Discussiones Mathematicae
Graph Theory 44 (2024) 459–473
https://doi.org/10.7151/dmgt.2455

THE DECYCLING NUMBER OF A PLANAR GRAPH COVERED BY K_4-SUBGRAPHS

Dengju Ma

School of Sciences
Nantong University
Jiangsu Province, China, 226019

e-mail: madengju@ntu.edu.cn

Mingyuan Ma

and

Han Ren

Department of Mathematics
East China Normal University
Shanghai, China, 200062

e-mail: ming623@163.com
hren@math.ecnu.edu.cn

Abstract

Let G be a planar graph of n vertices. The paper shows that the decycling number of G is at most $\frac{n-1}{2}$ if G has not any K_4-minor. If the maximum degree of G is at most four and G is not 4-regular, the paper proves that the decycling number of G is $\frac{n}{2}$ if and only if G is covered by K_4-subgraphs. In addition, the decycling number of G covered by octahedron-subgraphs or icosahedron-subgraphs is studied.

Keywords: decycling number, planar graph, K_4-minor.

2020 Mathematics Subject Classification: 05C10, 05C38.

1. Introduction

All graphs in the paper are simple. Let G be a graph. A subset S of $V(G)$ is said to be a *decycling set* of G if $G - S$ is acyclic. The cardinality of a minimum decycling set of G is said to be the *decycling number* of G, which is denoted by

$\nabla(G)$. A decycling set is also called a feedback vertex set. It has application in areas such as circuit design and deadlock prevention [6].

Determining the decycling number of a graph is not an easy work. It has been proved that the problem of determining the decycling number of a graph is NP-hard [8]. So an upper bound for the decycling number of a graph, especially a planar graph, attracts people's attention. The decycling number of every planar graph of n vertices is at most $\frac{3n}{5}$, which follows from the result that planar graphs are acyclically 5-colorable shown by Borodin [4]. For an outerplanar graph of n vertices, Hosono [7] proved that its decycling number is at most $\frac{n}{3}$. However, the following conjecture is challenging, which is still open now.

Conjecture 1 [1, 5]**.** *If G is a planar graph of n vertices, then $\nabla(G) \leq \frac{n}{2}$.*

In the paper we firstly study the decycling number of a planar graph without K_4-minor, and then we obtain the following result.

Theorem 2. *If G is a planar graph of order n without K_4-minor, then $\nabla(G) \leq \frac{n-1}{2}$.*

For a graph H of n vertices with maximum degree at most four, Ren *et al.* [13] showed that $\nabla(H) \leq \frac{n+1}{2}$ if H is 4-regular, or $\nabla(G) \leq \frac{n}{2}$, otherwise. Punnim [12] proved the following result.

Theorem 3 [12]**.** *Let H be a graph of n vertices with maximum degree at most four. If H is K_5-free graph, then $\nabla(H) \leq \frac{n}{2}$.*

By Theorem 3, the decycling number of a planar graph G of n vertices with maximum degree at most four is at most $\frac{n}{2}$. What is the structure of G if $\nabla(G) = \frac{n}{2}$? We will show the theorem below in the paper.

Theorem 4. *Let G be a planar graph of n vertices with maximum degree at most four which is not 4-regular. Then $\nabla(G) = \frac{n}{2}$ if and only if G is covered by K_4-subgraphs.*

The arrangement of the paper is as follows. Section 2 gives the proof of Theorem 2. In Section 3 we mainly study the decycling number of a planar graph with maximum degree at most four covered by K_4-subgraphs. In the end of the section Theorem 4 is proved. In Section 4 we show that the decycling number of a planar graph G of order n is $\frac{n}{2}$ if it is covered by O_6-subgraphs, or I_{12}-subgraphs such that the outer degree of each of O_6 and I_{12} is at most five, where O_6 and I_{12} denote the octahedron and the icosahedron, respectively.

The remainder of the section is contributed for some terminologies of graphs. The other undefined can be found in [3].

The complete graph of n vertices is denoted by K_n. Let G be a graph. The number of vertices of G is called its *order*. The maximum degree and minimum degree of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. For a vertex v in G, a vertex being adjacent to v in G is called a *neighbor* of v. The set of all neighbors of v is denoted by $N(v)$. Let X be a vertex subset of G with k elements. If $G-X$ has more components than that in G, then X is called a *k-vertex cut* of G. If X contains only one vertex, say v, then we call v a *cut-vertex* of G. An edge e of G is called a *cut-edge* if $G-e$ has more components than that in G. A connected graph H is said to be *k-connected* if the order of H is at least $k+1$ and $H-Y$ is connected for every $Y \subseteq V(G)$ with at most $k-1$ elements.

Let F and F' be two graphs. If F' is obtained from a subgraph of F by contracting several edges, then F' is called a *minor* of F, or we say F has an *F'-minor.* Let Q be a graph other than F and F'. If a subgraph of F is isomorphic to Q, then we say that F has a *Q-subgraph.* A *k-tree* is either the complete graph K_{k+1} or a graph obtained from a k-tree G by adding one vertex such that it joins to k pairwise adjacent vertices in G. The *tree-width* of a graph is the minimum k such that the graph is a subgraph of a k-tree. Obviously, every graph with tree-width k has a vertex of degree at most k.

