

## THE DECYCLING NUMBER OF A PLANAR GRAPH COVERED BY $K_4$ -SUBGRAPHS

DENGJU MA

*School of Sciences*  
*Nantong University*  
*Jiangsu Province, China, 226019*

**e-mail:** madengju@ntu.edu.cn

MINGYUAN MA

AND

HAN REN

*Department of Mathematics*  
*East China Normal University*  
*Shanghai, China, 200062*

**e-mail:** ming623@163.com  
hren@math.ecnu.edu.cn

### Abstract

Let  $G$  be a planar graph of  $n$  vertices. The paper shows that the decycling number of  $G$  is at most  $\frac{n-1}{2}$  if  $G$  has not any  $K_4$ -minor. If the maximum degree of  $G$  is at most four and  $G$  is not 4-regular, the paper proves that the decycling number of  $G$  is  $\frac{n}{2}$  if and only if  $G$  is covered by  $K_4$ -subgraphs. In addition, the decycling number of  $G$  covered by octahedron-subgraphs or icosahedron-subgraphs is studied.

**Keywords:** decycling number, planar graph,  $K_4$ -minor.

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### 1. INTRODUCTION

All graphs in the paper are simple. Let  $G$  be a graph. A subset  $S$  of  $V(G)$  is said to be a *decycling set* of  $G$  if  $G - S$  is acyclic. The cardinality of a minimum decycling set of  $G$  is said to be the *decycling number* of  $G$ , which is denoted by

$\nabla(G)$ . A decycling set is also called a feedback vertex set. It has application in areas such as circuit design and deadlock prevention [6].

Determining the decycling number of a graph is not an easy work. It has been proved that the problem of determining the decycling number of a graph is NP-hard [8]. So an upper bound for the decycling number of a graph, especially a planar graph, attracts people's attention. The decycling number of every planar graph of  $n$  vertices is at most  $\frac{3n}{5}$ , which follows from the result that planar graphs are acyclically 5-colorable shown by Borodin [4]. For an outerplanar graph of  $n$  vertices, Hosono [7] proved that its decycling number is at most  $\frac{n}{3}$ . However, the following conjecture is challenging, which is still open now.

**Conjecture 1** [1, 5]. *If  $G$  is a planar graph of  $n$  vertices, then  $\nabla(G) \leq \frac{n}{2}$ .*

In the paper we firstly study the decycling number of a planar graph without  $K_4$ -minor, and then we obtain the following result.

**Theorem 2.** *If  $G$  is a planar graph of order  $n$  without  $K_4$ -minor, then  $\nabla(G) \leq \frac{n-1}{2}$ .*

For a graph  $H$  of  $n$  vertices with maximum degree at most four, Ren *et al.* [13] showed that  $\nabla(H) \leq \frac{n+1}{2}$  if  $H$  is 4-regular, or  $\nabla(G) \leq \frac{n}{2}$ , otherwise. Punnim [12] proved the following result.

**Theorem 3** [12]. *Let  $H$  be a graph of  $n$  vertices with maximum degree at most four. If  $H$  is  $K_5$ -free graph, then  $\nabla(H) \leq \frac{n}{2}$ .*

By Theorem 3, the decycling number of a planar graph  $G$  of  $n$  vertices with maximum degree at most four is at most  $\frac{n}{2}$ . What is the structure of  $G$  if  $\nabla(G) = \frac{n}{2}$ ? We will show the theorem below in the paper.

**Theorem 4.** *Let  $G$  be a planar graph of  $n$  vertices with maximum degree at most four which is not 4-regular. Then  $\nabla(G) = \frac{n}{2}$  if and only if  $G$  is covered by  $K_4$ -subgraphs.*

The arrangement of the paper is as follows. Section 2 gives the proof of Theorem 2. In Section 3 we mainly study the decycling number of a planar graph with maximum degree at most four covered by  $K_4$ -subgraphs. In the end of the section Theorem 4 is proved. In Section 4 we show that the decycling number of a planar graph  $G$  of order  $n$  is  $\frac{n}{2}$  if it is covered by  $O_6$ -subgraphs, or  $I_{12}$ -subgraphs such that the outer degree of each of  $O_6$  and  $I_{12}$  is at most five, where  $O_6$  and  $I_{12}$  denote the octahedron and the icosahedron, respectively.

The remainder of the section is contributed for some terminologies of graphs. The other undefined can be found in [3].

The complete graph of  $n$  vertices is denoted by  $K_n$ . Let  $G$  be a graph. The number of vertices of  $G$  is called its *order*. The maximum degree and minimum

degree of  $G$  are denoted by  $\Delta(G)$  and  $\delta(G)$ , respectively. For a vertex  $v$  in  $G$ , a vertex being adjacent to  $v$  in  $G$  is called a *neighbor* of  $v$ . The set of all neighbors of  $v$  is denoted by  $N(v)$ . Let  $X$  be a vertex subset of  $G$  with  $k$  elements. If  $G - X$  has more components than that in  $G$ , then  $X$  is called a *k-vertex cut* of  $G$ . If  $X$  contains only one vertex, say  $v$ , then we call  $v$  a *cut-vertex* of  $G$ . An edge  $e$  of  $G$  is called a *cut-edge* if  $G - e$  has more components than that in  $G$ . A connected graph  $H$  is said to be *k-connected* if the order of  $H$  is at least  $k + 1$  and  $H - Y$  is connected for every  $Y \subseteq V(G)$  with at most  $k - 1$  elements.

Let  $F$  and  $F'$  be two graphs. If  $F'$  is obtained from a subgraph of  $F$  by contracting several edges, then  $F'$  is called a *minor* of  $F$ , or we say  $F$  has an *F'-minor*. Let  $Q$  be a graph other than  $F$  and  $F'$ . If a subgraph of  $F$  is isomorphic to  $Q$ , then we say that  $F$  has a *Q-subgraph*. A *k-tree* is either the complete graph  $K_{k+1}$  or a graph obtained from a *k-tree*  $G$  by adding one vertex such that it joins to  $k$  pairwise adjacent vertices in  $G$ . The *tree-width* of a graph is the minimum  $k$  such that the graph is a subgraph of a *k-tree*. Obviously, every graph with tree-width  $k$  has a vertex of degree at most  $k$ .

