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LINEAR ARBORICITY OF 1-PLANAR GRAPHS

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Abstract

The linear arboricity $\mathrm{la}(G)$ of a graph G is the minimum number of linear forests that partition the edges of G. In 1981, Akiyama, Exoo and Harary conjectured that $\lceil \frac{\Delta(G)}{2} \rceil \leq \mathrm{la}(G) \leq \lceil \frac{\Delta(G)+1}{2} \rceil$ for any simple graph G. A graph G is 1-planar if it can be drawn in the plane so that each edge has at most one crossing. In this paper, we confirm the conjecture for 1-planar graphs G with $\Delta(G) \geq 13$.

Keywords: linear arboricity, 1-planar graph, linear coloring, 3-alternating cycle.

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1. Introduction

All graphs considered in this paper are simple unless otherwise stated. For a graph G, we use $V(G), E(G), \delta(G)$, and $\Delta(G)$ (for short, Δ) to denote the set of vertices, the set of edges, the minimum degree, and the maximum degree of G, respectively. A *linear forest* is a graph in which each component is a path. A mapping $\phi : E(G) \to \{1, 2, \ldots, k\}$ is called a *linear k-coloring* if each color class induces a linear forest. The *linear arboricity*, denoted by $\mathrm{la}(G)$, of a graph G is the minimum number k for which G has a linear k-coloring. This concept is due to Harary [11]. In 1981, Akiyama, Exoo and Harary [1] proposed the following famous *Linear Arboricity Conjecture*.

Conjecture 1. *For any graph G, $\lceil \frac{\Delta}{2} \rceil \leq \mathrm{la}(G) \leq \lceil \frac{\Delta+1}{2} \rceil$.*

Conjecture 1 has been confirmed for graphs with $\Delta \in \{3, 4\}$ in [1, 2], for graphs with $\Delta \in \{5, 6, 8\}$ in [7], and for graphs with $\Delta = 10$ in [10]. Suppose that G is a planar graph. Wu [15] first proved that G satisfies Conjecture 1 if $\Delta \neq 7$. The remaining case where $\Delta = 7$ was later settled in [16]. Furthermore, Cygan *et al.* [6] proved that if $\Delta \geq 9$, then $\mathrm{la}(G) = \lceil \frac{\Delta}{2} \rceil$, and conjectured that this is also true for the case of $5 \leq \Delta \leq 8$. For a general graph G, Alon [3] proved that, for each $\varepsilon > 0$, there exists a constant $C(\epsilon)$ such that every graph G with $\Delta \geq C(\varepsilon)$ has $\mathrm{la}(G) \leq (\frac{1}{2} + \varepsilon)\Delta$. Recently, Ferber, Fox, and Jain [9] showed that there exist absolute constants $\eta, C > 0$ such that every graph G has $\mathrm{la}(G) \leq \frac{\Delta}{2} + C\Delta^{\frac{2}{3}-\eta}$.

A 1-*planar* graph is a graph that can be drawn in the Euclidean plane such that each edge crosses at most one edge. A number of interesting results about structures and parameters of 1-planar graphs have been obtained in recent years. Fabrici and Madaras [8] proved that every 1-planar graph G has $|E(G)| \leq 4|V(G)| - 8$, which implies that $\delta(G) \leq 7$, and constructed a 7-regular 1-planar graph. Borodin [4] showed that every 1-planar graph is vertex 6-colorable. Zhang and Wu [19] showed that every 1-planar graph G with $\Delta \geq 10$ is of Class One. A proper vertex coloring of a graph G is *acyclic* if G contains no bicolored cycle. Borodin *et al.* [5] proved that every 1-planar graph is acyclically 20-colorable. Yang, Wang and Wang [17] improved this result by reducing 20 to 18.

The *linear* 2-*arboricity* $\mathrm{la}_2(G)$ of a graph G is the least integer k such that G can be partitioned into k edge-disjoint forests, whose component trees are paths of length at most 2. Liu *et al.* [13] proved that every 1-planar graph G has $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 14$. This result was recently improved to that $\mathrm{la}_2(G) \leq \lceil \frac{\Delta+1}{2} \rceil + 7$ by Liu *et al.* [14]. In 2011, Zhang, Liu and Wu [18] considered the linear arboricity of 1-planar graphs and showed that if G is a 1-planar graph with $\Delta \geq 33$, then $\mathrm{la}(G) = \lceil \frac{\Delta}{2} \rceil$.

In this paper, we will show the following result.

Theorem 1. *If G is a 1-planar graph with $\Delta \geq 13$, then $\mathrm{la}(G) \leq \lceil \frac{\Delta+1}{2} \rceil$.*

The proof of Theorem 1 is based on the following key Theorem 2. Actually this theorem is of some interest itself.

Theorem 2. *Let G be a 1-planar graph with $\delta(G) \geq 3$. Then at least one of the following statements holds.*

(1) *There is an edge uv such that $d_G(u) + d_G(v) \leq 15$.*

(2) *There is a* 3*-cycle* $uvwu$ *such that* $d_G(u) + d_G(v) \le 16$.

(3) *There is a* 4*-cycle* $v_1v_2v_3v_4v_1$ *with* $d_G(v_1) = d_G(v_3) = 3$.

The organization of this paper is as follows. In Section 2, we will establish the proof of Theorem 1 by considering three cases. The proof of structural lemma, Theorem 2, will be postponed to Section 3. In the final Section 4, we give some remarks on the results obtained in the paper and put forward an open problem.

2. PROOF OF THEOREM 1

Suppose that G is a graph. For $v \in V(G)$, we use $E(v)$ to denote the set of edges incident to v in G. Given a linear k-coloring ϕ with a color set $C = \{1, 2, \ldots, k\}$, let $C(v)$ denote the set of colors that ϕ uses in $E(v)$. For $0 \le i \le 2$, let

$$I_i(v) = \{c \in C \mid c \text{ appears exactly } i \text{ times in } E(v)\}.$$

A path $P = v_1v_2 \cdots v_n$ is called a $(v_1, v_n)_c$*-path* in G if all edges of P are colored with same color c.

Instead of showing Theorem 1, we only need to prove the following equivalent statement.

Theorem 1*. *If* G *is a* 1*-planar graph and* $k = \max\left\{7, \lceil\frac{\Delta+1}{2}\rceil\right\}$*, then* $\mathrm{la}(G) \le k$.

Proof. The proof is processed by induction on the edge number $|E(G)|$. If $|E(G)| \le 7$, then the result holds trivially since we may color all edges of G with distinct colors. Suppose that G is a 1-planar graph with $|E(G)| \ge 8$. If G contains an edge xy such that $d_G(x) \le d_G(y)$ and $d_G(x) \le 2$, then let $H = G - xy$. By the induction hypothesis, H admits a linear k-coloring ϕ using the color set $C = \{1, 2, \ldots, k\}$. Let $S = I_2(y) \cup (I_1(x) \cap I_1(y))$. Then $|S| = |I_2(y) \cup (I_1(x) \cap I_1(y))| \le \left\lfloor\frac{1}{2}(d_H(x) + d_H(y))\right\rfloor = \left\lfloor\frac{1}{2}(d_G(x) + d_G(y))\right\rfloor - 1 \le \left\lfloor\frac{1}{2}(2 + \Delta)\right\rfloor - 1 = \left\lfloor\frac{1}{2}\Delta\right\rfloor$. Since $|C| \ge \lceil\frac{\Delta+1}{2}\rceil$, there exists a color $a \in C \setminus S$. Based on ϕ, we color xy with a to extend ϕ to the graph G.

Now assume that $\delta(G) \ge 3$. By Theorem 2, our proof can be split into the following three cases.

Case 1. G contains an edge uv such that $d_G(u) + d_G(v) \le 15$. Consider the graph $H = G - uv$, which admits a linear k-coloring ϕ using the color set C by the induction hypothesis. Let $S = I_2(u) \cup I_2(v) \cup (I_1(u) \cap I_1(v))$. Then $|S| = |I_2(u) \cup I_2(v) \cup (I_1(u) \cap I_1(v))| \le \left\lfloor\frac{1}{2}(d_H(u) + d_H(v))\right\rfloor = \left\lfloor\frac{1}{2}(d_G(u) + d_G(v))\right\rfloor - 1 \le \left\lfloor\frac{15}{2}\right\rfloor - 1 = 6$. Noting that $|C| \ge 7$, we can color uv with some color in $C \setminus S$ to extend ϕ to G.

Case 2. G contains a 3-cycle $uvwu$ such that $d_G(u) + d_G(v) \le 16$. If $d_G(u) + d_G(v) \le 15$, the proof is given in Case 1. So assume that $d_G(u) + d_G(v) = 16$.

Consider the graph $H = G - uv$, which admits a linear k-coloring ϕ using the color set C by the induction hypothesis. Let $S = I_2(u) \cup I_2(v) \cup (I_1(u) \cap I_1(v))$. Then, with a similar analysis, $|S| \leq |I_2(u) \cup I_2(v) \cup (I_1(u) \cap I_1(v))| \leq 7$. If $|S| \leq 6$, then we can color uv with some color in $C \setminus S$ to extend ϕ to G. If $\Delta \geq 14$, then it is easy to derive that $|C| \geq 8$ and hence uv can be colored with some color in $C \setminus S$. Otherwise, assume that $\Delta \leq 13$ and $|S| = 7$. This implies that $|C| = 7$ and every color in C appears exactly twice in $E(u) \cup E(v)$. To complete the proof, we consider two subcases as follows by symmetry.

Case 2.1. $\phi(uw) = 1$ and $\phi(vw) = 2$. By symmetry, we discuss two possibilities as follows.

Case 2.1.1. $1 \in I_2(u)$ and $2 \in I_2(v)$. We exchange the colors of uw and vw and then color uv with 1.

Case 2.1.2. $1 \in I_2(u)$ and $2 \in I_1(u) \cap I_1(v)$. We claim that there exists a $(u, v)_2$-path in H, for otherwise we may color uv with 2. This implies that $2 \in I_2(w)$. If $1 \in I_1(w)$, then we color uv with 2 and recolor vw with 1. Otherwise, $1 \in I_2(w)$. So there exists $a \in \{3, 4, \ldots, 7\} \cap I_1(w)$, say $a = 3$, by the assumption that $d_G(w) \leq 13$. If $3 \in I_2(u)$, then we color uv with 2 and recolor vw with 3. If $3 \in I_2(v)$, then we color uv with 1 and recolor uw with 3. Assume that $3 \in I_2(u) \cap I_1(v)$. If there does not exist a $(u, w)_3$-path, then we color uv with 1 and recolor uw with 3. Otherwise, no $(v, w)_3$-path may exist in H. It suffices to color uv with 2 and recolor vw with 3.

