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# THE LINEAR ARBORICITY OF GRAPHS WITH LOW TREEWIDTH

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**Abstract**

Let $G$ be a graph with treewidth $k$. In the paper, it is proved that if $k \leq 3$ and maximum degree $\Delta \geq 5$, or $k = 4$ and $\Delta \geq 9$, or $\Delta \geq 4k - 3$ and $k \geq 5$, then the linear arboricity $la(G)$ of $G$ is $\lceil \frac{\Delta}{2} \rceil$.

**Keywords:** graph, minor, linear arboricity, linear forest, treewidth.

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## 1. Introduction

In this paper, all graphs considered are simple and undirected, and all undefined notation and definitions follow [7]. Let $G = (V, E)$ be a graph, where $V(G)$ is the vertex set and $E(G)$ is the edge set of $G$. For $v \in V(G)$, let $N(v) = \{u : uv \in E(G)\}$. The degree $d(v)$ of a vertex $v$ is $|N(v)|$, $\Delta(G)$ (or simply $\Delta$) is the maximum degree of $G$ and $\delta(G)$ (or simply $\delta$) is the minimum degree of $G$. For a subset $W \subseteq V$, $N(W) = \bigcup_{w \in W} N(w)$. For a real number $x$, we use $\lceil x \rceil$ to denote the least integer not less than $x$.

A *linear forest* is a graph in which each component is a path. A *t-linear coloring* is a map from $E(G)$ to $\{1, 2, \ldots, t\}$ such that the edges using the same color $i$ induce a linear forest for any $i$ $(1 \leq i \leq t)$. The *linear arboricity* $la(G)$ of a graph $G$ is the minimum number $t$ for which $G$ has a $t$-linear coloring. It is easy to see that $la(G) \geq \left\lceil \frac{\Delta(G)}{2} \right\rceil$ for any graph $G$. At the same time, it is easy to check that for any regular graph, we have $la(G) \geq \lceil \frac{\Delta+1}{2} \rceil$, and in [1] Akiyama, Exoo and Harary conjectured the equality holds. Their conjecture is equivalent to the following linear arboricity conjecture (LAC).

**Conjecture A.** *For any graph $G$, $\lceil \frac{\Delta}{2} \rceil \leq\ la(G) \leq \lceil \frac{\Delta+1}{2} \rceil$.*

Conjecture A was proved for complete graphs, complete bipartite graphs, trees and graphs with $\Delta \in \{3,4,5,6,8,10\}$ [1, 2, 8, 9]. In [11, 12], it is also proved for all planar graphs.

In the paper, we consider the linear arboricity of graphs with bounded treewidth. The notion of treewidth was first introduced by Robertson and Seymour [10]. For a graph $G$, a *tree decomposition* $(T, \mathcal{V})$ consists of a tree $T$ and a collection $\mathcal{V} = \{V_t \subseteq V(G) : t \in V(T)\}$ of bags such that

- $V(G) = \bigcup_{t\in V(T)} V_t$,
- for each $vw \in E(G)$ there exists a $t \in V(T)$ such that $v, w \in V_t$, and
- if $v \in V_{t_1} \cap\ V_{t_2}$, then $v \in V_t$ for all vertices $t$ that lie on the path connecting $t_1$ and $t_2$ in $T$.

A tree decomposition $(T, \mathcal{V})$ of a graph $G$ has width $k$, if all bags have size at most $k+1$. The *treewidth* of $G$, denoted by $tw(G)$, is the smallest number $k$ for which there exists a width $k$ tree decomposition of $G$. Treewidth plays a crucial role in the studies on graph minors. For every fixed $k$, denote by $TW_k$ the set of graphs with treewidth at most $k$, which can be characterised by a finite set of forbidden minors [3].

Let $G$ be a graph of the treewidth $k$. In [5], it is proved that if $\Delta \geq \frac{(k+3)^2}{2}$, then the list chromatic index $ch'(G) = \Delta$, or if $\Delta \geq 3k-3$ and $k \geq 3$, then the total chromatic $\chi''(G) = \Delta + 1$. In this paper, we consider the linear arboricity of $G$ associated with its treewidth and get the following Theorem 1.

**Theorem 1.** *Let $G$ be a graph with $tw(G) \leq k$ and $\Delta \leq 2t$ for some integer $t$. Then $G$ has a $t$-linear coloring if one of the following conditions holds.*

(1) $k \leq 3$ *and* $t \geq 3$;

(2) $k \leq 4$ *and* $t \geq 5$;

(3) $k \geq 5$ *and* $t \geq 2k-1$.

By the theorem, it is easy to check the following corollary.

**Corollary 2.** *Let $G$ be a graph with treewidth $k$. Then $la(G) = \lceil \frac{\Delta}{2} \rceil$ if $k \leq 3$ and $\Delta \geq 5$, or $k = 4$ and $\Delta \geq 9$, or $k \geq 5$ and $\Delta \geq 4k-3$.*

Since the graph $G = K_5 - e$, the complete graph of order 5 minus one edge, has $tw(G) = 3$ and $la(G) = 3$, Theorem 1(1) is sharp. Moreover, Wu determined completely the linear arboricity of series-parallel graphs [13] and Halin graphs [14]. It is known that these two classes of graphs both have the treewidth at most 3 [3, 4]. So we generalize these results.