Let H be a graph. For $i = 1, 2, \ldots, k$, let V_i be a subset of $V(H)$, and let H_i be the subgraph of H induced by V_i. If $V_1, V_2, \ldots, V_k$ are pairwise disjoint sets such that $\cup_{i=1}^{k} V_i = V(H)$, then H is said to be *covered* by H_1-subgraph, H_2-subgraph, $\ldots$, H_k-subgraph. In particular, if H_i is isomorphic to the graph R for $i = 1, 2, \ldots, k$, then we say that H is covered by R-subgraphs. For a vertex x in V_i, the number of all edges incident with x which are not in H_i is said to be the *outer degree* of x in H_i. The sum of the outer degrees of all vertices in V_i is called the *outer degree* of H_i.

An *embedding* of a planar graph G in the plane is a drawing of G in the plane which has no edge-crossings. By the stereographic projection, a graph is embeddable in the plane if and only if it can be embedded in the sphere. By a well-known result of Whitney [15], a 3-connected planar graph has a unique embedding in the plane (or the sphere). Let H be a connected planar graph with at least four vertices which has an embedding Π in the sphere Σ. Then a connected component in $\Sigma - \Pi$ is a *face* of Π. If the boundary of a face f is a cycle C, then C is called a *facial cycle.* Sometimes, f is said to be the *interior* of C.

2. The Proof of Theorem 2

We now give the proof of Theorem 2.

***Proof of Theorem* 2.** We use the induction on n to show the theorem. If $n \leq 3$, then the theorem is true. If $n = 4$, then G is a proper subgraph of K_4. It can be checked that $\nabla(G) \leq 1$. Assume that the theorem holds for $n \leq k-1$, where $k \geq 5$. We now consider the case that $n = k$.

It is well-known that every graph that has no K_4-minor is planar and has tree-width at most two. In addition, every graph with tree-width k has a vertex of degree at most k. So $\delta(G) = 1$ or 2.

If $\delta(G) = 1$, let x be a vertex of degree one in G. Let G' be the graph obtained from G by deleting x. Clearly, G' has not any K_4-minor. So $\nabla(G') \leq \frac{n-2}{2}$ by the inductional assumption. Since $\nabla(G) = \nabla(G')$, we have that $\nabla(G) \leq \frac{n-2}{2}$.

If $\delta(G) = 2$, let y be a vertex of degree two in G. Let y_1 and y_2 be two neighbors of y. We now delete y_1 from G. Then the degree of y is one in the present graph. Next, y is deleted. Let H be the obtained graph, which has not any K_4-minor. Then $\nabla(G) \leq \nabla(H)+1$. By the inductional assumption, we have that $\nabla(H) \leq \frac{n-3}{2}$. So $\nabla(G) \leq \frac{n-1}{2}$. ■

The result below follows from Theorem 2 directly.

Corollary 5. *Let G be a planar graph with $n \geq 3$ vertices. If $\nabla(G) \geq \frac{n}{2}$, then G has a K_4-minor.*

3. The Proof of Theorem 4

In the section we will prove Theorem 4.

Lemma 6. *Let G be a planar graph of order n with $\Delta(G) \leq 4$. If $\delta(G) \leq 2$, then $\nabla(G) \leq \frac{n-1}{2}$.*

Proof. Let G be a minimal counterexample with respect to the number of vertices. Let x be a vertex of degree at most two in G. If $d_G(x) = 1$, then $\nabla(G) = \nabla(G-x)$. Since $\Delta(G-x) \leq 4$, we have that $\nabla(G-x) \leq \frac{n-1}{2}$ by Theorem 3. So $\nabla(G) \leq \frac{n-1}{2}$, a contradiction. If $d_G(x) = 2$, let $N(x) = \{x_1, x_2\}$. If x_1 is not adjacent to x_2, then the path x_1xx_2 is replaced with x_1x_2. Let G' be the obtained graph. Then $\nabla(G) = \nabla(G') \leq \frac{n-1}{2}$ by Theorem 3, a contradiction. If x_1 is adjacent to x_2, then we delete x_1, then x. Let G'' be the obtained graph in which the degree of x_2 is at most two. Then $\nabla(G'') \leq \frac{n-2}{2}$ by Theorem 3. If $\nabla(G'') = \frac{n-2}{2}$, then it violates the minimality of G. If $\nabla(G'') \leq \frac{n-3}{2}$, then $\nabla(G) \leq \nabla(G'')+1 \leq \frac{n-1}{2}$, a contradiction. So $\nabla(G) \leq \frac{n-1}{2}$. ■

Theorem 7. *Let G be a planar graph of order $n \geq 4$ with $\Delta(G) \leq 4$. If G is not a 4-regular graph and if $\nabla(G) = \frac{n}{2}$, then G is covered by K_4-subgraphs.*

Proof. Suppose that G is a minimal counterexample with respect to the number of vertices. We now suppose that G has been embedded in the plane.

Claim 1. *G has not any cut-edge.*

Proof. Otherwise, let $e = uv$ be a cut-edge in G. Suppose that e is an edge in a component G'_1 of G. Let $G'_{1,1}$ and $G'_{1,2}$ be two components of $G'_1 - e$. Let $G_1 = G'_{1,1}$, and let G_2 be the union of all other components in $G - e$. Then $\nabla(G_1) \leq \frac{|V(G_1)|}{2}$ and $\nabla(G_2) \leq \frac{|V(G_2)|}{2}$ by Theorem 3. If one of G_1 and G_2, say G_1, is such that $\nabla(G_1) < \frac{|V(G_1)|}{2}$, then $\nabla(G) = \nabla(G_1) + \nabla(G_2) < \frac{n}{2}$, a contradiction. So $\nabla(G_1) = \frac{|V(G_1)|}{2}$ and $\nabla(G_2) = \frac{|V(G_2)|}{2}$. Thus both G_1 and G_2 are covered by K_4-subgraphs. Hence G is covered by K_4-subgraphs, a contradiction. □

We now consider $\delta(G)$. If $\delta(G) \leq 2$, then $\nabla(G) \leq \frac{n-1}{2}$ by Lemma 6, a contradiction. So $\delta(G) \geq 3$. Since G is not a 4-regular graph, we have that $\delta(G) = 3$. Let x be a vertex of degree three in G, and let $N(x) = \{x_1, x_2, x_3\}$. Let Q_1 be the subgraph of G induced by x_1, x_2 and x_3.