Let  $H$  be a graph. For  $i = 1, 2, \dots, k$ , let  $V_i$  be a subset of  $V(H)$ , and let  $H_i$  be the subgraph of  $H$  induced by  $V_i$ . If  $V_1, V_2, \dots, V_k$  are pairwise disjoint sets such that  $\cup_{i=1}^k V_i = V(H)$ , then  $H$  is said to be *covered* by  $H_1$ -subgraph,  $H_2$ -subgraph,  $\dots$ ,  $H_k$ -subgraph. In particular, if  $H_i$  is isomorphic to the graph  $R$  for  $i = 1, 2, \dots, k$ , then we say that  $H$  is covered by  $R$ -subgraphs. For a vertex  $x$  in  $V_i$ , the number of all edges incident with  $x$  which are not in  $H_i$  is said to be the *outer degree* of  $x$  in  $H_i$ . The sum of the outer degrees of all vertices in  $V_i$  is called the *outer degree* of  $H_i$ .

An *embedding* of a planar graph  $G$  in the plane is a drawing of  $G$  in the plane which has no edge-crossings. By the stereographic projection, a graph is embeddable in the plane if and only if it can be embedded in the sphere. By a well-known result of Whitney [15], a 3-connected planar graph has a unique embedding in the plane (or the sphere). Let  $H$  be a connected planar graph with at least four vertices which has an embedding  $\Pi$  in the sphere  $\Sigma$ . Then a connected component in  $\Sigma - \Pi$  is a *face* of  $\Pi$ . If the boundary of a face  $f$  is a cycle  $C$ , then  $C$  is called a *facial cycle*. Sometimes,  $f$  is said to be the *interior* of  $C$ .

## 2. THE PROOF OF THEOREM 2

We now give the proof of Theorem 2.

**Proof of Theorem 2.** We use the induction on  $n$  to show the theorem. If  $n \leq 3$ , then the theorem is true. If  $n = 4$ , then  $G$  is a proper subgraph of  $K_4$ . It can be checked that  $\nabla(G) \leq 1$ . Assume that the theorem holds for  $n \leq k - 1$ , where  $k \geq 5$ . We now consider the case that  $n = k$ .

It is well-known that every graph that has no  $K_4$ -minor is planar and has tree-width at most two. In addition, every graph with tree-width  $k$  has a vertex of degree at most  $k$ . So  $\delta(G) = 1$  or  $2$ .

If  $\delta(G) = 1$ , let  $x$  be a vertex of degree one in  $G$ . Let  $G'$  be the graph obtained from  $G$  by deleting  $x$ . Clearly,  $G'$  has not any  $K_4$ -minor. So  $\nabla(G') \leq \frac{n-2}{2}$  by the inductual assumption. Since  $\nabla(G) = \nabla(G')$ , we have that  $\nabla(G) \leq \frac{n-2}{2}$ .

If  $\delta(G) = 2$ , let  $y$  be a vertex of degree two in  $G$ . Let  $y_1$  and  $y_2$  be two neighbors of  $y$ . We now delete  $y_1$  from  $G$ . Then the degree of  $y$  is one in the present graph. Next,  $y$  is deleted. Let  $H$  be the obtained graph, which has not any  $K_4$ -minor. Then  $\nabla(G) \leq \nabla(H) + 1$ . By the inductual assumption, we have that  $\nabla(H) \leq \frac{n-3}{2}$ . So  $\nabla(G) \leq \frac{n-1}{2}$ . ■

The result below follows from Theorem 2 directly.

**Corollary 5.** *Let  $G$  be a planar graph with  $n \geq 3$  vertices. If  $\nabla(G) \geq \frac{n}{2}$ , then  $G$  has a  $K_4$ -minor.*

### 3. THE PROOF OF THEOREM 4

In the section we will prove Theorem 4.

**Lemma 6.** *Let  $G$  be a planar graph of order  $n$  with  $\Delta(G) \leq 4$ . If  $\delta(G) \leq 2$ , then  $\nabla(G) \leq \frac{n-1}{2}$ .*

**Proof.** Let  $G$  be a minimal counterexample with respect to the number of vertices. Let  $x$  be a vertex of degree at most two in  $G$ . If  $d_G(x) = 1$ , then  $\nabla(G) = \nabla(G - x)$ . Since  $\Delta(G - x) \leq 4$ , we have that  $\nabla(G - x) \leq \frac{n-1}{2}$  by Theorem 3. So  $\nabla(G) \leq \frac{n-1}{2}$ , a contradiction. If  $d_G(x) = 2$ , let  $N(x) = \{x_1, x_2\}$ . If  $x_1$  is not adjacent to  $x_2$ , then the path  $x_1xx_2$  is replaced with  $x_1x_2$ . Let  $G'$  be the obtained graph. Then  $\nabla(G) = \nabla(G') \leq \frac{n-1}{2}$  by Theorem 3, a contradiction. If  $x_1$  is adjacent to  $x_2$ , then we delete  $x_1$ , then  $x$ . Let  $G''$  be the obtained graph in which the degree of  $x_2$  is at most two. Then  $\nabla(G'') \leq \frac{n-2}{2}$  by Theorem 3. If  $\nabla(G'') = \frac{n-2}{2}$ , then it violates the minimality of  $G$ . If  $\nabla(G'') \leq \frac{n-3}{2}$ , then  $\nabla(G) \leq \nabla(G'') + 1 \leq \frac{n-1}{2}$ , a contradiction. So  $\nabla(G) \leq \frac{n-1}{2}$ . ■

**Theorem 7.** *Let  $G$  be a planar graph of order  $n \geq 4$  with  $\Delta(G) \leq 4$ . If  $G$  is not a 4-regular graph and if  $\nabla(G) = \frac{n}{2}$ , then  $G$  is covered by  $K_4$ -subgraphs.*

**Proof.** Suppose that  $G$  is a minimal counterexample with respect to the number of vertices. We now suppose that  $G$  has been embedded in the plane.

