. This could also be proven using Theorem 3.6 by showing that for any graph G, the complement of the Mycielski graph $\overline{M}(G)$ has a perfect 4-star matching, which means by Theorem 3.6(a) that $\lambda(M^2(G)) \leq |M^2(G)| - 1$. Then the result follows from Observation 4.2 for any $t \geq 2$.

Corollary 4.6. *Let G_1 and G_2 be two graphs of the same order $|G_1| = |G_2| \geq 2$. Then for any $t \geq 2$, we have $\lambda(M^t(G_1)) + 2^t \leq \lambda(M^{t+1}(G_2))$.*

Proof. For $t \geq 2$, let G_1 and G_2 be two graphs such that $|G_1| = |G_2| = n \geq 2$. By Theorem 4.1 and Theorem 4.5, we have $\lambda(M^t(G_1)) \leq 2^t(n+1) - 2$ and $\lambda(M^{t+1}(G_2)) \geq 2^t(n+2) - 2$. Hence $\lambda(M^t(G_1)) + 2^t \leq \lambda(M^{t+1}(G_2))$. ∎

Let us denote $\overline{M^t}(G)$ the complement graph of $M^t(G)$. The close relation between Hamiltonicity and the $L(2,1)$-labeling allow us to prove the following.

Corollary 4.7. *For any graph G and any $t \geq 2$, $\overline{M^t}(G)$ is a Hamiltonian graph.*

Proof. Let G be a graph of order n. First we show that $\overline{M^2}(G)$ is Hamiltonian.

Let χ_i with $2 \leq i \leq n$ be a sequence of vertices in $\overline{M^2}(G)$, where $\chi_i = v_i^2 v_i^0 v_i^1$ if i is odd, and $\chi_i = v_i^1 v_i^0 v_i^2$ if i is even. Take the vertices of $\overline{M^2}(G)$ in the following order, $v_1^0 v_1^1 \chi_2 \chi_3 \cdots \chi_n v_n^3 v_{n-1}^3 \cdots v_1^3 v_1^2 u_{11} u_{10} u_{20} v_1^0$.

Notice that this is similar to the order proposed in Theorem 4.5 for labeling $M^2(K_n)$. Since every two consecutive vertices are non-adjacent in $M^2(G)$, then the vertices of $\overline{M^2}(G)$ taken in the above order form a Hamiltonian cycle. Thus, for any graph G we have $\overline{M^2}(G)$ is Hamiltonian. For $t \geq 2$, since $M^t(G) \cong M^2(M^{t-2}(G))$, $\overline{M^t}(G)$ is a Hamiltonian graph for any $t \geq 2$. ∎

Next we characterize the graphs with $\lambda(M^t(G)) = 2^t(n+1) - 2$, for $t \geq 2$.

Theorem 4.8. *Let G be a graph of order $n \geq 2$. Then for $t \geq 2$, we have $\lambda(M^t(G)) = 2^t(n+1) - 2$ if and only if $G \cong K_n$ or* $\mathrm{diam}(G) = 2$.

Proof. For $t \geq 2$, if $G \cong K_n$, then by Theorem 4.5 we have $\lambda(M^t(G)) = 2^t(n+1) - 2$. If $\mathrm{diam}(G) = 2$, from Theorem 4.5 we have $\lambda(M^t(G)) \leq 2^t(n+1) - 2$. By Lemma 2.5, $\mathrm{diam}(M^t(G)) = 2$, the vertices must be assigned distinct labels, hence $\lambda(M^t(G)) = 2^t(n+1) - 2$.

Conversely, suppose that G is a graph of order $n \geq 2$, with $\mathrm{diam}(G) \geq 3$. So there are at least two vertices at distance greater or equal to 3, one from another. Without loss of generality, we suppose that $d_G(v_1^0, v_n^0) \geq 3$. For $t = 2$, let χ_i with $2 \leq i \leq n-1$ be a sequence of vertices in $M^2(G)$, where $\chi_i = v_i^2 v_i^0 v_i^1$ if i is odd, and $\chi_i = v_i^1 v_i^0 v_i^2$ if i is even. The labeling f assigns consecutive labels to the vertices beginning with 0 in the following order, $v_1^0 v_1^1 \chi_2 \chi_3 \cdots \chi_{n-1} v_{n-1}^3 v_{n-2}^3 \cdots v_1^3 v_1^2$.

This is similar to the order in Theorem 4.5. The maximum label assigned is $f(v_1^2) = 4n-5$. We have $d_G(v_1^0, v_n^0) \geq 3$, so by Lemma 2.4 we have $d_{M^2}(v_1^2, v_n^0) \geq 3$, and $d_{M^2}(v_1^2, v_n^1) = 3$. We label $f(v_n^0) = f(v_1^2) = 4n-5$, $f(v_n^1) = 4n-4$, $f(v_n^2) = 4n-3$, $f(v_n^3) = 4n-2$, $f(u_{11}) = 4n-1$, $f(u_{10}) = 4n$, $f(u_{20}) = 4n+1$. This is a valid $L(2,1)$-labeling of $M^2(G)$ with span $4n+1$. Hence $\lambda(M^2(G)) \leq 4n+1 = 4(n+1) - 3$. From the upper bound of Theorem 3.1 and Theorem 4.1, for all $t \geq 3$, we have $\lambda(M^t(G)) \leq (2^{t-2} - 1)(|M^2(G)| + 1) + \lambda(M^2(G))$. Since $|M^2(G)| = 4(n+1) - 1$, it follows that for all $t \geq 2$, $\lambda(M^t(G)) \leq 2^t(n+1) - 3$. ∎