## 2. Proof of Theorem 1

For a positive integer $k$, we use $[k]$ to denote the set $\{1, 2, \ldots, k\}$. Suppose $\varphi$ is a $t$-linear coloring of $G$, and the color set is $[t]$. For a color $i \in [t]$, we call an edge colored with $i$ an $i$-edge. Let $v$ be a vertex of $G$, we use $C^i_\varphi(v)$ to denote the set of colors appear $i$ times at vertex $v$, where $i \in \{0, 1, 2\}$. Then $|C^0_\varphi(v)|+|C^1_\varphi(v)|+|C^2_\varphi(v)| = t$ and $|C^1_\varphi(v)|+2|C^2_\varphi(v)| = d(v)$. For any two vertices of $u$ and $v$, let $C_\varphi(u, v) = C^2_\varphi(u) \cup C^2_\varphi(v) \cup (C^1_\varphi(u) \cap C^1_\varphi(v))$, that is, $C_\varphi(u, v)$ is the set of colors that appear at least twice at $u$ and $v$. A *monochromatic path* is a path whose edges receive the same color. We use the notation $(u, i) \leftrightarrow (v, i)$ to denote that there is a monochromatic path from $u$ to $v$ receives the same color $i$. Let $x_1y_1 \in E(G)$, we use $x_1y_1 \leftrightarrow (u, i)$ to denote that there is a monochromatic path from $x_1$ to $u$ receiving the same color $i$ such that $y_1$ is an internal vertex in the path, and $x_1y_1 \not\leftrightarrow (u, i)$ to denote such monochromatic path does not exist.

***Proof of Theorem* 1.** We prove the theorem by contradiction. Let $G = (V, E)$ be a counterexample to Theorem 1 with $|V(G)| + |E(G)|$ as small as possible.

First, we describe some known lemmas for $G$. Note that proofs of Lemmas 3, 5 and 6 in [6] do not use planarity, so the results can apply to general graphs. The proof of Lemma 3 can be found in [6, Lemma 4], Lemma 5 can be found in [6, Lemma 5 and Lemma 6].

**Lemma 3** [6]**.** *For every edge $uv \in E(G)$, $d(u) + d(v) \geq 2t + 2 \geq \Delta + 2$.*

By the lemma, we have $\delta(G) \geq 2$. At the same time, we may apply Lemma 3 in [5] with parameters $\Delta_0 = 2t$, and obtain the following result.

**Lemma 4** [5]**.** *There are disjoint vertex sets $U, W \subseteq V(G)$ and a vertex $x \in U$, such that*

(a) *$W$ is stable with $N(W) \subseteq U$;*

(b) *$d(w) \leq k$ for every $w \in W$;*

(c) *$W \subseteq N(x) \subseteq W \cup U$; and*

(d) *$|U| \leq k + 1$ and $|W| \geq 2t + 2 - 2k$.*

In Lemma 4, $W$ is *stable* means that $W$ is a *vertex independent set*, that is, the vertices of $W$ are pairwise nonadjacent.

**Lemma 5** [6]**.** *Every vertex is adjacent to at most one* 2*-vertex, and for any* 2*-vertex of* $G$*, its two neighbors are adjacent.*

***Proof of* (1).** We begin to prove (1). According to [8], if $\Delta(G) \leq 5$, then $G$ has a 3-linear coloring. Henceforth, $\Delta(G) \geq 6$. In the following figures, the vertices marked by • have no other edge incident with it and any edge marked by broken line means that it does not exist.

**Lemma 6** [6]**.** $G$ *contains no subgraph isomorphic to one of configurations depicted in Figure* 1*.*

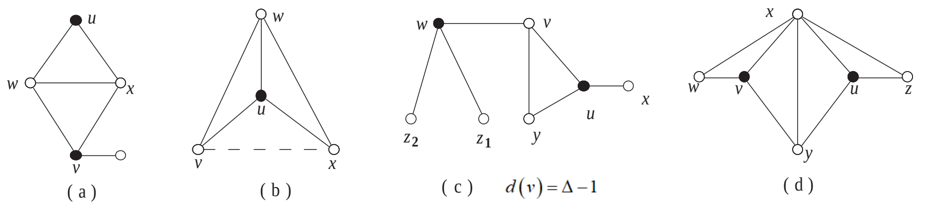


Figure 1. Forbidden configurations in Lemma 6.

The proof of (a) can be found in [6, Lemma 8], (b) can be found in [6, Lemma 7], (c) can be extracted from [6, Lemma 11], (d) can be found in [6, Corollary 13].

**Lemma 7** [15]**.** $G$ *contains no subgraph isomorphic to one of configurations depicted in Figure* 2*. In configuration* (b), $d(w) \leq 3$ *and* $w$ *is incident with a* 3*-cycle.*

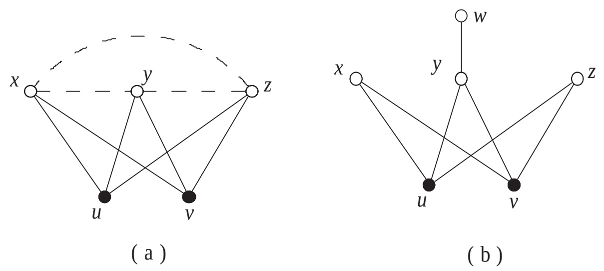


Figure 2. Forbidden configurations in Lemma 7.

***Proof.*** (a) Suppose $G$ has a configuration as depicted in Figure 2(a). Then $G' = G - \{u, v\} + \{xy, yz, xz\}$ has a $t$-coloring $\varphi$. Without loss of generality, assume that $\varphi(xy) \neq \varphi(xz)$. We can recolor $ux$, $uy$ with $\varphi(xy)$, $vy$, $vz$ with $\varphi(yz)$ and $uz$, $vx$ with $\varphi(xz)$ to obtain a $t$-linear coloring of $G$, a contradiction.

(b) The detailed proof of (b) can be found in [15]. The following is a sketch of the proof.

Suppose $G$ has a configuration as depicted in Figure 2(b). By Lemma 6(b), $|\{xy, yz, xz\} \cap E(G)| = 1$ or 3, then we consider the following two cases:

*Case* 1. $|\{xy, yz, xz\} \cap E(G)| = 1$. Without loss of generality, we assume that $xz \in E(G)$ and $xy$, $yz \notin E(G)$. Then $G' = G - \{u, v\} + \{xy, yz\}$ has a $t$-coloring $\varphi$. And we can obtain a $t$-linear coloring of $G$ by the method of color exchange, a contradiction.