Case 2.1.3. $1, 2 \in I_1(u) \cap I_1(v)$. Suppose that there exist both $(u, v)_1$-path and $(u, v)_2$ in H, otherwise the proof can be given easily. This implies that $1, 2 \in I_2(w)$ and so there exists some color $a \in \{3, 4, \ldots, 7\} \cap I_1(w)$, say $a = 3$. With a similar analysis as in Case 2.1.2, we can extend ϕ to the graph G.

Case 2.2. $\phi(uw) = \phi(vw) = 1$. Since $d_H(w) \leq 13$, there exists $a \in \{2, 3, \ldots, 7\} \cap (I_0(w) \cup I_1(w))$, say $a = 2$. If $2 \notin C(u)$, then we recolor uw with 2 and reduce the proof to Case 2.1. If $2 \notin C(v)$, we have a similar proof. Otherwise, $2 \in I_1(u) \cap I_1(v)$. Since at most one of $(u, w)_2$-path and $(v, w)_2$-path exists in H, we recolor uw or vw with 2 and reduce the proof to Case 2.1.1.

Case 3. G contains a 4-cycle $B = v_1v_2v_3v_4v_1$ such that $d_G(v_1) = d_G(v_3) = 3$. By Case 2, we may assume that $v_1v_2 \notin E(G)$. For $i = 1, 3$, let v_i' denote the neighbor of v_i other than v_2 and v_4. Let $H = G - E(B)$. By the induction hypothesis, H admits a linear k-coloring ϕ with the color set C. Suppose that $\phi(v_1v_1') = a$ and $\phi(v_3v_3') = b$. By symmetry, we have three subcases as follows.

Case 3.1. $|I_0(v_2)| = |I_0(v_4)| = 0$. For $= 2, 4$, since $|C| \geq \lceil \frac{\Delta+1}{2} \rceil$ and $d_H(v_i) = d_G(v_i) - 2 \leq \Delta - 2$, it follows easily that $|I_1(v_2)| \geq 3$ and $|I_1(v_4)| \geq 3$. Define a list assignment L for $E(B)$ as follows: $L(v_1v_2) = I_2(v_2) \setminus \{a\}$, $L(v_2v_3) = I_2(v_2) \setminus \{b\}$,

$L(v_1v_4) = I_2(v_4) \setminus \{a\}$, and $L(v_3v_4) = I_2(v_4) \setminus \{b\}$. Then $|L(e)| \geq 2$ for each edge $e \in E(B)$. So, $E(B)$ is L-colorable, and therefore ϕ is extended to G.

Case 3.2. $|I_0(v_2)| \geq 1$ and $|I_0(v_4)| = 0$. Then $|I_1(v_4)| \geq 3$. If $a \neq b$, then we color $\{v_1v_2, v_2v_3\}$ with a color $c \in I_0(v_2)$, v_1v_4 with a color $d \in I_1(v_4) \setminus \{a, c\}$, and v_3v_4 with some color in $I_1(v_4) \setminus \{b, c\}$. Otherwise, suppose that $a = b = 1$. If there exists $c \in I_0(v_2) \setminus \{1\}$, then we color $\{v_1v_2, v_2v_3\}$ with c, v_1v_4 with a color $d \in I_1(v_4) \setminus \{1, c\}$ and v_3v_4 with a color in $I_1(v_4) \setminus \{1, d\}$. Otherwise, $I_0(v_2) = \{1\}$. If there does not exist a $(v_1, v_3)_1$-path in H, then we can define a similar coloring. Otherwise, it follows that $1 \in I_2(v_1') \cap I_2(v_3')$, and hence there exists a color $c \in I_0(v_3') \cup I_1(v_3')$. Recolor v_3v_3' with c and the proof is reduced to the previous case.

Case 3.3. $|I_0(v_2)| \geq 1$ and $|I_0(v_4)| \geq 1$. First assume that there exist $c \in I_0(v_2)$ and $d \in I_0(v_4)$ such that $c \neq d$. Color $\{v_1v_2, v_2v_3\}$ with c and $\{v_1v_4, v_3v_4\}$ with d. If $a \neq b$, or $a = b$ and $a \notin \{c, d\}$, then the current coloring is available. Otherwise, suppose that $a = b = c = 1$ and $d = 2$. If there does not exist a $(v_1, v_3)_1$-path in H, we are done. Otherwise, it follows easily that $1 \in I_2(v_3')$ and so there is $j \in I_0(v_3') \cup I_1(v_3')$ with $j \neq 1$. Recoloring v_3v_3' with j, we reduce the proof to the previous case.

Next assume that $I_0(v_2) = I_0(v_4) = \{1\}$. In this case, it is easy to verify that $|I_1(v_2)| \geq 1$ and $|I_1(v_4)| \geq 1$. Let $p \in I_1(v_2)$ and $q \in I_1(v_4)$. Then $p \neq 1$ and $q \neq 1$. If $a \neq b$, assuming $q \neq a$, then we color $\{v_1v_2, v_2v_3, v_3v_4\}$ with 1 and v_1v_4 with q. Otherwise, $a = b$. If $a = 1$, then we color v_1v_2 with p, v_3v_4 with q, and $\{v_2v_3, v_1v_4\}$ with 1. Otherwise, suppose that $a = 2$. If $q \neq 2$, then we color $\{v_1v_2, v_2v_3, v_3v_4\}$ with 1 and v_1v_4 with q. Otherwise, $q = 2$. At most one of $(v_1, v_4)_2$-path and $(v_3, v_4)_2$-path exists in H. A similar coloring can be established. ■

3. PROOF OF THEOREM 2

To complete the proof of Theorem 2, we need to introduce a few concepts and known results. Suppose that G is a plane graph with the face set $F(G)$. For $f \in F(G)$, we use $\partial(f)$ to denote the boundary walk of f and write $f = [u_1u_2 \cdots u_n]$ if $u_1, u_2, \ldots, u_n$ are the vertices of $\partial(f)$ in a cyclic order. Let $V(f) = V(\partial(f))$ and $E(f) = E(\partial(f))$. A vertex of degree k (at most k, at least k, respectively) is called a k-vertex (k^--vertex, k^+-vertex, respectively). Similarly, we can define k-face, k^--face and k^+-face. A cycle C in a plane graph G is called *separating* if both its interior and exterior contain at least one vertex of G. Let $V_{\text{int}}(C)$ denote the set of vertices in G that lie interior to C.

Suppose that G is a 1-planar graph which is drawn in the plane such that each edge has at most one crossing and the number of crossings are as few as possible. Let $X(G)$ denote the set of crossings in G. For each $x \in X(G)$, there is a pair of crossing edges $y_1y_2, z_1z_2 \in E(G)$ with x as crossing. Define an *associated plane graph* $G^\times$ as follows:

$V(G^\times) = V(G) \cup X(G)$,
$E(G^\times) = E_0(G) \cup E_1(G)$,

where $E_0(G)$ is the set of non-crossed edges in G and

$$E_1(G) = \{yx, xz \mid yz \in E(G) \setminus E_0(G) \text{ and } x \text{ is a crossing on } yz\}.$$

The vertices in $V(G)$ are called *true vertices* of $G^\times$, and the vertices in $X(G)$ are called *false vertices* of $G^\times$. Observe that $d_{G^\times}(v) = d_G(v)$ if $v \in V(G)$, and $d_{G^\times}(v) = 4$ if $v \in X(G)$. A 3-face f of $G^\times$ is said to be *false* if $V(f)$ contains a false vertex, and otherwise it is *true*.

Zhang and Wu [19] gave the following useful properties for the associated plane graph of a 1-planar graph.

Lemma 3. *Let $G^\times$ be the associated plane graph of a 1-planar graph G. Then the following assertions hold.*

(1) *No two false vertices are adjacent in $G^\times$.*
(2) *If $uv \in E(G^\times)$ such that $d_{G^\times}(u) = 3$ and $v \in X(G)$, then uv is incident to at most one 3-face in $G^\times$.*
(3) *If a 3-vertex v is incident to two 3-faces and adjacent to two false vertices, then v is incident to a 5^+-face.*

***Proof of Theorem* 2.** Suppose that the theorem is false. Let G be a connected counterexample such that every edge is crossed by at most one other edge and crossings are as few as possible. The proof is divided into four parts as follows. In Part 1, we construct a plane graph H from the initial graph G by a series of operations. In Part 2, we investigate the structural properties of H by summarizing several claims. To derive a contradiction on H, we need to employ the Euler's formula and discharging method. Both Part 3 and Part 4 are contributed to deal with this long part.

Part 1. Forming plane graph H

Let $G^\times$ denote the associated plane graph of G. If $G^\times$ has a 4^+-face f with $x, y \in V(f)$ and $xy \notin E(f)$ such that $d_{G^\times}(x) + d_{G^\times}(y) \geq 15$, then we carry out the following operation.

($\star$) **Add an edge xy to the interior of f**

Let $(G^\times)_1$ denote the resultant graph, which is a plane graph obviously. If $(G^\times)_1$ contains a 4^+-face f which is incident to two non-adjacent vertices x_1 and y_1 in $\partial(f)$ satisfying similar property, then we carry out ($\star$) for x_1 and y_1 in

$(G^{\times})_1$. Let $(G^{\times})_2$ denote the new resultant graph. Repeat this process until the final graph, denoted G', has no 4^+-face satisfying the requirement.

From Lemma 3(1) and the definition of G', Observation 1 below holds evidently.

Observation 1. *Let $f = [u_1u_2 \cdots u_k]$ be a k-face of G'.*

(1) *If $k \geq 5$, then $d_{G'}(u_i) \leq 11$ for all $i = 1, 2, \ldots, k$.*

(2) *If $k = 4$ and u_1, u_3 are true, then $d_{G'}(u_1) + d_{G'}(u_3) \leq 14$.*

It is easy to see that G' is the associated plane graph of some 1-planar graph which is obtained from G by adding some edges according to Operation ($\star$). Note that, if $xy \in E(G')$ with x, y being true, or $xz, zy \in E(G')$ where z is a crossing in G, then the following statements (P1) and (P2) hold automatically.

(P1) $d_{G'}(x) + d_{G'}(y) \geq 16$;

(P2) If $xywx$ is a 3-cycle of G', then $d_{G'}(x) + d_{G'}(y) \geq 17$.

Moreover, we have the following.

(P3) There is no 4-cycle with two nonadjacent 3-vertices in G'.

