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ON THE EQUALITY OF DOMINATION NUMBER AND 2-DOMINATION NUMBER

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Abstract

The 2-domination number $\gamma_2(G)$ of a graph G is the minimum cardinality of a set $D \subseteq V(G)$ for which every vertex outside D is adjacent to at least two vertices in D. Clearly, $\gamma_2(G)$ cannot be smaller than the domination number $\gamma(G)$. We consider a large class of graphs and characterize those members which satisfy $\gamma_2 = \gamma$. For the general case, we prove that it is NP-hard to decide whether $\gamma_2 = \gamma$ holds. We also give a necessary and sufficient condition for a graph to satisfy the equality hereditarily.

Keywords: domination number, 2-domination number, hereditary property, computational complexity.

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1. Introduction

In this paper, we continue to expand on the study of graphs that satisfy the equality $\gamma(G) = \gamma_2(G)$, where $\gamma(G)$ and $\gamma_2(G)$ stand for the domination number and the 2-domination number of a graph G, respectively. If $\gamma(G) = \gamma_2(G)$ holds for a graph G, then we call it a (γ, γ_2)*-graph*. We prove that the corresponding recognition problem is NP-hard and there is no forbidden subgraph characterization for (γ, γ_2)-graphs in general. On the other hand, in one of our main results, we consider a large graph class $\mathcal{H}$ and give a special type of forbidden subgraph characterization for (γ, γ_2)-graphs over $\mathcal{H}$. Although the number of these forbidden subgraphs is infinite, we prove that the recognition problem is solvable in polynomial time on $\mathcal{H}$. Putting the question into another setting, we give a complete characterization for (γ, γ_2)-perfect graphs, that is, we characterize the graphs for which all induced subgraphs with minimum degree at least two satisfy the equality of domination number and 2-domination number.

1.1. Terminology and notation

Let G be a simple undirected graph, where $V(G)$ and $E(G)$ denote the set of vertices and the set of edges of G, respectively. The (*open*) *neighborhood* of a vertex v is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of v is given by the cardinality of $N_G(v)$, that is, $\deg_G(v) = |N_G(v)|$. We will write $N(v)$, $N[v]$ and $\deg(v)$ instead of $N_G(v)$, $N_G[v]$ and $\deg_G(v)$, if G is clear from the context. An edge uv is a *pendant edge* if $\deg(u) = 1$ or $\deg(v) = 1$, otherwise the edge is *non-pendant*. The minimum and maximum vertex degrees of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively. For a subset $S \subseteq V(G)$, let $G[S]$ denote the subgraph induced by S. We say that S is *independent* if $G[S]$ does not contain any edges. For disjoint subsets $U, W \subseteq V(G)$, we let $E[U, W]$ denote the set of edges between U and W.

For a positive integer k, the k^{th} *power* of a graph G, denoted by G^k, is the graph on the same vertex set as G such that uv is an edge if and only if the distance between u and v is at most k in G. An edge $uv \in E(G)$ is *subdivided* by deleting the edge uv, then adding a new vertex x and two new edges ux and xv. Let K_n, C_n and P_n denote the complete graph, the cycle and the path, all of order n, respectively; and let S_n denote the star of order $n+1$. For any positive integer n, let $[n]$ be the set of positive integers not exceeding n. For notation and terminology not defined here, we refer the reader to [31].

For a positive integer k, a subset $D \subseteq V(G)$ is a *k-dominating set* of the graph G if $|N_G(v) \cap D| \geq k$ for every $v \in V(G) \setminus D$. The *k-domination number* of G, denoted by $\gamma_k(G)$, is the minimum cardinality among the k-dominating sets of G. Note that the 1-domination number, $\gamma_1(G)$, is the classical domination number $\gamma(G)$.

A graph G is called *F-free* if it does not contain any induced subgraph isomorphic to F. More generally, let $\mathcal{F}$ be a (finite or infinite) class of graphs, then G is $\mathcal{F}$-free if it is F-free for all $F \in \mathcal{F}$. On the other hand, let G^D denote a graph G with a specified subset $D \subseteq V(G)$. Then, $F^{D'}$ is an (induced) subgraph of G^D if F is an (induced) subgraph of G and $D' = V(F) \cap D$. We say that $F_1^{D_1}$

is isomorphic to $F_2^{D_2}$ if there is an edge-preserving bijection between $V(F_1)$ and $V(F_2)$ which maps D_1 onto D_2. Analogously, we may define the $F^{D'}$-freeness of G^D and forbidden (induced) subgraph characterization with a specified vertex subset D.

1.2. Preliminary results

The concept of k-domination in graphs was introduced by Fink and Jacobson [15, 16] and it has been studied extensively by many researchers (see for example [5–8, 10, 13, 14, 18, 19, 26, 30, 32]). For more details, we refer the reader to the books on domination by Haynes, Hedetniemi and Slater [23, 24] and to the survey on k-domination and k-independence by Chellali *et al.* [9].

Fink and Jacobson [15] established the following basic theorem.

Theorem 1 [15]**.** *For any graph G with $\Delta(G) \geq k \geq 2$, $\gamma_k(G) \geq \gamma(G) + k - 2$.*

Although it is proved that the above inequality is sharp for every $k \geq 2$, the characterization of graphs attaining the equality is still open, even for the case when $k = 2$. The corresponding characterization problem was studied in [18, 20, 21], while similar problems involving different domination-type graph and hypergraph invariants were considered for example in [3, 4, 22, 26, 29].

In this paper, we study (γ, γ_2)-graphs, that is, graphs for which Theorem 1 holds with equality if $k = 2$. Note that G is a (γ, γ_2)-graph, that is $\gamma_2(G) = \gamma(G)$, if and only if every component of G is a (γ, γ_2)-graph. Thus, we only deal with connected graphs in the rest of the paper.

Hansberg and Volkmann [21] characterized the cactus graphs (i.e., graphs in which no two cycles share an edge) which are (γ, γ_2)-graphs and they also gave some general properties of the graphs attaining the equality. In 2016, the claw-free (i.e., S_3-free) (γ, γ_2)-graphs and the line graphs which are (γ, γ_2)-graphs were characterized by Hansberg *et al.* [20]. We will refer to the following basic lemmas proved in these papers.

Lemma 1 [21]**.** *If G is a connected nontrivial graph with $\gamma_2(G) = \gamma(G)$, then $\delta(G) \geq 2$.*

Lemma 2 [20]**.** *Let D be a minimum 2-dominating set of a graph G. If $\gamma_2(G) = \gamma(G)$, then D is independent.*

Lemma 3 [20]**.** *Let G be a connected nontrivial graph with $\gamma_2(G) = \gamma(G)$ and let D be a minimum 2-dominating set of G. Then, for each vertex $u' \in V \setminus D$ and $u, v \in D \cap N(u')$, there is a vertex $v' \in V \setminus D$ such that u, u', v and v' induce a C_4.*

We strengthen Lemma 3 by proving the following statement.

Lemma 4. *Let G be a connected nontrivial graph with $\gamma_2(G) = \gamma(G)$ and let D be a minimum 2-dominating set of G. For every pair $u, v \in D$, if $N_G(u) \cap N_G(v) \neq \emptyset$, then there exists a nonadjacent pair $u', v' \in V \setminus D$ such that $N_G(u') \cap D = N_G(v') \cap D = \{u, v\}$.*

Proof. For every vertex $x \in N_G(u) \cap N_G(v)$, there is a vertex y different from x such that $N_G(y) \cap D = \{u, v\}$ and $xy \notin E(G)$, since otherwise $(D \setminus \{u, v\}) \cup \{x\}$ would be a dominating set of G, a contradiction. This proves that we have at least two non-adjacent vertices u' and v' with the property $N_G(u') \cap D = N_G(v') \cap D = \{u, v\}$. ■

The following simple proposition demonstrates that (γ, γ_2)-graphs form a rich class and it indicates the possible difficulties in a general characterization.

