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BOUNDS ON DOMINATION PARAMETERS IN GRAPHS: A BRIEF SURVEY

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Abstract

In this paper we present a brief survey of bounds on selected domination parameters. We focus primarily on bounds on domination parameters in terms of the order and minimum degree of the graph. We present a list of open problems and conjectures that have yet to be solved in the hope of attracting future researchers to the field.

Keywords: bounds, domination parameters.

2010 Mathematics Subject Classification: 05C69.

1. Introduction

In this paper, we present a survey of upper bounds for various domination parameters, including the domination number, the total domination number, the independent domination number, the connected domination number, the paired domination number, the restrained domination number, the total restrained domination number, the semitotal domination number, and the semipaired domination number. For recent books on domination in graphs, we refer the reader to [33–35].

For notation and graph theory terminology, we in general follow [50]. Specifically, let G be a graph with vertex set $V(G)$ and edge set $E(G)$, and of order $n(G) = |V(G)|$ and size $m(G) = |E(G)|$. For a set of vertices $S \subseteq V(G)$, the subgraph induced by S is denoted by $G[S]$. Two vertices in G are *neighbors* if they are adjacent. The *open neighborhood* $N_G(v)$ of a vertex v in G is the set of neighbors of v, while the *closed neighborhood* of v is the set $N_G[v] = \{v\} \cup N(v)$. We denote the *degree* of v in G by $d_G(v) = |N_G(v)|$. The minimum and maximum degree in G is denoted by $\delta(G)$ and $\Delta(G)$, respectively. An *isolated vertex* is a vertex of degree 0. A graph is *isolate-free* if it contains no isolated vertex. For a set $S \subseteq V(G)$, its *open neighborhood* is the set $N_G(S) = \cup_{v\in S} N_G(v)$, and its *closed neighborhood* is the set $N_G[S] = N_G(S) \cup S$. If the graph G is clear from the context, we omit writing it in the above expressions. For example, we simply write V, E, n, m, $N(v)$ and $N(S)$ rather than $V(G)$, $E(G)$, $n(G)$, $m(G)$, $N_G(v)$ and $N_G(S)$, respectively.

A set D is a *packing* in G if the closed neighborhoods of vertices in D are pairwise disjoint. The *packing number* $\rho(G)$ is the maximum cardinality of a packing in G. A *perfect packing* in G is a packing that is also a dominating set of G, and so the closed neighborhoods of vertices in a perfect packing partition $V(G)$.

An *open packing* in a graph G is a set of vertices whose open neighborhoods are pairwise disjoint. The *open packing number* $\rho^o(G)$ is the maximum cardinality of an open packing in G. A *perfect open packing* in G is an open packing that is also a TD-set of G, and so the open neighborhoods of vertices in a perfect open packing partition $V(G)$.

2. The Domination Number

A set D of vertices is a *dominating set* of a graph G if every vertex in $V \setminus D$ has a neighbor in D. The *domination number* $\gamma(G)$ of G is the minimum cardinality of a dominating set of G. A *γ-set of G* is a dominating set of G of cardinality $\gamma(G)$. A *dominating vertex* of G is a vertex adjacent to every vertex of G. The packing number is a trivial lower bound on the domination number of a graph.

Observation 1. *If G is a graph, then $\gamma(G) \ge \rho(G)$.*

Lower bounds on the domination number can also readily be established in terms of the diameter and radius.

Theorem 2. *If G is a connected graph, then $\gamma(G) \ge \frac{1}{3}(\mathrm{diam}(G)+1)$ and $\gamma(G) \ge \frac{2}{3}\mathrm{rad}(G)$.*

The matching number $\alpha'(G)$ of G, which is the maximum cardinality of a matching in G, is an upper bound on the domination number of an isolate-free graph as first shown in 1979 by Bollobás and Cockayne [7].

Theorem 3 [7]**.** *If G is an isolate-free graph, then $\gamma(G) \le \alpha'(G)$.*

Upper and lower bounds on the domination number of a graph in terms of its maximum degree were presented in 1973 by Berge [4] and in 1979 by Walikar, Acharya, and Sampathkumar [73], respectively.

Theorem 4 [4, 73]**.** *If G is a graph of order n, then*

$$\frac{n}{1+\Delta(G)} \leq \gamma(G) \leq n - \Delta(G).$$

We remark that the bounds presented in Observation 1 and Theorems 2, 3, and 4 are all tight. However, we will not discuss the extremal graphs that achieve these bounds since in this brief survey we will focus primarily on bounds on domination parameters in terms of the order and minimum degree of the graph, and the associated extremal graphs. The first such bound, referred to as Ore's Theorem, is a classical result in domination theory due to Ore [63] in 1962.

Theorem 5 [63]**.** *If G is an isolate-free graph of order n, then $\gamma(G) \leq \frac{1}{2}n$.*

The *corona* $G \circ K_1$ of a graph G, also denoted $\mathrm{cor}(G)$ in the literature, is the graph obtained from G by adding for each vertex $v \in V(G)$ a new vertex v' and the edge vv', called a *pendant edge*. For example, the corona $G = K_{1,4} \circ K_1$ of a star $K_{1,4}$, shown in Figure 1, has order $n = 10$ and $\gamma(G) = 5 = \frac{1}{2}n$, and the five black vertices form a γ-set of G.

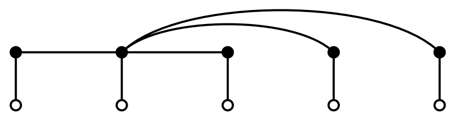


Figure 1. The corona $K_{1,4} \circ P_1$ of a star $K_{1,4}$.

In 1982 Payan and Xuong [65] characterized the connected graphs achieving equality in Theorem 5, and showed that the class of coronas of connected graphs form the extremal graphs, with the 4-cycle as the only exception.

Theorem 6 [65]**.** *If G is a connected graph of order $n \geq 2$, then $\gamma(G) = \frac{1}{2}n$ if and only if G is a 4-cycle or G is a corona $F \circ K_1$ for some connected graph F.*

In 1973 Blank [5] showed that if the minimum degree of a connected graph is at least 2, then the $\frac{1}{2}n$-bound on the domination number given in Ore's Theorem can be improved to a $\frac{2}{5}n$-bound, provided the order $n \geq 8$.

Theorem 7 [5]**.** *If G is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma(G) \leq \max\left\{\frac{n+2}{3}, \frac{2}{5}n\right\}$.*

As an immediate consequence of Theorem 7, we have the following result.

Theorem 8 [5]**.** *If G is a connected graph of order $n \geq 8$ with $\delta(G) \geq 2$, then $\gamma(G) \leq \frac{2}{5}n$.*

In 1989 McCuaig and Shepherd [62] provided a new insightful proof of Theorem 8. Their proof identified the exceptional graphs of small order $n \le 7$ that do not achieve the $\frac{2}{5}n$-bound. Let $\mathcal{B}_{\rm dom} = \{B_1, B_2, \ldots, B_7\}$ be the family of seven graphs (one of order four and six of order seven) shown in Figure 2. Each of the seven graphs $G \in \mathcal{B}_{\rm dom}$ of order n satisfies $\delta(G) = 2$ and $\gamma(G) > \frac{2}{5}n$. The following result, referred to as the McCuaig-Shepherd Theorem, shows that these seven graphs are the only exceptional graphs to the $\frac{2}{5}n$-bound for the domination number of a connected graph with minimum degree at least 2.

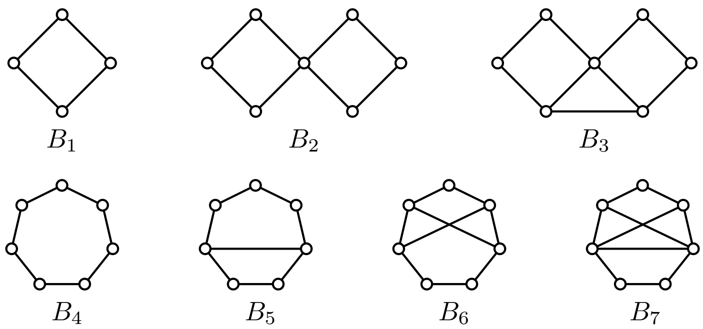


Figure 2. The family $\mathcal{B}_{\rm dom}$.

Theorem 9 [62]**.** *If G is a connected graph of order n with $\delta(G) \ge 2$, then $\gamma(G) \le \frac{2}{5}n$, unless G is one of the seven exceptional graphs in the family $\mathcal{B}_{\rm dom}$.*

Moreover, the proof due to McCuaig and Shepherd [62] provided sufficient structure to characterize the extremal graphs of large order, namely $n > 10$, that achieve equality in the $\frac{2}{5}n$-bound. The family of extremal graphs they constructed partitioned the vertex set into sets, each of which induces a subgraph isomorphic to a 5-cycle or to a 5-key, which is a graph of order 5 obtained from a 4-cycle by adding a new vertex and joining this vertex to exactly one vertex of the cycle. The resulting subgraphs are called *units*. From each 5-cycle unit two non-adjacent vertices are selected and designated as the *link vertices* of the unit, while in each key unit the vertex of degree 1 is designated as the (unique) *link vertex* of the unit. Let $\mathcal{G}_{{\rm dom},\ge 2}$ be the family of all graphs G that can be obtained from the disjoint union of at least three units, by adding edges between link vertices so that the resulting graph is connected. A graph G in the family $\mathcal{G}_{{\rm dom},\ge 2}$ with five units is shown in Figure 3 with the link vertices of G given by the black vertices. Every graph $G \in \mathcal{G}_{{\rm dom},\ge 2}$ of order n satisfies $\delta(G) = 2$ and $\gamma(G) = \frac{2}{5}n$.

Theorem 10 [5]**.** *If G is a connected graph of order $n > 10$ with $\delta(G) \ge 2$, then $\gamma(G) \le \frac{2}{5}n$, with equality if and only if $G \in \mathcal{G}_{{\rm dom},\ge 2}$.*

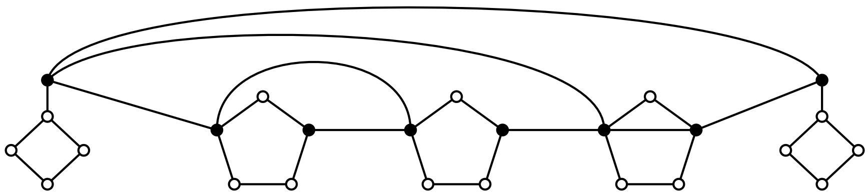
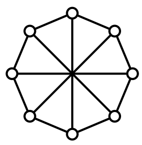


Figure 3. A graph $G \in \mathcal{G}_{\text{dom},\geq 2}$ with five units.

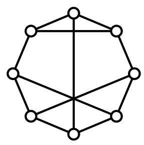
In 1996 Reed [66] provided an elegant proof that the domination number of a graph with minimum degree at least 3 is at most three-eights its order.

Theorem 11 [66]**.** *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma(G) \leq \frac{3}{8}n$.*

If G is one of the two non-planar cubic graphs $G_{8.1}$ and $G_{8.2}$ of order $n = 8$ shown in Figure 4(a) and (b), respectively, then $\gamma(G) = 3 = \frac{3}{8}n$. We remark that the graph $G_{8.1}$ is the Möbius ladder M_8.



(a) $G_{8.1}$



(b) $G_{8.2}$

Figure 4. The two non-planar cubic graphs of order $n = 8$.

In 2009 Kostochka and Stocker [60] proved that if $G \notin \{G_{8.1}, G_{8.2}\}$ is a connected cubic graph of order n, then $\gamma(G) \leq \frac{5}{14}n$. Hence, the two cubic graphs $G_{8.1}$ and $G_{8.2}$ are the only connected cubic graphs that achieve the three-eights bound in Reed's Theorem 11.

Reed [66] constructed a family of extremal graphs that achieve the $\frac{3}{8}n$-bound in Theorem 11. The vertex set of such graphs are partitioned into so-called *units*, where the subgraph induced by each unit is isomorphic to the nonplanar cubic graph $G_{8.2}$, illustrated in Figure 4(b), of order 8 that contains a triangle. In his construction, from each unit a vertex that belongs to a triangle is selected and designated as the (unique) *link vertex* of the unit. Let $\mathcal{G}_{\text{dom},\geq 3}$ be the family of all graphs that can be obtained from the disjoint union of $k \geq 1$ such units, by adding edges between link vertices so that the resulting graph is connected. Every graph $G \in \mathcal{G}_{\text{dom},\geq 3}$ with k units has order $n = 8k$ and satisfies $\delta(G) = 3$ and $\gamma(G) = k = \frac{3}{8}n$. A graph G in the family $\mathcal{G}_{\text{dom},\geq 3}$ with $k = 4$ units is shown in Figure 5 with the link vertices of G given by the black vertices.

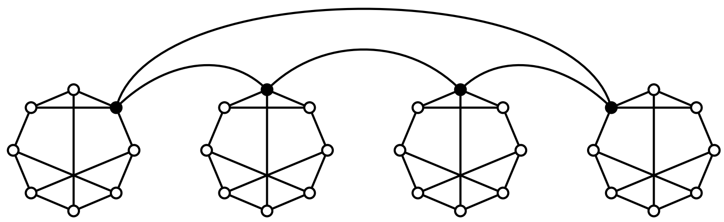


Figure 5. A graph G in the family $\mathcal{G}_{\mathrm{dom},\geq 3}$.