**Claim 1.**  *$G$  has not any cut-edge.*

**Proof.** Otherwise, let  $e = uv$  be a cut-edge in  $G$ . Suppose that  $e$  is an edge in a component  $G'_1$  of  $G$ . Let  $G'_{1,1}$  and  $G'_{1,2}$  be two components of  $G'_1 - e$ . Let  $G_1 = G'_{1,1}$ , and let  $G_2$  be the union of all other components in  $G - e$ . Then  $\nabla(G_1) \leq \frac{|V(G_1)|}{2}$  and  $\nabla(G_2) \leq \frac{|V(G_2)|}{2}$  by Theorem 3. If one of  $G_1$  and  $G_2$ , say  $G_1$ , is such that  $\nabla(G_1) < \frac{|V(G_1)|}{2}$ , then  $\nabla(G) = \nabla(G_1) + \nabla(G_2) < \frac{n}{2}$ , a contradiction. So  $\nabla(G_1) = \frac{|V(G_1)|}{2}$  and  $\nabla(G_2) = \frac{|V(G_2)|}{2}$ . Thus both  $G_1$  and  $G_2$  are covered by  $K_4$ -subgraphs. Hence  $G$  is covered by  $K_4$ -subgraphs, a contradiction.  $\square$

We now consider  $\delta(G)$ . If  $\delta(G) \leq 2$ , then  $\nabla(G) \leq \frac{n-1}{2}$  by Lemma 6, a contradiction. So  $\delta(G) \geq 3$ . Since  $G$  is not a 4-regular graph, we have that  $\delta(G) = 3$ . Let  $x$  be a vertex of degree three in  $G$ , and let  $N(x) = \{x_1, x_2, x_3\}$ . Let  $Q_1$  be the subgraph of  $G$  induced by  $x_1, x_2$  and  $x_3$ .

If  $Q_1$  is a cycle, then the subgraph of  $G$  induced by vertices in  $V(Q_1) \cup \{x\}$  is isomorphic to  $K_4$ . We now delete  $x_1$  and  $x_2$  from  $G$ . Then  $x$  is of degree one and  $x_3$  is of degree at most two in the present graph. Next,  $x$  is deleted, then  $x_3$ . Let  $G'$  be the obtained graph, which has  $n - 4$  vertices. So  $\nabla(G') \leq \frac{n-4}{2}$  by Theorem 3. If  $\nabla(G') < \frac{n-4}{2}$ , then  $\nabla(G) \leq \nabla(G') + 2 < \frac{n}{2}$ , a contradiction. If  $\nabla(G') = \frac{n-4}{2}$ , then  $G'$  is covered by  $K_4$ -subgraphs. Hence  $G$  is covered by  $K_4$ -subgraphs, a contradiction.

If  $Q_1$  is not a cycle, then there are two vertices in  $N(x)$ , say  $x_1$  and  $x_2$ , such that they are not adjacent to each other in  $G$ .

**Claim 2.**  $x_1$  is not any vertex in any  $K_4$ -subgraph of  $G$ .

**Proof.** Otherwise, suppose that  $x_1$  is in some  $K_4$ -subgraph  $Q_2$  of  $G$ . Let  $V(Q_2) = \{x_1, y_1, y_2, y_3\}$ . Obviously,  $x$  is not in  $V(Q_2)$ . Otherwise, both  $x_2$  and  $x_3$  are in  $V(Q_2)$ , a contradiction.

If there is some vertex in  $\{y_1, y_2, y_3\}$ , say  $y_1$ , such that its degree is three in  $G$ , then we delete  $y_2$  and  $y_3$  from  $G$ . So  $y_1$  is of degree one and  $x_1$  is of degree at most two in the present graph. Next,  $y_1$  is deleted, then  $x_1$ . Let  $H_1$  be the obtained graph in which the degree of  $x$  is two. So  $\nabla(H_1) \leq \frac{n-5}{2}$  by Lemma 6. Since  $\nabla(G) \leq \nabla(H_1) + 2$ , we have that  $\nabla(G) \leq \frac{n-1}{2}$ , a contradiction. Hence the degree of each of  $y_1, y_2$ , and  $y_3$  is four in  $G$ .

Let  $y'_i$  be the neighbor of  $y_i$  which is not in  $V(Q_2)$  for  $i = 1, 2, 3$ . Since  $K_4$  is 3-connected planar graph, it has a unique embedding in the plane, say  $\Pi$ . Without loss of generality, suppose that  $C = x_1 y_1 y_2 x_1$  is the boundary of the outer face of  $\Pi$ . We claim that  $y'_1, y'_2$  and  $y'_3$  are not in the exterior of the cycle  $C$ . If not, then the edge  $y_3 y'_3$  crosses some edge of  $C$  by Jordan's curve theorem, since  $y_3$  is not in the exterior of  $C$ . Thus there is a contradiction. Similarly,  $y'_1, y'_2$  and  $y'_3$  are not in the interior of the same inner face of  $\Pi$ . Without loss of generality, suppose that  $y'_1$  and  $y'_2$  are in the interior of two distinct inner faces of  $\Pi$ . So  $y'_1$  is not adjacent to  $y'_2$ . We now delete  $x_1$  and  $y_3$  from  $G$ . Then the

degree of  $y_i$  is two in the present graph for  $i = 1, 2$ . The path  $y'_1 y_1 y_2 y'_2$  is now replaced with  $y'_1 y'_2$ . Let  $H_2$  be the obtained graph in which the degree of  $x$  is two. So  $\nabla(H_2) \leq \frac{n-5}{2}$  by Lemma 6. Since  $\nabla(G) \leq \nabla(H_2) + 2$ , we have that  $\nabla(G) \leq \frac{n-1}{2}$ , a contradiction.  $\square$

Similarly, we have the following claim.