4.3. Graphs with $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$

Lemma 4.9. *Let $t \geq 2$ and $1 \leq i, j \leq n$. Then for $1 \leq k \leq 2^{t-1} - 1$, we have $d_{M^t}\left(v_i^k, v_j^{2^{t-1}+k}\right) = 2$, and for $2^{t-1}+1 \leq k \leq 2^t - 1$, we have $d_{M^t}\left(v_i^k, v_j^{2^{t-1}-1}\right) = 2$.*

Proof. For $1 \leq k \leq 2^{t-1} - 1$, we have $v_j^{2^{t-1}+k}$ is the copy of v_j^k in $M^t(G)$. Since $d_{M^{t-1}}\left(v_i^k, v_j^k\right) = 2$, by Lemma 2.4 we have $d_{M^t}\left(v_i^k, v_j^{2^{t-1}+k}\right) = 2$.

For $t \geq 2$, v_i^3 is the copy of v_i^1. So by Lemma 2.4 $d_{M^2}\left(v_i^3, v_j^1\right) = 2$. Since $d_{M^2}\left(v_i^3, v_j^2\right) = 2$, by using Lemma 2.4 inductively, we can show that for $2^{t-1}+1 \leq$

$k \leq 2^t - 1$, we have $d_{M^t}\left(v_i^k, v_j^{2^{t-1}-1}\right) = 2$. ■

Lemma 4.10. *If v_i^0 and v_j^0 are not isolated vertices, then for $0 \leq k \leq 2^{t-1} - 1$, we have $d_{M^t}\left(v_i^k, v_j^{2^t-k-1}\right) = \min\left\{3, d_G\left(v_i^0, v_j^0\right)\right\}$.*

Proof. We have $v_i^{2^t-k-1}$ is the copy of $v_i^{2^{t-1}-k-1}$ in $M^t(G)$. Based on Lemma 2.4, we have $d_{M^t}\left(v_i^k, v_j^{2^t-k-1}\right) = \min\left\{3, d_{M^{t-1}}\left(v_i^k, v_j^{2^{t-1}-k-1}\right)\right\}$. If $0 \leq k \leq 2^{t-2} - 1$, we have $d_{M^{t-1}}\left(v_i^k, v_j^{2^{t-1}-k-1}\right) = \min\left\{3, d_{M^{t-2}}\left(v_i^k, v_j^{2^{t-2}-k-1}\right)\right\}$. Otherwise, if $2^{t-2} \leq k \leq 2^{t-1} - 1$, by symmetry $k = 2^{t-1} - m - 1$ where $0 \leq m \leq 2^{t-2} - 1$, so $d_{M^{t-1}}\left(v_i^k, v_j^{2^{t-1}-k-1}\right) = d_{M^{t-1}}\left(v_i^{2^{t-1}-m-1}, v_j^m\right) = \min\left\{3, d_{M^{t-2}}\left(v_i^{2^{t-2}-m-1}, v_j^m\right)\right\}$. By recursively using Lemma 2.4, we get $d_{M^t}\left(v_i^k, v_j^{2^t-k-1}\right) = \min\left\{3, d_G\left(v_i^0, v_j^0\right)\right\}$. ■

In the case where v_i^0 or v_j^0 are isolated vertices, for $1 \leq k \leq 2^{t-1} - 1$, we have $d_{M^t}\left(v_i^k, v_j^{2^t-k-1}\right) = 3$.

The direct product $G \times K_2$, called the *canonical double cover* (or *Kronecker double cover*) is a bipartite graph with two partition sets $X = V \times \{x\}$ and $Y = V \times \{y\}$, where $(v_i, x)(v_j, y) \in E(G \times K_2)$ if and only if $v_i v_j \in E(G)$.

From Lemma 4.10, $v_i^{2^{t-1}-1} v_j^{2^{t-1}} \in E(M^t(G))$ if and only if $v_i^0 v_j^0 \in E(G)$. Since two copies of the same vertex or copies from the same level are non-adjacent, we have the following result.

Observation 4.11. *For $t \geq 2$, let $S = \left\{v_i^{2^{t-1}-1}, v_i^{2^{t-1}} : 1 \leq i \leq n\right\}$. In $M^t(G)$, the subgraph induced by the vertices in S is isomorphic to $G \times K_2$.*

A *matching* in a graph G is a collection of vertex-disjoint edges in G, a *perfect matching* is a matching that covers all the vertices of G. The following theorem known as *the Marriage Theorem*, gives a criterion for any bipartite graph $G = (X, Y)$ to have a perfect matching.