Proposition 5. *There is no forbidden (induced) subgraph for the graphs satisfying the equality of domination number and 2-domination number.*

Proof. Consider an arbitrary graph F and a four-cycle C_4, which is vertex-disjoint to F. Let u and v be two non-adjacent vertices of C_4. Construct the graph G_F by joining each vertex of F to both u and v. Since, for any F, the graph G_F contains F as an induced subgraph and it satisfies the equality $\gamma_2(G_F) = \gamma(G_F) = 2$, there is no forbidden induced subgraph for (γ, γ_2)-graphs. ■

As a consequence of Lemmas 1–4, we will prove that all (γ, γ_2)-graphs belong to the following graph class $\mathcal{G}$ that we define together with its subclasses $\mathcal{G}_1$ and $\mathcal{G}_2$.

Definition 1. Given an arbitrary simple graph F with vertex set $V(F) = D = \{v_1, \ldots, v_d\}$, a graph G belongs to the class $\mathcal{G}(F)$ if G can be obtained from F by the following rules.

(i) Define a pair of vertices $X_{i,j} = \{x^1_{i,j}, x^2_{i,j}\}$ for every edge v_iv_j of F, and further, let Y be an arbitrary (possibly empty) set of vertices, such that D, Y and all the pairs $X_{i,j}$ are mutually disjoint sets of vertices. Define $V(G) = D \cup X \cup Y$, where $X = \bigcup_{v_iv_j \in E(F)} X_{i,j}$.

(ii) The edges between D and $X \cup Y$ are defined such that $N_G(x^s_{i,j}) \cap D = \{v_i, v_j\}$ for every vertex $x^s_{i,j} \in X$, and the set $N_G(u) \cap D$ contains at least two vertices and induces a complete subgraph in F for any $u \in Y$. The induced subgraph $G[D]$ cannot contain edges.

(iii) The edges inside $X \cup Y$ can be chosen arbitrarily, but each $X_{i,j}$ must remain independent.

Moreover, G belongs to $\mathcal{G}_1(F)$ if $|N_G(y) \cap D| = 2$ for each $y \in Y$; and G belongs to $\mathcal{G}_2(F)$ if $Y = \emptyset$. The graph classes $\mathcal{G}$, $\mathcal{G}_1$, $\mathcal{G}_2$ contain those graphs G for which there exists a graph F such that G belongs to $\mathcal{G}(F)$, $\mathcal{G}_1(F)$, $\mathcal{G}_2(F)$, respectively (see Figure 1 for an example).

We will prove in Section 2 that if a graph $G \in \mathcal{G}_1$ is obtained by starting with the underlying graph F, then $D = V(F)$ is a minimum 2-dominating set of G. The vertices in X make sure that the necessary condition from Lemma 4 is satisfied.

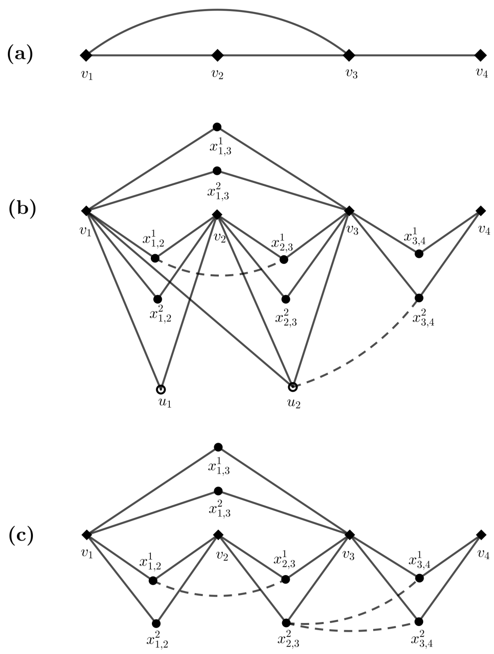

Figure 1. (a) Graph F which is the underlying graph of G and G'. (b) A graph G from $\mathcal{G}(F)$. (c) A graph G' from $\mathcal{G}_2(F)$.

For $G \in \mathcal{G}(F)$ with the fixed partition $V(G) = D \cup X \cup Y$ as per above definition, a vertex v is a *D-vertex* (or *original vertex*) if $v \in D$; v is a *subdivision vertex* if $v \in X$; and v is a *supplementary vertex* if $v \in Y$. The edges inside $G[X \cup Y]$ are called *supplementary edges*, and F is said to be the *underlying graph* of G. In Section 5, we will show that the underlying graph is not necessarily unique by presenting a (γ, γ_2)-graph having two non-isomorphic underlying graphs. Note that the construction in the proof of Proposition 5 always belongs to the class $\mathcal{G}_1$. Hence, Proposition 5 remains true under the condition $G \in \mathcal{G}_1$. This motivates us to focus on the smaller class $\mathcal{G}_2$.

In Figure 1, two different graphs obtained from the same underlying graph F are illustrated, namely $G \in \mathcal{G}(F)$ and $G' \in \mathcal{G}_2(F)$. The supplementary edges are shown by the dashed lines. It is worth to note that $Y = \{u_1, u_2\}$ in Figure 1(a) and $Y = \emptyset$ in Figure 1(c).

Alternatively, we may define the graph class $\mathcal{G}_2(F)$ in the following constructive way. Let F be a simple graph with vertex set $V(F)$ and edge set $E(F)$. Consider the *double subdivision graph* F^* obtained by substituting each edge v_iv_j by two parallel edges and subdividing each edge once by adding the vertices $x^1_{i,j}$ and $x^2_{i,j}$. Let $X_{i,j} = \{x^1_{i,j}, x^2_{i,j}\}$ and define the set of subdivision vertices $X = \bigcup_{v_iv_j \in E(F)} X_{i,j}$. The graph class $\mathcal{G}_2(F)$ consists of the graphs obtained by adding some (maybe zero) supplementary edges between subdivision vertices of F^* such that each $X_{i,j}$ remains independent (see Figure 2 for an example).

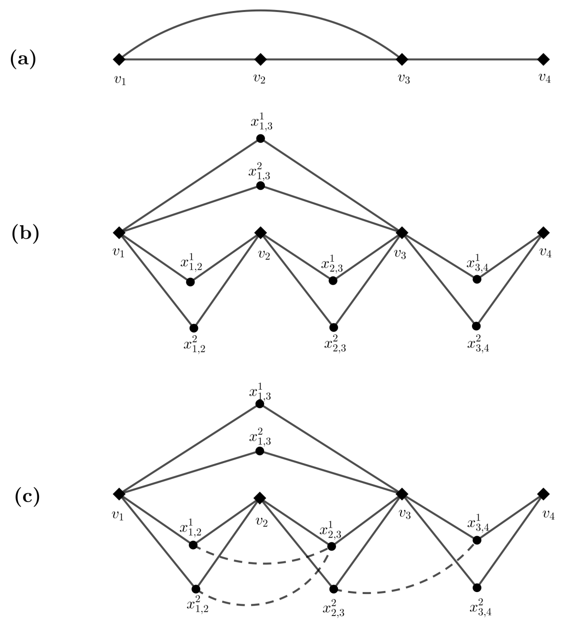

Figure 2. (a) The underlying graph F. (b) The double subdivision graph F^*. (c) The graph $G \in \mathcal{G}_2(F)$ obtained by adding three supplementary edges between subdivision vertices of F^*.