*Case* 2. $|\{xy, yz, xz\} \cap E(G)| = 3$. Then $G' = G - \{u, v\}$ has a $t$-coloring $\varphi$. In the same way, we can prove that $G$ has a $t$-linear coloring, a contradiction. □

**Lemma 8.** *For every vertex $w \in W$, $d(w) = 3$.*

***Proof.*** Suppose there exists a vertex $w^* \in W$ such that $d(w^*) = 2$. Let $N(w^*) = \{x, u_1\} \subseteq U$. Then $xu_1 \in E(G)$ by Lemma 5. By Lemma 3, we have $d(x) \geq 2t \geq 6$. Since $|U| \leq 4$ and $N(x) \subseteq U \cup W$, we have $|W| \geq 3$. Let $\{w^*, w_1, w_2\} \subseteq W$. Then $d(w_1) = d(w_2) = 3$ by Lemma 5 and $w_1u_1, w_2u_1 \notin E(G)$ by Figure 1(a). Since $|U| \leq 4$, $N(w_1) = N(w_2)$. Hence $G$ has a configuration as depicted in Figure 2(b), a contradiction. □

By Lemma 8 and Lemma 4 (b), the result of Theorem 1(1) is clear when $k \leq 2$.

**Lemma 9.** $|U| = 4$.

***Proof.*** By Lemma 4 and Lemma 8, $3 \leq |U| \leq 4$. Suppose $|U| = 3$ and $U = \{x, y, z\}$. Since $d(x) \geq 2t-1 \geq 5$ and $N(x) \subseteq U \cup W$, $|W| \geq 3$. Let $\{u, v, w\} \subseteq W$. Then $d(u) = d(v) = d(w) = 3$ and $N(u) = N(v) = N(w) = U$. If $\{xy, yz, xz\} \cap E(G) = \emptyset$, then $G$ has a configuration as depicted in Figure 2(a); otherwise $G$ has a configuration as depicted in Figure 2(b), a contradiction. Hence $|U| = 4$. □

By Lemma 9, let $U = \{x, u_1, u_2, u_3\}$. By Lemma 3 and Lemma 4, $|W| \geq 2t - 1 - |N(x) \cap U| \geq 5 - |N(x) \cap U| \geq 2$. We consider the following four cases.

*Case* 1. $|\{xu_1, xu_2, xu_3\} \cap E(G)| = 3$. Without loss of generality, assume that $w \in W$ and $N(w) = \{x, u_1, u_2\}$. Then $u_1u_2 \in E(G)$ by Figure 1(b). Since $x$ with two 3-neighbors, and the 3-neighbor $w$ is incident with a triangle $xu_1w$, we have $d(x) = \Delta \geq 6$. For otherwise, $G$ has a configuration as depicted in 1(c), a contradiction. Since $|U| = 4$, $|W| \geq 3$. Let $\{w, w_1, w_2\} \subseteq W$. Then for each $i (1 \leq i \leq 2)$, $w_i$ is incident with at least one 3-cycle. If $N(w) = N(w_1)$, then $G$ has a configuration as depicted in Figure 2(b), a contradiction. So $N(w) \neq N(w_1)$. It follows that $w_1u_3 \in E(G)$. Since $|U| = 4$, $|N(w) \cap N(w_1)| = 2$. Without loss of generality, assume that $N(w) \cap N(w_1) = \{x, u_1\}$. These implies that $G$ has two 3-vertices $w$ and $w_1$ such that $N(w) = \{x, u_1, u_2\}$, $N(w_1) = \{x, u_1, u_3\}$ and $\{wu_2, w_1u_3, xu_1, xu_2, xu_3\} \subseteq E(G)$. Thus Figure 1(d) appears, a contradiction.

*Case* 2. $|\{xu_1, xu_2, xu_3\} \cap E(G)| = 2$. Without loss of generality, assume $xu_1$, $xu_2 \in E(G)$ and $xu_3 \notin E(G)$. Since $x$ has at least two 3-neighbors, and each 3-neighbor $w \in W$ is incident with a triangle $xu_1w$ or $xu_2w$, we have $d(x) = \Delta \geq 6$ by Figure 1(c). It follows that $|W| \geq 4$. Since $W \subseteq N(x)$, $|N(w') \cap \{u_1, u_2, u_3\}| = 2$ for any $w' \in W$. It follows that there are two vertices $u, v \in W$ such that $N(u) = N(v)$. Note that any vertex in $W$ is incident with at least one 3-cycle. So $G$ has a configuration as depicted in Figure 2(b), a contradiction.

*Case* 3. $|\{xu_1, xu_2, xu_3\} \cap E(G)| = 1$. Without loss of generality, assume that $xu_1 \in E(G)$, $xu_2 \notin E(G)$, $xu_3 \notin E(G)$. If there exists a vertex $w \in W$ such that $wu_1 \in E(G)$, without loss of generality, suppose $w^*u_1 \in E(G)$. Then $d(x) = \Delta \geq 6$ by Figure 1(c). Since $|\{xu_1, xu_2, xu_3\} \cap E(G)| = 1$, we have $|W| \geq 5$. Thus at least two vertices of $W \backslash \{w^*\}$ have the same neighbors. And because $w^*$ is incident with a 3-face, $G$ has a configuration as depicted in Figure 2(b), a contradiction. Otherwise, if each vertex $w \in W$ is not adjacent to $u_1$, then $N(w) = \{x, u_2, u_3\}$. Since $|\{xu_1, xu_2, xu_3\} \cap E(G)| = 1$ and $d(x) \geq 5$, we have $|W| \geq 4$. Thus at least four vertices of degree 3 have the same neighbors. Since $xu_2, xu_3 \notin E(G)$, we have $u_2u_3 \in E(G)$ by Figure 2(a). Then $G$ has a configuration as depicted in Figure 2(b), a contradiction.