Theorem 4.12 (The Marriage Theorem)**.** *Let $G = (X, Y)$ be a bipartite graph. Then G has a perfect matching if and only if $|X| = |Y|$ and for any $S \subseteq X$, $|N_G(S)| \geq |S|$.*

A 2-*matching* of a graph G is an assignment of weights 0, 1, or 2 to the edges of G such that the sum of weights of edges incident to any vertex in G is less or equal to 2 (see Chapter 6 in [18]). A 2-matching of a graph G can be seen as components with degree vertex at most 2. The sum of weights in a 2-matching is called the *size*. The maximum size of a 2-matching is denoted by $\nu_2(G)$, which can be computed in polynomial time [21]. A *perfect* 2-*matching* is a 2-matching where the sum of weights incident to any vertex in G is exactly 2. Tutte in [21], provides a characterization for the existence of perfect 2-matching of a graph.

Theorem 4.13 [21]. *A graph G has a perfect 2-matching if and only if for any independent set $S \subseteq V$, $|N_G(S)| \geq |S|$.*

A perfect 2-matching can be seen as a spanning subgraph in which each component is a single edge K_2 or a cycle. Since every even cycle has a perfect matching, a graph with a perfect 2-matching has a spanning subgraph in which each component is a single edge or an odd cycle. It is easy to see from the two preceding Theorem 4.12 and Theorem 4.13, that the existence of perfect 2-matching in a graph G is equivalent to that $G \times K_2$ admits a perfect matching.

Theorem 4.14. *Let G be a graph without isolated vertices of order $n \geq 2$. Then for $t \geq 2$, $\lambda(M^t(G)) = 2^{t-1}(n+2)-2$ if and only if for any $S \subseteq V$, $|D_2(S)| \geq |S|$, where $D_2(S) = \{x \in V : \exists\, v \in S, d_G(x,v) > 2\}$.*

Proof. Let G be a graph without isolated vertices of order $n \geq 2$ such that for $t \geq 2$, $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$. Let f be a λ-labeling of $M^t(G)$, using labels from the set $L = \{0, \ldots, 2^{t-1}(n+2) - 2\}$. From Lemma 4.3, we have $d_{M^t}(v_i^k, u) \leq 2$ and $d_{M^t}(u, u') \leq 2$, for all $v_i^k \in V^t$ and all $u, u' \in U_t$. The roots are assigned distinct labels, different from the labels assigned to the vertices in V^t. So for $2^{t-1} \leq k \leq 2^t - 1$, we have $f(v_i^k) \in L \setminus f(U_t)$ and $|L \setminus f(U_t)| = 2^{t-1}n$. For $1 \leq i, j \leq n$, we have $d_{M^t}\left(v_i^k, v_j^m\right) = 2$, where $2^{t-1} \leq k, m \leq 2^t - 1$. It follows that the $2^{t-1}n$ vertices v_i^k where $2^{t-1} \leq k \leq 2^t - 1$, and $1 \leq i \leq n$, have distinct labels and use all the labels in $L \setminus f(U_t)$. By Lemma 4.9, we have $d_{M^t}\left(v_i^k, v_j^{2^{t-1}-1}\right) = 2$, for $2^{t-1} + 1 \leq k \leq 2^t - 1$. The only labels remaining in $L \setminus f(U_t)$, for the vertices $v_j^{2^{t-1}-1}$, are those assigned to the vertices $v_i^{2^{t-1}}$. Since $d_{M^t}\left(v_i^{2^{t-1}-1}, v_j^{2^{t-1}-1}\right) = 2$ and $d_{M^t}\left(v_i^{2^{t-1}}, v_j^{2^{t-1}}\right) = 2$, we have $f\left(v_i^{2^{t-1}-1}\right) \neq f\left(v_j^{2^{t-1}-1}\right)$ and $f\left(v_i^{2^{t-1}}\right) \neq f\left(v_j^{2^{t-1}}\right)$. It follows that for any vertex $v_j^{2^{t-1}}$, there is one and only one vertex $v_i^{2^{t-1}-1}$ such that $f\left(v_i^{2^{t-1}-1}\right) = f\left(v_j^{2^{t-1}}\right)$. Let (v_i, x) and (v_j, y), $1 \leq i, j \leq n$, denote the vertices of $G \times K_2$, where $(v_i, x)(v_j, y) \in E(G \times K_2)$ if and only if $v_i^0 v_j^0 \in E(G)$. Let $M = \left\{(v_i, x)(v_j, y) : f\left(v_i^{2^{t-1}-1}\right) = f\left(v_j^{2^{t-1}}\right)\right\}$. Since $f\left(v_i^{2^{t-1}-1}\right) = f\left(v_j^{2^{t-1}}\right)$, we have by Lemma 4.10 that $d_G\left(v_i^0, v_j^0\right) \geq 3$. From Observation 4.11, M is a perfect matching of the graph $\overline{G^2} \times K_2$, then by Theorem 4.12 we get the necessity.

Conversely, suppose that for any $S \subseteq V$, we have $|D_2(S)| \geq |S|$. This means by Theorem 4.13 that the graph $\overline{G^2}$ has a perfect 2-matching, which means that $\overline{G^2}$ has a spanning subgraph H, whose connected components are vertex-disjoint edges or odd cycles. Let $E^1, E^2, \ldots, E^r$ be the K_2 components, and let $C^1, C^2, \ldots, C^s$ be the odd cycle components of H.