Observe that G' may contain multi-edges, but have no 2-faces. Thus, every 2-cycle of G' is separating. To obtain a contradiction by applying discharging method, we need to define a new graph H from G' in such a way: if there is no separating 2-cycle in G', let $H = G'$; otherwise, choose a separating 2-cycle C such that $|V_{\text{int}}(C)|$ is as small as possible, and let $H = G'[V_{\text{int}}(C) \cup V(C)]$. We call the vertices in $V(C)$ *external vertices* of H and the vertices in $V_{\text{int}}(C)$ *internal* vertices of H and write $V^0(H) = V_{\text{int}}(C)$. Let f_0 denote the outer face of H, and let $F^0(H) = F(H) \setminus \{f_0\}$. Sometimes, the faces in $F^0(H)$ are also called *internal faces* of H. For $v \in V^0(H)$, it holds obviously that $d_H(v) = d_{G'}(v)$. By the choice of C, H contains no 2-cycles other than C.

Let $v \in V(H)$ be a true vertex. If $vv' \in E(H)$ and v' is true, then we call v' a *direct-neighbor* of v in H. If $vw \in E(H)$ and w is false such that $wv' \in E(H)$ and $vv' \in E(G)$, then we call v' an *indirect-neighbor* of v in H. A true vertex $v \in V^0(H)$ is *small* if $d_H(v) \leq 7$ and *big* otherwise. A vertex v is *fat* if $v \in V(C)$ or $v \in V^0(H)$ is big. For a vertex $v \in V(H)$ and an integer $i \geq 1$, we use $m_i(v), m_i^+(v), m_3^*(v)$ to denote the number of i-faces, i^+-faces, and false 3-faces which are incident to v, respectively.

A 6-face $f = [x_1x_2 \cdots x_5x_6]$ is *special* if x_2, x_4, x_6 are small such that $d_H(x_2) = 3$ and $m_3^*(x_2) = 2$. By (P3) and Lemma 3(1), at most one of x_2, x_4, x_6 is a 3-vertex which is incident to two false 3-faces. An internal 4-vertex v is *good* if $m_3^*(v) \leq 1$, or $m_3^*(v) = 2$ and $m_5^+(v) \geq 1$.

An internal 3-vertex u is *rich* if one of the following holds:

(1) $m_3^*(u) = 0$;

(2) $m_3^*(u) = 1$ and $m_5^+(u) \geq 1$;

(3) $m_3^*(u) = 2$ and u is incident to a face f such that

(3.1) $d_H(f) \geq 6$; or

(3.2) $d_H(f) = 5$, and f is incident to only one small vertex, i.e., u.

Otherwise, u is called a *poor* 3-vertex.

Part 2. Structural properties

For a k-vertex $v \in V^0(H)$, let $v_0, v_1, \ldots, v_{k-1}$ denote the neighbors of v in H in a cyclic order, and let $f_0, f_1, \ldots, f_{k-1}$ denote the faces of H incident to v with $vv_i, vv_{i+1} \in \partial(f_i)$ for $i = 0, 1, \ldots, k-1$, where the indices are taken as modulo k. If v_i is false, then we assume that v_i is a crossing of the edges vu_i and $x_i y_i$ of G.

Claims 1–3 below have been established in [13].

Claim 1. *If v is incident to three consecutively adjacent 3-faces f_{i-1}, f_i, f_{i+1}, then at least one of $v_{i-1}, v_i, v_{i+1}, v_{i+2}$ is fat.*

Claim 2. *If u is a small vertex with $d_H(u) \in \{3, 4, 5, 7\}$, then u is incident to at most $d_H(u) - 1$ false 3-faces.*

Claim 3. *If an edge vv_i lies on two false 3-faces $[vv_{i-1}v_i]$ and $[vv_iv_{i+1}]$ such that v is fat, v_{i-1}, v_{i+1} are false, u_{i-1}, u_{i+1} are small and $4 \leq d_H(v_i) \leq 6$, then $m_3^*(v_i) \leq d_H(v_i) - 2$.*

Claim 4. *Assume that v_i is a false vertex which is crossed by the edges vu_i and x_iy_i in G. If $8 \leq d_H(v) \leq 11$, then x_i or y_i is fat so that $[vx_iv_i]$ or $[vy_iv_i]$ is a 3-face. If $d_H(v) \geq 12$, then f_{i-1}, f_i are 3-faces.*

Proof. If $8 \leq d_H(v) \leq 11$, then x_i or y_i is fat by (P1), say x_i. So, by the definition of H, we derive that $vx_i \in E(H)$ and hence $[vx_iv_i]$ is a 3-face. If $d_H(v) \geq 12$, then both f_{i-1} and f_i must be 3-faces by Observation 1. □

Claim 5. *If v_i is a small t-vertex and $d_H(v) = 16 - t$, then vv_i is not incident to any 3-face.*

Proof. Suppose that vv_i is incident to a 3-face, say $f_i = [vv_iv_{i+1}]$. By (P2), f_i is false, so v_{i+1} is a false vertex which is a crossing of the edges vu_{i+1} and zv_i in G. By (P1), $d_{G'}(z) \geq 16-t$, which implies that $d_{G'}(v)+d_{G'}(z) \geq (16-t)+(16-t) = 32 - 2t \geq 18$ as $t \leq 7$. Thus, $zv \in E(H)$ by the structure of H. Now, a 3-cycle vv_izv with vv_i as an edge is obtained, which contradicts (P2). □

Part 3. Discharging rules

To derive a contradiction, we make use of the discharging method. Since G' is connected, so is H. First, by Euler's formula $|V(H)| - |E(H)| + |F(H)| = 2$, we can derive the following identity:

$$\sum_{v\in V(H)}(d_H(v)-4)+\sum_{f\in F(H)}(d_H(f)-4)=-8. \tag{1}$$

Define an initial weight function c on H by $c(x)=d_H(x)-4$ for each $x\in V(H)\cup F(H)$. Redistribute the weight between vertices and faces in H and keep the sum of all weights unchanged so that the resultant weight function c' satisfies

(I) $c'(x)\geq 0$ for all $x\in V^0(H)\cup F^0(H)$; and

(II) $c'(f_0)+\sum_{x\in V(C)}c'(x)\geq -7$.

Hence we obtain a contradiction

$$-7\leq \sum_{x\in V(H)\cup F(H)}c'(x)=\sum_{x\in V(H)\cup F(H)}c(x)=-8 \tag{2}$$

and the proof is completed.

For a k-vertex $v\in V^0(H)$, let $v_0,v_1,\ldots,v_{k-1}$ denote the neighbors of v in H in a cyclic order, and let $f_0,f_1,\ldots,f_{k-1}$ denote the faces of H incident to v with $vv_i,vv_{i+1}\in\partial(f_i)$ for $i=0,1,\ldots,k-1$, where the indices are taken as modulo k. If v_i is false, then we assume that v_i is a crossing of the edges vu_i and x_iy_i of G.

We first design several discharging rules as follows.

(R0) Let $v\in V(C)$. Then v sends $\frac{1}{2}$ to each internal (direct or indirect) neighbor and to each incident internal face. Then we carry out the following additional subrules (if any).

(AR0.1) If $v_i\in V^0(H)$ with $3\leq d_H(v_i)\leq 4$ such that $f_i=[vv_iv_{i+1}]$ is a 3-face, v_{i+1} is false, u_{i+1} is an external vertex or an internal 6^+-vertex, then u_{i+1} sends $\frac{1}{3}$ to v_i through $u_{i+1}v_{i+1}$ and v_iv_{i+1}.

(AR0.2) If $v_i\in V^0(H)$ is a poor 3-vertex, and $f_i=[vv_iwv_{i+1}]$ is a 4-face such that w is false, then f_i sends $\frac{1}{2}$ to v_i.

(R1) Let $f=[x_1x_2x_3]\in F^0(H)$ be a 3-face of H.

(R1.1) Suppose that $|V(f)\cap V(C)|=1$, say $x_1\in V(C)$. If f is false, say x_2 is false, then f gets $\frac{1}{2}$ from x_3. Otherwise, f gets $\frac{1}{2}$ from each of its incident fat vertices.

(R1.2) Suppose that $|V(f)\cap V(C)|=0$. If f is false, then f gets $\frac{1}{2}$ from each of its incident true vertices. Otherwise, f gets $\frac{1}{2}$ from each of its incident big vertices.

(R2) Let f be a 5^+-face. If $f=[x_1x_2\cdots x_5x_6]$ is a special 6-face, say x_2 is an internal 3-vertex with $m_3^*(x_2)=2$ and x_4,x_6 are small vertices, then f gives 1 to x_2, and $\frac{1}{2}$ to each of x_4 and x_6. Otherwise, f divides equally $d_H(f)-4$ to its incident small vertices.

Given a small vertex $v \in V^0(H)$, we use $\alpha(v)$ to denote the sum of weights that v gets from all incident 5^+-faces according to (R2).

(R3) Let $v \in V^0(H)$ be a 3-vertex.

(R3.1) If v is rich and $b_1 = 1 + \frac{1}{2}m_3^*(v) - \alpha(v) > 0$, then v gets $\frac{b_1}{3}$ from each internal (direct or indirect) neighbor in $V^0(H)$.

(R3.2) Assume that v is poor.

(R3.2.1) Suppose that $m_3^*(v) = 1$. Let f_1 be a false 3-face with v_1 being false, let $f_0 = [vv_0xv_1]$ be a 4-face, and $3 \le d_H(f_2) \le 4$.

(R3.2.1.1) If v_0 is true, then v gets $\frac{10}{21}$ from v_2, $\frac{2}{3}$ from v_0, and $\frac{5}{14}$ from u_1.

Remark. If some of v_0, u_1, v_2 in (R3.2.1) are external vertices, then we need to carry out (R0), whereas the discharging operation here for them will be ignored. The similar convention is valid for other cases below.

(R3.2.1.2) If v_0 is false, then v gets $\frac{11}{14}$ from v_2, and $\frac{5}{14}$ from each of u_0 and u_1.

(R3.2.2) Suppose that $m_3^*(v) = 2$, say, f_0, f_2 are false 3-faces. Then v_1, v_2 are false and $f_1 = [vv_1wyv_2]$ is a 5-face. In this case, both w and y are external vertices or internal 6^+-vertices. Then v gets $\frac{5}{6}$ from v_0, and $\frac{1}{3}$ from each of u_1 and u_2.

(R4) Let $v \in V^0(H)$ be a true 4-vertex.

(R4.1) If v is good and $b_2 = \frac{1}{2}m_3^*(v) - \alpha(v) > 0$, then v gets $\frac{b_2}{4}$ from each (direct or indirect) neighbor in $V^0(H)$.

(R4.2) Assume that $m_3^*(v) = 2$ and $m_5^+(v) = 0$.

(R4.2.1) Suppose that f_1 and f_2 are false 3-faces. If v_2 is false, then v gets $\frac{1}{2}$ from each of v_1 and v_3. If v_2 is true (it follows that v_0 is true), then v gets $\frac{1}{2}$ from v_0, and $\frac{1}{6}$ from each of u_1, v_2 and u_3.