*Case* 4. $|\{xu_1, xu_2, xu_3\} \cap E(G)| = 0$. Then $|W| \geq 5$. Let $\{w_1, w_2, w_3, w_4, w_5\} \subseteq W$. At the same time, at least two vertices of $W$ have the same neighbors. Without loss of generality, assume that $N(w_1) = N(w_2) = \{x, u_1, u_2\}$. Since $xu_1$, $xu_2 \notin E(G)$, we have $u_1u_2 \in E(G)$ by Lemma 7. Similarly, $|N(w_i) \cap \{u_1, u_2\}| = 1$ for any $i \in \{3, 4, 5\}$. It follows that there are at least two vertices in $\{w_3, w_4, w_5\}$ having the same neighbors. Hence $G$ has a configuration as depicted in Figure 2(b), a contradiction too.

All these contradictions imply that (1) holds.

***Proof of* (2).** Next, we begin to prove (2). By (1), we assume that $tw(G) = 4$. According to [8], if $\Delta \leq 8$, then $G$ has a 5-linear coloring. Henceforth $\Delta \geq 9$. Let $w^* \in W \subseteq N(x)$. Then $G' = G - w^*x$ has a $t$-linear coloring $\varphi$. Denote $n_i = |\{\varphi(uw) = i : u \in U \backslash \{x\}, w \in W\}|$ for any $i \in [t]$, and $W' = \{w \in W : \varphi(xw) \in C^0_\varphi(w^*)\}$. We have the five fundamental facts.

① $C_\varphi(w^*, x) = [t]$;

② If $i \in C^1_\varphi(w^*)$, then $i \in C^2_\varphi(x)$, or $i \in C^1_\varphi(x)$ and $(w^*, i) \leftrightarrow (x, i)$;

③ $|W'| \geq 2|C^0_\varphi(w^*)| - (|U| - 1) \geq 2|C^0_\varphi(w^*)| - 4$;

④ $|W| = |N(x) \backslash U| \geq d(x) - (|U| - 1) \geq 2t - 2 - d(w^*)$;

⑤ $n_i \leq 2(|U| - 1) \leq 2k = 8$ for each $i \in [t]$.

In the following, we will use the structure properties of $G$ and the method of color exchange to obtain a contradiction to prove (2). We consider the following three cases.

*Case* 1. $d(w^*) = 2$, that is, $d_{G'}(w^*) = 1$. Without loss of generality, assume that $C^1_\varphi(w^*) = \{1\}$. Then $C^0_\varphi(w^*) = C^2_\varphi(x) = [t] \setminus \{1\}$, $1 \in C^1_\varphi(x)$ and $(x, 1) \leftrightarrow (w^*, 1)$ by ① and ②. Since $d(x)+d(w^*) \geq 2t+2 \geq 12$ and $|U| \leq 5$, $|W \setminus w^*| \geq 5$. It follows that $|W'| \geq 4$. For any $w \in W'$, if $1 \notin C^2_\varphi(w)$, we can recolor $xw$ with 1 and color $w^*x$ with $\varphi(xw)$ to obtain a $t$-linear coloring of $G$, a contradiction. So $1 \in C^2_\varphi(w)$ for any $w \in W'$ and it follows from $C^1_\varphi(w^*) = \{1\}$ that $n_1 \geq 2 \times |W'| + 1 \geq 9$, a contradiction with ⑤.

*Case* 2. $d(w^*) = 3$.

*Subcase* 2.1. $C^2_\varphi(w^*) \neq \emptyset$. Without loss of generality, assume that $C^2_\varphi(w^*) = \{1\}$. Then $C^0_\varphi(w^*) = C^2_\varphi(x) = [t] \setminus \{1\}$ and $1 \in C^0_\varphi(x) \cup C^1_\varphi(x)$. Since $t \geq 5$ and $|U \backslash x| \leq 4$, $|W'| \geq 4$. If there is a vertex $w \in W'$ such that $1 \in C^0_\varphi(w)$, then we can recolor $xw$ with 1 and color $w^*x$ with $\varphi(xw)$ to obtain a $t$-linear coloring of $G$, a contradiction. So $1 \in C^1_\varphi(w) \cup C^2_\varphi(w)$ for any $w \in W'$. At the same time, if there are two vertices $w', w'' \in W$ such that $1 \in C^1_\varphi(w') \cap C^1_\varphi(w'')$, then it is impossible that $(w, 1) \leftrightarrow (x, 1)$ for any $w \in \{w', w''\}$(if $(x, 1)$ exists). So there is at most one element $w \in W'$ such that $1 \in C^1_\varphi(w)$, and it follows that $n_1 \geq 2 \times (1 + |W'| - 1) + 1 \geq 9$, a contradiction with ⑤.

*Subcase* 2.2. $C^2_\varphi(w^*) = \emptyset$. Without loss of generality, assume that $C^1_\varphi(w^*) = \{1, 2\}$. Then $\{3, 4, \ldots, t\} \subseteq C^2_\varphi(x)$ and $\{1, 2\} \subset C^1_\varphi(x) \cup C^2_\varphi(x)$. Since $d_{G'}(x) \leq 2t - 1$, $|\{1, 2\} \cap C^1_\varphi(x)| \geq 1$. Without loss of generality, assume that $1 \in C^1_\varphi(x)$. Then $(w^*, 1) \leftrightarrow (x, 1)$ by ②. Since $t \geq 5$, $|W'| \geq 2$. Let $w_1, w_2 \in W'$. Then $\{\varphi(xw_1), \varphi(xw_2)\} \cap \{1, 2\} = \emptyset$ by the definition of $W'$. If $1 \notin C^2_\varphi(w_1)$, then we can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. So $1 \in C^2_\varphi(w_1)$. By the same argument, we have $1 \in C^2_\varphi(w_2)$.