If Q_1 is a cycle, then the subgraph of G induced by vertices in $V(Q_1) \cup \{x\}$ is isomorphic to K_4. We now delete x_1 and x_2 from G. Then x is of degree one and x_3 is of degree at most two in the present graph. Next, x is deleted, then x_3. Let G' be the obtained graph, which has $n - 4$ vertices. So $\nabla(G') \leq \frac{n-4}{2}$ by Theorem 3. If $\nabla(G') < \frac{n-4}{2}$, then $\nabla(G) \leq \nabla(G') + 2 < \frac{n}{2}$, a contradiction. If $\nabla(G') = \frac{n-4}{2}$, then G' is covered by K_4-subgraphs. Hence G is covered by K_4-subgraphs, a contradiction.

If Q_1 is not a cycle, then there are two vertices in $N(x)$, say x_1 and x_2, such that they are not adjacent to each other in G.

Claim 2. *x_1 is not any vertex in any K_4-subgraph of G.*

Proof. Otherwise, suppose that x_1 is in some K_4-subgraph Q_2 of G. Let $V(Q_2) = \{x_1, y_1, y_2, y_3\}$. Obviously, x is not in $V(Q_2)$. Otherwise, both x_2 and x_3 are in $V(Q_2)$, a contradiction.

If there is some vertex in $\{y_1, y_2, y_3\}$, say y_1, such that its degree is three in G, then we delete y_2 and y_3 from G. So y_1 is of degree one and x_1 is of degree at most two in the present graph. Next, y_1 is deleted, then x_1. Let H_1 be the obtained graph in which the degree of x is two. So $\nabla(H_1) \leq \frac{n-5}{2}$ by Lemma 6. Since $\nabla(G) \leq \nabla(H_1) + 2$, we have that $\nabla(G) \leq \frac{n-1}{2}$, a contradiction. Hence the degree of each of y_1, y_2, and y_3 is four in G.

Let y'_i be the neighbor of y_i which is not in $V(Q_2)$ for $i = 1, 2, 3$. Since K_4 is 3-connected planar graph, it has a unique embedding in the plane, say Π. Without loss of generality, suppose that $C = x_1y_1y_2x_1$ is the boundary of the outer face of Π. We claim that y'_1, y'_2 and y'_3 are not in the exterior of the cycle C. If not, then the edge $y_3y'_3$ crosses some edge of C by Jordan's curve theorem, since y_3 is not in the exterior of C. Thus there is a contradiction. Similarly, y'_1, y'_2 and y'_3 are not in the interior of the same inner face of Π. Without loss of generality, suppose that y'_1 and y'_2 are in the interior of two distinct inner faces of Π. So y'_1 is not adjacent to y'_2. We now delete x_1 and y_3 from G. Then the degree of y_i is two in the present graph for $i = 1, 2$. The path $y'_1y_1y_2y'_2$ is now replaced with $y'_1y'_2$. Let H_2 be the obtained graph in which the degree of x is two. So $\nabla(H_2) \leq \frac{n-5}{2}$ by Lemma 6. Since $\nabla(G) \leq \nabla(H_2) + 2$, we have that $\nabla(G) \leq \frac{n-1}{2}$, a contradiction. □

Similarly, we have the following claim.

Claim 3. *x_2 is not any vertex in any K_4-subgraph of G.*

We now delete x_3 from G. Then the degree of x is two in the present graph. Next, the path x_1xx_2 is replaced with the edge x_1x_2. Let H_3 be the obtained graph, whose maximum degree is at most four. By Theorem 3, $\nabla(H_3) \leq \frac{n-2}{2}$. If $\nabla(H_3) < \frac{n-2}{2}$, then $\nabla(G) \leq \nabla(H_3) + 1 < \frac{n}{2}$, a contradiction. If $\nabla(H_3) = \frac{n-2}{2}$, then it is covered by K_4-subgraphs. Hence x_i must be a vertex in some K_4-subgraph, say R_i, for $i = 1, 2$. Since the maximum degree of G is at most four, x_1x_2 can not be a common edge of R_1 and R_2. If R_1 and R_2 are two different subgraphs, then one of R_1 and R_2, say R_1, does not contain the edge x_1x_2. Thus R_1 is a K_4-subgraph of G which contains x_1, which violates Claim 2. So both x_1 and x_2 are two vertices in the same K_4-subgraph, say Q_3, in H_3. Suppose that $V(Q_3) = \{x_1, x_2, z_1, z_2\}$.

If the degree of one of z_1 and z_2, say z_1, is three, then we delete x_1 and x_2 instead of x_3 from G. Then the degree of z_1 is one and the degree of z_2 is at most two in the present graph. Next, z_1 is deleted, then z_2. Let H_4 be the obtained graph in which the degree of x is one. So $\nabla(H_4) \leq \frac{n-5}{2}$ by Lemma 6. Since $\nabla(G) \leq \nabla(H_4)+2$, we have that $\nabla(G) \leq \frac{n-1}{2}$, a contradiction. Hence each of z_1 and z_2 is of degree four in G. Suppose that w_i is the neighbor of z_i which is not in $V(Q_3)$ for $i = 1, 2$.