In 2009 Sohn and Yuan [68] established the best upper bound to date on the domination number of a graph with minimum degree at least 4.

Theorem 12 [68]**.** *If G is a graph of order n with $\delta(G) \geq 4$, then $\gamma(G) \leq \frac{4}{11}n$.*

In 2006 Xing, Sun, and Chen [74] proved that if G is a graph of order n with $\delta(G) \geq 5$, then $\gamma(G) \leq \frac{5}{14}n < 0.3572n$. In 2016 Bujtás and Klavžar [12] improved this bound to $\gamma(G) \leq \frac{2671}{7766}n < 0.343935n$. In 2021 Bujtás [9] proved a magnificent result that the bound can be improved to the magical threshold of $\gamma(G) \leq \frac{1}{3}n$.

Theorem 13 [9]**.** *If G is a graph of order n with $\delta(G) \geq 5$, then $\gamma(G) \leq \frac{1}{3}n$.*

In 2016 Bujtás and Klavžar [12] proved that if G is a graph of order n with $\delta(G) \geq 6$, then $\gamma(G) \leq \frac{1702}{5389}\,n \approx 0.315829\,n$. In 2021 Bujtás and Henning [10] improved this result, and established the best upper bound to date on the domination number of a graph with minimum degree at least 6.

Theorem 14 [10]**.** *If G is a graph of order n with $\delta(G) \geq 6$, then $\gamma(G) \leq \frac{127}{418}\,n \approx 0.30382775\,n$.*

In 2016 Bujtás and Klavžar [12] proved a powerful result that enabled them to compute best known upper bounds to date on the domination number of a graph when the minimum degree is in the integer range $[7, 50]$. However, it is unlikely that their bounds, which as mentioned are currently the best known bounds, are achievable. We mention here only a small sample of their bounds, namely when the minimum degree is in the integer range $[7, 12]$. The associated upper bounds on the domination number in these cases are given in Table 1.

One of the earliest bounds on the domination number was due to Arnautov [3] in 1974 and Payan [64] in 1975.

Theorem 15 [3,64]**.** *If G is a graph of order n with minimum degree $\delta \geq 1$, then*

$$\gamma(G) \leq \frac{n}{\delta+1}\sum_{j=1}^{\delta+1}\frac{1}{j}.$$

$\delta(G)$	7	8	9	10	11	12
$\gamma(G)$	$\leq 0.292678\,n$	$\leq 0.273213\,n$	$\leq 0.256566\,n$	$\leq 0.242128\,n$	$\leq 0.229463\,n$	$\leq 0.218244\,n$

Table 1. Upper bounds on $\gamma(G)$ in terms of its order n with given minimum degree $\delta(G)$.

Since the kth harmonic number, $H_k = \sum_{j=1}^{k} \frac{1}{j}$, of the first k natural numbers is approximately $\gamma + \ln(k) + \frac{1}{2k}$, where $\gamma = 0.57721\ldots$ is the Euler-Mascheroni constant, as an immediate consequence of Theorem 15 we have the following upper bound on the domination number.

Theorem 16 [3,64]. *If G is a graph of order n with minimum degree $\delta \geq 1$, then*

$$\gamma(G) \leq \left(\frac{1 + \ln(\delta + 1)}{\delta + 1} \right) n.$$

The bound on the domination number in Theorem 16 is not very good for small values of δ. Indeed, the bounds presented earlier for small $\delta \leq 50$ are much better bounds. However as δ increases, the bound in Theorem 16 gets increasingly sharp. In 1990 Alon provided a probabilistic proof in [1] that shows that the bound in Theorem 16 is asymptotically optimal, that is, when $\delta \to \infty$. Many probabilistic bounds on the domination number of a graph have been established over the past few decades. The best probabilistic bound to date on the domination number of a graph is due to Rad [55] in 2019.

Theorem 17 [55]. *If G is a graph of order n with minimum degree $\delta > 1$ and maximum degree Δ, then for every integer $k \geq 1$,*

$$\gamma(G) \leq \left(\frac{1 + \ln(\delta + 1)}{\delta + 1} \right) n - \left((\delta - \ln(1 + \delta)) \sum_{i=1}^{k} \left(\frac{\ln(1 + \delta)}{1 + \delta} \right)^{i(1+\Delta)} \right) \cdot \frac{n}{\delta + 1}.$$

Let $\mathcal{G}_{n,m}$ denote the class of all connected graphs of order $n \geq 3$ and size m. As a special case of a more general hypergraph result, in 2012 Bujtás, Henning and Tuza [11] proved the following general upper bound on the domination number that involves both the order and size.

Theorem 18 [11]. *All upper bounds on the domination number of a graph $G \in \mathcal{G}_{n,m}$ in terms of its order n and size m can be written in the unified form*

$$\gamma(G) \leq \frac{an + bm}{2a + b}$$

for constants $a, b \in \mathbb{R}$ where $b \geq 0$ and $a > -\frac{b}{2}$.

An equivalent formulation of Theorem 18 gives us the following general upper bound on the domination number of a graph.

Theorem 19 [11]**.** *The bound $\gamma(G) \le an + nm$ is valid for every graph $G \in \mathcal{G}_{n,m}$ if and only if both $2a + b \ge 1$ and $b \ge 0$ hold.*

For example, to illustrate Theorem 19, taking $(a, b) = \left(\frac{1}{2}, 0\right)$, yields Ore's Theorem 5. Taking $(a, b) = \left(\frac{1}{3}, \frac{1}{3}\right)$, we have $\gamma(G) \le \frac{1}{3}(n + m)$. Taking $(a, b) = \left(\frac{1}{4}, \frac{1}{2}\right)$, we have $\gamma(G) \le \frac{1}{4}n + \frac{1}{2}m$.

3. The Total Domination Number

A set D of vertices is a *total dominating set*, abbreviated TD-set, of a graph G if every vertex has a neighbor in D. Equivalently, D is a TD-set if D is a dominating set and the subgraph $G[D]$ induced by D has no isolates. The *total domination number* $\gamma_t(G)$ of G is the minimum cardinality of a TD-set of G. A *γ_t-set of G* is a TD-set of G of cardinality $\gamma_t(G)$. In 1979 Bollobás and Cockayne [7] were the first to observe that the total domination number of an isolate-free graph is squeezed between the domination number and twice the domination number.

Theorem 20 [7]**.** *If G is an isolate-free graph, then $\gamma(G) \le \gamma_t(G) \le 2\gamma(G)$.*

The open packing number is a trivial lower bound on the total domination number of a graph.

Theorem 21. *If G is an isolate-free graph, then $\gamma_t(G) \ge \rho^0(G)$.*

DeLaViña, Liu, Pepper, Waller, and West [20] established lower bounds on the total domination number in terms of the diameter and radius.

Theorem 22 [20]**.** *If G is a nontrivial connected graph, then $\gamma_t(G) \ge \frac{1}{2}(\mathrm{diam}(G) + 1)$ and $\gamma_t(G) \ge \mathrm{rad}(G)$.*

The total domination number and matching number are not related in the sense that there exist graphs G and H such that the difference $\gamma_t(G) - \alpha'(G)$ and the difference $\alpha'(H) - \gamma_t(H)$ can be made arbitrarily large. Since the ends of the edges in a maximum matching in a graph form a TD-set in the graph, if G is an isolate-free graph, then $\gamma_t(G) \le 2\alpha'(G)$. A *path covering* of a graph G is a collection of vertex disjoint paths of G that partition $V(G)$. The minimum cardinality of a path covering of G is the *path covering number* of G, denoted $\mathrm{pc}(G)$. In 2007 DeLaViña, Liu, Pepper, Waller, and West [20] showed that the total domination number of an isolate-free graph is at most the matching number plus the path covering number.

Theorem 23 [20]**.** *If G is a nontrivial connected graph, then $\gamma_t(G) \le \alpha'(G) + \mathrm{pc}(G)$.*

Upper and lower bounds on the total domination number of a graph in terms of its maximum degree were presented in 1980 by Cockayne, Dawes, and Hedetniemi [18] and in 1979 by Walikar, Acharya, and Sampathkumar [73], respectively.

Theorem 24 [18, 73]**.** *If G is a connected graph of order n, then following hold.*

(a) $\gamma_t(G) \ge \frac{n}{\Delta(G)}$.

(b) *If G does not contain a dominating vertex, then $\gamma_t(G) \le n - \Delta(G)$.*

As mentioned in the previous section, in this brief survey we will focus primarily on bounds on domination parameters in terms of the order and minimum degree of the graph, and the associated extremal graphs. The first such bound on the total domination number is a classical result in domination theory due to Cockayne, Dawes, and Hedetniemi [18] in 1980.

Theorem 25 [18]**.** *If G is a connected graph of order $n \ge 3$, then $\gamma_t(G) \le \frac{2}{3}n$.*

The 2-*corona* $F \circ P_2$ of a connected graph F is the graph of order $3|V(F)|$ obtained from F by attaching a path of length 2 to each vertex of F so that the resulting paths are vertex-disjoint. For example, the 2-corona $G = K_{1,4} \circ P_2$ of a star $K_{1,4}$, shown in Figure 6, has order $n = 15$ and $\gamma_t(G) = 10 = \frac{2}{3}n$, and the ten black vertices form a γ_t-set of G.

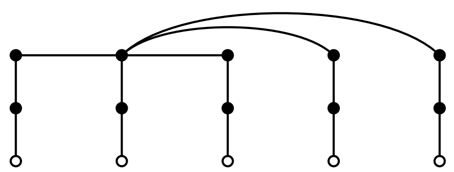


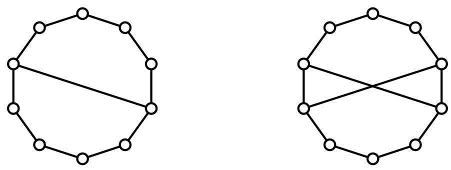
Figure 6. The 2-corona $K_{1,4} \circ P_2$ of a star $K_{1,4}$.

In 2000 Brigham, Carrington, and Vitray [8] characterized the connected graphs that achieve equality in the upper bound of Theorem 25.

Theorem 26 [8]**.** *If G is a connected graph of order $n \ge 3$, then $\gamma_t(G) = \frac{2}{3}n$ if and only if G is a 3-cycle C_3, a 6-cycle C_6, or the 2-corona of a connected graph, that is, $F \circ P_2$ for some connected graph F.*

The upper bound in Theorem 25 is also best possible for graphs of minimum degree 2, as may be seen by considering a 3-cycle or a 6-cycle. In 1995 Sun [69] showed that if G is a connected graph of order n with $\delta(G) \ge 2$, then

$\gamma_t(G) \le \frac{4}{7}(n+1)$. This upper bound can be improved slightly if we forbid six graphs of small orders. Let C'_{10} and C''_{10} be the two graphs that are obtained from a 10-cycle by adding one or two chords as shown in Figure 7(a) and 7(b), respectively, and let $\mathcal{B}_{\rm tdom} = \{C_3, C_5, C_6, C_{10}, C'_{10}, C''_{10}\}$.



(a) C'_{10} (b) C''_{10}

Figure 7. The graphs C'_{10} and C''_{10}.

Theorem 27 [42]**.** *If G is a connected graph of order n with $\delta(G) \ge 2$, then $\gamma_t(G) \le \frac{4}{7}n$, unless G is one of the six exceptional graphs in the family $\mathcal{B}_{\rm tdom}$.*

Let $\mathcal{G}_{\rm tdom}$ be the family of graphs G that can be constructed from a connected graph F of order at least 2 as follows. For each vertex v of F, add a 6-cycle C_v and join v either to exactly one vertex of C_v or to two vertices at distance 2 apart on the cycle C_v. The graph F is called the *underlying graph* of G. An example of a graph G in the family $\mathcal{G}_{\rm tdom}$ whose underlying graph F is a star $K_{1,4}$ is shown in Figure 9. In this example, G has order $n = 35$ and $\gamma_t(G) = 4 \times 5 = \frac{4}{7}n$.

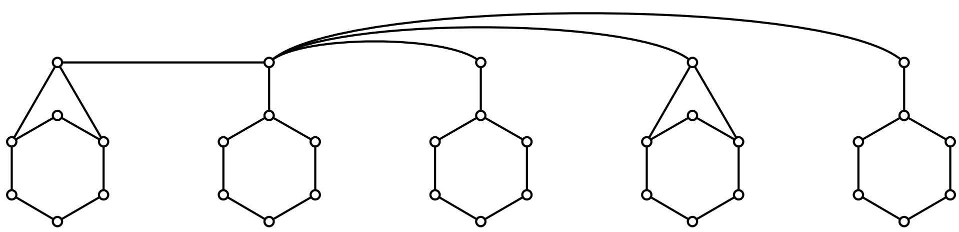
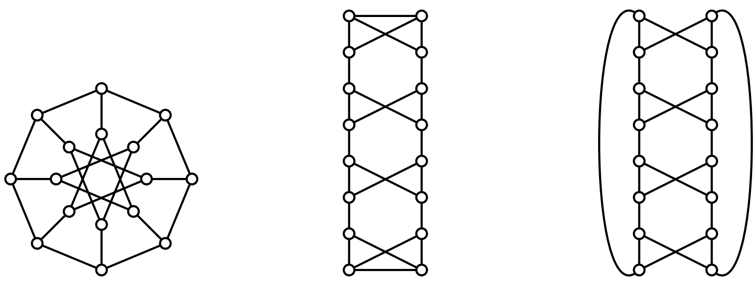


Figure 8. A graph in the family $\mathcal{G}_{\rm tdom}$.