**Claim 3.**  $x_2$  is not any vertex in any  $K_4$ -subgraph of  $G$ .

We now delete  $x_3$  from  $G$ . Then the degree of  $x$  is two in the present graph. Next, the path  $x_1 x x_2$  is replaced with the edge  $x_1 x_2$ . Let  $H_3$  be the obtained graph, whose maximum degree is at most four. By Theorem 3,  $\nabla(H_3) \leq \frac{n-2}{2}$ . If  $\nabla(H_3) < \frac{n-2}{2}$ , then  $\nabla(G) \leq \nabla(H_3) + 1 < \frac{n}{2}$ , a contradiction. If  $\nabla(H_3) = \frac{n-2}{2}$ , then it is covered by  $K_4$ -subgraphs. Hence  $x_i$  must be a vertex in some  $K_4$ -subgraph, say  $R_i$ , for  $i = 1, 2$ . Since the maximum degree of  $G$  is at most four,  $x_1 x_2$  can not be a common edge of  $R_1$  and  $R_2$ . If  $R_1$  and  $R_2$  are two different subgraphs, then one of  $R_1$  and  $R_2$ , say  $R_1$ , does not contain the edge  $x_1 x_2$ . Thus  $R_1$  is a  $K_4$ -subgraph of  $G$  which contains  $x_1$ , which violates Claim 2. So both  $x_1$  and  $x_2$  are two vertices in the same  $K_4$ -subgraph, say  $Q_3$ , in  $H_3$ . Suppose that  $V(Q_3) = \{x_1, x_2, z_1, z_2\}$ .

If the degree of one of  $z_1$  and  $z_2$ , say  $z_1$ , is three, then we delete  $x_1$  and  $x_2$  instead of  $x_3$  from  $G$ . Then the degree of  $z_1$  is one and the degree of  $z_2$  is at most two in the present graph. Next,  $z_1$  is deleted, then  $z_2$ . Let  $H_4$  be the obtained graph in which the degree of  $x$  is one. So  $\nabla(H_4) \leq \frac{n-5}{2}$  by Lemma 6. Since  $\nabla(G) \leq \nabla(H_4) + 2$ , we have that  $\nabla(G) \leq \frac{n-1}{2}$ , a contradiction. Hence each of  $z_1$  and  $z_2$  is of degree four in  $G$ . Suppose that  $w_i$  is the neighbor of  $z_i$  which is not in  $V(Q_3)$  for  $i = 1, 2$ .

It is clear that  $x_3$  is neither  $z_1$  nor  $z_2$  in  $G$ . We observe that  $x_3, w_1$  and  $w_2$  are not the same vertex. Otherwise, the subgraph of  $G$  induced by  $\{x, x_1, x_2, x_3\} \cup \{z_1, z_2, w_1, w_2\}$  has a  $K_{3,3}$ -minor, a contradiction. Similarly,  $x_3$  is not adjacent to both  $w_1$  and  $w_2$ , or  $x_3$  is one of  $w_1$  and  $w_2$  but  $x_3$  is adjacent to the other vertex. So  $x_3$  is not adjacent to one of  $w_1$  and  $w_2$ . We now suppose that  $x_3$  is not adjacent to  $w_2$  in  $G$ .

**Claim 4.** The degree of each of  $x_1$  and  $x_2$  is four in  $G$ .

**Proof.** Without loss of generality, suppose on the contrary that the degree of  $x_1$  is three in  $G$ . We now delete  $x_2$  and  $z_1$  instead of  $x_3$  from  $G$ . Then the degree of each of  $x, x_1$  and  $z_2$  is two in the present graph. Next, the path  $x_3 x x_1 z_2 w_2$  is replaced with  $x_3 w_2$ . Let  $H_5$  be the obtained graph. Then  $\nabla(H_5) \leq \frac{n-5}{2}$  by Lemma 6. Since  $\nabla(G) \leq \nabla(H_5) + 2$ , we have that  $\nabla(G) \leq \frac{n-1}{2}$ , a contradiction.  $\square$

Suppose that the neighbor of  $x_i$  which is not in  $\{x\} \cup V(Q_3)$  is  $u_i$  in  $G$  for  $i = 1, 2$ . A local structure of  $G$  is shown in Figure 1. Need to say that several

vertices in  $u_1, u_2, w_1, w_2$  may be the same vertex. Let  $F$  be the subgraph of  $G$  induced by the vertices  $x, x_1, x_2, z_1, z_2$ . Suppose that  $F$  is a subgraph of a component of  $G$ , say  $B$ . Then  $x_3, u_1, u_2, w_1$  and  $w_2$  are vertices of  $B$ . Let  $F'$  be the graph obtained from  $B$  by deleting all vertices in  $F$ . We claim that  $F'$  is not connected (a graph with one vertex is viewed as a connected one). Otherwise, all vertices in  $F'$  are contracted into a vertex. Then  $G$  has a  $K_5$ -minor, since  $F$  is isomorphic to a subdivision of  $K_4$ . Thus there is a contradiction. If  $F'$  has at least three components, then there is a component which contains only one vertex in  $x_3, u_1, u_2, w_1$  and  $w_2$ , say  $w_1$ . In this case, the edge  $z_1w_1$  is a cut-edge of  $G$ , which violates Claim 1. So  $F'$  has exactly two components, say  $F'_1$  and  $F'_2$ .

If  $w_1$  and  $w_2$  are in  $V(F'_1)$  and  $V(F'_2)$ , respectively, then we delete both  $x_1$  and  $x_2$  instead of  $x_3$  from  $G$ . Let  $H_6$  be the obtained graph in which the degree of  $x$  is one and the degree of each of  $z_1$  and  $z_2$  is two. The path  $w_1z_1z_2w_2$  is now replaced with the edge  $w_1w_2$ . Let  $H'_6$  be the obtained graph. Then  $\nabla(H'_6) \leq \frac{n-5}{2}$  by Lemma 6. So  $\nabla(G) \leq \nabla(H'_6) + 2 \leq \frac{n-1}{2}$ , a contradiction. So both  $w_1$  and  $w_2$  are in one of  $V(F'_1)$  and  $V(F'_2)$ , say  $V(F'_1)$ . In this case, there are at least two vertices in  $\{x_3, u_1, u_2\}$  which are in  $V(F'_2)$ . We claim that  $x_3$  must be in  $V(F'_2)$ . Otherwise,  $x_3$  is in  $V(F'_1)$ . So the graph  $F'_1 \cup F$  has a minor isomorphic to  $K_{3,3}$  if all vertices in  $V(F'_1)$  are contracted into a vertex, a contradiction.