Proposition 6. *If G is a graph with $\gamma_2(G) = \gamma(G)$, then $G \in \mathcal{G}$.*

Proof. Assuming $\gamma_2(G) = \gamma(G)$, choose a minimum 2-dominating set D of G and define the graph $F = G^2[D]$; i.e., we take the 2^{nd} power of G (as defined in the Introduction) and then consider the subgraph induced by D in G^2. We first note that, by Lemma 2, D is independent in G. Since D is a 2-dominating set, every $u \in V(G) \setminus D$ has at least two neighbors in D and, by the definition of F, the set $N_G(u) \cap D$ induces a complete subgraph in F. By Lemma 4, for every edge v_iv_j of F, there exist at least two different and non-adjacent vertices $u, u' \in V(G) \setminus D$ such that $N_G(u) \cap D = N_G(u') \cap D = \{v_i, v_j\}$. If we select such a pair and define $X_{i,j} = \{u, u'\}$ for every $v_iv_j \in E(F)$, and let $Y = V(G) \setminus (D \cup X)$, then G can be obtained from the underlying graph F with the vertex partition $V(G) = D \cup X \cup Y$, proving that $G \in \mathcal{G}(F)$. ■

In a follow-up paper of the present work [12], we studied the analogous problem for each $k \geq 3$. There we gave a characterization for connected bipartite graphs satisfying $\gamma_k(G) = \gamma(G) + k - 2$ and $\Delta(G) \geq k$. This result is based on the notion of the k-uniform "underlying hypergraph" that corresponds to the underlying graph, as defined here, if $k = 2$.

1.3. Structure of the paper

In Section 2, we define the class $\mathcal{H}$ of those graphs which are contained in $\mathcal{G}_2$ with an underlying graph of girth at least 5 and we give a characterization for (γ, γ_2)-graphs over $\mathcal{H}$. Then, in Section 3, we discuss algorithmic complexity questions. First, we prove that the recognition problem of (γ, γ_2)-graphs is NP-hard on $\mathcal{G}_1$ (even if a minimum 2-dominating set is given together with the problem instance). Then, on the positive side, we show that there is a polynomial-time algorithm which recognizes (γ, γ_2)-graphs over the class $\mathcal{H}$ if the instance is given together with the minimum 2-dominating set $D = V(F)$. The algorithm is based on our characterization theorem and Edmond's Blossom Algorithm. In Section 4, we consider the hereditary version of the property and characterize (γ, γ_2)-perfect graphs. As a direct consequence, we get that (γ, γ_2)-perfect graphs are easy to recognize. In the concluding section, we put remarks on the underlying graphs and discuss some open problems.

2. Characterization of (γ, γ_2)-Graphs Over $\mathcal{H}$

To formulate the main result of this section, we will refer to the following definitions.

Definition 2. Let $\mathcal{H}$ be the union of those graph classes $\mathcal{G}_2(F)$ where the underlying graph F is (C_3, C_4)-free.

When we consider a graph $G \in \mathcal{H}$, we will always assume that a fixed (C_3, C_4)-free underlying graph F and a corresponding partition $V(G) = D \cup X$ are given. In order to indicate this structure, we will use the notation G^D.

Definition 3. For a positive integer $k \geq 2$, let $A_k^{W_k}$ be the graph on the vertex set

$$V(A_k) = \{v, w_1, \ldots, w_k, x_1^1, \ldots, x_k^1, x_1^2, \ldots, x_k^2\}$$

and with the edge set

$$E(A_k) = \{vx_i^1, vx_i^2, w_ix_i^1, w_ix_i^2 : 1 \leq i \leq k\} \cup \{x_i^1x_{i+1}^2 : 1 \leq i \leq k\} \cup \{x_k^1x_1^2\}.$$

The specified vertex set is $W_k = \{v\} \cup \{w_i : 1 \leq i \leq k\}$ (for illustration see Figure 3).

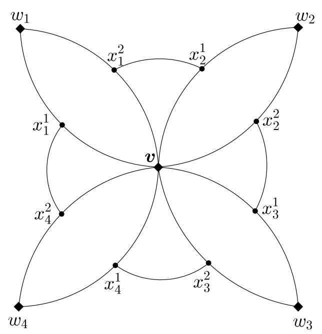

Figure 3. The graph A_4.

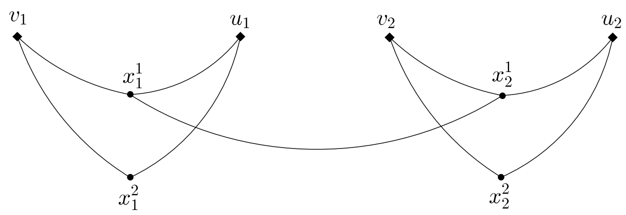

Figure 4. The graph B.

Definition 4. Let B^W be the graph of order 8 with

$$V(B) = \{v_1, u_1, v_2, u_2, x_1^1, x_1^2, x_2^1, x_2^2\},$$

$$E(B) = \{v_ix_i^1, v_ix_i^2, u_ix_i^1, u_ix_i^2 : 1 \leq i \leq 2\} \cup \{x_1^1x_2^1\}.$$

The specified vertex set is $W = \{v_1, u_1, v_2, u_2\}$ (for illustration see Figure 4).

Note that $A_k \in \mathcal{G}_2(S_k)$ and $B \in \mathcal{G}_2(2K_2)$.

We first prove a lemma which will be referred to in the proof of our main theorem and also in later sections.

Lemma 7. *If $G^D \in \mathcal{G}_1(F)$, then D is a minimum 2-dominating set of G.*

Proof. By definition, every vertex from X has two neighbors in D. Thus, D is a 2-dominating set in G. Suppose to the contrary that D' is a 2-dominating set of G such that $|D'| < |D|$. Let $D_1 = D \cap D'$ and $D_2 = D \setminus D'$. Since D is independent in G, the vertices in D_2 have to be 2-dominated by the vertices of $D' \setminus D$, that is, every vertex in D_2 has at least two neighbors in D'. Then we have

$$|E[D', D_2]| \geq 2|D_2|.$$

Moreover, by the definition of $\mathcal{G}_1(F)$, every vertex in $D' \setminus D$ has exactly two neighbors in D, so we have

$$2|D' \setminus D| \geq |E[D', D_2]|.$$

Thus, $|D' \setminus D| \geq |D_2|$. Since $D' = (D' \setminus D) \cup D_1$, we conclude $|D'| \geq |D_2| + |D_1| = |D|$, a contradiction. ■

Theorem 2. *Let G^D be a graph from $\mathcal{H}$. Then $\gamma(G) = \gamma_2(G)$ holds if and only if G^D contains no subgraph isomorphic to B^W and no subgraph isomorphic to $A_k^{W_k}$ for any $k \geq 2$.*

Proof. Throughout the proof, we assume that $G \in \mathcal{H}$ and hence there exists a (C_3, C_4)-free underlying graph F such that $G \in \mathcal{G}_2(F)$. By Lemma 7, $D = V(F)$ is a minimum 2-dominating set of G.

First assume that G^D contains a (not necessarily induced) subgraph which is isomorphic to B^W. We may assume, without loss of generality, that this subgraph contains the vertices $S = \{v_1, u_1, v_2, u_2, x_1^1, x_1^2, x_2^1, x_2^2\}$, the edges correspond to those in Figure 4, and $S \cap D = \{v_1, u_1, v_2, u_2\}$. Since F is $\{C_3, C_4\}$-free, the induced subgraph $F[S \cap D]$ is $\{C_3, C_4\}$-free as well. Therefore, as $|S \cap D| = 4$, $F[S \cap D]$ is a forest. It contains at least two edges, namely $v_1 u_1$ and $v_2 u_2$. Hence, $F[S \cap D]$ contains a leaf, say v_1. Consider the set $D' = (D \setminus S) \cup \{u_1, x_1^1, x_2^2\}$. Observe that D' dominates all the vertices in D; the vertex $x_1^1 \in D'$ dominates x_2^1; the vertex u_1 dominates x_1^2. By the choice of v_1 and u_1, $F[\{v_1, v_2, u_2\}]$ contains only the edge $v_2 u_2$. Hence, all the subdivision vertices different from $\{x_1^1, x_1^2, x_2^1, x_2^2\}$ are dominated either by $D \setminus S$ or u_1. Hence, D' is a dominating set in G and $|D'| < |D|$. These imply $\gamma(G) < \gamma_2(G)$.