(R4.2.2) Suppose that f_0 and f_2 are false 3-faces. If v_0 and v_2 are false, then v gets $\frac{1}{3}$ from each of v_1 and v_3, $\frac{1}{6}$ from each of u_0 and u_2. If v_0 and v_3 are false, then v gets $\frac{1}{3}$ from each of v_1 and v_2, $\frac{1}{6}$ from each of u_0 and u_3.

(R4.3) Assume that $m_3^*(v) = 3$, say f_0, f_1, f_2 are false 3-faces and v_0, v_2 are false. Now it is easy to derive that $d_H(f_3) = 4$. Then v gets $\frac{2}{3}$ from v_3, $\frac{5}{13}$ from u_2, $\frac{1}{3}$ from v_1, and $\frac{3}{26}$ from u_0.

(R5) If v is an internal 5-vertex such that $b_3 = \frac{1}{2}m_3^*(v) - 1 - \alpha(v) > 0$, then v gets $\frac{b_3}{5}$ from each (direct or indirect) neighbor in $V^0(H)$.

(R6) If v is an internal 6-vertex such that $b_4 = \frac{1}{2}m_3^*(v) - 2 - \alpha(v) > 0$, then v gets $\frac{b_4}{6}$ from each (direct or indirect) neighbor in $V^0(H)$.

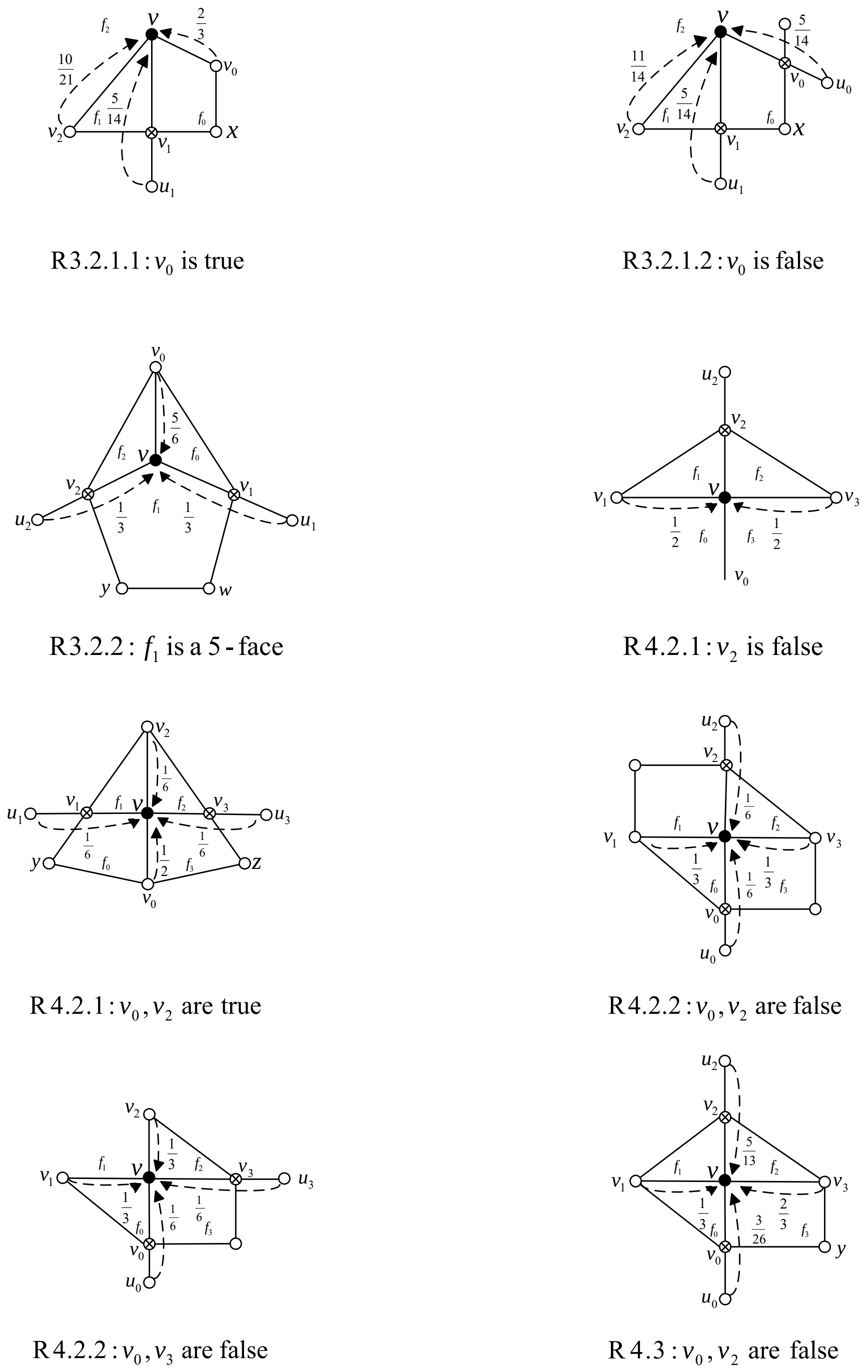

Figure 1. Rules (R3) and (R4).

In Figure 1, vertices marked $\bullet$ have no edges of G incident to them other than those shown, vertices marked $\circ$ may have edges connected to other vertices of H not in the configuration, and vertices marked $\otimes$ are false vertices of H.

Observation 2. *$b_2 \le \frac{1}{2}$; $b_i \le 1$ for $i = 1, 3, 4$.*

Proof. (1) To show that $b_1 \le 1$, we need to carry out (R3.1) for a rich 3-vertex v. First notice that $\alpha(v) \ge 0$ by its definition. If $m_3^*(v) = 0$, then $b_1 = 1 - \alpha(v) \le 1$. If $m_3^*(v) = 1$, then $m_5^+(v) \ge 1$. By (R2), $\alpha(v) \ge \frac{1}{2}$, and hence $b_1 = 1 + \frac{1}{2}m_3^*(v) - \alpha(v) \le 1$. If $m_3^*(v) = 2$, then $m_5^+(v) = 1$, and it is easy to check that $\alpha(v) \ge 1$ by (R2). Consequently, $b_1 = 1 + \frac{1}{2} \times 2 - \alpha(v) \le 1$.

(2) To show that $b_2 \le \frac{1}{2}$, we need to carry out (R4.1) for a good 4-vertex v. By the definition, $m_3^*(v) \le 2$. If $m_3^*(v) \le 1$, then $b_2 = \frac{1}{2}m_3^*(v) - \alpha(v) \le \frac{1}{2}m_3^*(v) \le \frac{1}{2}$. If $m_3^*(v) = 2$, then $m_5^+(v) \ge 1$. By (R2), $\alpha(v) \ge \frac{1}{2}$, and therefore $b_2 = \frac{1}{2}m_3^*(v) - \alpha(v) \le \frac{1}{2} \times 2 - \frac{1}{2} = \frac{1}{2}$.

(3) To show that $b_3 \le 1$, it suffices to note that $m_3^*(v) \le 4$ by Claim 2. Thus, $b_3 = \frac{1}{2}m_3^*(v) - 1 - \alpha(v) \le \frac{1}{2} \times 4 - 1 \le 1$.

(4) Because $m_3^*(v) \le 6$, it is immediate to deduce that $b_4 = \frac{1}{2}m_3^*(v) - 2 - \alpha(v) \le \frac{1}{2} \times 6 - 2 \le 1$. $\square$

Part 4. Computation of weights

Let c' denote the resultant weight function after (R0)–(R6) are carried out on H. Let us first show that $c'(x) \ge 0$ for all $x \in V^0(H) \cup F^0(H)$.

Suppose that $f \in F^0(H)$. Then $d_H(f) \ge 3$. If $d_H(f) = 3$, then $c(f) = -1$. If f is false, then f is incident to two true vertices by Lemma 3(1). By (R0) and (R1), $c'(f) = -1 + \frac{1}{2} \times 2 = 0$. Assume that f is true. If $|V(f) \cap V(C)| = 2$, then $c'(f) \ge -1 + \frac{1}{2} \times 2 = 0$ by (R0). If $|V(f) \cap V(C)| = 1$, then $c'(f) \ge -1 + \frac{1}{2} + \frac{1}{2} = 0$ by (R0) and (R1.1). If $|V(f) \cap V(C)| = 0$, then $c'(f) \ge -1 + \frac{1}{2} \times 2 = 0$ by (R1.2). If $d_H(f) = 4$, then $c'(f) = c(f) = 0$. If $d_H(f) \ge 5$, then (R2) implies that $c'(f) \ge 0$.

Suppose that $v \in V^0(H)$ is a k-vertex. Then $k \ge 3$. Let $v_0, v_1, \ldots, v_{k-1}$ be the neighbors of v in cyclic order, and $f_0, f_1, \ldots, f_{k-1}$ be the faces of H incident to v with $vv_i, vv_{i+1} \in \partial(f_i)$ for $i = 0, 1, \ldots, k-1$, where indices are taken modulo k. Moreover, if v_i is a false vertex, then we assume that v_i is a crossing of G lying on the edge vu_i.

According to the size of k, we consider several cases as follows.

Case 1. $k = 3$. Then $c(v) = k - 4 = -1$, and every (direct or indirect) neighbor of v is fat by (P1).

First assume that v is rich. Then v gets $\min\left\{\frac{1}{2}, \frac{b_1}{3}\right\}$ from each (direct or indirect) neighbor by (R0) and (R3.1), where $b_1 = 1 + \frac{1}{2}m_3^*(v) - \alpha(v) > 0$, and receives $\alpha(v)$ by (R2). Since $b_1 \le 1$ by Observation 2, it follows that $\min\left\{\frac{1}{2}, \frac{b_1}{3}\right\} \ge \frac{b_1}{3}$. Thus, by (R1), $c'(v) \ge -1 - \frac{1}{2}m_3^*(v) + \frac{b_1}{3} \times 3 + \alpha(v) = 0$.

Next assume that v is poor. There are two possibilities to be discussed.

(1.1) $m_3^*(v) = 1$ and $m_5^+(v) = 0$. Let f_1 be a false 3-face such that v_1 is false, $f_0 = [vv_0xv_1]$ is a 4-face, and $3 \le d_H(f_2) \le 4$. By (R1), v needs to send $\frac{1}{2}$ to f_1. If v_0 is true, then v gets $\frac{10}{21}$ from v_2, $\frac{2}{3}$ from v_0, and $\frac{5}{14}$ from u_1 by (R0) and (R3.2.1.1). Hence, $c'(v) \ge -1 - \frac{1}{2} + \frac{10}{21} + \frac{2}{3} + \frac{5}{14} = 0$. If v_0 is false, then v gets $\frac{11}{14}$ from v_2, and $\frac{5}{14}$ from each of u_0 and u_1 by (R0) and (R3.2.1.2). Hence, $c'(v) \ge -1 - \frac{1}{2} + \frac{11}{14} + \frac{5}{14} \times 2 = 0$.