Suppose that $2 \in C^1_\varphi(x)$. Then $(x, 2) \leftrightarrow (w^*, 2)$ by ②. Since $d_G(w_1) \leq 4$, $2 \notin C^2_\varphi(w_1)$. Thus we can recolor $xw_1$ with 2 and color $w^*x$ with $\varphi(xw_1)$, a contradiction, too. Hence $2 \in C^2_\varphi(x)$, that is, $\{2, 3, \ldots, t\} = C^2_\varphi(x)$.

Since $|U \backslash x| \leq 4$, there exist $w_3, w_4 \in N(x) \cap (W \setminus \{w^*, w_1, w_2\})$ such that $\varphi(xw_3) \neq 1$ and $\varphi(xw_4) \neq 1$. Similarly, we also have that for any $w_i (i = 3$ or $4)$, if $\varphi(xw_i) \neq 2$, that is, $\varphi(xw_i) \in \{3, \ldots, t\}$, then $1 \in C^2_\varphi(w_i)$. At the same time, if $1 \in C^2_\varphi(w_3) \cap C^2_\varphi(w_4)$, then $n_1 \geq 9$, a contradiction. So we assume, without loss of generality, that $1 \notin C^2_\varphi(w_4)$. It follows that $\varphi(xw_4) = 2$.

Suppose that $\varphi(xw_3) \neq 2$. Then first we recolor $xw_4$ with 1. Next, if $(w^*, 2) \leftrightarrow (x, 2)$, we recolor $xw_1$ with 2, and color $w^*x$ with $\varphi(xw_1)$; otherwise, we color $w^*x$ with 2. Thus we obtain a $t$-linear coloring of $G$, a contradiction. So $\varphi(xw_3) = 2$.

Thus $\varphi(xw_3) = \varphi(xw_4) = 2$ and $1 \notin C^2_\varphi(w_4)$. Suppose that $1 \notin C^2_\varphi(w_3)$. First, we recolor $xw_3$ with 1. Then, if $xw_4 \leftrightarrow (w^*, 2)$, we can recolor $xw_1$ with 2 and color $w^*x$ with $\varphi(xw_1)$; otherwise we color $w^*x$ with 2. Thus a $t$-linear coloring of $G$ is obtained, a contradiction. So $1 \in C^2_\varphi(w_3)$.

Finally, we obtain a $t$-linear coloring of $G$ as follows. First, we recolor $xw_4$ with 1, color $w^*x$ with 2. Then, if $xw_3 \leftrightarrow (w^*, 2)$, then $2 \in C^2_\varphi(w_3)$ and we exchange the coloring of $xw_1$ and $xw_3$.

*Case* 3. $d(w^*) = 4$.

*Subcase* 3.1. $C^2_\varphi(w^*) \neq \emptyset$. Without loss of generality, assume that $C^2_\varphi(w^*) = \{1\}$ and $C^1_\varphi(w^*) = \{2\}$. Then $|W'| \geq 2$. Let $w_1, w_2 \in W'$. It follows from $2 \in C^1_\varphi(w^*)$ and ② that $2 \in C^1_\varphi(x)$ and $(w^*, 2) \leftrightarrow (x, 2)$, or $2 \in C^2_\varphi(x)$.

*Subcase* 3.1.1. $2 \in C^1_\varphi(x)$ and $(w^*, 2) \leftrightarrow (x, 2)$. Then it is similar to prove that $2 \in C^2_\varphi(w_1) \cap C^2_\varphi(w_2)$. This implies that $1 \notin C^2_\varphi(w_1) \cup C^2_\varphi(w_2)$.

Suppose $1 \in C^0_\varphi(x)$. We can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$ to obtain a $t$-linear coloring of $G$, a contradiction.

Suppose $1 \in C^1_\varphi(x)$. Then $1 \in C^1_\varphi(w_1)$ and $(w_1, 1) \leftrightarrow (x, 1)$. For otherwise, we can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. Since $1 \notin C^2_\varphi(w_2)$, we can recolor $xw_2$ with 1 and color $w^*x$ with $\varphi(xw_2)$ to obtain a $t$-linear coloring of $G$, a contradiction.

Suppose $1 \in C^2_\varphi(x)$. Then $d_{G'}(x) = 2t - 1 \geq 9$ and we can get that $|W \backslash \{w^*\}| = m \geq 5$. Assume that $W = \{w^*, w_1, \ldots, w_m\}$. If $|W'| \geq 4$, without loss of generality, assume that $\varphi(xw_3), \varphi(xw_4) \notin \{1, 2\}$. It is easy to see that $2 \in C^2_\varphi(w_3)$. For otherwise, we can recolor $xw_3$ with 2 and color $w^*x$ with $\varphi(xw_3)$, a contradiction. Similarly, $2 \in C^2_\varphi(w_4)$. Thus $n_2 \geq 9$, a contradiction. If $|W'| = 3$, since $|U| \leq 5$ and $t \geq 5$, there must exist an edge $xw_i$ which colored by 1. Without loss of generality, assume that $\varphi(xw_3) \notin \{1, 2\}$ and $\varphi(xw_4) = 1$. Then $2 \in C^2_\varphi(w_i)$, $i = 1, 2, 3$. If $2 \notin C^2_\varphi(w_4)$, first we can recolor $xw_4$ with 2. Next, if $(x, 1) \leftrightarrow (w_1, 1)$, we can recolor $xw_2$ with 1 and color $w^*x$ with $\varphi(xw_2)$ to obtain a $t$-linear coloring of $G$. Otherwise, we can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. Thus $2 \in C^2_\varphi(w_4)$. We have $n_2 \geq 9$, a contradiction. If $|W'| = 2$, then we can assume that $\varphi(xw_3) = \varphi(xw_4) = 1$. If $2 \notin C^2_\varphi(w_3)$, first we can recolor $xw_3$ with 2. Next, if $(x, 1) \leftrightarrow (w_1, 1)$, we can recolor $xw_2$ with 1 and color $w^*x$ with $\varphi(xw_2)$ to obtain a $t$-linear coloring of $G$. Otherwise, we can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. Thus $2 \in C^2_\varphi(w_3)$. Similarly $2 \in C^2_\varphi(w_4)$. Then $n_2 \geq 9$, a contradiction.