Next assume that G^D contains a subgraph which is isomorphic to $A_k^{W_k}$. We may assume, without loss of generality, that the vertices of this subgraph are named as given in the definition of $A_k^{W_k}$. Let $W = W_k$. Consider the set $D' = (D \setminus W) \cup \{x_1^1, \ldots, x_k^1\}$. Observe that D' dominates all the vertices in D; the set $\{x_1^1, \ldots, x_k^1\} \subseteq D'$ dominates all the vertices of the form x_i^s ($i \in [k]$, $s \in [2]$). Since F is assumed to be C_3-free, for any further subdivision vertex $x_{i,j}^s$ of G, at least one of its neighbors which is a D-vertex, namely at least one of v_i and v_j, is not included in W. Thus, $x^s_{i,j}$ is dominated by a vertex in $D \setminus W$. We may conclude that D' is a dominating set in G. Since $|W| = k+1$, we have $|D'| < |D|$ from which $\gamma(G) < \gamma_2(G)$ follows. This finishes the proof of one direction of our theorem.

For the converse, we assume that G contains no subgraph isomorphic to B^W and no subgraph isomorphic to $A_k^{W_k}$ for any $k \geq 2$, and then prove that $\gamma(G) = \gamma_2(G)$. In particular, having no subgraph isomorphic to B^W means that every supplementary edge is inside a neighborhood of a D-vertex and, therefore, $N[x^s_{i,j}] \subseteq N[v_i] \cup N[v_j]$ holds for each supplementary vertex $x^s_{i,j}$. Now, suppose for a contradiction that $\gamma(G) < \gamma_2(G)$. Let D' be a minimum dominating set of G such that $|D' \cap D|$ is maximum under this condition. It is clear that $|D'| = \gamma(G) < \gamma_2(G) = |D|$.

We first prove that no pair $x^1_{i,j}$, $x^2_{i,j}$ are contained together in D'. Suppose, to the contrary, that $\{x^1_{i,j}, x^2_{i,j}\} \subseteq D'$. Then, since $N[x^1_{i,j}] \cup N[x^2_{i,j}] \subseteq N[v_i] \cup N[v_j]$, the set $D'' = (D' \setminus \{x^1_{i,j}, x^2_{i,j}\}) \cup \{v_i, v_j\}$ would be a dominating set of G. This contradicts either the minimality of $|D'|$ or the maximality of $|D' \cap D|$.

If we have some edges $v_iv_j \in E(F)$ such that $|X_{i,j} \cap D'| = 0$, then we delete all these $X_{i,j}$ pairs from G, delete all the associated edges from F and obtain G' and F'. Note that, by definition, $G' \in \mathcal{G}_2(F')$ and F' is still (C_3, C_4)-free. As D' contains exactly one vertex from each remaining pair $X_{i,j}$, we infer that $|E(F')| \leq |D'|$. By Lemma 7, $\gamma_2(G')$ remains $|D|$ (we did not delete the possibly arising isolated vertices). We deleted only subdivision vertices not contained in $D \cup D'$ and D' contains exactly one vertex from each pair $X_{i,j}$ corresponding to an edge $v_iv_j \in E(F')$. Therefore,

$$|E(F')| \leq |D' \cap V(G')| < |D \cap V(G')| \tag{1}$$

holds and $D' \cap V(G')$ is a dominating set in G'. By Lemma 7, $D \cap V(G')$ remains a minimum 2-dominating set in G'.

G' might contain several components. By the inequality (1), there is a component, say G'', such that $|D' \cap V(G'')| < |D \cap V(G'')| = \gamma_2(G'')$. It is clear that G'' is not an isolated vertex. Recall that $N_G[x^s_{i,j}] \subseteq N_G[v_i] \cup N_G[v_j]$ holds for each supplementary vertex $x^s_{i,j}$ in G and hence, by construction, the analogous statement remains true in G''. Thus, the connectivity of the underlying graph F'' of G'' follows from the connectivity of G''. It also holds that $V(F'') = D \cap V(G'')$. Moreover, as $D' \cap V(G'')$ intersects each pair $X_{i,j}$ from G'', we have $|E(F'')| \leq |D' \cap V(G'')|$. We may conclude

$$|E(F'')| \leq |D' \cap V(G'')| < |V(F'')|. \tag{2}$$

The underlying graph F'' is therefore a tree and

$$|E(F'')| = |D' \cap V(G'')| = |V(F'')| - 1 \tag{3}$$

holds. By the first equality in (3), $D' \cap D \cap V(G'') = \emptyset$. Note that F'' is not necessarily an induced subgraph of F but, as F is C_3-free, all the star-subgraphs of F'' are induced stars in F.

Consider a non-pendant edge v_iv_j in F'' (if there exists). We know that $D' \cap V(G'')$ is a dominating set in G'' and it contains exactly one vertex from X_{ij}. Renaming the vertices if necessary, we may suppose $x^1_{i,j} \in D'$. Then the vertex $x^2_{i,j}$ must be dominated by a vertex from D', which is a neighbor of either v_i or v_j. Without loss of generality, assume that $x^2_{i,j}$ is dominated by a neighbor of v_i. Let $S = V(G'') \setminus (N_{G''}(v_j) \setminus X_{ij})$ and consider the induced subgraph $G''[S]$. Let H be the component of the resulting graph, which contains both v_i and v_j.

Recall that $D' \cap V(G'')$ dominates all vertices in G''. By construction, $N_{G''}[v_p] \subseteq V(H)$ is true for every vertex $v_p \neq v_j$ from $D' \cap V(H)$ and

$$N_{G''}\left[x^s_{p,q}\right] \subseteq N_{G''}[v_p] \cup N_{G''}[v_q] \subseteq V(H)$$

holds for every $x^s_{p,q} \in X \cap V(H)$ if $p \neq j \neq q$. The set $D' \cap V(H)$ therefore dominates all vertices from $V(H) \setminus N_H[v_j]$. As $N_H[v_j] = \{v_j, x^1_{i,j}, x^2_{i,j}\}$, it can be readily seen that $D' \cap V(H)$ is a dominating set in H.

Repeate sequentially this procedure of deleting non-pendant edges in the underlying graph. At the end we obtain a graph H_r with an underlying graph F_r such that F_r is isomorphic to a star graph $K_{1,m}$. Then the set $D_r = V(H_r) \cap D'$ is a dominating set of H_r and it contains exactly one vertex from each pair $X_{i,j}$ of subdivision vertices.

We will construct a directed graph R as follows. We create a vertex $x_{i,j}$ corresponding to each pair $X_{i,j} \subset V(H_r)$ of subdivision vertices. Then, we add a directed edge from $x_{i,j}$ to $x_{k,\ell}$ in R, if the vertex in $X_{i,j} \setminus D_r$ is dominated by the vertex in $X_{k,\ell} \cap D_r$. As D_r has exactly one vertex from each pair $X_{i,j}$, the outdegree of each vertex $x_{i,j} \in V(R)$ is at least one. Thus, there is a directed cycle of order at least $t \geq 2$, which corresponds to a subgraph isomorphic to $A^{W_t}_t$ in $H_r^{D \cap V(H_r)} \subseteq G^D$. This contradicts our assumption and finishes the proof of the theorem. ■

3. Algorithmic Complexity

Since there are infinitely many forbidden subgraphs, Theorem 2 does not give directly a polynomial time recognition algorithm for (γ, γ_2)-graphs on $\mathcal{H}$. However, based on this characterization, we can design a polynomial time algorithm to check whether $\gamma(G) = \gamma_2(G)$ holds for a general instance $G^D \in \mathcal{H}$.

Theorem 3. *Let $G^D \in \mathcal{H}$ be given. It can be decided in polynomial time whether the graph G^D satisfies the equality $\gamma(G) = \gamma_2(G)$.*

Proof. By Theorem 2, $\gamma(G) = \gamma_2(G)$ holds if and only if G^D contains no subgraph isomorphic to B^W and no subgraph isomorphic to $A_k^{W_k}$ for any $k \geq 2$.