Further, we denote $x_i^0 y_i^0$ is the edge E^i and $c_{1,i}^0 c_{2,i}^0 \cdots c_{n_i,i}^0$ is the odd cycle

C^i, where $n_i = |C^i|$. We define an $L(2,1)$-labeling f to the vertices of $M^t(G)$ as follows.

Suppose that $r \geq 2$. First we label the vertices x_1^k, y_1^k with $0 \leq k \leq 2^t - 1$, where x_1^k and y_1^k are the vertices x_1^0 and y_1^0 and their consecutive copies. The labeling f assigns in descending order the labels $2^{t-1}-1, 2^{t-1}-2, \ldots, 0$, respectively, to $x_1^0, x_1^1, \ldots, x_1^{2^{t-1}-1}$ and the labels $2^t - 1, 2^t - 2, \ldots, 2^{t-1}$, respectively, to $x_1^{2^{t-1}}, x_1^{2^{t-1}+1}, \ldots, x_1^{2^t-1}$. Then assign the same list of consecutive labels, now in ascending order $0, 1, \ldots, 2^{t-1}-1$, respectively, to the vertices $y_1^{2^{t-1}}, y_1^{2^{t-1}+1}, \ldots, y_1^{2^t-1}$ and the labels $2^{t-1}, 2^{t-1}+1, \ldots, 2^t - 1$, respectively, to $y_1^0, y_1^1, \ldots, y_1^{2^{t-1}-1}$.

- For $0 \leq k \leq 2^{t-1} - 1$, $f(x_1^k) = 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(x_1^k) = 3 \times 2^{t-1} - k - 1$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f(y_1^k) = k + 2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f(y_1^k) = k - 2^{t-1}$.

We have $f\left(x_1^k\right) = f\left(y_1^m\right)$ if $m = 2^t - k - 1$. Since $x_1^0 y_1^0 \in E(\overline{G^2})$, we have $d_G\left(x_1^0, y_1^0\right) \geq 3$, so by Lemma 4.10 $d_{M^t}\left(x_1^k, y_1^{2^t-k-1}\right) = 3$. Otherwise $f\left(x_1^k\right) \neq f\left(y_1^m\right)$. Since x_1^0 and y_1^0 are not adjacent in G, we have $d_{M^t}\left(x_1^k, y_1^m\right) \geq 2$, for all $0 \leq k, m \leq 2^t - 1$. Also $d_{M^t}\left(x_1^k, x_1^m\right) = d_{M^t}\left(y_1^k, y_1^m\right) = 2$, $f\left(x_1^k\right) \neq f\left(x_1^m\right)$ and $f\left(y_1^k\right) \neq f\left(y_1^m\right)$. The smallest label is $f\left(x_1^{2^{t-1}-1}\right) = f\left(y_1^{2^{t-1}}\right) = 0$, the maximum label is $f\left(x_1^{2^{t-1}}\right) = f\left(y_1^{2^{t-1}-1}\right) = 2^t - 1$.

For $2 \leq i \leq r$, we have $d_G\left(x_i^0, y_i^0\right) \geq 3$, so a vertex in E^{i-1} cannot be adjacent in G to both x_i^0 and y_i^0. Since in every E^i the vertices x_i^0 and y_i^0 are symmetric, we rearrange the vertices of each E^i depending on the cases.

(i) If x_{i-1}^0 is adjacent in G to a vertex in E^i, we consider without loss of generality that x_{i-1}^0 is adjacent to y_i^0.

(ii) If x_{i-1}^0 is not adjacent to E^i and y_{i-1}^0 is adjacent, we let $d_G(y_{i-1}^0, x_i^0) = 1$. Otherwise the vertices in E^{i-1} and E^i are mutually non-adjacent. This means that $d_G(x_{i-1}^0, x_i^0) \geq 2$, and $d_G\left(y_{i-1}^0, y_i^0\right) \geq 2$, for all $2 \leq i \leq r$.

With respect to the above assumptions, we label the vertices x_i^k and y_i^k with $2 \leq i \leq r$, as following.

- For $2 \leq i \leq r - 1$, and $0 \leq k \leq 2^t - 1$, $f\left(x_i^k\right) = (i-1)2^t + f\left(x_1^k\right)$, and $f\left(y_i^k\right) = (i-1)2^t + f\left(y_1^k\right)$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f\left(x_r^k\right) = (r-1)2^t + f\left(x_1^k\right)$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f\left(x_r^k\right) = (r-1)2^t + k$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f\left(y_r^k\right) = r2^t - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f\left(y_r^k\right) = (r-1)2^t + f\left(y_1^k\right)$.