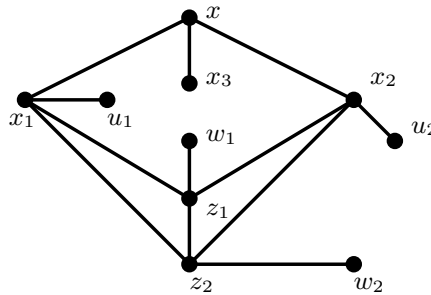


Figure 1. A local structure of  $G$ .

If  $u_1$  is in  $V(F'_1)$ , then it is not adjacent to  $x_3$ . We now delete  $x_2$  instead of  $x_3$  from  $G$ . Let  $H_7$  be the obtained graph in which the degree of  $x$  is two. Let  $H'_7$  be the graph obtained from  $H_7$  by replacing the path  $x_3xx_1$  with the edge  $x_3x_1$ . Then  $\nabla(H'_7) \leq \frac{n-2}{2}$  by Theorem 3. If  $\nabla(H'_7) \leq \frac{n-3}{2}$ , then  $\nabla(G) \leq \nabla(H_7) + 2 \leq \frac{n-1}{2}$ , a contradiction. So  $\nabla(H'_7) = \frac{n-2}{2}$ . Considering that the degree of  $z_1$  is three in  $H'_7$  and the order of  $H'_7$  is less than that of  $G$ , we have that  $H'_7$  is covered by  $K_4$ -subgraphs. Thus  $x_1$  is a vertex in some  $K_4$ -subgraph. Hence there are three vertices in  $N(x_1)$  such that they are adjacent to each other. Since  $N(x_1) = \{x_3, u_1, z_1, z_2\}$  and  $x_3$  is not adjacent to  $u_1$ , we have that either  $u_1, z_1, z_2$  are adjacent to each other or  $x_3, z_1, z_2$  are adjacent to each other. If the former

occurs, then  $x_1$  is a vertex in some  $K_4$ -subgraph in  $G$ , which violates Claim 2. If the latter occurs, then  $G$  has a subgraph isomorphic to  $K_{3,3}$ , whose vertex set is  $\{x, z_1, z_2\} \cup \{x_1, x_2, x_3\}$ , a contradiction. So  $u_1$  is not in  $V(F'_1)$ . Similarly,  $u_2$  is not in  $V(F'_2)$ . In other words, both  $u_1$  and  $u_2$  are in  $V(F'_2)$ . Thus  $u_1$  is not adjacent to  $w_1$  in  $G$ .

**Claim 5.**  $x_3$ ,  $u_1$  and  $u_2$  are not the same vertex in  $G$ .

**Proof.** Otherwise, we delete  $x_1$  and  $x_2$  instead of  $x_3$  from  $G$ . So the degree of  $x$  is one, the degree of  $x_3$  is at most two, and degree of each of  $z_1$  and  $z_2$  is two in the present graph. Next,  $x$  is deleted, then  $x_3$ . Let  $H_8$  be the obtained graph whose minimum degree is at most two. Then  $\nabla(H_8) \leq \frac{n-5}{2}$  by Lemma 6. Since  $\nabla(G) \leq \nabla(H_8) + 2$ , we have that  $\nabla(G) \leq \frac{n-1}{2}$ , a contradiction.  $\square$

By Claim 5, without loss of generality, suppose that  $x_3$  and  $u_1$  are not the same vertex in  $G$ . We now delete the two vertices  $x_2$  and  $z_2$  instead of  $x_3$  from  $G$ . Then the degree of each of  $x$  and  $z_1$  is two in the present graph. Next, the path  $x_1xx_3$  is replaced with  $x_1x_3$ , and the path  $x_1z_1w_1$  is replaced with  $x_1w_1$ . Let  $H_9$  be the obtained graph whose maximum degree is at most four. If  $\nabla(H_9) \leq \frac{n-5}{2}$ , then  $\nabla(G) \leq \frac{n-1}{2}$ , a contradiction. If  $\nabla(H_9) = \frac{n-4}{2}$ , then  $H_9$  is covered by  $K_4$ -subgraphs. In this case,  $x_1$  is a vertex in a unique  $K_4$ -subgraph, say  $Q_4$ . So  $Q_4$  contains  $x_1, x_3, w_1$  and  $u_1$ . Hence  $u_1$  must be adjacent to  $w_1$  in  $G$ , a contradiction. Thus, the proof is completed.  $\blacksquare$

**Theorem 8.** Let  $G$  be a planar graph of order  $n \geq 4$ . If  $G$  is covered by  $K_4$ -subgraphs and the outer degree of each  $K_4$ -subgraph is at most five, then  $\nabla(G) = \frac{n}{2}$ .

**Proof.** Suppose that  $G$  is covered by  $K_4$ -subgraphs  $Q_1, Q_2, \dots, Q_k$ . So  $n = 4k$ . Since  $\nabla(K_4) = 2$ , we have that  $\nabla(G) \geq 2k$ , i.e.,  $\nabla(G) \geq \frac{n}{2}$ . We can suppose that  $G$  is connected. Otherwise, each component of  $G$  is considered in the similar way.

We now use the induction on  $k$  to show that  $\nabla(G) \leq 2k$ . If  $k = 1$ , then  $G$  is exactly  $K_4$ . So  $\nabla(G) = 2$ . Assume that the inequality is true for  $k \leq l - 1$ , where  $l \geq 2$ . We now consider the case that  $k = l$ . For  $i = 1, 2, \dots, l$ , suppose that  $V(Q_i) = \{v_{i,1}, v_{i,2}, v_{i,3}, v_{i,4}\}$ .