In 2004 Archdeacon, *et al.* [2] provided an elegant graph theory proof using Brooks' Coloring Theorem that the total domination number of a graph with minimum degree at least 3 is at most one-half its order. This result can also be proven as an application of a hypergraph result due to Tuza [72] and Chvátal and McDiarmid [16].

Theorem 28 [2, 16, 72]**.** *If G is a graph of order n with $\delta(G) \ge 3$, then $\gamma_t(G) \le \frac{1}{2}n$.*

Let GP_{16} be the generalized Petersen graph shown in Figure 9(a). For $k \geq 1$, let G_k be the graph constructed as follows. Consider two copies of the path P_{2k} with respective vertex sequences $a_1b_1a_2b_2 \cdots a_kb_k$ and $c_1d_1c_2d_2 \cdots c_kd_k$. For each $i \in [k]$, join a_i to d_i and b_i to c_i. To complete the construction of the graph G_k join a_1 to c_1 and b_k to d_k. Let $\mathcal{G}_{\mathrm{cubic}} = \{G_k : k \geq 1\}$. For $k \geq 2$, let H_k be obtained from G_k by deleting the two edges a_1c_1 and b_kd_k and adding the two edges a_1b_k and c_1d_k. Let $\mathcal{H}_{\mathrm{cubic}} = \{H_k : k \geq 2\}$. The graphs $G_4 \in \mathcal{G}_{\mathrm{cubic}}$ and $H_4 \in \mathcal{H}_{\mathrm{cubic}}$ are illustrated in Figure 9(b) and 9(c), respectively. In 2008 Henning and Yeo [49] characterized the extremal graphs that achieve equality in the bound of Theorem 28. In particular, we note that every extremal graph G is a cubic graph of order n where $n \equiv 0 \pmod 4$.



(a) GP_{16} (b) G_4 (c) H_4

Figure 9. The graph GP_{16}, and the graphs $G_4 \in \mathcal{G}_{\mathrm{cubic}}$ and $H_4 \in \mathcal{H}_{\mathrm{cubic}}$.

Theorem 29 [49]**.** *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_t(G) = \frac{1}{2}n$ if and only if $G \in \mathcal{G}_{\mathrm{cubic}} \cup \mathcal{H}_{\mathrm{cubic}}$ or G is the generalized Petersen graph* GP_{16}.

In 2007 Thomassé and Yeo [71] established a best possible upper bound on the total domination number of a graph with minimum degree at least 4.

Theorem 30 [71]**.** *If G is a graph of order n with $\delta(G) \geq 4$, then $\gamma_t(G) \leq \frac{3}{7}n$.*

The bound of Theorem 30 is best possible, but is only achieved by the bipartite complement of the Heawood graph, shown in Figure 10, which is the bipartite graph formed by taking the two partite sets of the Heawood graph and joining a vertex from one partite set to a vertex from the other partite set by an edge whenever they are not joined in the Heawood graph. A proof of this result can be found in [50, Theorem 5.18]. We remark that the bipartite complement of the Heawood graph is the incidence bipartite graph of the complement of the Fano plane.

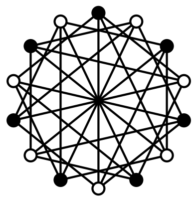


Figure 10. The bipartite complement of the Heawood graph.

Theorem 31 [50, Theorem 5.18]. *If G is a connected graph of order n with $\delta(G) \ge 4$, then $\gamma_t(G) \le \frac{3}{7}n$, with equality if and only if G is the bipartite complement of the Heawood graph.*

In 2016 Eustis, Henning, and Yeo [24] established the best known upper bound to date on the total domination number of a graph with minimum degree at least 5.

Theorem 32 [24]. *If G is a graph of order n with $\delta(G) \ge 5$, then $\gamma_t(G) \le \frac{2453}{6500}n \approx 0.3773846\, n$.*

In 2021 Henning and Yeo [52] proved that if G is a graph of order n with $\delta(G) \ge 6$, then $\gamma_t(G) \le \frac{5138}{14145}n \approx 0.363238\, n$. This upper bound was recently improved slightly by the same authors in [53].

Theorem 33 [53]. *If G is a graph of order n with $\delta(G) \ge 6$, then $\gamma_t(G) \le \frac{4549}{13299}n \approx 0.342056\, n$.*

In 2007 Henning and Yeo [48] presented a heuristic algorithm that yields an upper bound on the total domination of a graph in terms of its order and minimum degree.

Theorem 34 [48]. *If G is a graph of order n with minimum degree $\delta \ge 1$, then*

$$\gamma_t(G) \le \left(\frac{1 + \ln \delta}{\delta} \right) n.$$

Further, using a greedy algorithm we can in time complexity $\mathrm{O}(n + \delta\, n)$ find a total dominating set T in the graph G such that $|T| \le \left(\frac{1+\ln \delta}{\delta}\right) n$.

Although the bound on the total domination in Theorem 34 is not very good for small values of δ, it is asymptotically (that is, when $\delta \to \infty$) optimal as shown by Alon [1] and Thomasse and Yeo [71]. In 2019 Henning and Rad [45] gave the following slightly improved probabilistic upper bound on the total domination number.

Theorem 35 [45]**.** *If G is a graph with minimum degree $\delta \geq 2$ and maximum degree Δ, then*

$$\gamma_t(G) \leq n\left(\frac{1+\ln\delta}{\delta}\right) - n\left(1-\frac{1}{\delta}\right)\left(\frac{1}{\Delta}\right)^{\frac{1}{\delta-1}}\left(\frac{\ln\delta}{\delta}\right)^{1+\Delta(\Delta-1)}.$$

In 2013 Henning and Yeo [51] showed that the domination and total domination numbers enjoy a tight concentration in the Erdős-Rényi random graph $\mathcal{G}(n,p)$. In this model for a positive integer n and real number p with $0 < p < 1$, $\mathcal{G}(n,p)$ denotes the probability space whose elements G are all possible graphs of order n where an edge is chosen to be in G with probability p and independently of the choice for any other edge. An element $G \in \mathcal{G}(n,p)$ is called a *random graph.*

Theorem 36 [51]**.** *Let $G \in \mathcal{G}(n,p)$ be a random graph on n vertices with*

$$p = (1+\epsilon')\sqrt{\frac{2\ln n}{n}}.$$

For every $0 < \epsilon' < \epsilon$, asymptotically almost surely the graph G has diameter two and

$$\left(\frac{1}{2\sqrt{2}} - \epsilon\right)\sqrt{n\,\ln(n)} < \gamma(G) \leq \gamma_t(G) < \left(\frac{1}{\sqrt{2}} + \epsilon\right)\sqrt{n\,\ln(n)}.$$

Recall that $\mathcal{G}_{n,m}$ denote the class of all connected graphs of order $n \geq 3$ and size m. In 2018 Henning [44] proved that all the upper bounds on the total domination number of a graph in the family $\mathcal{G}_{n,m}$ in terms of its order and size can be written in the following unified form.

Theorem 37 [44]**.** *All upper bounds on the total domination number of a graph $G \in \mathcal{G}_{n,m}$ in terms of its order n and size m can be written in the unified form*

$$\gamma_t(G) \leq \frac{2an+2bm}{3a+2b}$$

for constants $a, b \in \mathbb{R}$ where $b \geq 0$ and $a > -\frac{2}{3}b$.

An equivalent formulation of Theorem 37 gives us the following general upper bound on the total domination number of a graph in terms of its order and size.

Theorem 38 [44]**.** *Let $a, b \in \mathbb{R}$. Then, the bound $\gamma_t(G) \leq an + bm$ is valid for every graph $G \in \mathcal{G}_{n,m}$ if and only if both $3a + 2b \geq 2$ and $b \geq 0$ hold.*

For example, to illustrate Theorem 38, taking $(a,b) = \left(\frac{2}{3}, 0\right)$ yields Theorem 25. Taking $(a,b) = \left(\frac{1}{2}, \frac{1}{4}\right)$, yields the upper bound $\gamma_t(G) \leq \frac{1}{4}(2n+m)$, while taking $(a,b) = \left(\frac{1}{3}, \frac{1}{2}\right)$, yields the upper bound $\gamma_t(G) \leq \frac{1}{6}(2n+3m)$.

More generally, for $k \geq 2$ let $\mathcal{G}_{n,m,k}$ denote the class of all connected graphs of order n and size m with minimum degree at least k. The following unified forms of upper bounds for the total domination number of a graph in the family $\mathcal{G}_{n,m,k}$ for small $k \in \{2,3,4\}$ are given in [44]

Theorem 39 [44]**.** *For $a, b \in \mathbb{R}$, the following hold.*

(a) *The bound $\gamma_t(G) \leq an + bm$ is valid for every graph $G \in \mathcal{G}_{n,m,2}$ if and only if both $b \geq 0$ and $a \geq \frac{2}{3} - b$ hold.*

(b) *The bound $\gamma_t(G) \leq an + bm$ is valid for every graph $G \in \mathcal{G}_{n,m,3}$ if and only if both $b \geq 0$ and $a \geq \frac{1}{2}(1-3b)$ hold.*

(c) *The bound $\gamma_t(G) \leq an + bm$ is valid for every graph $G \in \mathcal{G}_{n,m,4}$ if and only if both $b \geq 0$ and $a \geq \frac{3}{7} - 2b$ hold.*

For example, taking $(a,b) = (\frac{1}{3}, \frac{1}{3})$ in Theorem 39(a) we have that $\gamma_t(G) \leq \frac{1}{3}(n+m)$ for every graph $G \in \mathcal{G}_{n,m,2}$. Taking $(a,b) = \left(\frac{1}{2}, 0\right)$ in Theorem 39(b) yields Theorem 28. Taking $(a,b) = \left(\frac{3}{7}, 0\right)$ in Theorem 39(c) yields Theorem 30.

4. The Independent Domination Number

A set D of vertices is an *independent dominating set*, abbreviated ID-set, of a graph G if D is both a dominating set and an independent set. Equivalently, an ID-set of G is a maximal independent set in G. The *independent domination number* $i(G)$ of G is the minimum cardinality of an ID-set of G. An *i-set of G* is an ID-set of G of cardinality $i(G)$. A trivial upper bound on the independent domination number in terms of the order and maximum degree is given by the following observation.

Observation 40. *If G is a graph of order n, then $i(G) \leq n - \Delta(G)$.*

The following observation is immediate from the definition of an ID-set.

Observation 41. *If G is a graph, then $\gamma(G) \leq i(G)$.*

In 1979 Bollobás and Cockayne [7] proved the following upper bound on the independent domination number.

Theorem 42 [7]**.** *If G is an isolate-free graph of order n with domination number γ, then $i(G) \leq n + 2 - \gamma - \frac{n}{\gamma}$.*

Treating n as fixed, the function $f(\gamma) = n + 2 - \gamma - \frac{n}{\gamma}$ is maximized at $\gamma = \sqrt{n}$. Thus since $f(\sqrt{n}\,) = n + 2 - 2\sqrt{n}$, Favaron [25] was the first to observe the following upper bound on the independent domination number of a graph in terms of its order.

Theorem 43 [25]**.** *If G is an isolate-free graph of order n, then $i(G) \leq n + 2 - 2\sqrt{n}$.*

Moreover, in 1988 Favaron [25] conjectured a more general upper bound on the independent domination number of a graph in terms of its order n and minimum degree δ. Her conjecture was proven for $\delta = 2$ in 1998 by Glebov and Kostochka [26], and finally proven in 1999 for all δ by Sun and Wang [70].

Theorem 44 [70]**.** *If G is a graph of order n with minimum degree δ, then $i(G) \leq n + 2\delta - 2\sqrt{\delta n}$.*

Theorem 43 is a special case of Theorem 44 when $\delta \geq 1$. Favaron [25] showed that for every positive integer δ, the bound in Theorem 44 is attained for infinitely many graphs as follows. For $\delta \geq 1$ and $\ell \geq 2$, let G_i be the complete bipartite graph $K_{\delta,\delta(\ell-1)}$ with partite sets X_i and Y_i where $|X_i| = \delta$ and $|Y_i| = \delta(\ell - 1)$ for $i \in [\ell]$. Let $G_{\delta,\ell}$ be the graph obtained from the disjoint union of the graphs $G_1, G_2, \ldots, G_\ell$ by adding all edges between the sets X_i and X_j for all i and j, where $i, j \in [\ell]$ and $i \neq j$. The resulting graph $G = G_{\delta,\ell}$ has order $n = \delta\ell^2$ and satisfies $i(G) = \delta + \delta(\ell - 1)^2 = n + 2\delta - 2\sqrt{\delta n}$. Thus the bound in Theorem 44 is tight.