The labeling f uses distinct labels from $(i-1)2^t, \ldots, i2^t - 1$, for every pair of x_i^k,y_i^m, where $m = 2^t - k - 1$, by using the same pattern for x_1^k, y_1^m (except for x_r^k,y_r^k). In the case where $r = 1$, let for $0 \leq k \leq 2^{t-1} - 1$, $f\left(x_1^k\right) = 2^{t-1} - k - 1$, for $2^{t-1} \leq k \leq 2^t - 1$, $f\left(x_1^k\right) = k$, for $0 \leq k \leq 2^{t-1} - 1$, $f\left(y_1^k\right) = 2^t - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f\left(y_1^k\right) = k - 2^{t-1}$. The only vertices from two different components, with the difference between the labels equal to 1, are for $x_{i-1}^{2^{t-1}}$ and $y_{i-1}^{2^{t-1}-1}$, with both $x_i^{2^{t-1}-1}$ and $y_i^{2^{t-1}}$. This does not violate the distance two conditions, since $d_G\left(x_{i-1}^0, x_i^0\right) \geq 2$, and $d_G\left(y_{i-1}^0, y_i^0\right) \geq 2$, for all $2 \leq i \leq r$. The maximum label assigned is $f\left(x_r^{2^t-1}\right) = f(y_r^0) = r2^t - 1$.

If $s \geq 1$, next we label the vertices of the odd cycle components C^i. We make the following claim.

Claim 4.15. *For a vertex v in G not in the odd cycle component $C^i = c_{1,i}^0 c_{2,i}^0 \cdots c_{n_i,i}^0$, there is at least one edge $c_{p,i}^0 c_{q,i}^0 \in C^i$ such that v is not adjacent in G to both $c_{p,i}^0$ and $c_{q,i}^0$.*

Proof. We prove this by using contradiction. We suppose that v is adjacent to at least one endpoint of any $c_{p,i}^0 c_{q,i}^0 \in C^i$. We may assume that v is adjacent to $c_{1,i}^0$. Since $d_G\left(c_{1,i}^0, c_{2,i}^0\right) \geq 3$, v is not adjacent to $c_{2,i}^0$, so v is adjacent to $c_{3,i}^0$, and so forth. Hence, if j is odd, then v is adjacent to $c_{j,i}^0$, and if j is even, then v is not adjacent to $c_{j,i}^0$. Since v is adjacent to $c_{1,i}^0$, v is not adjacent to $c_{n_i,i}^0$. It follows that n_i is even, a contradiction. $\square$

Since the cycles C^i are symmetric, we may consider that $d_G\left(y_r, c_{1,1}^0\right) \geq 2$, and $d_G\left(y_r, c_{n_1,1}^0\right) \geq 2$, and for $1 \leq i \leq s-1$, $d_G\left(c_{n_i,i}^0, c_{1,i+1}^0\right) \geq 2$, and $d_G\left(c_{n_i,i}^0, c_{n_{i+1},i+1}^0\right) \geq 2$. We label the vertices $c_{j,i}^k$ where $1 \leq j \leq n_i$, $1 \leq i \leq s$ and $0 \leq k \leq 2^t - 1$, with respect to the above assumptions.

- For $0 \leq k \leq 2^{t-1} - 1$, $f\left(c_{1,1}^k\right) = r2^t + 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f\left(c_{1,1}^k\right) = r2^t + k$.
- For $2 \leq j \leq n_1 - 1$ and all $0 \leq k \leq 2^t - 1$, $f\left(c_{j,1}^k\right) = f\left(c_{1,1}^k\right) + (j-1)2^{t-1}$.
- For $0 \leq k \leq 2^{t-1} - 1$, $f\left(c_{n_1,1}^k\right) = f\left(c_{1,1}^k\right) + (n_1 - 1)2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f\left(c_{n_1,1}^k\right) = f\left(c_{1,1}^{2^t-k-1}\right)$.

The smallest label for the vertices $c_{i,1}^k$ is $f\left(c_{1,1}^{2^{t-1}-1}\right) = f\left(c_{n_1,1}^{2^{t-1}}\right) = r2^t$, and the maximum is $f\left(c_{n_1,1}^0\right) = f\left(c_{n_1-1,1}^{2^t-1}\right) = r2^t + n_1 2^{t-1} - 1$. Now let $\varphi_i =$

$r2^t + \sum_{j=1}^{i-1} n_j 2^{t-1}$. For $2 \leq i \leq s$, we label $f\left(c_{1,i}^{2^{t-1}-1}\right) = f\left(c_{n_i,i}^{2^{t-1}}\right) = \varphi_i$, then we label vertices in the following way.

- For $0 \leq k \leq 2^{t-1}-1$, $f\left(c_{1,i}^{k}\right) = \varphi_i + 2^{t-1} - k - 1$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f\left(c_{1,i}^{k}\right) = \varphi_i + k$.
- For $2 \leq j \leq n_i - 1$ and all $0 \leq k \leq 2^t - 1$, $f(c_{j,i}^k) = f\left(c_{1,i}^k\right) + (j-1)2^{t-1}$.
- For $0 \leq k \leq 2^{t-1}-1$, $f\left(c_{n_i,i}^{k}\right) = f\left(c_{1,i}^{k}\right) + (n_i - 1)2^{t-1}$, and for $2^{t-1} \leq k \leq 2^t - 1$, $f\left(c_{n_i,i}^{k}\right) = f\left(c_{1,i}^{2^t-k-1}\right)$.