(1.2) $m_3^*(v) = 2$, say f_0, f_2 are false 3-faces. It follows that v_1 and v_2 are false vertices, and f_1 is a 5-face, say $f_1 = [vv_1wyv_2]$, where both w and y are external vertices or internal 6^+-vertices by (P1), (P2) and Observation 1. By (R1), v sends $\frac{1}{2}$ to each of f_0 and f_2. By (R0) and (R3.2.2), v gets $\frac{5}{6}$ from v_0, and $\frac{1}{3}$ from each of u_1 and u_2. By (R2), f_1 gives $\frac{1}{2}$ to v. Consequently, $c'(v) \ge -1 - \frac{1}{2} \times 2 + \frac{5}{6} + \frac{1}{2} + \frac{1}{3} \times 2 = 0$.

Case 2. $k = 4$. Then $c(v) = 0$. By Claim 2, $m_3^*(v) \le 3$. If v is good, then v gets $\min\{\frac{1}{2}, \frac{b_2}{4}\}$ from each (direct or indirect) neighbor in H by (R0) and (R4.1). Hence, $c'(v) \ge \frac{b_2}{4} \times 4 + \alpha(v) - \frac{1}{2}m_3^*(v) = 0$ by (R2). Otherwise, v is not good. If $m_3^*(v) = 3$, then $c'(v) \ge \frac{2}{3} + \frac{5}{13} + \frac{1}{3} + \frac{3}{26} - \frac{1}{2} \times 3 = 0$ by (R0) and (R4.3). Suppose that $m_3^*(v) = 2$. We have to consider two subcases by symmetry. If f_0, f_2 are false 3-faces, then $c'(v) \ge \frac{1}{3} \times 2 + \frac{1}{6} \times 2 - \frac{1}{2} \times 2 = 0$ by (R0) and (R4.2.2). Otherwise, assume that f_1, f_2 are false 3-faces. If v_2 is true, then $c'(v) \ge \frac{1}{2} + \frac{1}{6} \times 3 - \frac{1}{2} \times 2 = 0$ by (R0) and (R4.2.1). If v_2 is false, then $c'(v) \ge \frac{1}{2} \times 2 - \frac{1}{2} \times 2 = 0$ by (R0) and (R4.2.1).

Case 3. $k = 5$. Then $c(v) = 1$. By Claim 3, $m_3^*(v) \le 4$. By Observation 2, $b_3 = \frac{1}{2}m_3^*(v) - 1 - \alpha(v) \le 1$ and so $\frac{b_3}{5} \le \frac{1}{5}$. By (R5) and (R0), v gets at least $\frac{b_3}{5}$ from each (direct or indirect) neighbor in H and $\alpha(v)$ by (R2). Hence, $c'(v) \ge 1 + \alpha(v) + \frac{b_3}{5} \times 5 - \frac{1}{2}m_3^*(v) = 0$.

Case 4. $k = 6$. Then $c(v) = 2$. By (R6), v gets at least $\frac{b_4}{6}$ from each (direct or indirect) neighbor in H and $\alpha(v)$ by (R2). Hence, $c'(v) \ge 2 + \alpha(v) + \frac{b_4}{6} \times 6 - \frac{1}{2}m_3^*(v) = 0$.

Case 5. $7 \le k \le 9$. If $k = 7$, then $m_3^*(v) \le 6$ by Claim 2, so that $c'(v) \ge 3 - \frac{1}{2} \times 6 = 0$ by (R1). If $8 \le k \le 9$, then the neighbors of v are external vertices or internal 7^+-vertices by (P1). Hence, $c'(v) \ge k - 4 - \frac{1}{2}k \ge 0$.

Case 6. $k = 10$. Then $c(v) = 6$. By (P1), the (direct or indirect) neighbors of v are external vertices or internal 6^+-vertices. Let $i \in \{0, 1, \dots, k-1\}$. If v_i or u_i is an external vertex or an internal 7^+-vertex, then v gives nothing to v_i or u_i by our rules. Otherwise, there are two possibilities below. If v_i is an internal 6-vertex, then f_{i-1} and f_i are 4^+-faces by Claim 5, and hence $m_3^*(v_i) \le 4$. So, $b_4 = \frac{1}{2}m_3^*(v_i) - 2 - \alpha(v_i) \le 0$, and v gives nothing to v_i by (R6). If u_i is an internal

6-vertex, then by Claim 4, at least one of f_{i-1} and f_i is a 3-face. This implies that $m_3^*(u_i) \leq 5$ by (P2). Now $b_4 = \frac{1}{2}m_3^*(u_i) - 2 - \alpha(u_i) \leq \frac{1}{2}$ and hence v gives u_i at most $\frac{1}{12}$ by (R6). In a word, v gives at most $\frac{1}{12}$ to each of (direct or indirect) neighbors in every possible situation. Consequently, $c'(v) \geq 6 - \frac{1}{2} \times 10 - \frac{1}{12} \times 10 = 0$ by (R1).

Case 7. $k = 11$. Then $c(v) = 7$. By (P1), the (direct or indirect) neighbors of v are external vertices or internal 5^+-vertices. Let $i \in \{0, 1, \ldots, k-1\}$. Assume that v_i or u_i is small. If v_i or u_i is an internal 6^+-vertex, then v gives at most $\frac{1}{6}$ to v_i or u_i by (R6). Otherwise, we consider two possibilities. If v_i is an internal 5-vertex, then f_{i-1} and f_i are 4^+-faces by Claim 5, which implies that $m_3^*(v_i) \leq 3$. So, $b_3 = \frac{1}{2}m_3^*(v_i) - 1 - \alpha(v_i) \leq \frac{1}{2}$, and henceforth v gives at most $\frac{1}{10}$ to v_i by (R5). If u_i is an internal 5-vertex, then Claim 4 asserts that at least one of f_{i-1} and f_i is a 3-face, say $f_{i-1} = [vv_{i-1}v_i]$. If f_i is also a 3-face, then it is easy to derive that $m_3^*(u_i) \leq 3$ by (P2). Otherwise, f_i is a 4^+-face. Let v_i be the crossing of two edges vu_i and $v_{i-1}x$ in G. Since $xv \notin E(H)$, it follows that x is a small vertex of G by the structure of H. Thus, $xu_i \notin E(H)$ by (P1). Again, we obtain that $m_3^*(u_i) \leq 3$. Thus we always have that $b_3 = \frac{1}{2}m_3^*(u_i) - 1 - \alpha(u_i) \leq \frac{1}{2}$ and hence v gives u_i at most $\frac{1}{10}$ by (R5). If $m_3(v) \leq 10$, then $c'(v) \geq 7 - \frac{1}{2} \times 10 - \frac{1}{6} \times 11 = \frac{1}{6}$ by (R1). If $m_3(v) = 11$, then v is adjacent to at least two fat vertices by Claim 1, and therefore $c'(v) \geq 7 - \frac{1}{2} \times 11 - \frac{1}{6} \times 9 = 0$ by (R1).

Now suppose that $d_H(v) \geq 12$. By Observation 1, every face incident to v is of degree at most 4. Moreover, if $f_i = [vv_izv_{i+1}]$ is a 4-face incident to v, then z must be a false vertex. For the sake of convenience, we relabel the neighbors of v in H in a cyclic order as $y_0; x_0^1, x_0^2, \ldots, x_0^{m_0}; y_1; x_1^1, x_1^2, \ldots, x_1^{m_1}; y_2, \ldots, y_{t-1}; x_{t-1}^1, x_{t-1}^2, \ldots, x_{t-1}^{m_{t-1}}$, where $y_0, y_1, \ldots, y_{t-1}$ are fat vertices, and other vertices are false or small. Set

$Y = \{y_0, y_1, \ldots, y_{t-1}\}$,

$X_s = \{x_s^1, x_s^2, \ldots, x_s^{m_s}\}$ for $s = 0, 1, \ldots, t-1$.

Without loss of generality, we assume that $y_0 = v_0, x_0^1 = v_1, \ldots, x_0^{m_0} = v_{p-1}, y_1 = v_p$, where $p = m_0 + 1 \geq 1$. In particular, when $|Y| = 1$, we have that $y_0 = y_1 = v_0$ and $p = k$. It is easy to check that $(m_0 + 1) + (m_1 + 1) + \cdots + (m_{t-1} + 1) = m_0 + m_1 + \cdots + m_{t-1} + t = m_0 + m_1 + \cdots + m_{t-1} + |Y| = d_H(v)$.

Claim 6.

(1) *There is no index i with $2 \leq i \leq p-3$ such that $d_H(f_i) = 3$;*

(2) *There is no index i with $1 \leq i \leq p-2$ such that $d_H(f_i) = 3$ and $d_H(f_{i-1}) = d_H(f_{i+1}) = 4$.*

Proof. (1) Suppose that $f_i = [vv_iv_{i+1}]$ is a 3-face where $2 \leq i \leq p-3$. Without loss of generality, assume that v_i is small. If f_i is true, then v_{i+1} is fat by (P2), which contradicts the choice of i. Otherwise, v_{i+1} is false and so f_i is false. By the structure of H, f_{i+1} is a false 3-face with v_{i+2} being fat by Claim 4. Thus, v_{i+2} is just v_p and $i = p-2$, contradicting the assumption.

(2) The proof is analogous to that of the conclusion (1). □

By Claim 6, we may define the following three subsets of faces incident to v which lie between edges vv_0 and vv_p.

$T_0 = \{f_i \mid d_H(f_i) = 3 \text{ for } i = 0, 1, \ldots, q-1\}$,
$Q = \{f_i \mid d_H(f_i) = 4 \text{ for } i = q, q+1, \ldots, r-1\}$,
$T_p = \{f_i \mid d_H(f_i) = 3 \text{ for } i = r, r+1, \ldots, p-1\}$.

Note that some of T_0, Q, T_p may be empty. Let σ_0 denote the sum of weights that v sends to the elements in $\{f_0, f_1, \ldots, f_{p-1}, v_1 \text{ (or } u_1), v_2 \text{ (or } u_2), \ldots, v_{p-1} \text{ (or } u_{p-1})\}$ according to our rules (R0)–(R6). For $x, y \in V(H) \cup F(H)$, we use $\tau(x \to y)$ to denote the amount of weight that x sends to y according to our rules. Let

$\theta(f_0) = \tau(v \to f_0) + \tau(v \to v_1)$,
$\theta(f_{p-1}) = \tau(v \to f_{p-1}) + \tau(v \to v_{p-1})$,
$\theta(f_0, f_1) = \tau(v \to f_0) + \tau(v \to u_1) + \tau(v \to f_1) + \tau(v \to v_2)$,
$\theta(f_{p-1}, f_{p-2}) = \tau(v \to f_{p-1}) + \tau(v \to u_{p-1}) + \tau(v \to f_{p-2}) + \tau(v \to v_{p-2})$.