We remark that this infinite class of graphs constructed by Favaron are far from regular. If we impose a regularity requirement, then the upper bound in Theorem 44 can be significantly improved. In 1964 Rosenfeld [67] showed that every maximal independent set in a regular graph has cardinality at most one-half its order, implying that if G is a regular graph of order n, then $i(G) \leq \frac{1}{2}n$. Goddard *et al.* [29] showed that equality only holds for graphs with every component a balanced complete bipartite graph.

Theorem 45 [29]**.** *For $r \geq 1$, if G is a connected r-regular graph of order n, then $i(G) \leq \frac{1}{2}n$, with equality if and only if $G = K_{r,r}$.*

A natural question is to improve the upper bound in Theorem 45 for r-regular graphs different from $K_{r,r}$. If G is a connected 2-regular graph of order n different from $K_{2,2}$, then G is a cycle C_n where $n = 3$ or $n \geq 5$. Since $i(C_n) = \lceil \frac{1}{3}n \rceil$, this yields the following observation.

Observation 46. *If $G \neq K_{2,2}$ is a connected 2-regular graph of order n, then $i(G) \leq \frac{3}{7}n$, with equality if and only if $G = C_7$.*

In 1999 Lam, Shiu, and Sun [61] provided a best possible upper bound on the independent domination number of a 3-regular (cubic) graph different from $K_{3,3}$.

Theorem 47 [61]**.** *If $G \neq K_{3,3}$ is a connected 3-regular graph of order n, then $i(G) \leq \frac{2}{5}n$, and this bound is best possible.*

The bound in Theorem 47 is achieved by the 5-prism, $G = C_5 \,\square\, K_2$, namely the Cartesian product of a 5-cycle with a copy of K_2 shown in Figure 11. In this case, G is a connected 3-regular graph of order $n = 10$ satisfying $i(G) = 4 = \frac{2}{5}n$.

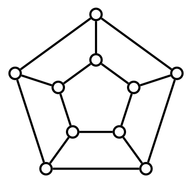


Figure 11. The 5-prism $C_5 \,\square\, K_2$.

In 2013 Goddard and Henning [28] posed the conjecture that if $G \neq K_{4,4}$ is a connected 4-regular graph of order n, then $i(G) \le \frac{3}{7}n$. In 2021 Cho, Choi and Park [15] announced they had settled this conjecture in the affirmative. For this purpose, they proved a much stronger result.

Theorem 48 [15]**.** *For $r \ge 3$, if $G \neq K_{r,r}$ is a connected r-regular graph of order n, then $i(G) \le \left(\frac{r-1}{2r-1}\right)n$.*

As a special case of Theorem 48 when $r = 3$ we have the result of Theorem 47, and when $r = 4$ we have the $\frac{3}{7}n$-bound conjectured in [28].

The bound in Theorem 48 in the case when $r = 4$ is achieved by the so-called expansion $G = \exp(C_7, 2)$ of a 7-cycle shown in Figure 12. In this case, G is a connected 4-regular graph of order $n = 14$ satisfying $i(G) = 6 = \frac{3}{7}n$. However, it is not yet known if the bound in Theorem 48 is achievable for any $r \ge 5$.

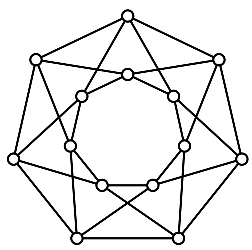


Figure 12. The expansion $\exp(C_7, 2)$.

5. The Connected Domination Number

A set D of vertices is a *connected dominating set*, abbreviated CD-set, of a connected graph G if D is a dominating set of G with the additional property that the induced subgraph $G[D]$ is connected. The *connected domination number* of

G, denoted by $\gamma_c(G)$, is the minimum cardinality of a CD-set of G. A *γ_c-set of G* is a CD-set of G of cardinality $\gamma_c(G)$. In 1984 Hedetniemi and Laskar [40] proved that D is a minimal CD-set of G if and only if D is the set of non-leaf vertices of a spanning tree of G. Thus if $\epsilon_T(G)$ denotes the number of leaves of a spanning tree T of G, then this yields the following result.

Theorem 49 [40]**.** *If G is a connected graph of order n, then*

$$\gamma_c(G) = n - \max_{T \in \mathcal{T}_G} \epsilon_T(G),$$

where $\mathcal{T}_G$ is the set of all spanning trees of G.

By Theorem 49, $n - \gamma_c(G)$ is the maximum number of leaves in a spanning tree of a graph G of order n. Thus to determine the connected domination number of a graph, it suffices to determine the maximum number of leaves among all spanning trees of G. The problem of finding a spanning tree with maximum number of leaves is NP-complete, even for 4-regular graphs. Since $\epsilon_T(G) \geq \Delta(G)$, this yields the following trivial upper bound on the connected domination number.

Observation 50. *If G is a connected graph of order n, then $\gamma_c(G) \leq n - \Delta(G)$.*

By definition every CD-set is a TD-set, yielding the following lower bound on the connected domination number.

Observation 51. *If G is a connected isolate-free graph with $\gamma(G) > 1$, then $\gamma_t(G) \leq \gamma_c(G)$.*

We construct next a class of graphs with large connected domination number. For integers $k \geq 3$ and $d \geq 2$, take d disjoint copies $D_1, D_2, \ldots, D_d$ of a clique $K_{k+1} - e$ with an edge removed, where $a_i b_i$ is the missing edge in D_i. Let $N_{k,d}$ be obtained from the disjoint union of these d graphs by adding the edges $\{a_i b_{i+1} \colon i \in [d-1]\}$ and adding the edge $a_d b_1$. We call $N_{k,d}$ a *k-diamond-necklace with d diamonds*. Let $\mathcal{N}_k = \{N_{k,d} \colon d \geq 2\}$. A 3-diamond-necklace $G = N_{3,6} \in \mathcal{N}_3$ with six diamonds is illustrated in Figure 13, where the 16 black vertices form a γ_c-set of G. More generally, if $G = N_{k,d} \in \mathcal{N}_k$, then G has order $n = (k+1)d$ and $\gamma_c(G) = 3(d-1) + 1 = \left(\frac{3}{k+1}\right)n - 2$.

Proposition 52. *For $k \geq 3$, if $G \in \mathcal{N}_k$ has order n, then G is a connected k-regular graph satisfying $\gamma_c(G) = \left(\frac{3}{k+1}\right)n - 2$.*

Let $\mathcal{G}_{n,k}$ denote the collection of connected graphs of order $n \geq 3$ with minimum degree at least k. Let $\ell(n,k)$ denote the maximum ℓ such that every graph in $\mathcal{G}_{n,k}$ contains a spanning tree with at least ℓ leaves.

Observation 53. *If $G \in \mathcal{G}_{n,k}$, then $\gamma_c(G) \leq n - \ell(n,k)$.*

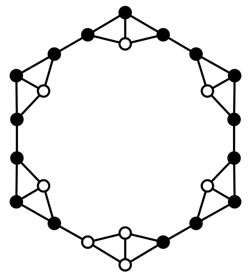


Figure 13. A 3-diamond-necklace $N_{3,6}$ with six diamonds.

We observe that $\ell(n,1) = \ell(n,2) = 2$. Thus by Observation 53, we have the following result.

Observation 54. *If G is a connected graph of order $n \geq 3$ with $\delta(G) = 1$ or $\delta(G) = 2$, then $\gamma_c(G) \leq n - 2$.*

In 1991 Kleitman and West [58] showed that $\ell(n,3) \geq \frac{1}{4}n + 2$ and $\ell(n,4) \geq \frac{2}{5}n + \frac{8}{5}$, yielding the following upper bounds on the connected domination number of a graph.

Theorem 55 [58]**.** *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_c(G) \leq \frac{3}{4}n - 2$.*

Theorem 56 [58]**.** *If G is a connected graph of order n with $\delta(G) \geq 4$, then $\gamma_c(G) \leq \frac{3}{5}n - \frac{8}{5}$.*

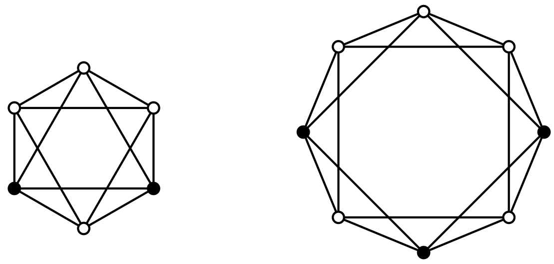
The case when the minimum degree is at least 5 was settled by Griggs and Lu [31].

Theorem 57 [31]**.** *If G is a connected graph of order n with $\delta(G) \geq 5$, then $\gamma_c(G) \leq \frac{1}{2}n - 2$.*

The bounds of Theorem 55 and 57 are tight, as shown by the family of graphs $\mathcal{N}_3$ and $\mathcal{N}_5$ (see, Proposition 52). The only known graph that achieves equality in the bound of Theorem 56 is the circulant $C_6\langle 1,2\rangle$, illustrated in Figure 14(a), of order $n = 6$ with jumps 1 and 2. The circulant $G = C_8\langle 1,2\rangle$, illustrated in Figure 14(b), of order $n = 8$ with jumps 1 and 2 satisfies $\gamma_c(G) = \frac{3}{5}n - \frac{9}{5}$.

Kleitman and West [58] conjectured that if we exclude the circulants $C_6\langle 1,2\rangle$ and $C_8\langle 1,2\rangle$, then the bound in Theorem 56 can be improved slightly to $\gamma_c(G) \leq \frac{3}{5}n - 2$, which as shown by the family of graphs $\mathcal{N}_4$ (see Proposition 52) is achieved by infinitely many graphs.

In 2000 Caro, West, and Yuster [13] proved the following result.



(a) $C_6\langle 1,2\rangle$ (b) $C_8\langle 1,2\rangle$

Figure 14. The circulants $C_6\langle 1,2\rangle$ and $C_8\langle 1,2\rangle$.

Theorem 58 [13]**.** *If G is a connected graph of order n with $\delta = \delta(G)$, then*

$$\gamma_c(G) \le \left(\frac{100 + 0.5\sqrt{\ln(\delta+1)} + \ln(\delta+1)}{\delta+1}\right) n.$$

As a consequence of Theorem 58, we have the following result, where by $\mathrm{o}_\delta(1)$ we mean some quantity that tends to 0 as δ gets large.

Theorem 59 [13]**.** *If G is a connected graph of order n with $\delta = \delta(G)$, then*

$$\gamma_c(G) \le (1 + \mathrm{o}_\delta(1)) \left(\frac{\ln(\delta+1)}{\delta+1}\right) n.$$

By Observation 51, and by a result of Thomasse and Yeo [71] on total domination in graphs, we have the following result.

Theorem 60 [71]**.** *For every $\epsilon > 0$ and for δ sufficiently large, there exists a bipartite δ-regular graph G of order n satisfying*

$$\gamma_c(G) > (1 - \epsilon) \left(\frac{\ln(\delta)}{\delta}\right) n.$$

Using a deterministic algorithm, Caro, West, and Yuster [13] proved the following result.

Theorem 61 [13]**.** *For every $\epsilon > 0$ and δ sufficiently large, if G is a connected graph of order n with $\delta = \delta(G)$, then*

$$\gamma_c(G) \le (1 + \epsilon) \left(\frac{\ln(\delta)}{\delta}\right) n.$$

6. The Paired Domination Number

A set D of vertices is a *paired dominating set*, abbreviated PD-set, of a graph G if D is a dominating set of G with the additional property that the induced subgraph $G[D]$ contains a perfect matching M (not necessarily induced). The *paired domination number* of G, denoted by $\gamma_{\rm pr}(G)$, is the minimum cardinality of a PD-set of G. As with total domination in graphs, paired domination is defined only for isolate-free graphs. Necessarily, the paired domination number of a graph is an even integer. The set of vertices that are incident with the edges of a maximal matching in an isolate-free graph G is a PD-set of G, which yields the following trivial upper bound on the paired domination number.

Observation 62. *If G is an isolate-free graph, then $\gamma_{\rm pr}(G) \le 2\alpha'(G)$.*

By definition every PD-set is a TD-set, yielding the following lower bound on the paired domination number.

Observation 63. *If G is an isolate-free graph, then $\gamma_t(G) \le \gamma_{\rm pr}(G)$.*

In 1998 Haynes and Slater [39] were the first to observe that using properties of a γ-set of a graph due to Bollobás and Cockayne [7], the paired domination number is at most twice the domination number.

Theorem 64 [39]**.** *If G is an isolate-free graph, then $\gamma_{\rm pr}(G) \le 2\gamma(G)$.*

Haynes and Slater [39] established the following lower bound on the paired domination number of a graph in terms of its order and maximum degree.