It is clear that x_3 is neither z_1 nor z_2 in G. We observe that x_3, w_1 and w_2 are not the same vertex. Otherwise, the subgraph of G induced by $\{x, x_1, x_2, x_3\} \cup \{z_1, z_2, w_1, w_2\}$ has a $K_{3,3}$-minor, a contradiction. Similarly, x_3 is not adjacent to both w_1 and w_2, or x_3 is one of w_1 and w_2 but x_3 is adjacent to the other vertex. So x_3 is not adjacent to one of w_1 and w_2. We now suppose that x_3 is not adjacent to w_2 in G.

Claim 4. *The degree of each of x_1 and x_2 is four in G.*

Proof. Without loss of generality, suppose on the contrary that the degree of x_1 is three in G. We now delete x_2 and z_1 instead of x_3 from G. Then the degree of each of x, x_1 and z_2 is two in the present graph. Next, the path $x_3xx_1z_2w_2$ is replaced with x_3w_2. Let H_5 be the obtained graph. Then $\nabla(H_5) \leq \frac{n-5}{2}$ by Lemma 6. Since $\nabla(G) \leq \nabla(H_5) + 2$, we have that $\nabla(G) \leq \frac{n-1}{2}$, a contradiction. □

Suppose that the neighbor of x_i which is not in $\{x\} \cup V(Q_3)$ is u_i in G for $i = 1, 2$. A local structure of G is shown in Figure 1. Need to say that several vertices in u_1, u_2, w_1, w_2 may be the same vertex. Let F be the subgraph of G induced by the vertices x, x_1, x_2, z_1, z_2. Suppose that F is a subgraph of a component of G, say B. Then x_3, u_1, u_2, w_1 and w_2 are vertices of B. Let F' be the graph obtained from B by deleting all vertices in F. We claim that F' is not connected (a graph with one vertex is viewed as a connected one). Otherwise, all vertices in F' are contracted into a vertex. Then G has a K_5-minor, since F is isomorphic to a subdivision of K_4. Thus there is a contradiction. If F' has at least three components, then there is a component which contains only one vertex in x_3, u_1, u_2, w_1 and w_2, say w_1. In this case, the edge z_1w_1 is a cut-edge of G, which violates Claim 1. So F' has exactly two components, say F'_1 and F'_2.

If w_1 and w_2 are in $V(F'_1)$ and $V(F'_2)$, respectively, then we delete both x_1 and x_2 instead of x_3 from G. Let H_6 be the obtained graph in which the degree of x is one and the degree of each of z_1 and z_2 is two. The path $w_1z_1z_2w_2$ is now replaced with the edge w_1w_2. Let H'_6 be the obtained graph. Then $\nabla(H'_6) \leq \frac{n-5}{2}$ by Lemma 6. So $\nabla(G) \leq \nabla(H'_6) + 2 \leq \frac{n-1}{2}$, a contradiction. So both w_1 and w_2 are in one of $V(F'_1)$ and $V(F'_2)$, say $V(F'_1)$. In this case, there are at least two vertices in $\{x_3, u_1, u_2\}$ which are in $V(F'_2)$. We claim that x_3 must be in $V(F'_2)$. Otherwise, x_3 is in $V(F'_1)$. So the graph $F'_1 \cup F$ has a minor isomorphic to $K_{3,3}$ if all vertices in $V(F'_1)$ are contracted into a vertex, a contradiction.

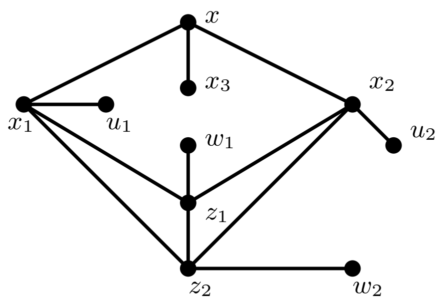

Figure 1. A local structure of G.

If u_1 is in $V(F'_1)$, then it is not adjacent to x_3. We now delete x_2 instead of x_3 from G. Let H_7 be the obtained graph in which the degree of x is two. Let H'_7 be the graph obtained from H_7 by replacing the path x_3xx_1 with the edge x_3x_1. Then $\nabla(H'_7) \leq \frac{n-2}{2}$ by Theorem 3. If $\nabla(H'_7) \leq \frac{n-3}{2}$, then $\nabla(G) \leq \nabla(H_7)+2 \leq \frac{n-1}{2}$, a contradiction. So $\nabla(H'_7) = \frac{n-2}{2}$. Considering that the degree of z_1 is three in H'_7 and the order of H'_7 is less than that of G, we have that H'_7 is covered by K_4-subgraphs. Thus x_1 is a vertex in some K_4-subgraph. Hence there are three vertices in $N(x_1)$ such that they are adjacent to each other. Since $N(x_1) = \{x_3, u_1, z_1, z_2\}$ and x_3 is not adjacent to u_1, we have that either u_1, z_1, z_2 are adjacent to each other or x_3, z_1, z_2 are adjacent to each other. If the former occurs, then x_1 is a vertex in some K_4-subgraph in G, which violates Claim 2. If the latter occurs, then G has a subgraph isomorphic to $K_{3,3}$, whose vertex set is $\{x, z_1, z_2\} \cup \{x_1, x_2, x_3\}$, a contradiction. So u_1 is not in $V(F'_1)$. Similarly, u_2 is not in $V(F'_1)$. In other words, both u_1 and u_2 are in $V(F'_2)$. Thus u_1 is not adjacent to w_1 in G.