Suppose that  $G$  has been embedded in the sphere. We claim that there is some  $K_4$ -subgraph  $Q_j$  which contains some 3-cycle  $C$  such that any vertex in  $V(Q_j) - V(C)$  is not adjacent to any vertex in  $V(G) - V(Q_j)$ . In fact,  $Q_i$  is a 3-connected planar graph for  $i = 1, 2, \dots, l$ . Then  $Q_i$  has a unique embedding in the sphere in which each facial cycle is a 3-cycle. We consider  $Q_1$  at first. If  $Q_1$  has a facial cycle such that all vertices in  $V(G) - V(Q_1)$  are in the interior of the cycle, then it is the desired. Otherwise, there are two facial cycles in  $Q_1$ , say  $C_1$  and  $C_2$ , such that at least one vertex in  $V(G) - V(Q_1)$  is adjacent to some vertex of  $C_i$  for  $i = 1, 2$ . We now select one of  $C_1$  and  $C_2$ , say  $C_1$ . We observe that if some vertex of  $Q_s$  ( $s \neq 1$ ) is in the interior of  $C_1$ , then all other vertices



in  $Q_s$  are in the interior of  $C_1$ . Otherwise, there is at least one edge-crossing, a contradiction. Without loss of generality, suppose that all vertices of  $Q_2$  are in the interior of  $C_1$ . Next, we argue  $Q_2$  in a similar way used for  $Q_1$ , and so on. Since  $l$  is a finite number, there is some  $j$  in  $\{1, 2, \dots, l\}$  such that  $Q_j$  has some cycle  $C$  satisfying the desired condition. Suppose that  $C = v_{j,1}v_{j,2}v_{j,3}v_{j,1}$ .

Since the outer degree of  $Q_j$  is at most five, there is some vertex in  $V(C)$ , say  $v_{j,3}$ , such that its outer degree is at most one. We now delete  $v_{j,1}$  and  $v_{j,2}$  from  $G$ . Then the degree of  $v_{j,4}$  is one and the degree of  $v_{j,3}$  is at most two in the present graph. Next,  $v_{j,4}$  is deleted, then  $v_{j,3}$ . Let  $G'$  be the obtained graph. Then  $G'$  is covered by  $(l-1)$   $K_4$ -subgraphs and  $\nabla(G) \leq \nabla(G') + 2$ . By the inductual assumption,  $\nabla(G') \leq 2l - 2$ . So  $\nabla(G) \leq 2l$ . Hence the proof is completed. ■

**Proof of Theorem 4.** The theorem follows from Theorem 7 and Theorem 8 directly. ■

#### 4. PLANAR GRAPHS WITH MINIMUM DEGREE AT LEAST FOUR

The decycling number of a planar graph with minimum degree at least four will be studied in the section. Let us start with a definition. A regular polyhedron is a convex one which satisfies the following conditions: (1) the polygons are congruent ones, and (2) each vertex is incident with the same number of polygons. It has been known that there are exactly five regular polyhedra which contains the tetrahedron, the hexahedron, the octahedron, the dodecahedron, and the icosahedron. (One can refer to [14] for the proof.) Let  $O_6$  and  $I_{12}$  denote the octahedron and the icosahedron, respectively, which are shown in Figures 2 and 3, respectively.

Let  $G$  be a planar graph with  $\delta(G) \geq 4$ . Then  $G$  has at least five vertices. If  $G$  has exactly five vertices, then it is the complete graph  $K_5$  which is not a planar graph. So  $G$  has at least six vertices. If  $G$  has six vertices and  $\delta(G) \geq 4$ , we have a result below.

**Theorem 9.** *If  $G$  is a planar graph with six vertices and  $\delta(G) \geq 4$ , then  $G$  is the graph  $O_6$ .*

**Proof.** We firstly claim that each vertex is of degree four in  $G$ . Otherwise, let  $x$  be a vertex in  $G$  of degree five. Then there is another vertex  $y$  of degree five in  $G$ . Otherwise,  $G$  has only one vertex of degree five and any other vertex has degree four, a contradiction. Since  $G$  has six vertices,  $x$  is adjacent to  $y$ . We now delete  $x$  and  $y$  from  $G$ . Let  $G'$  be the obtained graph. Then any vertex in  $G'$  is of degree at least two. So  $G'$  has a 4-cycle. Considering that each of  $x$  and  $y$  is adjacent to each vertex in  $G'$ ,  $G$  has a minor isomorphic to  $K_5$ , a contradiction.

Since the degree of each vertex in  $G$  is four, there are two vertices, say  $v_1$  and  $v_5$ , in  $G$ , such that  $v_1$  is not adjacent to  $v_5$ . Let  $H$  be the graph obtained from  $G$  by deleting  $v_1$  and  $v_5$  from  $G$ . Then each vertex in  $H$  is of degree two. So  $H$  is a 4-cycle. Since each of  $v_1$  and  $v_5$  is adjacent to each vertex in  $H$ ,  $G$  is the graph shown in Figure 2, which is exactly  $O_6$ . ■

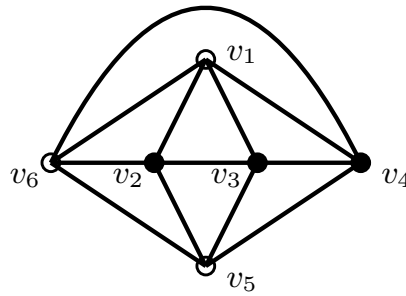


Figure 2. The graph  $O_6$  (the vertices depicted by solid circles form a decycling set).

**Theorem 10.**  $\nabla(O_6) = 3$ .

**Proof.** It is not hard to see that  $\nabla(O_6) \leq 3$  (refer to Figure 2). If  $\nabla(O_6) = 2$ , let  $S$  be a decycling set of  $O_6$  with two vertices. Since each vertex is of degree four in  $O_6$ , we have that each vertex is of degree at least two in  $O_6 - S$ . So  $O_6 - S$  contains a cycle, a contradiction. Hence  $\nabla(O_6) \geq 3$ . Thus  $\nabla(O_6) = 3$ . ■

**Theorem 11.** Let  $G$  be a planar graph with  $n \geq 6$  vertices. If  $G$  is covered by  $O_6$ -subgraphs and the outer degree of each  $O_6$ -subgraph is at most five, then  $\nabla(G) = \frac{n}{2}$ .