Theorem 65 [39]**.** *If G is an isolate-free graph of order n, then $\gamma_{\rm pr}(G) \ge \frac{n}{\Delta(G)}$.*

As in the previous section, we will focus on bounds on the paired domination number in terms of the order and minimum degree of the graph. Haynes and Slater [39] obtained the following upper bound on the paired domination number of a connected graph of order at least 3. For $k \ge 2$ the subdivided star $G = S(K_{1,k})$ is obtained from a star $K_{1,k}$ by subdividing every edge exactly once.

Theorem 66 [39]**.** *If G is a connected graph of order $n \ge 3$, then $\gamma_{\rm pr}(G) \le n-1$ with equality if and only if G is the cycle C_3, the cycle C_5 or a subdivided star $S(K_{1,k})$ for $k \ge 1$.*

If we exclude the 3-cycle and the 5-cycle, then the trivial bound of one less than the order in Theorem 66 can be improved if the minimum degree is at least 2, as shown by Huang and Shan [54].

Theorem 67 [54]**.** *If G is a connected graph of order $n \ge 6$ with minimum degree $\delta(G) \ge 2$, then $\gamma_{\rm pr}(G) \le \frac{2}{3}n$.*

The extremal graphs achieving equality in Theorem 67 were characterized by Henning [43], where $\mathcal{F}_{\text{pdom}} = \{F_1, F_2, \ldots, F_{10}\}$ is the family of ten graphs shown in Figure 15.

Theorem 68 [43]**.** *If G is a connected graph of order $n \geq 6$ with $\delta(G) \geq 2$, then $\gamma_{\text{pr}}(G) \leq \frac{2}{3}n$, with equality if and only if $G \in \mathcal{F}_{\text{pdom}}$.*

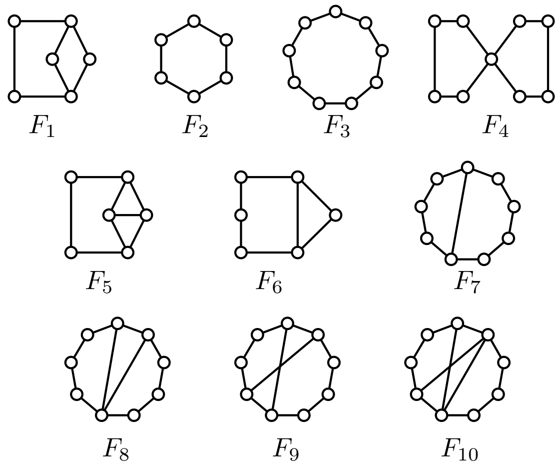


Figure 15. The ten graphs in the family $\mathcal{F}_{\text{pdom}}$.

Every graph in the family $\mathcal{F}_{\text{pdom}}$ has order equal to 6 or 9. If we restrict the order of a graph to at least 10, then the upper bound in Theorem 68 can only be improved slightly as shown by the bound in Theorem 69, which is tight in the sense that the bound is achieved for connected graphs of arbitrarily large order. The extremal graphs achieving equality in the bound of Theorem 69 were characterized in [43].

Theorem 69 [43]**.** *If G is a connected graph of order $n \geq 10$ with $\delta(G) \geq 2$, then $\gamma_{\text{pr}}(G) \leq \frac{2}{3}(n-1)$, and this bound is tight.*

Chen, Sun, and Xing [14] established a best possible upper bound on the paired domination number of a cubic graph.

Theorem 70 [14]**.** *If G is a cubic graph of order n, then $\gamma_{\text{pr}}(G) \leq \frac{3}{5}n$.*

Subsequently, Goddard and Henning [27] showed that the only connected graph achieving equality in Theorem 70 is the Petersen graph G_{10}, shown in Figure 16. In this case, if $G = G_{10}$, then G is a cubic graph of order $n = 10$ and $\gamma_{\text{pr}}(G) = 6 = \frac{3}{5}n$.

Henning, Pilśniak, and Tumidajewicz [47] provided the best known upper bound to date on the paired domination of a graph with minimum degree at least 3. However, it is unlikely that their bound (given in Theorem 71) is achievable.

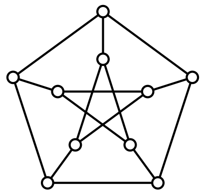


Figure 16. The Petersen graph G_{10}.

Theorem 71 [47]**.** *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_{\rm pr}(G) \leq \frac{19037}{30000} n < 0.634567\, n$.*

7. The Restrained Domination Number

A set D of vertices is a *restrained dominating set*, abbreviated RD-set, of a graph G if D is a dominating set of G with the additional property that the induced subgraph $G[V(G) \setminus D]$ is isolate-free. The *restrained domination number* of G, denoted by $\gamma_r(G)$, is the minimum cardinality of an RD-set of G. An upper bound on the restrained domination number of a graph in terms of its maximum degree was presented in 2007 by Dankelmann, Day, Hattingh, Henning, Markus, and Swart [19].

Theorem 72 [19]**.** *If G is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_r(G) \leq n - \Delta(G)$.*

In 1999 Domke, Hattingh, Hedetniemi, Laskar, and Markus [22] established the following upper bound on the restrained domination number of a connected graph.

Theorem 73 [22]**.** *If G is a connected graph of order $n \geq 2$, then $\gamma_r(G) \leq n-2$, unless G is a star $K_{1,n-1}$, in which case $\gamma_r(G) = n$.*

The infinite family of connected graphs of order $n \geq 2$ satisfying $\gamma_r(G) = n-2$ is characterized in [22].

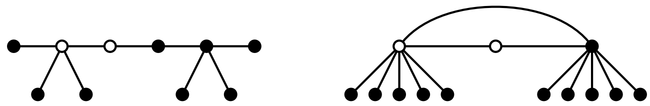
Theorem 74 [22]**.** *If G is a connected graph of order $n \geq 3$ satisfying $\gamma_r(G) = n - 2$, then one of the following hold.*

(a) $G \in \{P_4, P_5, C_4, C_5\}$.

(b) *G is obtained from P_6 by adding leaf neighbors, including the possibility of none, to the support vertices of the path.*

(c) *G is obtained from C_3 by adding leaf neighbors, including the possibility of none, to at most two of the vertices of the cycle.*

Graphs with properties in the statement of Theorem 74(b) and (c) are illustrated in Figures 17(a) and (b), respectively. In these two examples, the black vertices are examples of γ_r-sets of the respective graphs.



(a) (b)

Figure 17. Examples of graphs G of order n with $\gamma_r(G) = n - 2$.

Let $\mathcal{B}_{\rm rdom} = \{R_1, R_2, \ldots, R_8\}$ be the family of eight graphs shown in Figure 18.

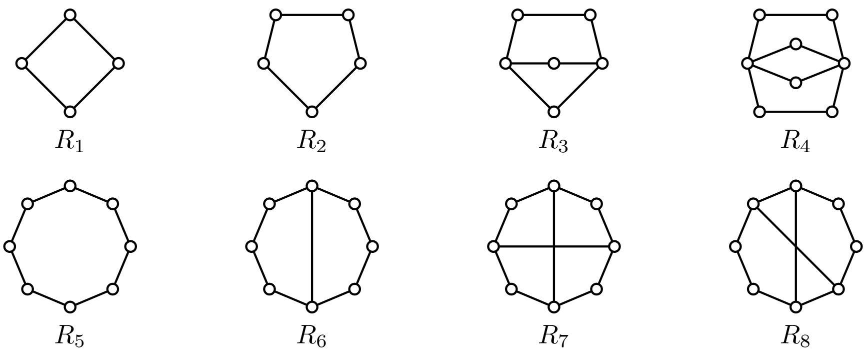


Figure 18. The family $\mathcal{B}_{\rm rdom}$.

In 2000 Domke, Hattingh, Henning, and Markus [23] established the following upper bound on the restrained domination number of a graph with minimum degree at least 2.

Theorem 75 [23]. *If G is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_r(G) \leq \frac{1}{2}(n-1)$, unless G is one of the eight exceptional graphs in the family $\mathcal{B}_{\rm rdom}$.*

The graphs achieving equality in the upper bound in Theorem 75 were characterized in [41]. We observe that if $G \in \mathcal{B}_{\rm rdom}$ has order n, then $\gamma_r(G) = \frac{1}{2}n$, unless G is the 5-cycle R_2, in which case $\gamma_r(G) = 3 = \frac{1}{2}(n+1)$. Thus as an immediate consequence of Theorem 75, we have the following result.

Theorem 76 [23]. *If $G \neq C_5$ is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_r(G) \leq \frac{1}{2}n$.*

In 2011 Hattingh and Joubert [32] established the best upper bound to date in the restrained domination number of a cubic graph.

Theorem 77 [32]**.** *If G is a cubic graph of order n, then $\gamma_r(G) \le \frac{5}{11}n$.*

In an unpublished manuscript in the last 1990s, Cockayne gave the following probabilistic upper bounds on the restrained domination number.

Theorem 78. *If G is a graph of order n and minimum degree $\delta = 3$, then $\gamma_r(G) \le 0.668\, n$.*

Theorem 79. *If G is a graph of order n and minimum degree $\delta \ge 4$, then*

$$\gamma_r(G) \le \left(1 - \frac{2\delta}{(2\delta+2)^{1+\frac{1}{\delta}}}\right) n.$$

8. The Total Restrained Domination Number

A set D of vertices is a *total restrained dominating set*, abbreviated TRD-set, of an isolate-free graph G if D is a TD-set of G with the additional property that the induced subgraph $G[V(G) \setminus D]$ is isolate-free. The *total restrained domination number* of G, denoted by $\gamma_{\rm tr}(G)$, is the minimum cardinality of a TRD-set of G. A best possible upper bound on the total restrained domination number of a graph in terms of its maximum degree was presented in 2008 by Henning and Maritz [46].

Theorem 80 [46]**.** *If G is a connected graph of order $n \ge 4$ with $\delta(G) \ge 2$ and $\Delta(G) \le n-2$, then $\gamma_{\rm tr}(G) \le n - \frac{1}{2}\Delta(G) - 1$.*

We note that if G is a star $K_{1,n-1}$, then $\gamma_{\rm tr}(G) = n$. This yields the following trivial sharp upper bound on the total restrained domination number.

Observation 81. *If G is a connected graph of order $n \ge 2$, then $\gamma_{\rm tr}(G) \le n$, and this bound is tight.*

If G is a graph of order n and every component of G is a copy of K_3, then $\gamma_{\rm tr}(G) = n$. In view of this observation, together with Observation 81, it is only of interest to determine upper bounds on the total restrained domination number of a connected graph of order $n \ge 4$ with minimum degree at least 2. For this class of graphs, in 2019 Joubert [56] proved the following upper bound.

Theorem 82 [56]**.** *If G is a connected graph of order $n \ge 4$ with $\delta(G) \ge 2$, then*

$$\gamma_{\rm tr}(G) \le n - \sqrt{\frac{n}{2}}.$$

As remarked in [46], for any integer $k \geq 2$, there exists an infinite family of connected graphs G with $\delta(G) = k$ such that $\gamma_{\rm tr}(G)/n(G)$ tends to one when $n(G)$ goes to infinity. For integers r and k where $r \geq 2k$ and $k \geq 2$, let $G = G_{r,k}$ be the bipartite graph formed by taking as one partite set a set A of r elements, and as the other partite set B all the k-element subsets of A, and joining each element of A to those subsets in B it is a member of. The resulting graph G has order $n = |A| + |B| = r + \binom{r}{k}$. Each vertex in B has degree k, and each vertex in A has degree $\binom{r-1}{k-1} > k$, and so $\delta(G) = k$. Let D be a TRD-set of G. In order to totally dominate the vertices in B, the set D contains at least $r - k + 1$ vertices in A. Let $A_1 = A \cap D$. Let B_1 be the set of vertices of B all of whose k neighbors belong to A_1, and so $|B_1| = \binom{|A_1|}{k} \geq \binom{r-k+1}{k}$. Further if $|A_1| < r$, then to totally dominate the vertices in $A \setminus A_1$, the set D contains at least one vertex of $B \setminus B_1$. These observations imply that $\gamma_{\rm tr}(G) \geq (r - k + 1) + 1 + \binom{r-k+1}{k} = r - k + 2 + \binom{r-k+1}{k}$. Moreover, choosing A_1 to be an arbitrary set of $r - k + 1$ vertices in A, and letting B_1 be defined as before, and letting v be a vertex in B that is adjacent to all $k - 1$ vertices in $A \setminus A_1$, we produce a TRD-set of G, implying that $\gamma_{\rm tr}(G) \leq r - k + 2 + \binom{r-k+1}{k}$. Consequently,

$$\gamma_{\rm tr}(G) = r - k + 2 + \binom{r-k+1}{k}.$$

Thus,

$$\frac{\gamma_{\rm tr}(G)}{n(G)} = \frac{r - k + 2 + \binom{r-k+1}{k}}{r + \binom{r}{k}},$$

which tends to one when r tends to infinity (treating k as fixed). In the special case when $k = 2$, we have $n = r + \binom{r}{2}$ and $\gamma_{\rm tr}(G) = r + \binom{r-1}{2} = n - r + 1$, and so

$$\gamma_{\rm tr}(G) = n + \frac{3}{2} - \sqrt{\frac{n}{2} + \frac{1}{4}}.$$

Thus, the bound of Theorem 82 is asymptotically sharp.