The algorithm below, first, determines whether $B^W \subseteq G^D$. If it holds, then the algorithm halts. It can be readily checked that this part of the algorithm requires polynomial time.

```
Input: A graph G^D ∈ H
Output: If γ(G) = γ2(G), then true; else false.
    for each supplementary edge uv in G
        if D ∩ (N_G(u) ∩ N_G(v)) = ∅, then return false
    for each vertex x in D
        X ← N_G(x) and G' ← G[X]
        k = (deg_G x)/2
        for i ← 1 to k do
            E ← E(G')
            for j ← 1 to k do
                if j ≠ i, then E ← E ∪ {x_j^1 x_j^2}
            μ ← the order of the maximum matching in E
            if μ = k, then return false
        end-for
    end-for
    return true
end.
```

Then, in the next steps of the algorithm, the existence of subgraphs isomorphic to $A_\ell^{W_\ell}$ is tested. In order to find such a subgraph (if it exists), the algorithm searches for an appropriate matching in $G[N_G(v_i)]$ for every vertex v_i from D. Since a subgraph $A_\ell^{W_\ell}$ does not necessarily contain all the neighbors of v_i, it is not enough to check the existence of a perfect matching in $G[N_G(v_i)]$. Instead, we define the edge set $E_i = \{x_{i,j}^1 x_{i,j}^2 : v_j \in N_F(v_i)\}$. Let G_i^* be the graph $G[N_G(v_i)]$ extended by the edges from E_i. Clearly, G_i^* contains a perfect matching which is E_i. On the other hand, G_i^* contains a perfect matching different from E_i if and only if $G[N_G(v_i)]$ has a subgraph isomorphic to $A_\ell^{W_\ell}$. Hence, the algorithm checks all possible $G_i^* - e$ graphs, where $e \in E_i$, and if any of them has a perfect matching, then there exists a subgraph isomorphic to $A_\ell^{W_\ell}$.

In order to find a maximum matching in $G_i^* - e$, we can use Edmond's Blossom Algorithm [11], which was improved by Micali and Vazirani in [28] to run in time $O(\sqrt{n}m)$ for any graph of order n and size m. The procedure will be repeated $(\deg_G(x)/2) = \deg_F(x)$ times for every vertex $x \in D$, that is, $\Sigma_{v \in V(F)} \deg(v) = 2|E(F)|$, in total. Thus, the second part of the algorithm requires polynomial-time. This finishes the proof. ■

We now show that the same problem is NP-hard even on the graph class $\mathcal{G}_1^D$.

Theorem 4. *Consider every graph $G \in \mathcal{G}_1$ together with a specified set D such that $G^2[D] \cong F$ and $G \in \mathcal{G}_1(F)$. Then, it is NP-complete to decide whether the inequality $\gamma(G) < \gamma_2(G)$ holds for a general instance $G \in \mathcal{G}_1$.*

Proof. By Lemma 7, we have $\gamma_2(G) = |D|$ and it can be checked in polynomial time whether a given set D' with $|D'| < |D|$ is a dominating set of G. Thus, the decision problem belongs to NP.

In order to prove the NP-hardness, we present a polynomial-time reduction from the well-known 3-SAT problem, which is proved to be NP-complete [17].

Let $X = \{x_1, x_2, \ldots, x_k\}$ be a set of Boolean variables. A truth assignment for X is a function $\varphi : X \to \{t, f\}$. If $\varphi(x_i) = t$ holds, then the variable x_i is called *true*; else if $\varphi(x_i) = f$ holds, then x_i is called $false$. If x_i is a variable in X, then x_i and $\neg x_i$ are literals over X. The literal x_i is true under φ if and only if the variable x_i is true under φ; the literal $\neg x_i$ is true if and only if the variable x_i is false. A clause over X is a set of three literals over X, represents the disjunction of those literals and it is satisfied by a truth assignment if and only if at least one of its members is true under that assignment. A collection $\mathcal{C}$ of clauses over X is *satisfiable* if and only if there exists some truth assignment for X that satisfies all the clauses in $\mathcal{C}$. Such a truth assignment is called a *satisfying truth assignment* for $\mathcal{C}$. The 3-SAT problem is specified as follows.

3-SATISFIABILITY (3-SAT) PROBLEM

Instance: A collection $\mathcal{C} = \{C_1, C_2, \ldots, C_\ell\}$ of clauses over a finite set X of variables such that $|\mathcal{C}_j| = 3$, for $1 \leq j \leq \ell\}$.

Question: Is there a truth assignment for X that satisfies all the clauses in $\mathcal{C}$?

Let $\mathcal{C}$ be a 3-SAT instance with clauses $C_1, C_2, \ldots, C_\ell$ over the Boolean variables $X = \{x_1, x_2, \ldots, x_k\}$. We may assume that for every three variables $x_{i_1}, x_{i_2}, x_{i_3}$ there exists a clause C_j, where $j \in [\ell]$, such that C_j does not contain any of the variables $x_{i_1}, x_{i_2}, x_{i_3}$(neither in positive form, nor in negative form). Otherwise, the problem could be reduced to at most eight (separated) 2-SAT problems, which are solvable in polynomial time.

We now construct a graph $G \in \mathcal{G}_1(F)$, where $F \cong S_{k+1}$, such that the given instance $\mathcal{C}$ of 3-SAT problem is satisfiable if and only if $\gamma(G) < \gamma_2(G)$. The construction is as follows.

For every variable x_i, we create three vertices $\{x_i^t, x_i^f, v_i\}$ and then we add the edges $x_i^t v_i$ and $x_i^f v_i$. For every clause $C_j \in \mathcal{C}$, we create a vertex c_j, and if x_i is a literal in C_j, then $x_i^t c_j \in E(G)$; if $\neg x_i$ is a literal in C_j, then $x_i^f c_j \in E(G)$. Moreover, we add a vertex c^* and the edges $c^* x_i^t$ and $c^* x_i^f$ for every $i \in [k]$. We also add a vertex v_{k+1} and the edge set $\{c_i v_{k+1} : 1 \leq i \leq \ell\} \cup \{c^* v_{k+1}\}$. Finally, we add a new vertex v_0, which is adjacent to every vertex in $V(G) \setminus \{v_1, v_2, \ldots, v_{k+1}\}$ (for an illustration of the construction see Figure 5). The order of G is obviously $3k + \ell + 3$ and this construction can be done in polynomial time. Note that $G \in \mathcal{G}_1(F)$, where F is a star with center v_0 and leaves $v_1, \ldots, v_{k+1}$. Thus, we have $\gamma_2(G) = k + 2$, by Lemma 7.

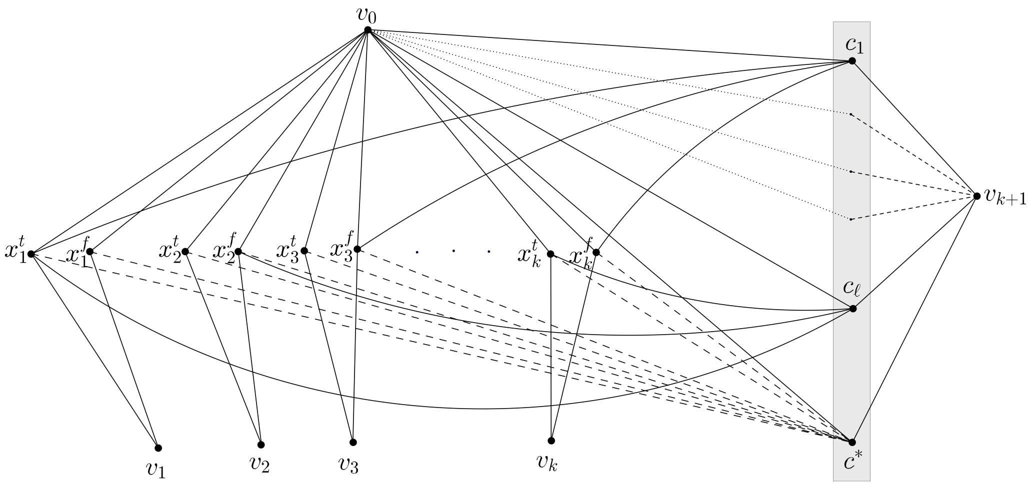

Figure 5. An illustration of the construction for 3-SAT reduction. The clauses C_1 and C_ℓ corresponding to the vertices c_1 and c_ℓ, respectively, are $C_1 = (x_1 \vee \neg x_3 \vee \neg x_k)$ and $C_\ell = (x_1 \vee \neg x_2 \vee x_k)$.