**Proof.** Suppose that  $G$  is covered by  $k$   $O_6$ -subgraphs. So  $n = 6k$ . By Theorem 10,  $\nabla(O_6) = 3$ . So  $\nabla(G) \geq 3k = \frac{n}{2}$ .

We suppose that  $G$  has been embedded in the sphere. We use the induction on  $k$  to show that  $\nabla(G) \leq 3k$ . If  $k = 1$ , then  $G$  is exactly  $O_6$ . So the inequality holds by Theorem 10. Assume that the inequality is true for  $k \leq l-1$ , where  $l \geq 2$ . We now consider the case that  $k = l$ . Since  $O_6$  is a 3-connected planar graph, it has a unique embedding in the sphere. By a similar argument to that in the proof of Theorem 8, there is some  $O_6$ -subgraph, denoted by  $Q$ , which contains some 3-cycle  $C$  such that any vertex in  $V(Q) - V(C)$  is not adjacent to any vertex in  $V(G) - V(Q)$ . Suppose that the vertex set of  $Q$  is  $\{v_i | i = 1, 2, 3, 4, 5, 6\}$ . Without loss of generality, suppose the cycle  $C = v_1v_2v_3v_1$  (one can refer to Figure 2).

Since the outer degree of  $Q$  is at most five, there is some vertex in  $C$ , say  $v_1$ , such that its outer degree is at most one. We now delete  $v_2$ ,  $v_3$  and  $v_6$  from  $G$ .

Let  $H$  be the obtained graph. Then the degree of  $v_5$  is one in  $H$ , and the degrees of  $v_4$  and  $v_1$  are two in  $H$ , respectively. We now delete  $v_5$ ,  $v_4$ , and  $v_1$  in this order from  $H$ . Let  $H'$  be the obtained graph. Then  $H'$  is covered by  $l - 1$   $O_6$ -graphs, and  $\nabla(G) \leq \nabla(H') + 3$ . By the inductual assumption,  $\nabla(H') \leq 3(l - 1)$ . So  $\nabla(G) \leq 3l$ . Thus  $\nabla(G) \leq \frac{n}{2}$ . Hence  $\nabla(G) = \frac{n}{2}$ . ■

Next, we consider the decycling number of a planar graph with minimum degree five. Let  $G$  be a planar graph with minimum degree five. Then  $G$  is not covered by  $K_4$ -subgraphs or  $O_6$ -subgraphs. We ask that whether there is a planar graph of order  $n$  with minimum degree five such that its decycling number is  $\frac{n}{2}$ . The following result gives a positive answer.

**Theorem 12.**  $\nabla(I_{12}) = 6$ .

**Proof.** It is easy to see that the set of vertices depicted by solid circle in Figure 3 is a decycling set of  $I_{12}$ . So  $\nabla(I_{12}) \leq 6$ .

We firstly claim that  $\nabla(I_{12}) \geq 5$ . Otherwise, let  $S_1$  be a decycling set of  $I_{12}$  with four vertices. Let  $H_1$  be the subgraph of  $I_{12}$  induced by all vertices in  $V(I_{12}) - S_1$ . Then  $H_1$  contains eight vertices. Considering that the deletion of any vertex in  $S_1$  destroys at most five edges and  $I_{12}$  has 30 edges,  $H_1$  contains at least ten edges. So  $H_1$  has a cycle, a contradiction.

We now suppose that  $\nabla(I_{12}) = 5$ . Without loss of generality, suppose that  $S_2 = \{v_1, v_2, v_3, v_4, v_5\}$  is a decycling set of  $I_{12}$  with five vertices. Let  $H_2$  be the subgraph of  $I_{12}$  induced by all vertices in  $V(I_{12}) - S_2$ . Then  $H_2$  contains seven vertices, and it has at most six edges. In this case the deletion of all vertices in  $S_2$  must destroy at least 24 edges of  $G$ . Thus the subgraph of  $I_{12}$  induced by  $S_2$  has at most one edge. In other words,  $I_{12}$  has an independent set of four vertices. It can be checked that any independent set of  $I_{12}$  has at most three vertices, a contradiction. So  $\nabla(I_{12}) \geq 6$ . Thus  $\nabla(I_{12}) = 6$ . ■

**Theorem 13.** Let  $G$  be a planar graph with  $n \geq 12$  vertices. If  $G$  is covered by  $I_{12}$ -subgraphs and the outer degree of each  $I_{12}$ -subgraph is at most five, then  $\nabla(G) = \frac{n}{2}$ .

**Proof.** Suppose that  $G$  is covered by  $k$   $I_{12}$ -subgraphs. So  $n = 12k$ . By Theorem 12,  $\nabla(I_{12}) = 6$ . So  $\nabla(G) \geq 6k = \frac{n}{2}$ .

We suppose that  $G$  has been embedded in the sphere. We use the induction on  $k$  to show that  $\nabla(G) \leq 6k$ . If  $k = 1$ , then  $G$  is exactly  $I_{12}$ . So the inequality holds by Theorem 12. Assume that the inequality is true for  $k \leq l - 1$ , where  $l \geq 2$ . We now consider the case that  $k = l$ . Since  $I_{12}$  is a 3-connected planar graph, it has a unique embedding in the sphere. By a similar argument to that in the proof of Theorem 8, there is some  $I_{12}$ -subgraph, denoted by  $Q$ , which contains some 3-cycle  $C$  such that any vertex in  $V(Q) - V(C)$  is not adjacent to any vertex

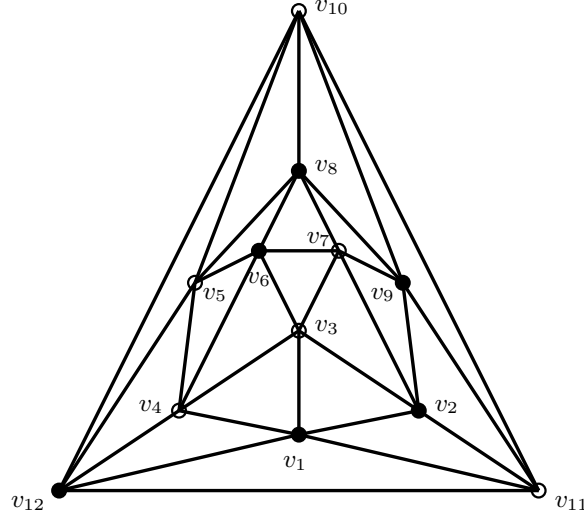


Figure 3. The graph  $I_{12}$  (the vertices depicted by solid circles form a decycling set).