Case 8. $k = 12$. Then $c(v) = 8$. By (P1), each of (direct or indirect) internal neighbors of v is of degree at least 4.

Assume that $|Y| = 0$. We claim that $m_3(v) = 0$. Suppose to the contrary that $f_i = [vv_iv_{i+1}]$ is a 3-face. If f_i is true, then at least one of v_i and v_{i+1} is fat by (P2), contradicting the fact that $|Y| = 0$. So, f_i is false, say, v_i is a false vertex and v_{i+1} is small. By Claim 4, f_{i-1} must be a 3-face. This implies that v_{i-1} is fat by (P1), also a contradiction. Now, by (R4)–(R6), v gives at most $\frac{1}{2}$ to each of its (direct or indirect) neighbors. Thus, $c'(v) \geq 8 - \frac{1}{2} \times 12 = 2$.

Assume that $|Y| \geq 1$. We first establish the following result.

Claim 7. *$\sigma_0 \leq \frac{2}{3}(m_0 + 1)$.*

Proof. Note that $m_0 + 1 = p$. The proof is split into some cases by symmetry, depending on the size of $|T_0|, |Q|, |T_p|$.

Case I. $|T_0| = |T_p| = 0$. All $f_0, f_1, \ldots f_{p-1}$ are 4-faces. By (R4)–(R6), v gives at most $\frac{1}{2}$ to v_i for each $i = 1, 2, \ldots, p-1$. Thus, $\sigma_0 \leq \frac{1}{2}p < \frac{2}{3}p$.

Case II. $|Q| = 0$. Note that $p \geq 1$. By Claims 1 and 6, $p \leq 4$. If $p = 1$, then $\sigma_0 \leq \frac{1}{2} < \frac{2}{3}p$.

Assume that $p = 2$. If v_1 is small, then $d_H(v_1) \geq 5$ by (P2), and v gives at most $\frac{1}{5}$ to v_1 by (R5) and (R6). Assume that v_1 is false. If u_1 is a 4-vertex, then it is easy to derive that $m_3^*(u_1) \leq 2$ by (P2), then v gives at most $\frac{1}{6}$ to u_1 by (R4). Thus, v gives at most $\frac{1}{5}$ to u_1 by (R4)–(R6). By (R1), we get that $\sigma_0 \leq \frac{1}{5} + 2 \times \frac{1}{2} = \frac{6}{5} < \frac{4}{3} = \frac{2}{3}p$.

Assume that $p = 3$. Suppose, without loss of generality, that v_1 is false and v_2 is small. Similarly, we can show that v gives at most $\frac{1}{5}$ to each of u_1 and v_2. Consequently, $\sigma_0 \le 3 \times \frac{1}{2} + 2 \times \frac{1}{5} = \frac{19}{10} < 2 = \frac{2}{3}p$.

Assume that $p = 4$. Then v_2 is small and v_1, v_3 are false. Similarly, v sends at most $\frac{1}{5}$ to each of u_1, v_2, u_3, and hence $\sigma_0 \le 4 \times \frac{1}{2} + 3 \times \frac{1}{5} = \frac{13}{5} < \frac{8}{3} = \frac{2}{3}p$.

Case III. $|T_p| = 0$ and $|T_0|, |Q| \ge 1$. Since $Q \neq \emptyset$, it is easy to deduce that $|T_0| \le 2$. First assume that $|T_0| = 1$, then $p \ge 2$. Namely, only f_0 is a 3-face and v_1 is small. By (P2), $d_H(v_1) \ge 5$. By (R5) and (R6), v gives at most $\frac{1}{5}$ to v_1. By (R4)–(R6), v gives at most $\frac{1}{2}$ to each of $v_2, v_3, \ldots, v_{p-1}$. So, by (R1), we get that $\sigma_0 \le \frac{1}{2} + \frac{1}{5} + \frac{1}{2}(p-2) = \frac{1}{2}p + \frac{3}{10} < \frac{2}{3}p$. Next assume that $|T_0| = 2$, then $p \ge 3$. Only f_0, f_1 are 3-faces. It is immediate to derive that v_1 is false and v_2 is small. Similarly, by (R4)–(R5), v gives at most $\frac{1}{5}$ to each of u_1 and v_2. Hence, by (R1), $\sigma_0 \le 2 \times \frac{1}{2} + 2 \times \frac{1}{5} + \frac{1}{2}(p-3) = \frac{2}{5} + \frac{1}{2}(p-1) \le \frac{2}{3}p$.

Case IV. $|T_0|, |Q|, |T_p| \ge 1$. By virtue of the above discussion, we have three possibilities by symmetry.

If $|T_0| = |T_p| = 1$, then $p \ge 3$, $\theta(f_0) \le \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$, $\theta(f_{p-1}) \le \frac{1}{2} + \frac{1}{5} = \frac{7}{10}$, and hence $\sigma_0 \le \theta(f_0) + \theta(f_{p-1}) + \frac{1}{2}(p-3) = 2 \times \frac{7}{10} + \frac{1}{2}(p-3) \le \frac{2}{3}p$.

If $|T_0| = 1$ and $|T_p| = 2$, then $p \ge 4$, $\theta(f_{p-2}, f_{p-1}) \le 2 \times \frac{1}{2} + 2 \times \frac{1}{5} = \frac{7}{5}$, and hence $\sigma_0 \le \theta(f_0) + \theta(f_{p-1}, f_{p-2}) + \frac{1}{2}(p-4) \le \frac{7}{10} + \frac{7}{5} + \frac{1}{2}(p-4) = \frac{1}{2}p + \frac{1}{10} \le \frac{2}{3}p$.

If $|T_0| = |T_p| = 2$, then $p \ge 5$ and $\sigma_0 \le \theta(f_0, f_1) + \theta(f_{p-1}, f_{p-2}) + \frac{1}{2}(p-5) \le \frac{7}{5} + \frac{7}{5} + \frac{1}{2}(p-5) \le \frac{2}{3}p$. □

For $i = 0, 1, \ldots, t-1$, we can similarly define the symbol σ_i on the set X_i, where indices are taken modulo t. Analogous to Claim 7, we can prove that $\sigma_i \le \frac{2}{3}(m_i + 1)$. So, the following inequalities hold:

$$c'(v) \ge 8 - \sum_{0 \le i \le t-1} \sigma_i \ge 8 - \frac{2}{3} \sum_{0 \le i \le t-1} (m_i + 1) \ge 8 - \frac{2}{3} \times 12 \ge 0.$$

Case 9. $k = 13$. Then $c(v) = 9$. Every (direct or indirect) internal neighbor of v is a 3^+-vertex.

If $|Y| = 0$, then it can be similarly shown that $m_3(v) = 0$. By (R3)–(R6), v gives at most $\frac{1}{2}$ to each of its (direct or indirect) neighbors. Consequently, $c'(v) \ge 9 - \frac{1}{2} \times 13 = \frac{5}{2}$.

Assume that $|Y| \ge 1$. We have the following useful result.

Claim 8. $\sigma_0 \le \frac{9}{13}(m_0 + 1)$.

Proof. Similarly to the proof of Claim 7, we consider four cases as follows.

Case I. $|T_0| = |T_p| = 0$. All $f_0, f_1, \ldots f_{p-1}$ are 4-faces. By (R3)–(R6), v gives at most $\frac{1}{2}$ to v_i for each $i = 1, 2, \ldots, p-1$. Thus, $\sigma_0 \le \frac{1}{2}p < \frac{9}{13}p$.

Case II. $|Q| = 0$. Note that $p \geq 1$. By Claims 1 and 6, $p \leq 4$. If $p = 1$, then $\sigma_0 \leq \frac{1}{2} < \frac{9}{13}p$ by (R1).

Assume that $p = 2$. If v_1 is small, then $d_H(v_1) \geq 4$ by (P2). Thus, (R4)–(R6) asserts that v gives at most $\frac{1}{3}$ to v_1. If v_1 is false, then v gives at most $\frac{5}{13}$ to u_1. Consequently, $\sigma_0 \leq \frac{5}{13} + 2 \times \frac{1}{2} = \frac{18}{13} = \frac{9}{13}p$ by (R1).

Assume that $p = 3$. Suppose, without loss of generality, that v_1 is false and v_2 is small. By (P2), $d_H(v_2) \geq 4$. Let β denote the sum of weights that v sends to u_1 and v_2 according to our rules. Let us compute the value of β. If u_1 is fat, then v gives at most $\frac{1}{2}$ to v_2 by (R4)–(R6) and hence $\beta \leq \frac{1}{2}$. Otherwise, assume that u_1 is small. If v_2 is a 5^+-vertex, or a good 4-vertex, then v gives at most $\frac{1}{5}$ to v_2, and at most $\frac{5}{14}$ to u_1 by (R3)–(R6), therefore $\beta \leq \frac{1}{5} + \frac{5}{14} = \frac{39}{70}$. Suppose that v_2 is a 4-vertex that is not good. Let f_1, f_2, g_1, g_2 denote the incident faces of v_2 in a cyclic order. Since $v_2u_1 \notin E(H)$ by (P1) and by the definition of a good 4-vertex, it follows that $g_1 = [v_2v_3z]$ is a false 3-face and $g_2 = [u_1v_1v_2z]$ is a 4-face, where z is a false vertex. On the one hand, since $m_3^*(v_2) = 2$, v gives at most $\frac{1}{3}$ to v_2 by (R4.2). On the other hand, vv_3u_1v forms a 3-cycle of G, which implies that $d_H(u_1) \geq 4$ by (P2). By (R4)–(R6), v gives at most $\frac{1}{5}$ to u_1 so that $\beta \leq \frac{1}{3} + \frac{1}{5} = \frac{8}{15}$. Hence, $\sigma_0 \leq 3 \times \frac{1}{2} + \beta \leq \frac{3}{2} + \frac{39}{70} = \frac{72}{35} < \frac{9}{13}p$.