We now prove that $\mathcal{C}$ is satisfiable if and only if $\gamma(G) < \gamma_2(G)$. First, consider a truth assignment $\varphi : x_i \to \{t, f\}$ which satisfies $\mathcal{C}$. Let $D_1 = \bigcup_{i \in [k]} \{x_i^t : \varphi(x_i) = t\}$ and let $D_2 = \bigcup_{i \in [k]} \{x_i^f : \varphi(x_i) = f\}$. Consider the set $D' = D_1 \cup D_2 \cup \{c^*\}$. It can be readily checked that D' is a dominating set of cardinality $k + 1$. Hence, $\gamma(G) < \gamma_2(G)$ follows.

Conversely, assume that $\gamma(G) < \gamma_2(G)$ and consider a minimum dominating set D' of cardinality at most $k+1$. In order to dominate v_i, the set D' contains at least one vertex from the set $\{x_i^t, x_i^f, v_i\}$, for each $i \in [k]$. Similarly, to dominate v_{k+1}, the set D' contains at least one vertex from the set $\{c_1, c_2, \ldots, c_\ell, c^*, v_{k+1}\}$. Since $|D'| \leq k + 1$, we have $|D' \cap \{x_i^t, x_i^f, v_i\}| = 1$ for every $i \in [k]$. Moreover, $|D' \cap \{c_1, c_2, \ldots, c_\ell, c^*, v_{k+1}\}| = 1$ and $v_0 \notin D'$.

Suppose that $v_{k+1} \in D'$. In order to dominate the vertices x_i^t and x_i^f, the set D' contains the vertex v_i for all $i \in [k]$. Hence, $N_G(v_0) \cap D' = \emptyset$. From the discussion above, we know that $v_0 \notin D'$. Thus, v_0 is not dominated by a vertex from D', a contradiction.

Suppose that $c_j \in D'$ for some $j \in [\ell]$. Let C_j be the corresponding clause containing the variables $x_{i_1}, x_{i_2}, x_{i_3}$. Consider any variable $x_s \in X \setminus \{x_{i_1}, x_{i_2}, x_{i_3}\}$. Since $|D' \cap \{x_i^t, x_i^f, v_i\}| = 1$ for each $i \in [k]$, D' contains v_s in order to dominate both of the vertices x_s^t and x_s^f. By our assumption, there exists a clause C_q not containing the variables $x_{i_1}, x_{i_2}, x_{i_3}$ neither in positive nor in negative form. Thus, c_q is not dominated by a vertex from D', a contradiction.

Since $|D' \cap \{c_1, c_2, \ldots, c_\ell, c^*\}| = 1$, the only remaining case is $c^* \in D'$. Under this assumption, every vertex c_i must be dominated by the vertices corresponding to the literals in C_i. Thus, the truth assignment

$$\varphi(x_i) = \begin{cases} t, & \text{if } x_i^t \in D', \\ f, & \text{if } x_i^f \in D' \text{ or if } v_i \in D' \end{cases}$$

satisfies $\mathcal{C}$. This finishes the proof. ■

Theorem 4 implies that it is coNP-complete to decide whether the equality $\gamma(G) = \gamma_2(G)$ holds for a general instance G from $\mathcal{G}_1$. On the other hand, we cannot prove that the problem belongs to NP. Instead, we will consider the complexity class Θ_2^p, which consists of those problems solvable by a polynomial-time deterministic algorithm using NP-oracle asked for only $O(\log n)$ times. (For a detailed introduction, please, see [27].)

Proposition 8. *The complexity of deciding whether $\gamma(G) = \gamma_2(G)$ holds for a general instance G is in the class Θ_2^p.*

Proof. Using binary search, the parameters $\gamma(G)$ and $\gamma_2(G)$ can be determined by asking the NP-oracle $O(\log n)$ times whether the inequalities $\gamma(G) \leq k$ and $\gamma_2(G) \leq k$ hold. Thus, the decision problem belongs to Θ_2^p. ■

Note that in [3], a similar statement was proved for the problem of deciding whether the transversal number $\tau(\mathcal{H})$ equals the domination number $\gamma(\mathcal{H})$ for a general instance hypergraph $\mathcal{H}$.

4. Characterization of (γ, γ_2)-Perfect Graphs

Recently, Alvarado, Dantas, Rautenbach [1, 2] and Henning, Jäger, Rautenbach [25] studied graphs for which the equality between two fixed domination-type invariants hereditarily holds. The analogous problem for transversal and domination numbers of graphs and hypergraphs was considered in [3].

In this section, we characterize (γ, γ_2)-perfect graphs, that is, we characterize the graphs for which the equality between the domination and the 2-domination numbers hereditarily holds. By Lemma 1, $\delta(G) \geq 2$ is a necessary condition for $\gamma(G) = \gamma_2(G)$. Hence, we define (γ, γ_2)-perfect graphs as follows.

Definition 5. Let G be a graph with $\delta(G) \geq 2$. Then G is a (γ, γ_2)-perfect graph if the equality $\gamma(H) = \gamma_2(H)$ holds for every induced subgraph H of minimum degree at least two.

Note that a disconnected graph G is (γ, γ_2)-perfect if and only if all of its components are (γ, γ_2)-perfect.

In order to formulate the results of this section we will define the following class.

Definition 6. Let S_k be the star with center vertex v and end vertices $\{v_1, v_2, \ldots, v_k\}$ such that $k \geq 1$. Denote the edge $vv_j \in E(S_k)$ by e_j for $j \in [k]$. Let $S(i_1, i_2, \ldots, i_k)$ be the graph obtained by substituting each edge e_j of S_k by i_j parallel edges $e_j^1, e_j^2, \ldots, e_j^{i_j}$, where $i_j \geq 2$, and then subdividing each edge e_j^r by adding the vertex x_j^r for all $r \in [i_j]$ and all $j \in [k]$ (see Figure 6). A graph G belongs to the class $\mathcal{S}$ if it is isomorphic to $S(i_1, i_2, \ldots, i_k)$ for some $k \geq 1$, where $i_j \geq 2$ for all $j \in [k]$.

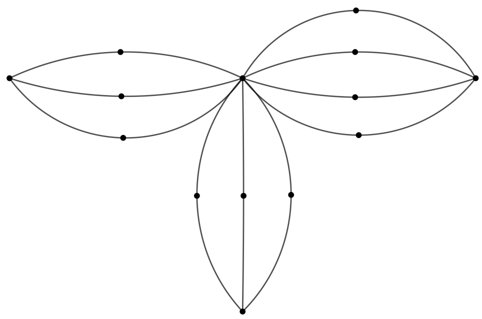

Figure 6. An illustration for the graph $S(3, 3, 4)$.

We clearly have $\mathcal{S} \subseteq \mathcal{G}_1$, since any $S(i_1, i_2, \ldots, i_k) \in \mathcal{G}_1(F)$, where $F \cong S_k$. On the other hand, if $G' \in \mathcal{G}(S_k)$, the underlying graph does not contain a clique of order larger than two and consequently, $|N(y) \cap D| = 2$ for every supplementary vertex y. This implies that $G' \in \mathcal{G}_1(S_k)$. By the definitions above, we have the following equivalence.