9. The Semitotal Domination Number

A set D of vertices is a *semitotal dominating set*, abbreviated semi-TD-set, of an isolate-free graph G if it is a dominating set of G and every vertex in D is within distance 2 of another vertex of D. The *semitotal domination number*, denoted by $\gamma_{t2}(G)$, is the minimum cardinality of a semi-TD-set of G. A γ_{t2}-*set of* G is a semi-TD-set of G of cardinality $\gamma_{t2}(G)$. Since every TD-set is a semi-TD-set, and since every semi-TD-set is a dominating set, we have the following statement/fact first observed by Goddard, Henning and McPillan [30].

Observation 83 [30]. *For every graph G with no isolated vertex, $\gamma(G) \le \gamma_{t2}(G) \le \gamma_t(G)$.*

An upper bound on the semitotal domination number was established in [30].

Theorem 84 [30]. *If G is a connected graph of order $n \ge 4$, then $\gamma_{t2}(G) \le \frac{1}{2}n$.*

Equality is, for example, obtained in Theorem 84 for any connected graph G of order $n \ge 4$ satisfying $\gamma(G) = \frac{1}{2}n$, that is, by Theorem 6, if G is a 4-cycle or G is a corona $F \circ K_1$ for some connected graph F of order at least 2.

The generalized *corona* $F \circledast H$ of a graph F with a graph H is the graph obtained from F by adding for each vertex $v \in V(F)$ a copy of H and identifying one vertex in the copy of H with the vertex v in F. For example, the generalized *corona* $F \circledast H$ when $F = K_{1,4}$ is a star and H is a 4-cycle is illustrated in Figure 19. The resulting graph G has order $n = 20$ and $\gamma_{t2}(G) = 10 = \frac{1}{2}n$, and the ten black vertices are an example of a γ_{t2}-set of G. For this graph G every semi-TD-set must contain at least two vertices from each copy of C_4.

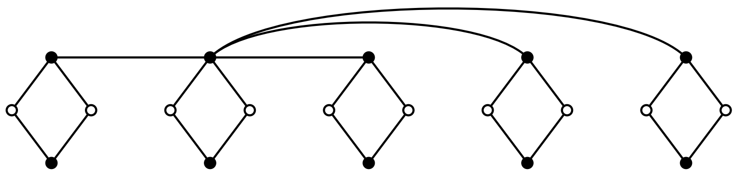


Figure 19. The corona $K_{1,4} \circledast C_4$.

The upper bound of Theorem 84 is therefore also sharp for graphs with minimum degree at least 2.

Theorem 85 [30]. *If G is a connected graph of order $n \ge 4$ with $\delta(G) \ge 2$, then $\gamma_{t2}(G) \le \frac{1}{2}n$, with equality if and only if G is C_6, C_8, a subgraph of K_4, or a generalized corona $F \circledast C_4$ where F is an arbitrary connected graph.*

For graphs with minimum degree $\delta \ge 3$, we have the same upper bound on the semitotal domination number as the domination number, and this bound is essentially best possible.

Theorem 86 [30]. *If G is a connected graph with order n and minimum degree δ, then*

$$\gamma_{t2}(G) \le (1 + o_\delta(1)) \left(\frac{\ln(\delta + 1)}{\delta + 1} \right) n.$$

10. THE SEMIPAIRED DOMINATION NUMBER

A set D of vertices is a *semipaired dominating set*, abbreviated semi-PD-set, of an isolate-free graph G if it is a dominating set of G and every vertex in D is paired with exactly one other vertex in D that is within distance 2 from it in G. In other words, the vertices in the dominating set D can be partitioned into 2-sets such that if $\{u, v\}$ is a 2-set, then $uv \in E(G)$ or the distance between u and v is 2. We say that u and v are *paired*. We call such a pairing a *semi-matching*. The *semipaired domination number*, denoted by $\gamma_{\rm pr2}(G)$, is the minimum cardinality of a semi-PD-set of G. A $\gamma_{\rm pr2}$-*set of* G is a semi-PD-set of G of cardinality $\gamma_{\rm pr2}(G)$. An example of a graph G with $\gamma_{\rm pr2}(G) = 6$ is illustrated in Figure 20, where the black vertices form a $\gamma_{\rm pr2}$-set of G and where vertices with the same label are paired.

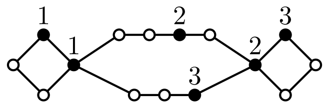


Figure 20. A graph G with $\gamma_{\rm pr2}(G) = 6$.

The semipaired domination number is squeezed between the domination number and the paired domination number.

Observation 87. *If G is an isolate-free graph, then $\gamma(G) \le \gamma_{\rm pr2}(G) \le \gamma_{\rm pr}(G) \le 2\gamma(G)$.*

In 2018 Haynes and Henning [36] established lower and upper bounds on the semipaired domination number of a graph in terms of its total domination number.

Theorem 88 [36]**.** *If G is an isolate-free graph, then*

$$\frac{2}{3}\gamma_t(G) \le \gamma_{\rm pr2}(T) \le \frac{4}{3}\gamma_t(G),$$

and these bounds are sharp.

The graph illustrated in Figure 21 that is obtained from a cycle C_4 by attaching a path of length 2 to one of its vertices is called the *stingray*, or just SR for short. The black vertices in Figure 21 form a $\gamma_{\rm pr2}$-set of G and vertices with the same label are paired.

In 2019 Haynes and Henning [37] showed that if G is a connected graph G of order $n \ge 3$, then $\gamma_{\rm pr2}(G) \le \frac{2}{3}n$, and they characterized the extremal graphs achieving equality in the bound.

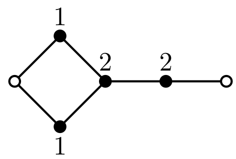


Figure 21. The stingray SR.

Theorem 89 [37]. *If G is a connected graph of order $n \geq 3$, then $\gamma_{\rm pr2}(G) \leq \frac{2}{3}n$, with equality if and only if one of the following hold.*

(a) *G is a cycle C_3 or a cycle C_6.*

(b) *G is the corona $P_3 \circ P_1$ of a path P_3.*

(c) *G is the corona $C_3 \circ P_1$ of a cycle C_3.*

(d) *G is the stingray SR.*

(e) *G is the 2-corona of a connected graph.*

As an immediate consequence of Theorem 89 we have the following result.

Corollary 90 [37]. *If G is a connected graph of order n with $\delta(G) \geq 2$, then $\gamma_{\rm pr2}(G) \leq \frac{2}{3}n$, with equality if and only if $G = C_3$ or $G = C_6$.*

In 2021 Haynes and Henning [38] improved the bound in Corollary 90.

Theorem 91 [37]. *If $G \neq C_6$ is a connected graph of order n with $\delta(G) \geq 2$, then*

$$\gamma_{\rm pr2}(G) \leq \begin{cases} \frac{n+1}{2} & \text{if } n \equiv 3 \pmod{4}, \\ \frac{n}{2} & \text{if } n \not\equiv 3 \pmod{4}. \end{cases}$$

Further, for every value of $n \geq 3$ where $n \equiv 3 \pmod{4}$, there exists a connected graph G of order n with $\delta(G) \geq 2$ satisfying $\gamma_{\rm pr2}(G) = \frac{1}{2}(n+1)$.

Theorem 92 [37]. *If G is a graph of order n with $\delta(G) \geq 3$, then $\gamma_{\rm pr2}(G) \leq \frac{1}{2}n$.*

The upper bound in Theorem 92 is best possible. For example, all connected cubic graphs G of order $n = 8$ satisfy $\gamma_{\rm pr2}(G) = 4 = \frac{1}{2}n$.

11. Open Problems and Conjectures

We close with a list of open problems and conjectures that, to the author's best knowledge, have yet to be solved. Throughout this section, for $k \geq 1$, let $\mathcal{G}_k$ denote the class of all connected graphs with minimum degree at least k containing no isolated edge. We note that $\mathcal{G}_1$ is the class of all connected graphs of order at least 3.

11.1. The domination number

In this section, we list open problems and conjectures pertaining to the domination number.

Problem 93. Characterize the connected graphs of order n with $\delta(G) \geq 3$ that satisfy $\gamma(G) = \frac{3}{8}n$.

Perhaps in Problem 93 it is even true that the extremal graphs that achieve equality in Reed's Theorem 11 all belong to the family $\mathcal{G}_{\mathrm{dom},\geq 3}$ constructed in Section 2, except for a finite exceptional family (which must include the graph $G_{8.1}$ in Figure 4(a)).

Problem 94. If $\mathcal{G}^n_{\mathrm{cubic}}$ denotes the family of all connected cubic graphs of order n, then determine limit of the supremum

$$\Phi^n_{\mathrm{cubic}} = \lim_{G \in \mathcal{G}^n_{\mathrm{cubic}}} \left(\sup_{n \to \infty} \frac{\gamma(G)}{n} \right).$$

It is known (see, [57, 59, 60]) that the supremum Φ^n_{cubic} in Problem 94 is sandwiched between the following values:

$$\frac{1}{3} + \frac{1}{60} \leq \Phi^n_{\mathrm{cubic}} \leq \frac{1}{3} + \frac{1}{42}.$$

It would be very interesting to determine precisely the value of the supremum, which would resolve a 25-year old problem in domination theory.

Problem 95. Determine or estimate the best possible constants $c_{\mathrm{dom},k}$ (which depends only on k) such that $\gamma(G) \leq c_{\mathrm{dom},k} \cdot n(G)$ for all $G \in \mathcal{G}_k$. These constants are given by

$$c_{\mathrm{dom},k} = \sup_{G \in \mathcal{G}_k} \frac{\gamma(G)}{n(G)}.$$

The constants $c_{\mathrm{dom},k}$ for small $k \in [3]$ are known. By Ore's Theorem 5, $c_{\mathrm{dom},1} = \frac{1}{2}$. By the McCuaig-Shepherd Theorem 9, $c_{\mathrm{dom},2} = \frac{1}{2}$, noting that if $G = B_1 \in \mathcal{B}_{\mathrm{dom}} \subset \mathcal{G}_2$, then G has order $n = 4$ and satisfies $\gamma(G) = 2 = \frac{1}{2}n$, while if $G \in \mathcal{B}_{\mathrm{dom}} \setminus \{B_1\} \subset \mathcal{G}_2$, then G has order $n = 7$ and satisfies $\gamma(G) = 3 = \frac{3}{7}n$. By Reed's Theorem 11, $c_{\mathrm{dom},3} = \frac{3}{8}$.

By Theorem 12, $c_{\mathrm{dom},4} \leq \frac{4}{11}$. However, it is not known if there is a graph of order n with $\delta(G) \geq 4$ satisfying $\gamma(G) = \frac{4}{11}n$. We observe that if G is the graph obtained from a complete graph K_6 by removing the edges of a perfect matching, then G is a 4-regular graph of order $n = 6$ satisfying $\gamma(G) = 2 = \frac{1}{3}n$.

By Theorem 13, $c_{\mathrm{dom},5} \leq \frac{1}{3}$. However, it is not known if there is a graph of order n with $\delta(G) \geq 5$ satisfying $\gamma(G) = \frac{1}{3}n$. We observe that if G is the complement of the graph $2K_3 \cup K_2$, then G is a graph of order $n = 8$ with $\delta(G) = 5$ satisfying $\gamma(G) = 2 = \frac{1}{4}n$.

By Theorem 14, $c_{\rm dom,6} \le \frac{127}{418}$. However, it is unlikely that the bound in Theorem 14 is achievable. We observe that if G is obtained from a complete graph K_8 by removing the edges of a perfect matching, then G is a 6-regular graph of order $n = 8$ satisfying $\gamma(G) = 2 = \frac{1}{4}n$. As a further example, if G is the 6-regular graph of order $n = 16$ shown in Figure 22, then $\gamma(G) = 4 = \frac{1}{4}n$, where the four black vertices form a γ-set of G.

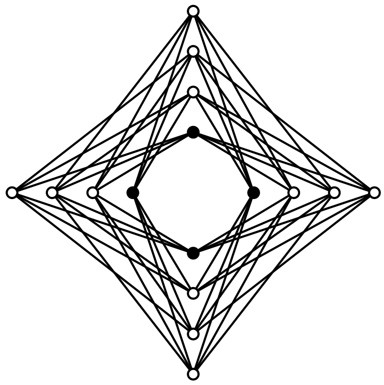


Figure 22. A 6-regular graph of order $n = 16$ with $\gamma(G) = 4 = \frac{1}{4}n$.

We summarize the above results and observations formally as follows.

Theorem 96. *The following hold.*

(a) $c_{\rm dom,1} = c_{\rm dom,2} = \frac{1}{2}$.

(b) $c_{\rm dom,3} = \frac{3}{8}$.

(c) $\frac{1}{3} \le c_{\rm dom,4} \le \frac{4}{11}$.

(d) $\frac{1}{4} \le c_{\rm dom,5} \le \frac{1}{3}$.

(e) $\frac{1}{4} \le c_{\rm dom,6} \le \frac{127}{418} < \frac{1}{4} + \frac{27}{500}$.