*Subcase* 3.1.2. $2 \in C^2_\varphi(x)$. Then $1 \in C^0_\varphi(x)$ or $1 \in C^1_\varphi(x)$ and $d_{G'}(x) \geq 2(t-1) \geq 8$. Since $|U| \leq 5$, we have $|W \backslash \{w^*\}| = m \geq 4$. Assume that $W = \{w^*, w_1, \ldots, w_m\}$ and $\varphi(xw_3) \neq 1$, $\varphi(xw_4) \neq 1$.

Suppose $1 \in C^0_\varphi(x)$. Then $1 \in C^2_\varphi(w_1)$. For otherwise, we can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. Similarly, $1 \in C^2_\varphi(w_2)$. If

$\varphi(xw_3) \neq 2$, similarly $1 \in C^2_\varphi(w_3)$. If $\varphi(xw_3) = 2$, we also have $1 \in C^2_\varphi(w_3)$. For otherwise, we can recolor $xw_3$ with 1 to obtain a new $t$-linear coloring $\varphi'$ of $G'$, where $2 \in C^1_{\varphi'}(x)$, which satisfies Subcase 3.1.1, a contradiction. Thus $1 \in C^2_\varphi(w_3)$. In the same way, we have $1 \in C^2_\varphi(w_4)$. Then $n_1 \geq 10$, a contradiction.

Now suppose $1 \in C^1_\varphi(x)$.

First we consider the case that $|W'| \geq 4$. Without loss of generality, assume that $\varphi(xw_3) \notin \{1,2\}$ and $\varphi(xw_4) \notin \{1,2\}$. Then $1 \in C^1_\varphi(w_1)$ and $(x,1) \leftrightarrow (w_1,1)$, or $1 \in C^2_\varphi(w_1)$. For otherwise, we can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. If $1 \in C^1_\varphi(w_1)$ and $(x,1) \leftrightarrow (w_1,1)$, then $1 \in C^2_\varphi(w_2)$. For otherwise, we can recolor $xw_2$ with 1 and color $w^*x$ with $\varphi(xw_2)$, a contradiction. Similarly, $1 \in C^2_\varphi(w_3)$, $1 \in C^2_\varphi(w_4)$. Then $n_1 \geq 9$, a contradiction. Thus $1 \in C^2_\varphi(w_1)$. Similarly, we have $1 \in C^2_\varphi(w_i)$, $i = 2,3,4$. Then $n_1 \geq 10$, a contradiction.

Secondly, we consider the case that $|W'| = 3$. Without loss of generality, assume that $\varphi(xw_3) \neq 2$, $\varphi(xw_4) = 2$. Then $1 \in C^1_\varphi(w_1)$ and $(x,1) \leftrightarrow (w_1,1)$, or $1 \in C^2_\varphi(w_1)$. If $1 \in C^1_\varphi(w_1)$ and $(x,1) \leftrightarrow (w_1,1)$, then $1 \in C^2_\varphi(w_2)$ and $1 \in C^2_\varphi(w_3)$. If $1 \notin C^2_\varphi(w_4)$, we can recolor $xw_4$ with 1 to obtain a new $t$-linear coloring $\varphi'$ of $G'$ which satisfies $2 \in C^1_{\varphi'}(x)$, by Subcase 3.1.1, a contradiction. Thus $1 \in C^2_\varphi(w_4)$. We have $n_1 \geq 9$, a contradiction. Thus $1 \in C^2_\varphi(w_1)$. Similarly, $1 \in C^2_\varphi(w_2)$ and $1 \in C^2_\varphi(w_3)$. If $1 \in C^0_\varphi(w_4)$, we can recolor $xw_4$ with 1 to get a new $t$-linear coloring $\varphi'$ of $G'$ which satisfies $2 \in C^1_{\varphi'}(x)$, a contradiction. Thus $1 \notin C^0_\varphi(w_4)$. We have $n_1 \geq 9$, a contradiction.

Finally, we consider the case that $|W'| = 2$. Without loss of generality, assume that $\varphi(xw_3) = \varphi(xw_4) = 2$. Then $1 \notin C^0_\varphi(w_1)$. If $1 \in C^1_\varphi(w_1)$, then $(x,1) \leftrightarrow (w_1,1)$. We can get $1 \in C^2_\varphi(w_2)$. If $1 \notin C^2_\varphi(w_3)$, we can recolor $xw_3$ with 1 to get a contradiction. Thus $1 \in C^2_\varphi(w_3)$. Similarly, $1 \in C^2_\varphi(w_4)$. Then $n_1 \geq 9$, a contradiction. Therefore $1 \in C^2_\varphi(w_1)$ and $1 \in C^2_\varphi(w_2)$. If $1 \in C^0_\varphi(w_3)$, or $1 \in C^1_\varphi(w_3)$ and $(x,1) \not\leftrightarrow (w_3,1)$, we can recolor $xw_3$ with 1 to get a new $t$-linear coloring $\varphi'$ of $G'$ such that $2 \in C^1_{\varphi'}(x)$, a contradiction. If $1 \in C^1_\varphi(w_3)$ and $(x,1) \leftrightarrow (w_3,1)$, then $1 \in C^2_\varphi(w_4)$. For otherwise, we can recolor $xw_4$ with 1 to get a contradiction. We have $n_1 \geq 9$, a contradiction. If $1 \in C^2_\varphi(w_3)$, since $1 \notin C^0_\varphi(w_4)$, we also have $n_1 \geq 9$, a contradiction.