Claim 5. *x_3, u_1 and u_2 are not the same vertex in G.*

Proof. Otherwise, we delete x_1 and x_2 instead of x_3 from G. So the degree of x is one, the degree of x_3 is at most two, and degree of each of z_1 and z_2 is two in the present graph. Next, x is deleted, then x_3. Let H_8 be the obtained graph whose minimum degree is at most two. Then $\nabla(H_8) \leq \frac{n-5}{2}$ by Lemma 6. Since $\nabla(G) \leq \nabla(H_8) + 2$, we have that $\nabla(G) \leq \frac{n-1}{2}$, a contradiction. □

By Claim 5, without loss of generality, suppose that x_3 and u_1 are not the same vertex in G. We now delete the two vertices x_2 and z_2 instead of x_3 from G. Then the degree of each of x and z_1 is two in the present graph. Next, the path x_1xx_3 is replaced with x_1x_3, and the path $x_1z_1w_1$ is replaced with x_1w_1. Let H_9 be the obtained graph whose maximum degree is at most four. If $\nabla(H_9) \leq \frac{n-5}{2}$, then $\nabla(G) \leq \frac{n-1}{2}$, a contradiction. If $\nabla(H_9) = \frac{n-4}{2}$, then H_9 is covered by K_4-subgraphs. In this case, x_1 is a vertex in a unique K_4-subgraph, say Q_4. So Q_4 contains x_1, x_3, w_1 and u_1. Hence u_1 must be adjacent to w_1 in G, a contradiction. Thus, the proof is completed. ■

Theorem 8. *Let G be a planar graph of order $n \geq 4$. If G is covered by K_4-subgraphs and the outer degree of each K_4-subgraph is at most five, then $\nabla(G) = \frac{n}{2}$.*

Proof. Suppose that G is covered by K_4-subgraphs $Q_1, Q_2, \ldots, Q_k$. So $n = 4k$. Since $\nabla(K_4) = 2$, we have that $\nabla(G) \geq 2k$, i.e., $\nabla(G) \geq \frac{n}{2}$. We can suppose that G is connected. Otherwise, each component of G is considered in the similar way.

We now use the induction on k to show that $\nabla(G) \leq 2k$. If $k = 1$, then G is exactly K_4. So $\nabla(G) = 2$. Assume that the inequality is true for $k \leq l - 1$, where $l \geq 2$. We now consider the case that $k = l$. For $i = 1, 2, \ldots, l$, suppose that $V(Q_i) = \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}\}$.

Suppose that G has been embedded in the sphere. We claim that there is some K_4-subgraph Q_j which contains some 3-cycle C such that any vertex in $V(Q_j) - V(C)$ is not adjacent to any vertex in $V(G) - V(Q_j)$. In fact, Q_i is a 3-connected planar graph for $i = 1, 2, \ldots, l$. Then Q_i has a unique embedding in the sphere in which each facial cycle is a 3-cycle. We consider Q_1 at first. If Q_1 has a facial cycle such that all vertices in $V(G) - V(Q_1)$ are in the interior of the cycle, then it is the desired. Otherwise, there are two facial cycles in Q_1, say C_1 and C_2, such that at least one vertex in $V(G) - V(Q_1)$ is adjacent to some vertex of C_i for $i = 1, 2$. We now select one of C_1 and C_2, say C_1. We observe that if some vertex of Q_s $(s \neq 1)$ is in the interior of C_1, then all other vertices in Q_s are in the interior of C_1. Otherwise, there is at least one edge-crossing, a contradiction. Without loss of generality, suppose that all vertices of Q_2 are in the interior of C_1. Next, we argue Q_2 in a similar way used for Q_1, and so on. Since l is a finite number, there is some j in $\{1, 2, \ldots, l\}$ such that Q_j has some cycle C satisfying the desired condition. Suppose that $C = v_{j,1}v_{j,2}v_{j,3}v_{j,1}$.

Since the outer degree of Q_j is at most five, there is some vertex in $V(C)$, say $v_{j,3}$, such that its outer degree is at most one. We now delete $v_{j,1}$ and $v_{j,2}$ from G. Then the degree of $v_{j,4}$ is one and the degree of $v_{j,3}$ is at most two in the present graph. Next, $v_{j,4}$ is deleted, then $v_{j,3}$. Let G' be the obtained graph. Then G' is covered by $(l-1)$ K_4-subgraphs and $\nabla(G) \leq \nabla(G') + 2$. By the inductional assumption, $\nabla(G') \leq 2l - 2$. So $\nabla(G) \leq 2l$. Hence the proof is completed. ■

***Proof of Theorem* 4.** The theorem follows from Theorem 7 and Theorem 8 directly. ■

4. Planar Graphs with Minimum Degree at Least Four

The decycling number of a planar graph with minimum degree at least four will be studied in the section. Let us start with a definition. A regular polyhedron is a convex one which satisfies the following conditions: (1) the polygons are congruent ones, and (2) each vertex is incident with the same number of polygons. It has been known that there are exactly five regular polyhedra which contains the tetrahedron, the hexahedron, the octahedron, the dodecahedron, and the icosahedron. (One can refer to [14] for the proof.) Let O_6 and I_{12} denote the octahedron and the icosahedron, respectively, which are shown in Figures 2 and 3, respectively.