Assume that $p = 4$. Then v_2 is small and v_1, v_3 are false. By (P2), $d_H(v_2) \geq 4$. By (R4)–(R6), v gives at most $\frac{1}{3}$ to v_2. Let η denote the sum of weights that v sends to u_1, v_2, u_3 according to our rules. It suffices to show that $\eta \leq \frac{10}{13}$ and henceforth $\sigma_0 \leq 4 \times \frac{1}{2} + \frac{10}{13} = \frac{9}{13}p$. In fact, if u_1 or u_3 is fat, then $\eta \leq \frac{1}{3} + \frac{5}{14} = \frac{29}{42}$ by (R3.2). Otherwise, assume that both u_1 and u_3 are small. Then $u_1v_2, u_3v_2 \notin E(H)$ by (P1). If $6 \leq d_H(v_2) \leq 7$, then v gives noting to v_2 and hence $\eta \leq 2 \times \frac{5}{14}$ by (R3)–(R6). In fact, this is evident if $d_H(v_2) = 7$. When $d_H(v_2) = 6$, it is easy to check that $m_3^*(v_2) \leq 4$ and the conclusion follows from (R6). Otherwise, $4 \leq d_H(v_2) \leq 5$. Then v_0 is a 12^+-vertex in G by (P2), implying $u_1v_0 \in E(H)$. By (P2), $d_H(u_1) \geq 4$, and v gives at most $\frac{1}{5}$ to u_1. Similarly, v gives at most $\frac{1}{5}$ to u_3. It follows consequently that $\eta \leq 2 \times \frac{1}{5} + \frac{1}{3} = \frac{11}{15}$.

Case III. $|T_p| = 0$ and $|T_0|, |Q| \geq 1$. Since $Q \neq \emptyset$, it is easy to deduce that $|T_0| \leq 2$. First assume that $|T_0| = 1$. So f_0 is only one 3-face with v_1 as small vertex. By (P2), $d_H(v_1) \geq 4$. Obviously, if $d_H(v_1) = 4$, then v_1 is a good 4-vertex. By (R4)–(R6) and Observation 2, v gives at most $\frac{1}{5}$ to v_1. Therefore, $\theta(f_0) \leq \frac{1}{5} + \frac{1}{2} = \frac{7}{10}$. It yields that $\sigma_0 \leq \theta(f_0) + \frac{1}{2}(p-2) \leq \frac{7}{10} + \frac{1}{2}(p-2) < \frac{9}{13}p$.

Next assume that $|T_0| = 2$. Only f_0, f_1 are 3-faces, v_1 is false and v_2 is small. By (P2), $d_H(v_2) \geq 4$. If we can show that $\theta(f_0, f_1) \leq \frac{71}{42}$, then $\sigma_0 \leq \theta(f_0, f_1) + \frac{1}{2}(p-3) \leq \frac{71}{42} + \frac{1}{2}(p-3) < \frac{9}{13}p$. In fact, by (R3)–(R6), v gives at most $\frac{5}{14}$ to u_2. If $\tau(v \to v_2) \leq \frac{1}{3}$, then $\theta(f_0, f_1) \leq 2 \times \frac{1}{2} + \frac{5}{14} + \frac{1}{3} = \frac{71}{42}$. Otherwise, it is easy to see that $d_H(v_2) = 4$ and $\tau(v \to v_2) \in \left\{\frac{1}{2}, \frac{2}{3}\right\}$ by (R4.2.3). However, in this case, $u_2v_2 \in E(H)$ and hence u_2 is fat by (P1). Thus, v gives nothing to u_2, and $\theta(f_0, f_1) \leq \frac{2}{3} + 2 \times \frac{1}{2} = \frac{5}{3} < \frac{71}{42}$.

Case IV. $|T_0|, |Q|, |T_p| \geq 1$. By the above discussion, we have three possibilities by symmetry.

If $|T_0| = |T_p| = 1$, then $\sigma_0 \leq \theta(f_0) + \theta(f_{p-1}) + \frac{1}{2}(p-3) \leq \frac{7}{10} + \frac{7}{10} + \frac{1}{2}(p-3) = \frac{1}{2}p - \frac{1}{10} < \frac{9}{13}p$.

If $|T_0| = 1$ and $|T_p| = 2$, then $p \geq 4$ and $\sigma_0 \leq \theta(f_0) + \theta(f_{p-1}, f_{p-2}) + \frac{1}{2}(p-4) \leq \frac{7}{10} + \frac{71}{42} + \frac{1}{2}(p-4) < \frac{9}{13}p$.

If $|T_0| = |T_p| = 2$, then $p \geq 5$ and $\sigma_0 \leq \theta(f_0, f_1) + \theta(f_{p-1}, f_{p-2}) + \frac{1}{2}(p-5) \leq \frac{71}{42} + \frac{71}{42} + \frac{1}{2}(p-5) = \frac{1}{2}p + \frac{37}{42} \leq \frac{9}{13}p$. □

Similarly we define σ_i on X_i for $i = 1, 2, \ldots, t-1$, and prove that $\sigma_i \leq \frac{9}{13}(m_i + 1)$. Therefore,

$$c'(v) \geq 9 - \sum_{0 \leq i \leq t-1} \sigma_i \geq 9 - \frac{9}{13} \sum_{0 \leq i \leq t-1} (m_i + 1) \geq 9 - \frac{9}{13} \times 13 \geq 0.$$

Case 9. $k \geq 14$. Then $c(v) = k - 4$. Every (direct or indirect) internal neighbor of v is 3^+-vertex.

If $|Y| = 0$, then $m_3(v) = 0$. In view of the structure of v, no poor 3-vertex gets $\frac{2}{3}$ or $\frac{11}{14}$ from v according to (R3.2.1). Thus, the amount of weight that v sends to each (direct or indirect) internal neighbor is at most $\frac{1}{2}$ by (R3)–(R6). It follows that $c'(v) \geq k - 4 - \frac{1}{2}k > 0$.

Assume that $|Y| \geq 1$. To complete the proof, we first establish the following claim.

Claim 9. $\sigma_0 \leq \frac{5}{7}(m_0 + 1)$.

Proof. Similarly to the proofs of Claims 7 and 8, we consider four cases as follows.

Case I. $|T_0| = |T_p| = 0$. All $f_0, f_1, \ldots f_{p-1}$ are 4-faces. Analogous to the foregoing discussion, v gives at most $\frac{1}{2}$ to v_i for each $i = 1, 2, \ldots, p-1$. Thus, $\sigma_0 \leq \frac{1}{2}(p-1) < \frac{5}{7}p$.

Case II. $|Q| = 0$. Note that $p \geq 1$. By Claims 1 and 6, $p \leq 4$. If $p = 1$, then $\sigma_0 \leq \frac{1}{2} < \frac{5}{7}p$ by (R0) and (R1).

Assume that $p = 2$. If v_1 is small, then (R3)–(R6) asserts that v gives at most $\frac{1}{3}$ to v_1. If v_1 is false, then v gives at most $\frac{5}{13}$ to u_1. Consequently, $\sigma_0 \leq \frac{5}{13} + 2 \times \frac{1}{2} = \frac{18}{13} < \frac{5}{7}p$.

Assume that $p = 3$. By symmetry, suppose that v_1 is false and v_2 is small. Let $\varepsilon = \tau(v \to u_1) + \tau(v \to v_2)$. It suffices to show that $\varepsilon \leq \frac{9}{14}$, and so we have that $\sigma_0 \leq 3 \times \frac{1}{2} + \frac{9}{14} = \frac{5}{7}p$. Note that v gives at most $\frac{5}{14}$ to u_1 by (R3)–(R6) and since v_2 is small. If $\tau(v \to u_1) = 0$, then it is easy to check that $\varepsilon = \tau(v \to v_2) \leq \frac{1}{2} < \frac{9}{14}$. Otherwise, assume that $\tau(v \to u_1) > 0$, which implies that u_1 is small. If $d_H(v_2) \geq 5$ or v_2 is a good 4-vertex, then $\tau(v \to v_2) \leq \frac{1}{5}$ and therefore $\varepsilon \leq \frac{5}{14} + \frac{1}{5} = \frac{39}{70}$. Otherwise, we have to handle two subcases as follows.

• $d_H(v_2) = 3$. Then $[v_3v_2v_1u_1]$ is a 4-face. By (R3.2.1.1), v gives at most $\frac{10}{21}$ to v_2. By (P3), $d_H(u_1) \geq 4$. If $d_H(u_1) \geq 6$, then $\tau(v \to u_1) \leq \frac{1}{6}$ by (R6). If $d_H(u_1) = 5$, then it is easy to check that $m_3^*(u_1) \leq 3$ and hence $\tau(v \to u_1) \leq \frac{1}{10}$ by (R5). If $d_H(u_1) = 4$, then $\tau(v \to u_1) \leq \frac{1}{6}$ by (R4). Hence $\varepsilon \leq \frac{1}{6} + \frac{10}{21} = \frac{9}{14}$.

• $d_H(v_2) = 4$ and v is not good. Let f_1, f_2, g_1, g_2 denote the incident faces of v_2 in a cyclic order. It is easy to derive that $g_1 = [v_2v_3z]$ is a false 3-face and $g_2 = [u_1v_1v_2z]$ is a 4-face, where z is a false vertex. Let z be the crossing of the edges u_1v_3 and v_2y in G. Since v_2 is small, it follows that y is fat. By the structure of H, we have that $u_1v_0, u_1y \in E(H)$, which implies that $d_H(u_1) \geq 4$. By (R4)–(R6), $\tau(v \to u_1) \leq \frac{1}{5}$, and $\tau(v \to v_2) \leq \frac{1}{3}$. Hence $\varepsilon \leq \frac{1}{5} + \frac{1}{3} = \frac{8}{15}$.

Assume that $p = 4$. Then v_2 is small and v_1, v_3 are false. It is easy to observe that each of u_1 and u_3 gets at most $\frac{5}{14}$ from v. Let $\xi = \tau(v \to u_1) + \tau(v \to v_2) + \tau(v \to u_3)$. It suffices to show that $\xi \leq \frac{6}{7}$, and so we get that $\sigma_0 \leq 4 \times \frac{1}{2} + \frac{6}{7} = \frac{5}{7}p$.

• Assume that $d_H(v_2) = 3$. By the definition of H, $v_0u_1, v_4u_3 \in E(H)$, and each of u_1 and u_3 is an external vertex or an internal 4^+-vertex. By (R4)–(R6), $\tau(v \to u_i) \leq \frac{1}{5}$ for $i = 1, 3$. Let f' be the third incident face of v_2 other than f_1, f_2. Then $d_H(f') \geq 5$. If $d_H(f') \geq 6$, then v_2 is a rich vertex. By (R3.1), v gives at most $\frac{1}{3}$ to v_2. It yields that $\xi \leq 2 \times \frac{1}{5} + \frac{1}{3} = \frac{11}{15}$. Assume that $d_H(f') = 5$. Then at least one of u_1 and u_3 is fat, say u_1. If u_3 is an external vertex or an internal 6^+-vertex, then $\tau(v \to u_3) = 0$. This is because if u_3 is an internal 6-vertex, then it is easy to compute that $b_4 \leq 0$. Now, since v gives at most $\frac{5}{6}$ to v_2 and hence $\xi \leq \frac{5}{6}$. Otherwise, $d_H(u_3) \leq 5$. By the structure of H, $u_1v_2 \in E(H)$, deriving a contradiction.