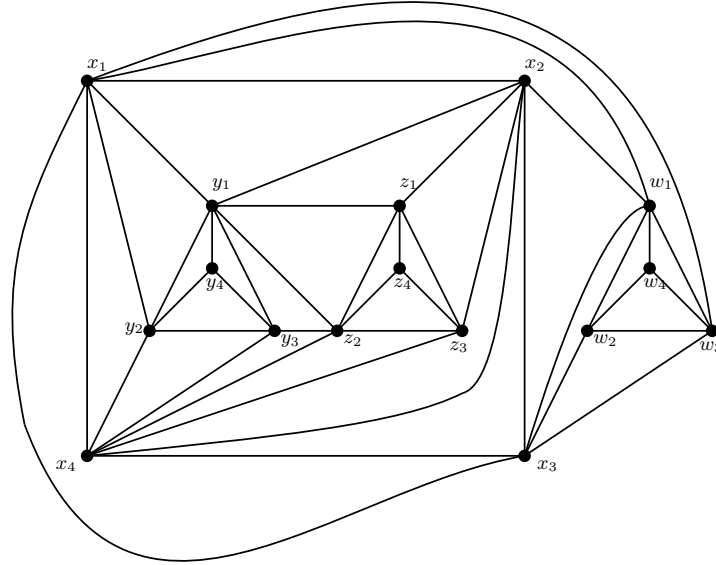
in  $V(G) - V(Q)$ . We suppose that the vertex set of  $Q$  is  $\{v_i | i = 1, 2, \dots, 12\}$ . See Figure 3. Since  $I_{12}$  is a regular polyhedron, the faces have symmetry. Without loss of generality, suppose that the cycle  $C = v_1 v_2 v_3 v_1$ .

Let  $S_1 = \{v_1, v_2, v_6, v_8, v_9, v_{12}\}$ ,  $S_2 = \{v_2, v_3, v_6, v_9, v_{10}, v_{12}\}$ , and  $S_3 = \{v_1, v_3, v_5, v_6, v_9, v_{11}\}$ . For  $i = 1, 2, 3$ , we observe that  $S_i$  is a decycling set of  $Q$  with six vertices, and that the graph obtained from  $Q$  by deleting all vertices in  $S_i$  is a path, say  $P_i$ . Note that  $P_1 = v_7 v_3 v_4 v_5 v_{10} v_{11}$ ,  $P_2 = v_{11} v_1 v_4 v_5 v_8 v_7$ , and  $P_3 = v_2 v_7 v_8 v_{10} v_{12} v_4$ .

Since the outer degree of  $Q$  is at most five, there is a vertex  $y$  in  $C$  such that its outer degree is at most one. If  $y$  is the vertex  $v_3$ , then we delete all vertices in  $S_1$ . Next, we firstly delete all vertices in  $V(P_1) - \{v_3\}$ , then  $x_3$ . Let  $H_1$  be the obtained graph, which is covered by  $l - 1$   $I_{12}$ -subgraphs. By the inductual assumption,  $\nabla(H_1) \leq 6(l - 1)$ . So  $\nabla(G) \leq 6l$ . Thus  $\nabla(G) \leq \frac{n}{2}$ . If  $y$  is the vertex  $v_1$  (or  $v_2$ ), then we delete all vertices in  $S_2$  (or  $S_3$ ). Subsequently, we delete all vertices in  $V(P_1) - \{v_1\}$  (or  $V(P_1) - \{v_2\}$ ), then  $v_1$  (or  $v_2$ ). Next, we proceed a similar argument to that for  $v_3$ . Then  $\nabla(G) \leq \frac{n}{2}$ . Hence  $\nabla(G) = \frac{n}{2}$ . ■

Using the methods in the proof of Theorems 8, 11 and 13, it is not hard to show the following result.

**Theorem 14.** *Let  $G$  be a planar graph with  $n \geq 12$  vertices. If  $G$  is covered by  $K_4$ -subgraphs,  $O_6$ -subgraphs, or  $I_{12}$ -subgraphs such that the outer degree of each of  $K_4$ -subgraph,  $O_6$ -subgraph, and  $I_{12}$ -subgraph is at most five, then  $\nabla(G) = \frac{n}{2}$ .*

Figure 4. A graph covered by  $K_4$ -subgraphs.

If a planar graph  $G$  of order  $n$  is covered by  $K_4$ -subgraphs,  $O_6$ -subgraphs, or  $I_{12}$ -subgraphs, then the decycling number of  $G$  is at least  $\frac{n}{2}$  by the facts that  $\nabla(K_4) = 2$ ,  $\nabla(O_6) = 3$ , and  $\nabla(I_{12}) = 6$ . In order to determine  $\nabla(G) = \frac{n}{2}$ , one needs to show that  $\nabla(G) \leq \frac{n}{2}$ . We can solve it if the outer degree of each of  $K_4$ -subgraph,  $O_6$ -subgraph, and  $I_{12}$ -subgraph is restricted to be at most five. If the condition on the outer degree is removed, it is not easy to solve it. For example, the graph  $H$  shown in Figure 4 is covered by  $K_4$ -subgraphs. Note that the subgraph induced by  $x_1, x_2, x_3$ , and  $x_4$  is isomorphic to  $K_4$ . We observe that any decycling set of  $H$  must contain two vertices in  $\{x_1, x_2, x_3, x_4\}$ , but we have not a general method to select them. We think that the condition on the outer degree in Theorem 14 can be removed. So we propose the following conjecture.

**Conjecture 15.** *Let  $G$  be a planar graph with  $n \geq 12$  vertices which is covered by  $K_4$ -subgraphs,  $O_6$ -subgraphs, or  $I_{12}$ -subgraphs. Then  $\nabla(G) = \frac{n}{2}$ .*

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