The labeling f uses $n_i 2^{t-1}$ distinct labels for the $n_i 2^t$ vertices of each component C^i and their copies. For $0 \leq k \leq 2^{t-1}-1$, we have $f\left(c_{1,i}^{k}\right) = f\left(c_{n_i,i}^{2^t-k-1}\right)$, and for $2 \leq j \leq n_i$ $f\left(c_{j,i}^{k}\right) = f\left(c_{j-1,i}^{2^t-k-1}\right)$. It is possible, since $d_G\left(c_{j,i}^0, c_{j-1,i}^0\right) \geq 3$, which means by Lemma 4.10 that $d_{M^t}\left(c_{j,i}^k, c_{j-1,i}^{2^t-k-1}\right) = 3$. For two vertices $c_{j,i}^k$, $c_{l,i}^m$ from the same component, the difference between the labels is equal to 1 in the following cases.

(i) The vertices are copies of the same vertex, or if $2^{t-1} \leq k, m \leq 2^t - 1$, in those two cases $d_{M^t}\left(c_{j,i}^k, c_{l,i}^m\right) = 2$.

(ii) For $l = j+1$, we have $d_G\left(c_{j,i}^0, c_{j+1,i}^0\right) \geq 3$, then $d_{M^t}\left(c_{j,i}^k, c_{j+1,i}^m\right) \geq 2$.

(iii) If $l = j+2$, $k = 2^t - 1$ and $m = 2^{t-1} - 1$, we have from Lemma 4.9 $d_{M^t}\left(c_{j,i}^{2^t-1}, c_{l,i}^{2^{t-1}-1}\right) = 2$. For two vertices from different odd cycle components, we have the difference between the labels assigned is equal to 1, it happens only for $c_{n_i,i}^0$ and $c_{n_{i-1},i}^{2^{t-1}}$ with $c_{1,i+1}^{2^{t-1}-1}$ and $c_{n_{i+1},i+1}^{2^{t-1}}$. For $1 \leq i \leq s-1$, we have $d_G\left(c_{n_i,i}^0, c_{n_{i+1},i+1}^0\right) \geq 2$ and $d_G\left(c_{n_i,i}^0, c_{1,i+1}^0\right) \geq 2$. Also from Lemma 4.9 the vertices are at distance greater or equal 2 in $M^t(G)$.

The maximum label assigned is $f\left(c_{n_s,s}^0\right) = f\left(c_{n_s-1,s}^{2^t-1}\right) = r2^t + \sum_{j=1}^{s} n_j 2^{t-1} - 1 = n2^{t-1} - 1$.

We finally label the remaining $2^t - 1$ roots with consecutive labels beginning with the label $n2^{t-1}$ in the following order

$$u_{1,2^{t-1}-1} u_{1,2^{t-1}-2} \cdots u_{1,0} u_{2,2^{t-2}-1} u_{2,2^{t-2}-2} \cdots u_{2,0} u_{3,2^{t-3}-1} \cdots u_{t,0}.$$

Since $d_{M^t}\left(u_{1,2^{t-1}-1}, c_{n_s,s}^0\right) = 2$, $d_{M^t}\left(u_{1,2^{t-1}-1}, c_{n_s-1,s}^{2^t-1}\right) = 2$, $d_{M^t}\left(u_{i,j}, u_{i,j-1}\right) = 2$, and $d_{M^t}(u_{i,0}, u_{i+1,2^{t-(i+1)}-1}) = 2$, this produces an $L(2,1)$-labeling with span $2^{t-1}(n+2) - 2$. In Figure 8, we show an $L(2,1)$-labeling with the same schema for $M^2(G)$, where $\overline{G^2}$ has a perfect 2-matching consisting of two K_2 components and two cycles of order 3 and 5, respectively. Hence from the lower bound of Theorem 4.1 for $t \geq 2$, we have $\lambda(M^t(G)) = 2^{t-1}(n+2) - 2$. ■

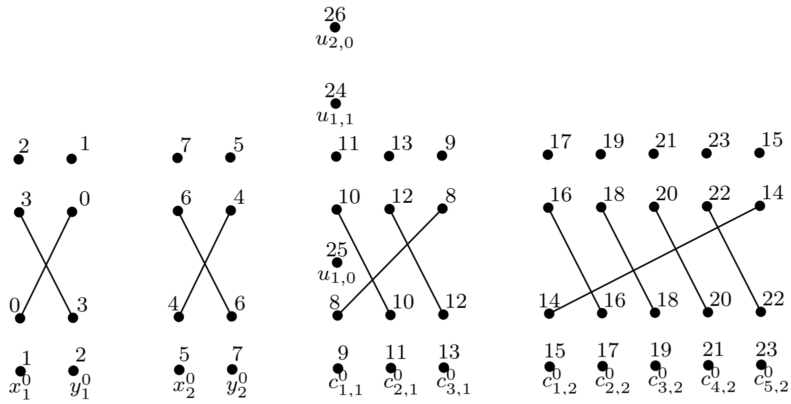

Figure 8. An $L(2,1)$-labeling of $M^2(G)$ as in Theorem 4.6, where $\overline{G^2}$ has a perfect 2-matching with two K_2 components and two cycles of order 3 and 5, here the edges represent a perfect matching of $\overline{G^2} \times K_2$.

The labeling defined in Theorem 4.14 is a valid $L(2,1)$-labeling for any graph G of order $n \geq 2$. If $\overline{G^2}$ has a perfect 2-matching, then we can label the vertices of $M^t(G)$ with a labeling having span $2^{t-1}(n+2)-2$. Next, we give an upper bound for $\lambda(M^t(G))$ in terms of the maximum size of a 2-matching of $\overline{G^2}$.