We believe the lower bounds in Theorem 96(c) and 96(e) are the correct values for $c_{\rm dom,4}$ and $c_{\rm dom,6}$, respectively, and pose the following conjectures.

Conjecture 97. $c_{\rm dom,4} = \frac{1}{3}$.

Conjecture 98. $c_{\rm dom,6} = \frac{1}{4}$.

11.2. The total domination number

Problem 99. Determine or estimate the best possible constants $c_{\rm tdom,k}$ (which depends only on k) such that $\gamma_t(G) \le c_{\rm tdom,k} \cdot n(G)$ for all $G \in \mathcal{G}_k$. These constants are given by

$$c_{\mathrm{tdom},k} = \sup_{G \in \mathcal{G}_k} \frac{\gamma_t(G)}{n(G)}.$$

The constants $c_{\mathrm{tdom},k}$ for small $k \in [4]$ are known. By Theorem 25, $c_{\mathrm{tdom},1} = \frac{2}{3}$. By Theorem 27, $c_{\mathrm{tdom},2} = \frac{2}{3}$, noting that if $G \in \{C_3, C_6\} \subset \mathcal{B}_{\mathrm{tdom}} \subset \mathcal{G}_2$, then $\gamma_t(G) = \frac{2}{3}n$, while if $G \in \mathcal{B}_{\mathrm{tdom}} \setminus \{C_3, C_6\} \subset \mathcal{G}_2$, then $\gamma_t(G) = \frac{3}{5}n$. By Theorem 28, $c_{\mathrm{dom},3} = \frac{1}{2}$. By Theorem 31, $c_{\mathrm{tdom},4}(G) = \frac{3}{7}$.

By Theorem 32, $c_{\mathrm{tdom},5}(G) \leq \frac{2453}{6500}$. It is not known if there is a graph G of order n with $\delta(G) \geq 5$ satisfying $\gamma_t(G) = \frac{2453}{6500}n$. Thomassé and Yeo [71] constructed a 5-regular, bipartite, graph G_{22}, illustrated in Figure 23, of order $n = 22$ satisfying $\gamma_t(G_{22}) = 8 = \frac{4}{11}n$. We remark that the graph G_{22} is constructed from the hypergraph with vertex set $V(H) = [10]_0$ and edge set $E(H) = \{e_0, e_1, \ldots, e_{10}\}$, where the edge $e_i = Q + i$ for $i \in [10]_0$ and where $Q = \{1, 3, 4, 5, 9\}$ is the set of non zero quadratic residues modulo 11.

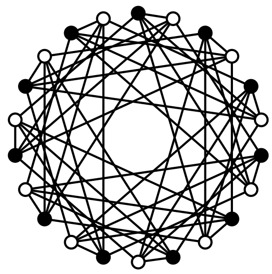


Figure 23. A bipartite graph G_{22} of order $n = 22$ and $\gamma_t(G_{22}) = 8 = \frac{4}{11}n$.

By Theorem 33, $c_{\mathrm{tdom},6} \leq \frac{4549}{13299}$. It is not known if there is a graph G of order n with $\delta(G) \geq 6$ satisfying $\gamma_t(G) = \frac{4549}{13299}n$. Henning and Yeo [52] constructed a 6-regular (bipartite) graph G_{26}, illustrated in Figure 24, of order $n = 26$ satisfying $\gamma_t(G_{26}) = 8 = \frac{4}{13}n$. We remark that the graph G_{262} is constructed from the hypergraph with vertex set $V(H) = [12]_0$ and edge set $E(H) = \{e_0, e_1, \ldots, e_{12}\}$, where the edge $e_i = Q+i$ for $i \in [12]_0$ and where Q is the set of non zero quadratic residues modulo 13, that is, $Q = \{1, 3, 4, 9, 10, 12\}$.

We summarize the above results and observations formally as follows.

Theorem 100. *The following hold.*

(a) $c_{\mathrm{tdom},1} = c_{\mathrm{tdom},2} = \frac{2}{3}$.

(b) $c_{\mathrm{tdom},3} = \frac{1}{2}$.

(c) $c_{\mathrm{tdom},4} = \frac{3}{7}$.

(d) $\frac{4}{11} \leq c_{\mathrm{tdom},5} \leq \frac{2453}{6500} < \frac{4}{11} + \frac{11}{800}$.

(e) $\frac{4}{13} \leq c_{\mathrm{tdom},6} \leq \frac{4549}{13299} < \frac{4}{13} + \frac{17}{494}$.

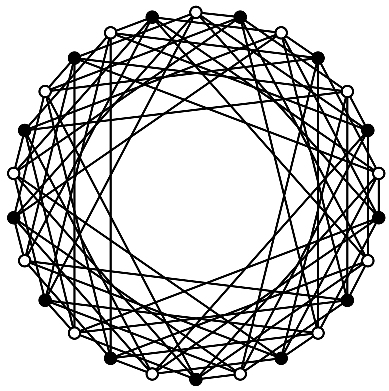


Figure 24. A bipartite graph G_{26} of order $n = 26$ satisfying $\gamma_t(G_{26}) = 8 = \frac{4}{13}n$.

The lower bounds in Theorem 100(d) and 100(e) are conjectured to be the correct values for $c_{\rm tdom,5}$ and $c_{\rm tdom,6}$, respectively.

Conjecture 101 [71]**.** $c_{\rm tdom,5} = \frac{4}{11}$.

Conjecture 102 [52]**.** $c_{\rm tdom,6} = \frac{4}{13}$.

Recall that $\mathcal{G}_{n,m,5}$ denotes the class of all connected graphs of order n and size m with minimum degree at least 5. The truth of the following conjecture would imply the truth of Conjecture 101.

Conjecture 103 [44]**.** *For $a, b \in \mathbb{R}$, the bound $\gamma_t(G) \le an+bm$ is valid for every graph $G \in \mathcal{G}_{n,m,5}$ if and only if both $b \ge 0$ and $a \ge \frac{4}{11} - \frac{5}{2}b$ hold.*

11.3. The independent domination number

For $r \ge 2$, let $\mathcal{G}_{{\rm reg},r}$ denote the family of all connected r-regular graphs different from $K_{r,r}$.

Problem 104. For $r \ge 3$ determine or estimate the best possible constant $c_{{\rm idom},r}$ (which depends only on r) such that $i(G) \le c_{{\rm idom},r} \cdot n(G)$ for all $G \in \mathcal{G}_{{\rm reg},r}$. These constants are given by

$$c_{{\rm idom},r} = \sup_{G \in \mathcal{G}_{{\rm reg},r}} \frac{i(G)}{n(G)}.$$

By Observation 46, and by Theorems 45, 47, and 48, and by our earlier observations, the known results on the constants $c_{{\rm idom},r}$ are summarized in Theorem 105. For $r \ge 5$, it remains an open problem to determine the exact value of the constant $c_{{\rm idom},r}$.

Theorem 105. *The following hold.*

(a) $c_{\mathrm{idom},3} = \frac{2}{5}$.

(b) $c_{\mathrm{idom},4} = \frac{3}{7}$.

(c) $c_{\mathrm{idom},r} \le \frac{r-1}{2r-1}$ *for all* $r \ge 5$.

Goddard and Henning [28] posed the question. Is it true that $c_{\mathrm{idom},r}$ tends to $\frac{1}{2}$ as $r \to \infty$? This question was answered in the affirmative in 2020 by Blumenthal [6] in his PhD thesis. For integers $r > k \ge 2$, let $K^k_{r,r}$ be the graph obtained from a complete bipartite graph $K_{r,r}$ by selecting an arbitrary vertex of the graph, which we call the *gluing vertex*, and removing $k-1$ edges incident with this vertex. For such r and k with k even, Blumenthal [6] constructed a family $\mathcal{G}_{\mathrm{idom},k,r}$ of connected r-regular graphs as follows. Let G be obtained from k vertex disjoint copies of $K^k_{r,r}$ by forming a clique K_k on the k gluing vertices, and let M be an arbitrary perfect matching in this clique. For each gluing vertex v, let G_v be the copy of $K^k_{r,r}$ that contains v, and let N_v be the set of vertices of degree $r-1$ in G_v different from v. We note that $|N_v| = k-1$. For each edge $uv \in M$, we add a perfect matching between the vertices of N_u and N_v. Let G be the resulting connected r-regular graph of order $2rk$, and let $\mathcal{G}_{\mathrm{idom},k,r}$ be the family of all such graphs G. A graph in the family $\mathcal{G}_{\mathrm{idom},4,5}$ is illustrated in Figure 25.

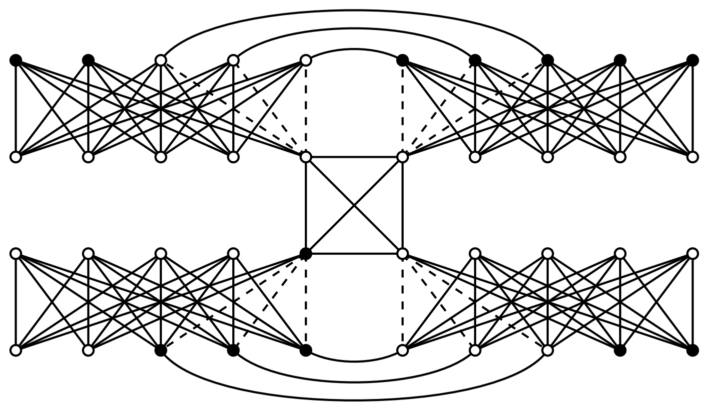


Figure 25. A graph in the family $\mathcal{G}_{\mathrm{idom},4,5}$.

Blumenthal [6] proved if $G \in \mathcal{G}_{\mathrm{idom},k,r}$, then $i(G) \ge k + (r-k-1)(k-1)$. With a more detailed analysis, the independent domination number of a graph in the family $\mathcal{G}_{\mathrm{idom},k,r}$ was determined precisely in [35].

Proposition 106 [35, Lemma 6.88]**.** *For integers r and k where $r-1 \ge k \ge 2$ with k even, if $G \in \mathcal{G}_{\mathrm{idom},k,r}$, then $i(G) = r(k-1) - \frac{1}{2}k(k-3)$.*

To illustrate Proposition 106, if $G \in \mathcal{G}_{\mathrm{idom},4,5}$, then $i(G) = 13$. An example of an i-set (of cardinality 13) in the graph $G \in \mathcal{G}_{\mathrm{idom},4,5}$ shown in Figure 25 is given by the set of black vertices. For integers r and k where $r-1 \ge k \ge 2$ where k is even, by Proposition 106 if $G \in \mathcal{G}_{\mathrm{idom},k,r}$, then

$$\frac{i(G)}{n(G)} = \frac{r(k-1) - \frac{1}{2}k(k-3)}{2rk} = f_r(k),$$

where $f_r(k)$ denote the function

$$f_r(k) = \frac{1}{2} - \frac{1}{2k} - \frac{k}{4r} + \frac{3}{4r}.$$

For real optimization with $r \ge 5$ a fixed integer and $k \in \mathbb{R}$ a real number satisfying $2 \le k \le r-1$, the function $f_r(k)$ is maximized when $k = \sqrt{2r}$ and $f_r(\sqrt{2r}) = \frac{1}{2} + \frac{3}{4r} - \frac{1}{2\sqrt{2r}}$. Thus, if $\sqrt{2r}$ is an even integer, then this yields

$$c_{\mathrm{idom},r} \ge \frac{1}{2} + \frac{3}{4r} - \frac{1}{\sqrt{2r}}. \tag{1}$$

For integer optimization with $r \ge 5$ a fixed integer and k an even integer satisfying $2 \le k \le r-1$, the optimal value of the function $f_r(k)$ is $\max\{k_1, k_2\}$, where k_1 and k_2 are even integers such that $k_1 \le \sqrt{2r} \le k_2$ and $k_2 = k_1 + 2$. This yields a lower bound of $c_{\mathrm{idom},r}$ of approximately $\frac{1}{2} + \frac{3}{4r} - \frac{1}{\sqrt{2r}}$, which tends to $\frac{1}{2}$ as $r \to \infty$. This yields the result of Blumenthal [6].

Theorem 107 [6]**.** *The constant $c_{\mathrm{idom},r}$ tends to $\frac{1}{2}$ as $r \to \infty$.*

It would be interesting to determine the exact value of the constant $c_{\mathrm{idom},r}$ for $r \ge 5$, even for the special case when $r = 5$.

11.4. The connected domination number

Problem 108. Determine or estimate the best possible constants $c_{\mathrm{cdom},k}$ (which depends only on k) such that $\gamma_{\mathrm{pr}}(G) \le c_{\mathrm{cdom},k} \cdot n(G)$ for all $G \in \mathcal{G}_k$. These constants are given by

$$c_{\mathrm{cdom},k} = \sup_{G \in \mathcal{G}_k} \frac{\gamma_c(G)}{n(G)}.$$

The constants $c_{\mathrm{cdom},k}$ for small $k \in [5]$ are known. By Observation 54, $c_{\mathrm{cdom},1} = c_{\mathrm{cdom},2} = 1$. By Theorems 55, 56 and 57, and by Proposition 52, we have $c_{\mathrm{cdom},3} = \frac{3}{4}$, $c_{\mathrm{cdom},4} = \frac{3}{5}$ and $c_{\mathrm{cdom},5} = \frac{1}{2}$. We summarize the known results as follows.