Proposition 9. *For any graph, $G \in \mathcal{S}$ holds if and only if $G \in \mathcal{G}_1(S_k)$ (or, equivalently, $G \in \mathcal{G}(S_k)$) for a non-trivial star S_k and G does not contain a supplementary edge.*

The main result of this section is a characterization theorem for (γ, γ_2)-perfect graphs.

Theorem 5. *G is a connected (γ, γ_2)-perfect graph if and only if $G \in \mathcal{S}$.*

Proof. We first prove that if $G \in \mathcal{S}$, then it is (γ, γ_2)-perfect graph. By Proposition 9, we know that $G \in \mathcal{G}_1(F)$, where $F \cong S_k$ for $k \geq 1$. Then, by Lemma 7, $\gamma_2(G) = |V(F)| = k + 1$. Since a minimal 2-dominating set is a dominating set, we have the inequality $\gamma(G) \leq k + 1$. In order to prove that $\gamma(G) = \gamma_2(G)$, it is enough to show that $\gamma(G) > k$. Suppose to the contrary that D' is a minimum dominating set of G such that $|D'| \leq k$.

Consider the vertices of G corresponding to the end vertices of the star S_k. Let $\{v_1, v_2, \ldots, v_k\} = V(F) \setminus \{v\} \subseteq V(G)$, where v is the center of $F \cong S_k$. Since D' is a dominating set, $|N_G[v_j] \cap D'| \geq 1$ for each $j \in [k]$. Note that the closed neighborhoods of any two vertices from the set $\{v_1, v_2, \ldots, v_k\}$ are disjoint. Since $|D'| \leq k$ by our assumption, we have $v \notin D'$ and $|N_G[v_j] \cap D'| = 1$, for every $j \in [k]$. Moreover, as the center v must also be dominated, there exists some $j \in [k]$ and $r \in [i_j]$ such that $x_j^r \in D'$. Then, $v_j \notin D'$ and the vertices in $(X_j \cup Y_j) \setminus \{x_j^r\}$ are not dominated by D', which is a contradiction. Consequently, k vertices are not enough to dominate all the vertices of G, that is, $\gamma(G) \geq k+1$. It follows that $\gamma(G) = \gamma_2(G)$ for any $G \in \mathcal{S}$.

Next, suppose that H is an induced subgraph of G with minimum degree at least two. If H does not contain any subdivision vertices, we have $\delta(H) = 0$, a contradiction. Thus, H contains a subdivision vertex. Let $x_p^q \in V(H)$ for some $p \in [k]$ and $q \in [i_p]$. Since $\deg_G(x_p^q) = 2$, then both of the neighbors of x_p^q must be in $V(H)$, i.e., $N_G(x_p^q) = \{v, v_p\} \subseteq V(H)$. Since $\delta(H) \geq 2$ by the assumption, using an argument similar to the above, we have $\deg_H(v_p) \geq 2$. Thus, $|(X_p \cup Y_p) \setminus \{x_p^q\}) \cap V(H)| \geq 1$. Consequently, $H \in \mathcal{S}$ and, as it was proved above, $\gamma(H) = \gamma_2(H)$ holds for every induced subgraph of G with minimum degree at least two.

To prove the converse, assume that G is a connected (γ, γ_2)-perfect graph. Note that $\gamma(C_n) = \lceil \frac{n}{3} \rceil$ and $\gamma_2(C_n) = \lceil \frac{n}{2} \rceil$, where $n \geq 3$. Thus, the (γ, γ_2)-perfect graph G does not contain an induced cycle C_n, where $n = 3$ or $n \geq 5$.

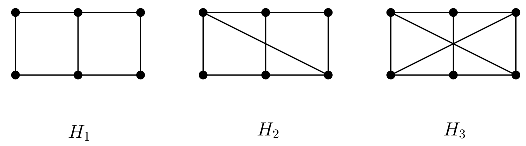

Figure 7. The graphs H_1, H_2 and H_3.

Now, suppose that G has a non-induced subgraph isomorphic to C_r, for some $r \geq 5$. Since all of its induced cycles are 4-cycles, G contains at least one of the three graphs H_1, H_2 and H_3, shown in Figure 7, as an induced subgraph. Observe that $\gamma(H_i) < \gamma_2(H_i)$ for all $i \in \{1, 2, 3\}$. This contradicts our assumption that G is a (γ, γ_2)-perfect graph. Thus, G does not contain a cycle C_r, where $r \neq 4$.

Since G is (γ, γ_2)-perfect by the assumption, then the equality $\gamma(G) = \gamma_2(G)$ holds. By Proposition 6, we know that $G \in \mathcal{G}$. Thus, if D is a minimum 2-dominating set of G, then D is independent and $F = G^2[D]$ is the underlying graph of G.

First, note that F does not contain a cycle C_r for $r \geq 3$. Otherwise, G would contain a subgraph isomorphic to C_{2r}, which is a contradiction. Thus, F is a forest and $G \in \mathcal{G}_1(F)$. Then suppose that F is not connected. Since G is connected, there is a supplementary edge $e = uv$, where u and v are two subdivision vertices of G such that $N(u) \cap V(F)$ and $N(v) \cap V(F)$ are in different components of F. By the definition of the graph class $\mathcal{G}_1$, there are two vertices u' and v' such that $N_G(u) \cap V(F) = N_G(u') \cap V(F)$ and $N_G(v) \cap V(F) = N_G(v') \cap V(F)$. Let $\{x_1, x_2\} = N_G(u) \cap V(F)$ and $\{x_3, x_4\} = N_G(v) \cap V(F)$, where the sets $\{x_1, x_2\}$ and $\{x_3, x_4\}$ are contained by different components of F. Consider the set $A = \{x_1, x_2, x_3, x_4, u, v, u', v'\}$ and the induced subgraph $G[A]$. It is easy to check that $\delta(G[A]) \geq 2$, $\gamma(G[A]) \leq 3$ and $\gamma_2(G[A]) = 4$, which is a contradiction. Thus, F is a tree.

Suppose that G has a supplementary edge $e = uv \in E(G)$, where $u, v \in V(G) \setminus V(F)$. Let $N_G(u) \cap V(F) = \{x_1, x_2\}$ and $N_G(v) \cap V(F) = \{x_3, x_4\}$. Note that $|\{x_1, x_2\} \cap \{x_3, x_4\}| \leq 1$, otherwise G would contain a subgraph isomorphic to C_3. By Lemma 4, there exist two further vertices u' and v' satisfying $N_G(u') \cap V(F) = \{x_1, x_2\}$ and $N_G(v') \cap V(F) = \{x_3, x_4\}$. If $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 1$, then without loss of generality, assume that $x_2 = x_3$. Then, there is a subgraph of G isomorphic to C_3 induced by the vertices u, v and x_2, a contradiction. If $\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset$, then let $S = \{x_1, x_2, x_3, x_4, u, v, v, u', v'\}$. A similar argument applied to the subgraph of G induced by the vertex set S yields the inequality $\gamma(G[S]) \leq 3 < \gamma_2(G[S]) = 4$. Thus, G does not have any supplementary edges.

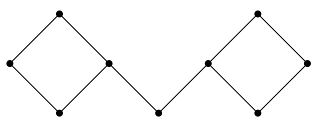

Figure 8. The graph H_4.

Suppose that F contains a subgraph isomorphic to P_4. Since G does not have a supplementary edge, it contains an induced subgraph isomorphic to H_4 given in Figure 8. Note that $\delta(H_4) \geq 2$ and $3 = \gamma(H_4) < \gamma_2(H_4) = 4$, which contradicts the assumption that G is (γ, γ_2)-perfect. Thus, F is a star, $G \in \mathcal{G}_1(F)$, and G does not contain supplementary edges. This finishes the proof by Proposition 9. ■

The graph obtained from an edge by attaching two pendant edges to both of its ends will be called T_6 (for illustration see Figure 9).