Theorem 4.16. *Let G be a graph of order $n \geq 2$, with $\nu_2(\overline{G^2}) = p$. Then for $t \geq 2$, we have $\lambda(M^t(G)) \leq 2^{t-1}(2n-p+2)-2$.*

Proof. Let G be a graph with $\nu_2(\overline{G^2}) = p$. So there is an induced subgraph H of $\overline{G^2}$ of order p such that H has a perfect 2-matching. Let V_H be the set of vertices of H. From Theorem 4.14, we can label the vertices of $M^t(G[V_H])$ with an $L(2,1)$-labeling f with span $2^{t-1}(p+2)-2$, where $f(u_{t,0}) = 2^{t-1}(p+2)-2$.

Now in $M^t(G)$, if $p < n$, then the vertices remaining unlabeled by f are the vertices in $V \setminus V_H$ and their copies. Let us denote v_i^k, where $1 \leq i \leq q$, and $0 \leq k \leq 2^t - 1$, such that $p + q = n$, the vertices of $V \setminus V_H$ and their consecutive copies. Let χ_i with $2 \leq i \leq q$ be a sequence of vertices in $M^t(G)$, where $\chi_i = v_i^2 v_i^0 v_i^1$ if i is odd, and $\chi_i = v_i^1 v_i^0 v_i^2$ if i is even. The only vertex labeled $2^{t-1}(p+2)-2$ by f is $u_{t,0}$. Using consecutive labels we label the vertices v_i^k, with $1 \leq i \leq q$ beginning with the label $2^{t-1}(p+2)-1$, in the following order $v_1^0 v_1^2 v_1^1 \chi_2 \cdots \chi_q v_q^3 v_{q-1}^3 \cdots v_1^3 v_1^4 \cdots v_q^4 v_q^5 \cdots v_1^{2^t-1}$.

This produces an $L(2,1)$-labeling with span $2^{t-1}(p+2) - 2 + 2^t(n-p) = 2^{t-1}(2n-p+2)-2$. ∎

Similarly to Subsection 3.3, we put interest in connected graphs, the path P_n and cycle C_n, which we use to determine some connected graphs with the smallest $\lambda(M^t(G))$.

Corollary 4.17. *For $t \geq 2$,*

$$\lambda(M^t(P_n)) = \begin{cases} 4 \times 2^t - 2 & \text{if } n = 3, 4, 5, \\ 2^{t-1}(n+2) - 2 & \text{if } n \geq 6. \end{cases}$$

Proof. For $n = 3$, we have $\operatorname{diam}(P_3) = 2$. By Theorem 4.8 for $t \geq 2$ we have $\lambda(M^t(P_3)) = 4 \times 2^t - 2$.

For $n = 4$, $\overline{P_4^2}$ consists of a single edge and 2 isolated vertices. So $\nu_2(\overline{P_4^2}) = 2$, it follows from Theorem 4.16 that $\lambda(M^t(P_4)) \leq 4 \times 2^t - 2$. Since $M^t(P_3)$ is a subgraph of $M^t(P_4)$, from above $\lambda(M^t(P_4)) = 4 \times 2^t - 2$.

For $n = 5$, $\overline{P_5^2}$ consists of 2 independent edges and one isolated vertex. Hence $\nu_2(\overline{P_5^2}) = 4$, so from Theorem 4.16, $\lambda(M^t(P_5)) \leq 4 \times 2^t - 2$. Also $M^t(P_3)$ is a subgraph of $M^t(P_5)$, then $\lambda(M^t(P_5)) = 4 \times 2^t - 2$.

For $n \geq 6$, it is easy to see that the path P_n verifies the condition of Theorem 4.14, thus $\lambda(M^t(P_n)) = 2^{t-1}(n+2) - 2$. ■

Corollary 4.18. *For $t \geq 2$,*

$$\lambda(M^t(C_n)) = \begin{cases} 4 \times 2^t - 2 & \text{if } n = 3, \\ 5 \times 2^t - 2 & \text{if } n = 4, \\ 6 \times 2^t - 2 & \text{if } n = 5, \\ 2^{t-1}(n+2) - 2 & \text{if } n \geq 6. \end{cases}$$

Proof. We have $\operatorname{diam}(C_3) = 1$, and $\operatorname{diam}(C_4) = \operatorname{diam}(C_5) = 2$. So by Theorem 4.8, for $t \geq 2$, we have $\lambda(M^t(C_3)) = 4 \times 2^t - 2$, $\lambda(M^t(C_4)) = 5 \times 2^t - 2$, and $\lambda(M^t(C_5)) = 6 \times 2^t - 2$. If $n \geq 6$, then the cycle C_n satisfies the condition of Theorem 4.14, thus $\lambda(M^t(C_n)) = 2^{t-1}(n+2) - 2$. ■

Corollary 4.19. *Let G be a connected graph, for $t \geq 2$ we have the following.*