Let G be a planar graph with $\delta(G) \geq 4$. Then G has at least five vertices. If G has exactly five vertices, then it is the complete graph K_5 which is not a planar graph. So G has at least six vertices. If G has six vertices and $\delta(G) \geq 4$, we have a result below.

Theorem 9. *If G is a planar graph with six vertices and $\delta(G) \geq 4$, then G is the graph O_6.*

Proof. We firstly claim that each vertex is of degree four in G . Otherwise, let x be a vertex in G of degree five. Then there is another vertex y of degree five in G. Otherwise, G has only one vertex of degree five and any other vertex has degree four, a contradiction. Since G has six vertices, x is adjacent to y. We now delete x and y from G. Let G' be the obtained graph. Then any vertex in G' is of degree at least two. So G' has a 4-cycle. Considering that each of x and y is adjacent to each vertex in G', G has a minor isomorphic to K_5, a contradiction.

Since the degree of each vertex in G is four, there are two vertices, say v_1 and v_5, in G, such that v_1 is not adjacent to v_5. Let H be the graph obtained from G by deleting v_1 and v_5 from G. Then each vertex in H is of degree two. So H is a 4-cycle. Since each of v_1 and v_5 is adjacent to each vertex in H, G is the graph shown in Figure 2, which is exactly O_6. ■

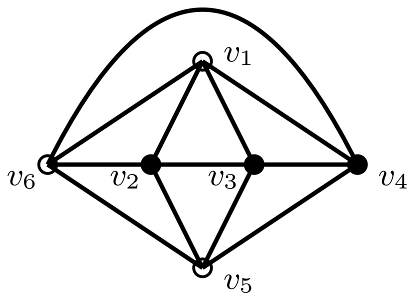

Figure 2. The graph O_6 (the vertices depicted by solid circles form a decycling set).

Theorem 10. $\nabla(O_6) = 3$.

Proof. It is not hard to see that $\nabla(O_6) \leq 3$ (refer to Figure 2). If $\nabla(O_6) = 2$, let S be a decycling set of O_6 with two vertices. Since each vertex is of degree four in O_6, we have that each vertex is of degree at least two in $O_6 - S$. So $O_6 - S$ contains a cycle, a contradiction. Hence $\nabla(O_6) \geq 3$. Thus $\nabla(O_6) = 3$. ■

Theorem 11. *Let G be a planar graph with $n \geq 6$ vertices. If G is covered by O_6-subgraphs and the outer degree of each O_6-subgraph is at most five, then* $\nabla(G) = \frac{n}{2}$.

Proof. Suppose that G is covered by k O_6-subgraphs. So $n = 6k$. By Theorem 10, $\nabla(O_6) = 3$. So $\nabla(G) \geq 3k = \frac{n}{2}$.

We suppose that G has been embedded in the sphere. We use the induction on k to show that $\nabla(G) \leq 3k$. If $k = 1$, then G is exactly O_6. So the inequality holds by Theorem 10. Assume that the inequality is true for $k \leq l-1$, where $l \geq 2$. We now consider the case that $k = l$. Since O_6 is a 3-connected planar graph, it has a unique embedding in the sphere. By a similar argument to that in the proof of Theorem 8, there is some O_6-subgraph, denoted by Q, which contains some 3-cycle C such that any vertex in $V(Q) - V(C)$ is not adjacent to any vertex in $V(G) - V(Q)$. Suppose that the vertex set of Q is $\{v_i | i = 1, 2, 3, 4, 5, 6\}$. Without loss of generality, suppose the cycle $C = v_1v_2v_3v_1$ (one can refers to Figure 2).

Since the outer degree of Q is at most five, there is some vertex in C, say v_1, such that its outer degree is at most one. We now delete v_2, v_3 and v_6 from G.

Let H be the obtained graph. Then the degree of v_5 is one in H, and the degrees of v_4 and v_1 are two in H, respectively. We now delete v_5, v_4, and v_1 in this order from H. Let H' be the obtained graph. Then H' is covered by $l-1$ O_6-graphs, and $\nabla(G) \leq \nabla(H') + 3$. By the inductional assumption, $\nabla(H') \leq 3(l-1)$. So $\nabla(G) \leq 3l$. Thus $\nabla(G) \leq \frac{n}{2}$. Hence $\nabla(G) = \frac{n}{2}$. ∎

Next, we consider the decycling number of a planar graph with minimum degree five. Let G be a planar graph with minimum degree five. Then G is not covered by K_4-subgraphs or O_6-subgraphs. We ask that whether there is a planar graph of order n with minimum degree five such that its decycling number is $\frac{n}{2}$. The following result gives a positive answer.

Theorem 12. $\nabla(I_{12}) = 6$.

Proof. It is easy to see that the set of vertices depicted by solid circle in Figure 3 is a decycling set of I_{12}. So $\nabla(I_{12}) \leq 6$.

We firstly claim that $\nabla(I_{12}) \geq 5$. Otherwise, let S_1 be a decycling set of I_{12} with four vertices. Let H_1 be the subgraph of I_{12} induced by all vertices in $V(I_{12}) - S_1$. Then H_1 contains eight vertices. Considering that the deletion of any vertex in S_1 destroys at most five edges and I_{12} has 30 edges, H_1 contains at least ten edges. So H_1 has a cycle, a contradiction.