Theorem 109. *The following hold.*

(a) $c_{\text{cdom},1} = c_{\text{cdom},2} = 1$.

(b) $c_{\text{cdom},3} = \frac{3}{4}$.

(c) $c_{\text{cdom},4} = \frac{3}{5}$.

(d) $c_{\text{cdom},5} = \frac{1}{2}$.

(e) $c_{\text{cdom},k} \geq \frac{3}{k+1}$ *for* $k \geq 6$.

(f) $c_{\text{cdom},k} \geq \frac{\ln(k)}{k}$ *for* k *sufficiently large.*

For $k \geq 6$, the exact value of $c_{\text{cdom},k}$ remains unknown, even in the special case when $k = 6$. We conjecture that the lower bound in Theorem 109(e) is the correct value for $c_{\text{cdom},6}$.

Conjecture 110. $c_{\text{cdom},6} = \frac{3}{7}$.

11.5. The paired domination number

Problem 111. Determine or estimate the best possible constants $c_{\text{pdom},k}$ (which depends only on k) such that $\gamma_{\text{pr}}(G) \leq c_{\text{pdom},k} \cdot n(G)$ for all $G \in \mathcal{G}_k$. These constants are given by

$$c_{\text{pdom},k} = \sup_{G \in \mathcal{G}_k} \frac{\gamma_{\text{pr}}(G)}{n(G)}.$$

The constants $c_{\text{pdom},k}$ are surprisingly only known for $k = 1$ and $k = 2$. By Theorem 66, $c_{\text{pdom},1} = 1$ and by Theorem 67, $c_{\text{pdom},2} = \frac{2}{3}$. By Theorem 71, $c_{\text{pdom},3}(G) \leq \frac{19037}{30000} n < 0.634567\, n$. It is not known if there is a graph G of order n with $\delta(G) \geq 3$ satisfying $\gamma_{\text{pr}}(G) = \frac{19037}{30000} n$. As observed earlier, if G is the Petersen graph, then G is a cubic graph of order $n = 10$ and $\gamma_{\text{pr}}(G) = 6 = \frac{3}{5} n$, implying that $c_{\text{pdom},3}(G) \geq \frac{3}{5}$. We summarize the above results and observations formally as follows.

Theorem 112. *The following hold.*

(a) $c_{\text{pdom},1} = 1$.

(b) $c_{\text{pdom},2} = \frac{2}{3}$.

(c) $\frac{3}{5} \leq c_{\text{pdom},3} \leq \frac{19037}{30000} < \frac{3}{5} + \frac{13}{375}$.

We pose the following conjecture.

Conjecture 113. $c_{\text{pdom},3} = \frac{3}{5}$.

Chen, Sun, and Xing [14] conjectured that if G is a connected graph of order $n \geq 11$ with $\delta(G) \geq 3$, then $\gamma_{\text{pr}}(G) \leq \frac{4}{7} n$. A slightly stronger conjecture was posed by Goddard and Henning [27].

Conjecture 114 [27]**.** *If G is a connected graph of order n with $\delta(G) \geq 3$, then $\gamma_{\rm pr}(G) \leq \frac{4}{7}n$, unless G is the Petersen graph, in which case $\gamma_{\rm pr}(G) = \frac{3}{5}n$.*

If Conjecture 114 is true, then this would imply that Conjecture 113 is true. Conjecture 114 has yet to be settled even in the special case of the class of cubic graphs. The following conjecture was posed by Desormeaux and Henning in [21].

Conjecture 115 [21]**.** *If G is a bipartite cubic graph of order n, then $\gamma_{\rm pr}(G) \leq \frac{1}{2}n$.*

By Theorem 64, if G is an isolate-free graph, then $\gamma_{\rm pr}(G) \leq 2\gamma(G)$, implying that $c_{\rm pdom,\mathit{k}} \leq 2c_{\rm dom,\mathit{k}}$ for all $k \geq 1$. As an application of Theorem 14 and the results in Table 1 due to Bujtás and Klavžar in [12], we therefore have the following trivial upper bounds on $c_{\rm pdom,\mathit{k}}$ for $k \geq 6$. For example, the resulting upper bounds on $c_{\rm pdom,\mathit{k}}$ for $k \in \{6, 7, 8, 9, 10\}$ as given in Table 2. These upper bounds are most likely far from optimal.

k	6	7	8	9	10
$c_{\rm pdom,\mathit{k}}$	≤ 0.60766	≤ 0.585356	≤ 0.546426	≤ 0.513131	≤ 0.484256

Table 2. Upper bounds on $c_{\rm pdom,\mathit{k}}$ for $k \in \{6, 7, 8, 9, 10\}$.

11.6. The restrained domination number

For $k \geq 1$ let $\mathcal{G}_{\rm rdom,\mathit{k}}$ denotes the class of all connected graphs with minimum degree at least k, where we forbid the 5-cycle C_5 to belong to the family $\mathcal{G}_{\rm rdom,2}$.

Problem 116. Determine or estimate the best possible constants $c_{\rm rdom,\mathit{k}}$ (which depends only on k) such that $\gamma_r(G) \leq c_{\rm rdom,\mathit{k}} \cdot n(G)$ for all $G \in \mathcal{G}_{\rm rdom,\mathit{k}}$. These constants are given by

$$c_{\rm rdom,\mathit{k}} = \sup_{G \in \mathcal{G}_{\rm rdom,\mathit{k}}} \frac{\gamma_r(G)}{n(G)}.$$

The constants $c_{\rm rdom,\mathit{k}}$ are surprisingly only known for $k = 1$ and $k = 2$. By our earlier observations, and by Theorem 73, $c_{\rm rdom,1} = 1$ and by Theorem 76, $c_{\rm rdom,2} = \frac{1}{2}$. The Petersen graph G, shown in Figure 16 has order $n = 10$ and $\gamma_r(G) = 4 = \frac{2}{5}n$, implying that $c_{\rm rdom,3} \geq \frac{2}{5}$. By Theorem 78, $c_{\rm rdom,3} \leq 0.668$. We summarize the known results as follows.

Theorem 117. *The following hold.*

(a) $c_{\rm rdom,1} = 1$.

(b) $c_{\mathrm{rdom},2} = \frac{1}{2}$.
(c) $\frac{2}{5} \le c_{\mathrm{rdom},3} \le 0.668$.

For $k \ge 3$, the exact value of $c_{\mathrm{rdom},k}$ remains unknown, even in the special case when $k = 3$. We conjecture that the lower bound in Theorem 117(c) is the correct value for $c_{\mathrm{rdom},3}$.

Conjecture 118. $c_{\mathrm{rdom},3} = \frac{2}{5}$.

11.7. The semitotal domination number

For $k \ge 1$ let $\mathcal{G}_{\gamma_{t2},k}$ denotes the class of all connected graphs of order at least 4 with minimum degree at least k.

Problem 119. Determine or estimate the best possible constants $c_{\gamma_{t2},k}$ (which depends only on k) such that $\gamma_{t2}(G) \le c_{\gamma_{t2},k} \cdot n(G)$ for all $G \in \mathcal{G}_{\gamma_{t2},k}$. These constants are given by

$$c_{\gamma_{t2},k} = \sup_{G \in \mathcal{G}_{\gamma_{t2},k}} \frac{\gamma_{t2}(G)}{n(G)}.$$

The constants $c_{\gamma_{t2},k}$ are only known for $k = 1$ and $k = 2$. By our earlier observations, and by Theorems 84 and 85, $c_{\gamma_{t2},1} = c_{\gamma_{t2},2} = \frac{1}{2}$.

For $k \ge 2$, let G be the graph obtained from a connected graph F of order at least 2 as follows. For each vertex $v \in V(F)$ add a copy of the graph $K_{k+1} - e$ where v_1 and v_2 are the two vertices that are not adjacent in this added copy (and so, the edge $e = v_1v_2$ is removed from $K_{k+1} - e$ in our construction), and add the edges vv_1 and vv_2. The resulting graph G has order $n = (k+2)|V(F)|$ and satisfies

$$\gamma_{t2}(G) = \left(\frac{2}{k+2}\right) n.$$

For example, when $k = 3$ and $F = K_{1,4}$ is a star, the resulting graph G of order $n = 25$ with $\gamma_{t2}(G) = 10 = \frac{2}{5}n$ is illustrated in Figure 26, where the ten black vertices are an example of a γ_{t2}-set of G.

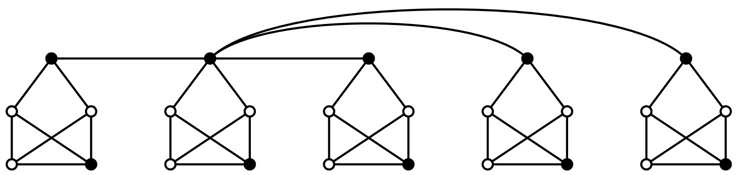


Figure 26. A graph G with $\delta(G) = 3$ and $\gamma_{t2}(G) = \frac{2}{5}n$.

We summarize the known results as follows.

Theorem 120. *The following hold.*

(a) $c_{\gamma_{t2},1} = c_{\gamma_{t2},2} = \frac{1}{2}$.

(b) $c_{\gamma_{t2},k} \ge \frac{2}{k+2}$ *for all* $k \ge 3$.

For $k \ge 3$, the exact value of $c_{\gamma_{t2},k}$ remains unknown, even in the special case when $k = 3$. We conjecture that in the case when $k = 3$ the lower bound in Theorem 120(b) is the correct value for $c_{\gamma_{t2},3}$.

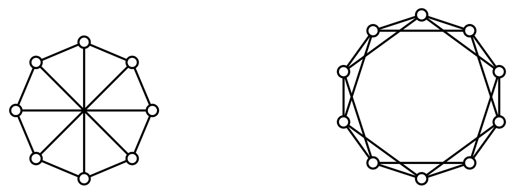
Conjecture 121. $c_{\gamma_{t2},3} = \frac{2}{5}$.

11.8. The semipaired domination number

Problem 122. Determine or estimate the best possible constants $c_{\gamma_{pr2},k}$ (which depends only on k) such that $\gamma_{pr2}(G) \le c_{\gamma_{pr2},k} \cdot n(G)$ for all $G \in \mathcal{G}_k$. These constants are given by

$$c_{\gamma_{pr2},k} = \sup_{G \in \mathcal{G}_k} \frac{\gamma_{pr2}(G)}{n(G)}.$$

The constants $c_{\gamma_{pr2},k}$ are known for $k \in \{1, 2, 3\}$. By Theorem 89 and Corollary 90, $c_{\gamma_{pr2},1} = c_{\gamma_{pr2},2} = \frac{2}{3}$. By Theorem 92, $c_{\gamma_{pr2},3} = \frac{1}{2}$. For $k \ge 2$ even, the circulant $C_{2k+2}\langle 1, \ldots, \frac{k}{2}\rangle$ of order $n = 2k + 2$ with jumps $1, \ldots, k$ satisfies $\gamma_{pr2}(G) = 4\big(\frac{2}{k+1}\big)n$, while for $k \ge 3$ odd, the circulant $C_{2k+2}\langle 1, \ldots, \frac{k-1}{2}, k+1\rangle$ of order $n = 2k + 2$ with jumps $1, \ldots, \frac{k-1}{2}, k+1$ satisfies $\gamma_{pr2}(G) = 4 = \big(\frac{2}{k+1}\big)n$. Therefore, $c_{\gamma_{pr2},k} \ge \frac{2}{k+1}$ for all $k \ge 2$. For example, the circulant $G = C_8\langle 1, 4\rangle$, illustrated in Figure 27(a), is a 3-regular graph of order $n = 8$ satisfying $\gamma_{pr2}(G) = 4 = \frac{1}{2}n$, and the circulant $G = C_{10}\langle 1, 2\rangle$, illustrated in Figure 27(b), is a 4-regular graph of order $n = 10$ satisfying $\gamma_{pr2}(G) = 4 = \frac{2}{5}n$.



(a) $C_8\langle 1, 4\rangle$ (b) $C_{10}\langle 1, 2\rangle$

Figure 27. The circulants $C_8\langle 1, 4\rangle$ and $C_{10}\langle 1, 2\rangle$.

We summarize the known results as follows.

Theorem 123. *The following hold.*

(a) $c_{\gamma_{pr2},1} = c_{\gamma_{pr2},2} = \frac{2}{3}$.

(b) $c_{\gamma_{\rm pr2},3} = \frac{1}{2}$.
(c) $c_{\gamma_{\rm pr2},k} \ge \frac{2}{k+1}$ *for all* $k \ge 4$.

For $k \ge 4$, the exact value of $c_{\gamma_{\rm pr2},k}$ remains unknown, even in the special case when $k = 4$. We conjecture that in the case when $k = 4$ the lower bound in Theorem 123(c) is the correct value for $c_{\gamma_{\rm pr2},4}$.

Conjecture 124. $c_{\gamma_{\rm pr2},4} = \frac{2}{5}$.

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