• Assume that $d_H(v_2) \geq 4$. If u_1 or u_3 is fat, then $\xi \leq \frac{1}{2} + \frac{5}{14} = \frac{6}{7}$. Otherwise, if v_2 is a 5^+-vertex or a good 4-vertex, then $\xi \leq \frac{1}{8} + \frac{5}{14} \times 2 = \frac{47}{56}$. Otherwise, v_2 is a 4-vertex that is not good. Let f_1, f_2, g_1, g_2 be the incident faces of v_2 in a cyclic order. Then both g_1 and g_2 are 4-faces by the definition of a good 4-vertex. Let $g_1 = [u_1v_1v_2z]$ and $g_2 = [zv_2v_3u_3]$ where z must be a true vertex. By (R4.2.1), v gives at most $\frac{1}{6}$ to v_2. By (P3), at least one of u_1 and u_3 is a 4^+-vertex, say u_1. So, $\tau(v \to u_1) \leq \frac{1}{5}$, and therefore $\xi \leq \frac{1}{5} + \frac{5}{14} + \frac{1}{6} = \frac{76}{105}$.

Case III. $|T_p| = 0$ and $|T_0|, |Q| \geq 1$. Since $Q \neq \emptyset$, it is easy to deduce that $|T_0| \leq 2$. First assume that $|T_0| = 1$, namely, only f_0 is a 3-face with v_1 as small vertex. By (R3)–(R6), v gives at most $\frac{2}{3}$ to v_1. Therefore, $\theta(f_0) \leq \frac{2}{3} + \frac{1}{2} = \frac{7}{6}$. It turns out that $\sigma_0 \leq \theta(f_0) + \frac{1}{2}(p-2) \leq \frac{7}{6} + \frac{1}{2}(p-2) < \frac{5}{7}p$.

Next assume that $|T_0| = 2$. Only f_0, f_1 are 3-faces, v_1 is false and v_2 is small. It suffices to show that $\theta(f_0, f_1) \leq \frac{25}{14}$ and so $\sigma_0 \leq \theta(f_0, f_1) + \frac{1}{2}(p-3) \leq \frac{25}{14} + \frac{1}{2}(p-3) \leq \frac{5}{7}p$. By (R3)–(R6), $\tau(v \to u_1) \leq \frac{5}{14}$. If $\tau(v \to u_1) = 0$, then since $\tau(v \to v_2) \leq \frac{11}{14}$, we obtain that $\theta(f_0, f_1) \leq 2 \times \frac{1}{2} + \frac{11}{14} = \frac{25}{14}$. Otherwise, $\tau(v \to u_1) > 0$, which implies that u_1 is small. If v_2 is a 4^+-vertex or a rich 3-vertex, then v gives at most $\frac{1}{3}$ to v_2 by (R3)–(R6). It follows that $\theta(f_0, f_1) \leq$

$2 \times \frac{1}{2} + \frac{1}{3} + \frac{5}{14} = \frac{71}{42}$. Otherwise, v_2 is a poor 3-vertex. Let f_1, f_2, g be the incident faces of v_2 in a cyclic order. Then both f_2 and g are 4-faces. Let $f_2 = [vv_2zv_3]$ and $g = [v_1u_1zv_2]$, where z is a false vertex. If $d_H(u_1) \le 5$, then (P2) implies that v_3 is fat and hence $v_2v_3 \in E(H)$ by the definition of H, which contradicts the fact that f_2 is a 4-face. Otherwise, $d_H(u_1) \ge 6$. When $d_H(u_1) = 6$, it is easy to inspect that $m_3^*(u_1) \le 4$. In this case, v gives noting to u_1, contradicting the assumption that $\tau(v \to u_1) > 0$.

Case IV. $|T_0|, |Q|, |T_p| \ge 1$. The proof is split into three subcases below by symmetry.

- $|T_0| = |T_p| = 1$. Note that $p \ge 3$. If $p \ge 4$, then $\sigma_0 \le \theta(f_0)+\theta(f_{p-1})+\frac{1}{2}(p-3) \le \frac{7}{6}+\frac{7}{6}+\frac{1}{2}(p-3) = \frac{1}{2}p+\frac{5}{6} < \frac{5}{7}p$ by the previous proof. Otherwise, $p = 3$. If both v_1 and v_2 are poor 3-vertices, then it is easy to find a 4-cycle with two nonadjacent 3-vertices in H, contradicting (P3). Otherwise, at least one of v_1 and v_2, say v_1, is a 4^+-vertex or a rich 3-vertex. Then v gives at most $\frac{1}{3}$ to v_1 by (R3)–(R6) and hence $\theta(f_0) \le \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$. Consequently, $\sigma_0 \le \theta(f_0) + \theta(f_{p-1}) \le \frac{5}{6} + \frac{7}{6} = 2 < \frac{5}{7}p$.
- $|T_0| = 1$ and $|T_p| = 2$. Then $p \ge 4$. If $p \ge 5$, then $\sigma_0 \le \theta(f_0) + \theta(f_{p-1}, f_{p-2}) + \frac{1}{2}(p-4) \le \frac{7}{6} + \frac{25}{14} + \frac{1}{2}(p-4) = \frac{1}{2}p + \frac{20}{21} < \frac{5}{7}p$. Otherwise, $p = 4$. Similarly to the previous discussion, at least one of v_1 and v_2 is not a poor 3-vertex. If v_1 is not, then $\sigma_0 \le \theta(f_0) + \theta(f_{p-1}, f_{p-2}) \le \frac{5}{6} + \frac{25}{14} = \frac{55}{21} < \frac{5}{7}p$. If v_2 is not, then $\theta(f_{p-1}, f_{p-2}) \le \frac{1}{2} \times 2 + \frac{1}{3} + \frac{5}{14} = \frac{71}{42}$, so that $\sigma_0 \le \theta(f_0)+\theta(f_{p-1}, f_{p-2}) \le \frac{7}{6} + \frac{71}{42} = \frac{20}{7} = \frac{5}{7}p$.
- $|T_0| = |T_p| = 2$. Then $p \ge 5$. It yields that $\sigma_0 \le \theta(f_0, f_1) + \theta(f_{p-1}, f_{p-2}) + \frac{1}{2}(p-5) \le \frac{25}{14} + \frac{25}{14} + \frac{1}{2}(p-5) = \frac{1}{2}p + \frac{15}{14} \le \frac{5}{7}p$. □

Similarly, we can define σ_i on X_i for $i = 1, 2, \ldots, t-1$, and establish $\sigma_i \le \frac{5}{7}(m_i + 1)$. Thus,

$$c'(v) \ge k - 4 - \sum_{0 \le i \le t-1} \sigma_i \ge k - 4 - \frac{5}{7} \sum_{0 \le i \le t-1} (m_i + 1) \ge k - 4 - \frac{5}{7}k \ge 0.$$

Up to now, the statement (I) has been proved. To show the statement (II), we first observe that $c'(f_0) = d_H(f_0) - 4 = 2 - 4 = -2$. Let $v \in V(C)$. By (R0), v needs to send the weight to $d_H(v) - 1$ incident internal faces and $d_H(v) - 2$ internal neighbors, so $c'(v) \ge d_H(v) - 4 - \frac{1}{2}(d_H(v) - 1) - \frac{1}{2}(d_H(v) - 2) = -\frac{5}{2}$. Consequently,

$$\sum_{v \in V(C)} c'(v) + c'(f_0) \ge -2 - \frac{5}{2} \times 2 = -7.$$

This completes the proof of the theorem. ■

4. Concluding Remarks

In this paper, we show that every 1-planar graph G with $\Delta \geq 13$ has $\mathrm{la}(G) \leq \lceil \frac{\Delta+1}{2} \rceil$. This fact together with some known results stated previously implies immediately the following.

Corollary 4.1. *Conjecture* 1 *holds for* 1*-planar graphs with* $\Delta \notin \{7, 9, 11, 12\}$.

It should be pointed out that, in a separate paper, we have further proved that Conjecture 1 holds for 1-planar graphs with $\Delta = 11$ or 12. However, it remains open for a 1-planar graph G with $\Delta = 7$ or 9 to have $\mathrm{la}(G) \leq \lceil \frac{\Delta+1}{2} \rceil$.

Suppose that G is a graph and $e = xy \in E(G)$. We say that e is an (i,j)*-edge* if $d_G(x) = i$ and $d_G(y) = j$, and an L*-light-edge* if $d_G(x) + d_G(y) \leq L$, where L is a constant. An even cycle $C = v_0v_1 \cdots v_{2m-1}v_0$ of G is called a k*-alternating-cycle* if $d_G(v_i) = k$ for $i = 0, 2, 4, \ldots, 2m-2$. A 3-alternating-cycle of length 4 is said to be a 3*-alternating* 4*-cycle*.

It was shown in [13] that every 1-planar graph G with $\delta(G) \geq 2$ contains a 29-light-edge or a 2-alternating-cycle. Our Theorem 2 shows that every 1-planar graph G with $\delta(G) \geq 3$ contains a 15-light-edge, or a 3-cycle with a 16-light-edge, or a 3-alternating 4-cycle. From this fact, the following two corollaries hold trivially.

Corollary 4.2. *Every* 1*-planar graph* G *with* $\delta(G) \geq 3$ *contains a* 16*-light-edge or a* 3*-alternating* 4*-cycle.*

Corollary 4.3. *Every* 1*-planar graph* G *with* $\delta(G) \geq 4$ *or without* 4*-cycles contains a* 16*-light-edge.*

It is unknown if the value 16 in Corollary 4.2 is best possible. Recall that Fabrici and Madaras [8] presented a 7-regular 1-planar graph. Hudák and Šugerek [12] constructed a 1-planar graph with only $(6,8)$-edges and $(8,8)$-edges, and a 1-planar graph with only $(5,9)$-edges, $(5,10)$-edges and $(9,10)$-edges. These examples assert that there exist 1-planar graphs G without 3-alternating 4-cycles contain a 14-light-edge and no edge $xy \in E(G)$ satisfies $d_G(x) + d_G(y) < 14$.

We conclude this paper by raising the following problem.

Problem 1. What is the least integer L such that every 1-planar graph G without 3-alternating 4-cycles contains an L-light-edge?

The above related discussion tells us that $14 \leq L \leq 16$.

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