We now suppose that $\nabla(I_{12}) = 5$. Without loss of generality, suppose that $S_2 = \{v_1, v_2, v_3, v_4, v_5\}$ is a decycling set of I_{12} with five vertices. Let H_2 be the subgraph of I_{12} induced by all vertices in $V(I_{12}) - S_2$. Then H_2 contains seven vertices, and it has at most six edges. In this case the deletion of all vertices in S_2 must destroy at least 24 edges of G. Thus the subgraph of I_{12} induced by S_2 has at most one edge. In other words, I_{12} has an independent set of four vertices. It can be checked that any independent set of I_{12} has at most three vertices, a contradiction. So $\nabla(I_{12}) \geq 6$. Thus $\nabla(I_{12}) = 6$. ∎

Theorem 13. *Let G be a planar graph with $n \geq 12$ vertices. If G is covered by I_{12}-subgraphs and the outer degree of each I_{12}-subgraph is at most five, then $\nabla(G) = \frac{n}{2}$.*

Proof. Suppose that G is covered by k I_{12}-subgraphs. So $n = 12k$. By Theorem 12, $\nabla(I_{12}) = 6$. So $\nabla(G) \geq 6k = \frac{n}{2}$.

We suppose that G has been embedded in the sphere. We use the induction on k to show that $\nabla(G) \leq 6k$. If $k = 1$, then G is exactly I_{12}. So the inequality holds by Theorem 12. Assume that the inequality is true for $k \leq l-1$, where $l \geq 2$. We now consider the case that $k = l$. Since I_{12} is a 3-connected planar graph, it has a unique embedding in the sphere. By a similar argument to that in the proof of Theorem 8, there is some I_{12}-subgraph, denoted by Q, which contains some 3-cycle C such that any vertex in $V(Q) - V(C)$ is not adjacent to any vertex

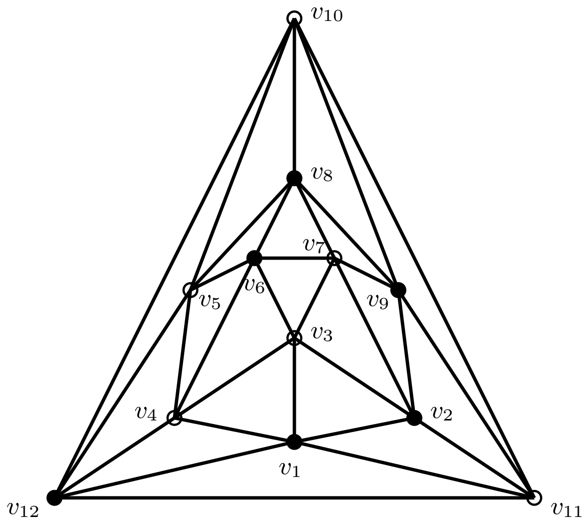

Figure 3. The graph I_{12} (the vertices depicted by solid circles form a decycling set).

in $V(G)-V(Q)$. We suppose that the vertex set of Q is $\{v_i|i=1,2,\ldots,12\}$. See Figure 3. Since I_{12} is a regular polyhedron, the faces have symmetry. Without loss of generality, suppose that the cycle $C=v_1v_2v_3v_1$.

Let $S_1=\{v_1,v_2,v_6,v_8,v_9,v_{12}\}$, $S_2=\{v_2,v_3,v_6,v_9,v_{10},v_{12}\}$, and $S_3=\{v_1,v_3,v_5,v_6,v_9,v_{11}\}$. For $i=1,2,3$, we observe that S_i is a decycling set of Q with six vertices, and that the graph obtained from Q by deleting all vertices in S_i is a path, say P_i. Note that $P_1=v_7v_3v_4v_5v_{10}v_{11}$, $P_2=v_{11}v_1v_4v_5v_8v_7$, and $P_3=v_2v_7v_8v_{10}v_{12}v_4$.

Since the outer degree of Q is at most five, there is a vertex y in C such that its outer degree is at most one. If y is the vertex v_3, then we delete all vertices in S_1. Next, we firstly delete all vertices in $V(P_1)-\{v_3\}$, then x_3. Let H_1 be the obtained graph, which is covered by $l-1$ I_{12}-subgraphs. By the inductional assumption, $\nabla(H_1)\leq 6(l-1)$. So $\nabla(G)\leq 6l$. Thus $\nabla(G)\leq \frac{n}{2}$. If y is the vertex v_1 (or v_2), then we delete all vertices in S_2 (or S_3). Subsequently, we delete all vertices in $V(P_1)-\{v_1\}$ (or $V(P_1)-\{v_2\}$), then v_1 (or v_2). Next, we proceed a similar argument to that for v_3. Then $\nabla(G)\leq\frac{n}{2}$. Hence $\nabla(G)=\frac{n}{2}$. ∎

Using the methods in the proof of Theorems 8, 11 and 13, it is not hard to show the following result.

Theorem 14. *Let G be a planar graph with $n\geq 12$ vertices. If G is covered by K_4-subgraphs, O_6-subgraphs, or I_{12}-subgraphs such that the outer degree of each of K_4-subgraph, O_6-subgraph, and I_{12}-subgraph is at most five, then $\nabla(G)=\frac{n}{2}$.*