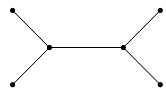

Figure 9. The graph T_6.

Proposition 10. *$G \in \mathcal{S}$ if and only if G is a connected graph with $\delta(G) \geq 2$ and it contains no subgraph isomorphic to any of T_6, P_8, or C_k where $k \neq 4$.*

Proof. If $G \in \mathcal{S}$, then it is easy to see that G is a connected graph with $\delta(G) \geq 2$ and it does not contain a subgraph isomorphic to T_6, P_8, or C_k where $k \neq 4$.

Now, assume that G is a connected graph of minimum degree at least two which does not contain a subgraph isomorphic to T_6, P_8, or C_k where $k \neq 4$. Note that G is bipartite. We further have $\min\{\deg_G(u), \deg_G(v)\} = 2$ for each edge $uv \in E(G)$, since $\delta(G) \geq 2$ and G does not contain a subgraph isomorphic to T_6 or C_3.

First, suppose that G contains an edge $e = uv \in E(G)$, which is a bridge. Then $G - e$ has two components, say G_1 and G_2. Since $\delta(G) \geq 2$, both G_1 and G_2 are non-trivial graphs and may contain at most one vertex, namely either u or v, which is of degree 1. Thus, both of the components contain a cycle. These cycles must be vertex-disjoint 4-cycles with a path between them. Hence, G contains a subgraph isomorphic to P_8 and this contradicts our assumption.

Since G does not contain a bridge, every edge of G lies on a 4-cycle. If all the vertices of G have degree two, then G is isomorphic to C_4 and $G \in \mathcal{S}$. If G is not isomorphic to C_4, then every 4-cycle contains a vertex of degree at least three. For a vertex v of degree two, we define the function $f(v)$ to denote the vertex opposite to v in a 4-cycle. Let $A = \{v \in V(G) : \deg(v) \geq 3$ or $\deg(f(v)) \geq 3\}$. Consider two vertices $u, v \in A$. If $uv \in E(G)$, then uv belongs to a 4-cycle, say $uvv'u'$. At least one of u and v is of degree two, without loss of generality, say $\deg(u) = 2$. Thus, u belongs only to this 4-cycle. Since $f(u) = v'$, by the definition of A, $\deg(v') \geq 3$. If $\deg(v) \geq 3$, then $vv' \in E(G)$, we have a contradiction. If $\deg(v) = 2$, then $v \in A$ and v belongs only to the 4-cycle $uvv'u'$. Thus, $f(v) = u'$, $\deg(u') \geq 3$ and $u'v' \in E(G)$, which is a contradiction. Hence, A is independent. Consider two vertices $u, v \in V(G) \setminus A$. If $uv \in E(G)$, then at least one of $f(u)$ or $f(v)$ is of degree at least three. Then, by the definition of the function f, we have $u \in A$ or $v \in A$, which is a contradiction. Hence, $V(G) \setminus A$ is independent.

Consequently, $(A, V(G) \setminus A)$ is a bipartition of $V(G)$. Note that every 4-cycle has exactly two vertices in A. Hence, $G^A \in \mathcal{G}_1(F)$ where $F \cong G^2[A]$, and there are no supplementary edges. Since G does not have a subgraph isomorphic to C_n

for $n \geq 6$, the underlying graph is a tree. If F contains a subgraph isomorphic to P_4, then G contains a subgraph isomorphic to P_8, which is a contradiction. Thus, F is a star, and Proposition 9 implies that $G \in \mathcal{S}$. ■

Thus, Proposition 10 allows us to state Theorem 5 in a different form as follows.

Theorem 6. *Let G be a connected graph with $\delta(G) \geq 2$. Then G is a (γ, γ_2)-perfect graph if and only if G contains no subgraph isomorphic to any of T_6, P_8, or C_k where $k \neq 4$.*

Note that for any $G \in \mathcal{S}$, the center of the underlying star can be chosen as a vertex v of degree $\Delta(G)$ and then, the subdivision vertices are exactly those contained in $N_G(v)$. Therefore, the characterization given in Theorem 5 directly yields a polynomial-time algorithm which recognizes (γ, γ_2)-perfect graphs.

5. Concluding Remarks and Open Problems

In Section 1, we defined the graph class $\mathcal{G}$ which contains all (γ, γ_2)-graphs. Then, in Section 2, we gave a characterization for (γ, γ_2)-graphs over a specified subclass $\mathcal{H}$ of $\mathcal{G}$. In the definition of $\mathcal{H}$ and in the proof of the main theorem, we referred to the properties of the underlying graph. We noted there that the underlying graph is not always unique when a graph G from $\mathcal{G}$ is given. In Figure 10, we show a (γ, γ_2)-graph having two non-isomorphic underlying graphs. Analogously, one can construct infinitely many graphs with the same property.

In the definition of the class $\mathcal{H}$, we forbid 3-cycles and 4-cycles in the underlying graph. The characterization given in Theorem 2 does not hold if 3-cycles are not forbidden in the underlying graph. This is shown by the graph $A_4^* \in \mathcal{G}_2(F)$ (see Figure 11), where the underlying graph F is a star supplemented by an edge. One can readily check that even if A_4^* contains an induced A_4^W subgraph, it remains a (γ, γ_2)-graph as $\gamma(A_4^*) = \gamma_2(A_4^*) = 5$. Similarly, it is possible to construct graphs whose underlying graphs are C_3-free but not C_4-free such that the statement of Theorem 2 does not remain valid for them. Therefore, the following problems are still open.

Problem 1. Characterize (γ, γ_2)-graphs over the following graph classes.

1. Over the subclass of $\mathcal{G}_2$ where the underlying graph does not contain any C_4 subgraphs.
2. Over the subclass of $\mathcal{G}_2$ where the underlying graph is C_3-free.
3. Over $\mathcal{G}_2$.

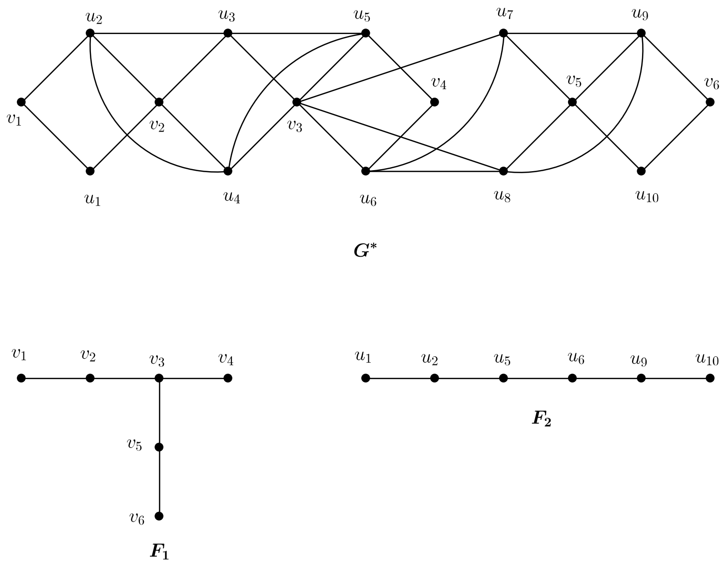

Figure 10. G^* is a graph with $\gamma(G^*) = \gamma_2(G^*) = 6$, which has two non-isomorphic underlying graphs and $G^* \in \mathcal{H}(F_1) \cap \mathcal{H}(F_2)$.

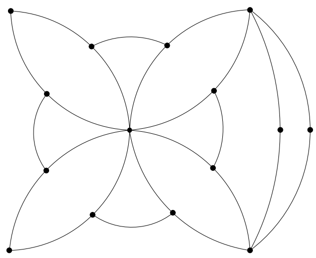

Figure 11. The graph A_4^*.

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