(1) $\lambda(M^t(G)) = 3 \times 2^t - 2$ *if and only if G is K_2.*

(2) $\lambda(M^t(G)) = 4 \times 2^t - 2$ *if and only if $G \in \{P_3, P_4, P_5, P_6, C_3, C_6\}$.*

(3) $\lambda(M^t(G)) = 9 \times 2^{t-1} - 2$ *if and only if $G \in \{P_7, C_7\}$.*

Proof. From the lower bound of Theorem 4.1, for $t \geq 2$, we have

$$\lambda(M^t(G)) \geq 2^{t-1} \max\{n+2, 2(\triangle + 2)\} - 2. \tag{3}$$

We have K_2 is the only connected graph with $\triangle = 1$, by Theorem 4.5 $\lambda(M^t(K_2)) = 3 \times 2^t - 2$. Based on inequality (3), if $\triangle \geq 2$, then $\lambda(M^t(G)) \geq 4 \times 2^t - 2$. Therefore, $\lambda(M^t(G)) = 3 \times 2^t - 2$ if and only if $G \cong K_2$.

If $\triangle = 2$, then G is either a path graph or a cycle. Then the graphs in Corollary 4.17 and Corollary 4.18 are the only connected graphs with $\triangle = 2$.

From inequality (3), if $\triangle \geq 3$, then $\lambda(M^t(G)) \geq 5 \times 2^t - 2$. Hence, based on Corollary 4.17 and Corollary 4.18, we can conclude that $\lambda(M^t(G)) = 4 \times 2^t - 2$ if and only if $G \in \{P_3, P_4, P_5, P_6, C_3, C_6\}$. Also, $\lambda(M^t(G)) = 9 \times 2^{t-1} - 2$ if and only if $G \in \{P_7, C_7\}$. ■

For any other non-trivial connected graph G not mentioned in Corollary 4.19 for $t \geq 2$, we have $\lambda(M^t(G)) \geq 5 \times 2^t - 2$.

5. Open Problems

From the statement of the $\triangle^2$-conjecture, and the upper bound of Theorem 3.1 and Theorem 4.1, we propose a weaker conjecture for the $L(2,1)$-labeling number of the Mycielski graph and the iterated Mycielski graph of graphs.

Conjecture 5.1. *For any graph G of order $n \geq 1$, with maximum degree $\triangle$, and for all $t \geq 1$, we have $\lambda(M^t(G)) \leq (2^t - 1)(n+1) + \triangle^2$.*

It is clear from Theorem 3.1 and Theorem 4.1 that if $\lambda(G) \leq \triangle^2$, then for any $t \geq 1$, $\lambda(M^t(G)) \leq (2^t - 1)(n+1) + \triangle^2$.

Remark 5.2. For any positive integers t, t' such that $t' > t \geq 1$, if $\lambda(M^t(G)) \leq (2^t - 1)(n+1) + \triangle^2$, then $\lambda(M^{t'}(G)) \leq (2^{t'} - 1)(n+1) + \triangle^2$.

Proof. From the definition of the iterated Mycielski graph of a graph G, for $t' > t \geq 1$, we have $M^{t'}(G) = M^{t'-t}(M^t(G))$. From the upper bound of Theorem 3.1 and Theorem 4.1, we get that $\lambda(M^{t'}(G)) \leq (2^{t'-t} - 1)(n+1) + \lambda(M^t(G))$. Therefore if $\lambda(M^t(G)) \leq (2^t - 1)(n+1) + \triangle^2$, then

$$\begin{aligned}
&\lambda(M^{t'}(G)) \leq (2^{t'-t} - 1)(n+1) + \lambda(M^t(G)) \\
&\leq (2^{t'-t} - 1)(n+1) + (2^t - 1)(n+1) + \triangle^2 = (2^{t'-t} + 2^t - 2)(n+1) + \triangle^2.
\end{aligned}$$

For $t' > t \geq 1$, we have

$$\begin{aligned}
&(2^{t'} - 1) - (2^{t'-t} + 2^t - 2) = 2^{t'} - 2^{t'-t} - 2^t + 1 \\
&= 2^t(2^{t'-t}) - 2^{t'-t} - (2^t - 1) = 2^{t'-t}(2^t - 1) - (2^t - 1) = (2^t - 1)(2^{t'-t} - 1) > 0.
\end{aligned}$$

It means that $\lambda(M^{t'}(G)) \leq (2^{t'} - 1)(n+1) + \triangle^2$. ■

Remark 5.2 shows that if Conjecture 5.1 is true for an iteration $t \geq 1$, then it is true for any iteration greater than t.

From our study, for any $t \geq 1$, the only graphs with at least one edge that we know having $\lambda(M^t(G)) = (2^t - 1)(n+1) + \triangle^2$, are the graph K_2, and the graphs achieving the bound in Corollary 3.2, which are the cycle C_5, the Petersen graph, the Hoffman-Singleton graph, and possibly a diameter two Moore graph of maximum degree 57, and order $57^2 + 1$ if such graph exists.

The complexity of the $L(2,1)$-labeling problem should be investigated more, whether for the Mycielski graph of graphs in general or the Mycielski graph of graphs not studied yet. For instance, trees, since the $L(2,1)$-labeling number can be determined in polynomial time for trees [6], we may ask if it is also the case for the Mycielski graphs generated from trees?

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