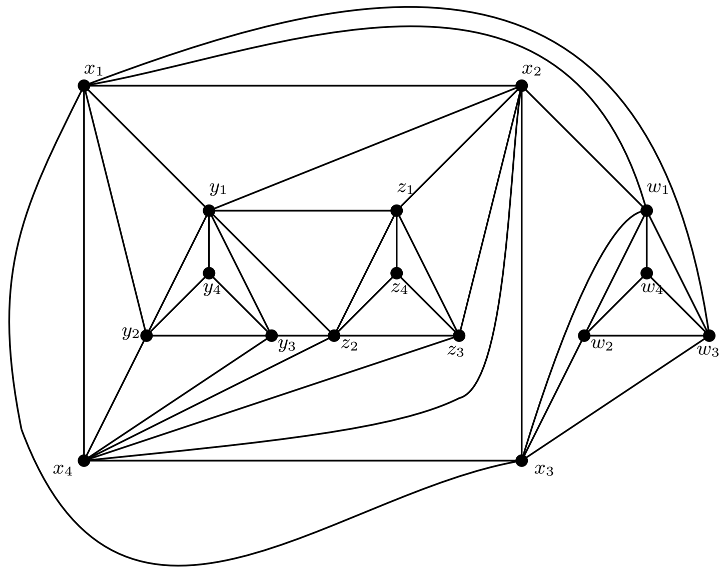

Figure 4. A graph covered by K_4-subgraphs.

If a planar graph G of order n is covered by K_4-subgraphs, O_6-subgraphs, or I_{12}-subgraphs, then the decycling number of G is at least $\frac{n}{2}$ by the facts that $\nabla(K_4) = 2$, $\nabla(O_6) = 3$, and $\nabla(I_{12}) = 6$. In order to determine $\nabla(G) = \frac{n}{2}$, one needs to show that $\nabla(G) \leq \frac{n}{2}$. We can solve it if the outer degree of each of K_4-subgraph, O_6-subgraph, and I_{12}-subgraph is restricted to be at most five. If the condition on the outer degree is removed, it is not easy to solve it. For example, the graph H shown in Figure 4 is covered by K_4-subgraphs. Note that the subgraph induced by x_1, x_2, x_3, and x_4 is isomorphic to K_4. We observe that any decycling set of H must contain two vertices in $\{x_1, x_2, x_3, x_4\}$, but we have not a general method to select them. We think that the condition on the outer degree in Theorem 14 can be removed. So we propose the following conjecture.

Conjecture 15. *Let G be a planar graph with $n \geq 12$ vertices which is covered by K_4-subgraphs, O_6-subgraphs, or I_{12}-subgraphs. Then $\nabla(G) = \frac{n}{2}$.*

Acknowledgements

The authors thank the referees for careful reading of the manuscript and their helpful suggestions.

The third author is supported by NNSFC under the granted number 11171114 and Science and Technology Commission of Shanghai Municipality (STCSM 13dz2260400)

References

[1] M.O. Albertson and D.M Berman, *The acyclic chromatic number*, Proceedings of the Seventh Southeastern Conference on Combinatorics, Graph Theory and Computing, Congr. Numer. **XVII** (1976) 51–69.

[2] L.W. Beineke and R.C. Vandell, *Decycling graphs*, J. Graph Theory **25** (1997) 59–77.
https://doi.org/10.1002/(SICI)1097-0118(199705)25:1¡59::AID-JGT4¿3.0.CO;2-H

[3] J.A. Bondy and U.S.R.Murty, Graph Theory (Springer, 2008).

[4] O.V. Borodin, *A proof of Grüngaum's conjecture on the acyclic* 5*-colorability of planar graphs*, Dokl. Akad. Nauk SSSR **231** (1976) 18–20, in Russian.

[5] P. Erdős, M. Saks and V. Sós, *Maximum induced trees in graphs*, J. Combin. Theory Ser. B **41** (1986) 61–79.
https://doi.org/10.1016/0095-8956(86)90028-6

[6] P. Festa, P.M. Pardalos and M.G.C. Reseude, *Feedback set problem*, in: Handbook of Combinatorial Optimization, Supplement Vol. A, (Kluwer, Dordrecht, 1999) 209–258.

[7] K. Hosono, *Induced forests in trees and outerplanar graphs*, Proc. Fac. Sci. Tokai Univ. **25** (1990) 27–29.

[8] R.M. Karp, *Reducibility among combinatorial problems*, in: Complexity of Computer Computations, R.E. Miller, J.W. Thatcher, J.D. Bohlinger (Ed(s)), (The IBM Research Symposia Series Springer, Boston, MA, 1972) 85–103.
https://doi.org/10.1007/978-1-4684-2001-2_9

[9] J.P. Liu and C. Zhao, *A new bound on the feedback vertex sets in cubic graphs*, Discrete Math. **184** (1996) 119–131.
https://doi.org/10.1016/0012-365X(94)00268-N

[10] S.D. Long and H. Ren, *The decycling number and maximum genus of cubic graphs*, J. Graph Theory **88** (2018) 375–384.
https://doi.org/10.1002/jgt.22218

[11] D.A. Pike and Y. Zou, *Decycling Cartesian products of two cycles*, SIAM J. Discrete Math. **19** (2005) 651–663.
https://doi.org/10.1137/S089548010444016X

[12] N. Punnim, *The decycling number of regular graphs*, Thai J. Math. **4** (2006) 145–161.

[13] H. Ren, C. Yang and T.-X. Zhao, *A new formula for the decycling number of regular graphs*, Discrete Math. **340** (2017) 3020–3031.
https://doi.org/10.1016/j.disc.2017.07.011

[14] A.T. White, Graphs, Groups and Surfaces (North-Holland, 1973).

[15] H. Whitney, *Two-isomorphic graphs*, Trans. Amer. Math. Soc. **34** (1932) 339–362.
https://doi.org/10.1090/S0002-9947-1932-1501641-2

Received 15 July 2020
Revised 10 April 2022
Accepted 10 April 2022
Available online 4 May 2022