*Subcase* 3.2. $C^2_\varphi(w^*) = \emptyset$. Without loss of generality, assume that $C^1_\varphi(w^*) = \{1,2,3\}$. Then $i \notin C^0_\varphi(x)$, $i = 1,2,3$, and at least one of them appears exactly one time on $x$. Without loss of generality, assume that $1 \in C^1_\varphi(x)$. Then $(x,1) \leftrightarrow (w^*,1)$.

Since $d_{G'}(x) \geq 2t-3 \geq 7$ and $|U| \leq 5$, we have $|W\backslash\{w^*\}| = m \geq 3$. Assume that $W = \{w^*, w_1, \ldots, w_m\}$.

*Subcase* 3.2.1. $2 \in C^1_\varphi(x)$ or $3 \in C^1_\varphi(x)$. Without loss of generality, assume that $2 \in C^1_\varphi(x)$. Then $(x,2) \leftrightarrow (w^*,2)$.

Suppose $|W'| \geq 1$. Without loss of generality, assume that $\varphi(xw_1) \notin \{1, 2, 3\}$. Then $1 \in C^2_\varphi(w_1)$. For otherwise, we can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. Since $d(w_1) \leq 4$, we have $2 \notin C^2_\varphi(w_1)$. Thus we can recolor $xw_1$ with 2 and color $w^*x$ with $\varphi(xw_1)$ to obtain a $t$-linear coloring of $G$, a contradiction.

Now suppose $|W'| = 0$. Without loss of generality, assume that $\varphi(xw_i) = i$, $i = 1, 2, 3$. If $1 \notin C^2_\varphi(w_2)$, we can recolor $xw_2$ with 1 and color $w^*x$ with 2, a contradiction. Thus $1 \in C^2_\varphi(w_2)$. Since $2 \in C^1_\varphi(x)$ and $(x, 2) \leftrightarrow (w^*, 2)$, we have $2 \in C^2_\varphi(w_2)$. Then $3 \in C^0_\varphi(w_2)$ for $d(w_2) \leq 4$. Similarly, $3 \in C^0_\varphi(w_1)$. If $1 \notin C^2_\varphi(w_3)$, we can recolor $xw_3$ with 1, $xw_1$ with 3 and color $w^*x$ with 1 to obtain a $t$-linear coloring of $G$, a contradiction. Thus $1 \in C^2_\varphi(w_3)$. Similarly $2 \in C^2_\varphi(w_3)$. But it is impossible since $d(w_3) \leq 4$.

*Subcase* 3.2.2. $2 \in C^2_\varphi(x)$ and $3 \in C^2_\varphi(x)$. Since $d_{G'}(x) \geq 2t - 1 \geq 9$ and $|U| \leq 5$, we have $|W \backslash \{w^*\}| = m \geq 5$.

Suppose $|W'| = 0$. Without loss of generality, assume that $\varphi(xw_1) = 1$, $\varphi(xw_2) = \varphi(xw_3) = 2$ and $\varphi(xw_4) = \varphi(xw_5) = 3$. If $1 \notin C^2_\varphi(w_2)$, first we can recolor $xw_2$ with 1. Then $xw_3 \leftrightarrow (w^*, 2)$. For otherwise, we can color $w^*x$ with 2 to obtain a $t$-linear coloring of $G$. If $2 \notin C^2_\varphi(w_1)$, we can recolor $xw_1$ with 2 and color $w^*x$ with 1. Thus $2 \in C^2_\varphi(w_1)$. Since $1 \in C^1_\varphi(x)$ and $(x, 1) \leftrightarrow (w^*, 1)$, we have $1 \in C^2_\varphi(w_1)$. Thus we can get that $3 \in C^0_\varphi(w_1)$ for $d(w_1) \leq 4$. If $2 \notin C^2_\varphi(w_4)$, we can recolor $xw_4$ with 2, $xw_1$ with 3 and color $w^*x$ with 1. Thus $2 \in C^2_\varphi(w_4)$. Now we can recolor $xw_4$ with 1, $xw_2$ with 2, $xw_1$ with 3 and color $w^*x$ with 1, a contradiction. Therefore $1 \in C^2_\varphi(w_2)$. Similarly we have $1 \in C^2_\varphi(w_i)$, $i = 3, 4, 5$. Then $n_2 \geq 10$, a contradiction.

Suppose $|W'| = 1$. Without loss of generality, assume that $\varphi(xw_1) \notin \{1, 2, 3\}$, $\varphi(xw_2) = 2$ and $\varphi(xw_3) = \varphi(xw_4) = 3$. Then $1 \in C^2_\varphi(w_1)$. For otherwise, we can recolor $xw_1$ with 1 and color $w^*x$ with $\varphi(xw_1)$. If $1 \notin C^2_\varphi(w_3)$, first we can recolor $xw_3$ with 1. We can get $xw_4 \leftrightarrow (w^*, 3)$. Next we can recolor $xw_1$ with 3 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. Thus $1 \in C^2_\varphi(w_3)$. Similarly $1 \in C^2_\varphi(w_2)$ and $1 \in C^2_\varphi(w_4)$. Then $n_1 \geq 9$, a contradiction.

Suppose $|W'| = 2$. Without loss of generality, assume that $\varphi(xw_1), \varphi(xw_2) \notin \{1, 2, 3\}$, $\varphi(xw_3) = \alpha$, $\varphi(xw_4) = \beta$, $\alpha, \beta \in \{2, 3\}$. Then $1 \in C^2_\varphi(w_1) \cap C^2_\varphi(w_2)$. If $1 \notin C^2_\varphi(w_3)$, first we recolor $xw_3$ with 1. If $(x, \alpha) \not\leftrightarrow (w^*, \alpha)$, we can color $w^*x$ with $\alpha$. Otherwise, we can recolor $xw_1$ with $\alpha$ and color $w^*x$ with $\varphi(xw_1)$ to get a $t$-linear coloring of $G$. Thus $1 \in C^2_\varphi(w_3)$. Similarly $1 \in C^2_\varphi(w_4)$. Therefore $n_1 \geq 9$, a contradiction.

Suppose $|W'| = 3$. Without loss of generality, assume that $\varphi(xw_1)$, $\varphi(xw_2)$, $\varphi(xw_3) \notin \{1, 2, 3\}$ and $\varphi(xw_4) = 2$. Then $1 \in C^2_\varphi(w_i)$, $i = 1, 2, 3$. If $1 \notin C^2_\varphi(w_4)$, first we can recolor $xw_4$ with 1. Then if $(x, 2) \not\leftrightarrow (w^*, 2)$, we can color $w^*x$ with 2. Otherwise, we can recolor $xw_1$ with 2 and color $w^*x$ with $\varphi(xw_1)$, a contradiction. Thus $1 \in C^2_\varphi(w_4)$. We have $n_1 \geq 9$, a contradiction.

Suppose $|W'| \geq 4$. Without loss of generality, assume that $\varphi(xw_i) \notin \{1,2,3\}$, $i = 1,2,3,4$. Then it is easy to get that $1 \in C^2_\varphi(w_i)$, $i = 1,2,3,4$. Thus $n_1 \geq 9$, a contradiction.

Hence, we complete the proof of Theorem 1(2).

***Proof of* (3).** Finally, we begin to prove (3). By the minimality of $G$, $G' = G - w^*x$ has a $t$-linear coloring $\varphi$. $|W'| \geq 2|C^0_\varphi(w^*)| - (|U|-1) \geq 2|C^0_\varphi(w^*)| - k \geq 2[2k-1-(k-1)] - k = k$. Without loss of generality, assume that $\varphi(xw_i) = \beta_i \in C^0_\varphi(w^*)$, $i = 1,2,\ldots,k$.

*Case* 1. $C^2_\varphi(w^*) = \emptyset$. Without loss of generality, assume that $C^1_\varphi(w^*) = \{\alpha_1, \alpha_2, \ldots, \alpha_m\}$, where $m = d(w^*) - 1 \leq k - 1$.

Then $\alpha_i \in C^1_\varphi(x) \cup C^2_\varphi(x)$, $i = 1,2,\ldots,m$. Since $d_{G'}(x) \leq 2t-1$, there must exist a color $\alpha \in C^1_\varphi(w^*)$ such that $\alpha \in C^1_\varphi(x)$. Then $(x,\alpha) \leftrightarrow (w^*,\alpha)$. If $\alpha \notin C^2_\varphi(w_1)$, we can recolor $xw_1$ with $\alpha$ and color $w^*x$ with $\beta_1$ to obtain a $t$-linear coloring of $G$, a contradiction. Thus $\alpha \in C^2_\varphi(w_1)$. Similarly, we have $\alpha \in C^2_\varphi(w_i)$, $i = 2,3,\ldots,k$. Then $n_\alpha \geq 2k+1$, a contradiction.

*Case* 2. $C^2_\varphi(w^*) \neq \emptyset$. If $\alpha \in C^1_\varphi(w^*)$, then $\alpha \in C^2_\varphi(x)$. For otherwise, similar to Case 1, we can get $n_\alpha \geq 2k+1$, a contradiction. And since $d_{G'}(x) \leq 2t-1$, there must exist a color $\beta \in C^2_\varphi(w^*)$ such that $\beta \in C^0_\varphi(x) \cup C^1_\varphi(x)$.

Suppose $\beta \in C^0_\varphi(x)$. Then $\beta \in C^2_\varphi(w_1)$. For otherwise, we can recolor $xw_1$ with $\beta$ and color $w^*x$ with $\beta_1$, a contradiction. Similarly, we have $\beta \in C^2_\varphi(w_i)$, $i = 2,3,\ldots,k$. Then $n_\beta \geq 2k+2$, a contradiction.

Suppose $\beta \in C^1_\varphi(x)$. If $\beta \in C^0_\varphi(w_1)$, or $\beta \in C^1_\varphi(w_1)$ and $(x,\beta) \not\leftrightarrow (w_1,\beta)$, we can recolor $xw_1$ with $\beta$ and color $w^*x$ with $\beta_1$, a contradiction. If $\beta \in C^1_\varphi(w_1)$ and $(x,\beta) \leftrightarrow (w_1,\beta)$, then $\beta \in C^2_\varphi(w_2)$. For otherwise, we can recolor $xw_2$ with $\beta$ and color $w^*x$ with $\beta_2$, a contradiction. Similarly, $\beta \in C^2_\varphi(w_i)$, $i = 3,\ldots,k$. Then $n_\beta \geq 2(k-1)+1+2 = 2k+1$, a contradiction. Thus $\beta \in C^2_\varphi(w_1)$. Similarly, we have $\beta \in C^2_\varphi(w_i)$, $i = 2,3,\ldots,k$. Then $n_\beta \geq 2k+2$, a contradiction.

This completes the proof of Theorem 1(3). ■

## 3. Conjecture and Open Question

In [5], it is proved that if $G$ is a graph with $\Delta \geq 3k-3$ and $k \geq 3$, then the total chromatic $\chi''(G) = \Delta + 1$. In this paper, we show that if $\Delta \geq 3k-3$ and $k = 3$ or $k = 4$, then the linear arboricity $la(G)$ is $\lceil \frac{\Delta}{2} \rceil$. Thus, we give the following conjecture.

**Conjecture B.** *If $G$ is a graph with $\Delta \geq 3k-3$ when $k$ is even, $\Delta \geq 3k-4$ when $k$ is odd, $k \geq 3$, then the linear arboricity $la(G)$ of $G$ is $\lceil \frac{\Delta}{2} \rceil$.*

We propose the following open question. Is the bound on $\Delta$ is sharp?

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