

## ON THE EQUALITY OF DOMINATION NUMBER AND 2-DOMINATION NUMBER

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### Abstract

The 2-domination number  $\gamma_2(G)$  of a graph  $G$  is the minimum cardinality of a set  $D \subseteq V(G)$  for which every vertex outside  $D$  is adjacent to at least two vertices in  $D$ . Clearly,  $\gamma_2(G)$  cannot be smaller than the domination number  $\gamma(G)$ . We consider a large class of graphs and characterize those members which satisfy  $\gamma_2 = \gamma$ . For the general case, we prove that it is NP-hard to decide whether  $\gamma_2 = \gamma$  holds. We also give a necessary and sufficient condition for a graph to satisfy the equality hereditarily.

**Keywords:** domination number, 2-domination number, hereditary property, computational complexity.

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### 1. INTRODUCTION

In this paper, we continue to expand on the study of graphs that satisfy the equality  $\gamma(G) = \gamma_2(G)$ , where  $\gamma(G)$  and  $\gamma_2(G)$  stand for the domination number and the 2-domination number of a graph  $G$ , respectively. If  $\gamma(G) = \gamma_2(G)$  holds

for a graph  $G$ , then we call it a  $(\gamma, \gamma_2)$ -graph. We prove that the corresponding recognition problem is NP-hard and there is no forbidden subgraph characterization for  $(\gamma, \gamma_2)$ -graphs in general. On the other hand, in one of our main results, we consider a large graph class  $\mathcal{H}$  and give a special type of forbidden subgraph characterization for  $(\gamma, \gamma_2)$ -graphs over  $\mathcal{H}$ . Although the number of these forbidden subgraphs is infinite, we prove that the recognition problem is solvable in polynomial time on  $\mathcal{H}$ . Putting the question into another setting, we give a complete characterization for  $(\gamma, \gamma_2)$ -perfect graphs, that is, we characterize the graphs for which all induced subgraphs with minimum degree at least two satisfy the equality of domination number and 2-domination number.

### 1.1. Terminology and notation

Let  $G$  be a simple undirected graph, where  $V(G)$  and  $E(G)$  denote the set of vertices and the set of edges of  $G$ , respectively. The *(open) neighborhood* of a vertex  $v$  is the set  $N_G(v) = \{u \in V(G) : uv \in E(G)\}$  and its *closed neighborhood* is  $N_G[v] = N_G(v) \cup \{v\}$ . The *degree* of  $v$  is given by the cardinality of  $N_G(v)$ , that is,  $\deg_G(v) = |N_G(v)|$ . We will write  $N(v)$ ,  $N[v]$  and  $\deg(v)$  instead of  $N_G(v)$ ,  $N_G[v]$  and  $\deg_G(v)$ , if  $G$  is clear from the context. An edge  $uv$  is a *pendant edge* if  $\deg(u) = 1$  or  $\deg(v) = 1$ , otherwise the edge is *non-pendant*. The minimum and maximum vertex degrees of  $G$  are denoted by  $\delta(G)$  and  $\Delta(G)$ , respectively. For a subset  $S \subseteq V(G)$ , let  $G[S]$  denote the subgraph induced by  $S$ . We say that  $S$  is *independent* if  $G[S]$  does not contain any edges. For disjoint subsets  $U, W \subseteq V(G)$ , we let  $E[U, W]$  denote the set of edges between  $U$  and  $W$ .

For a positive integer  $k$ , the  $k^{\text{th}}$  *power* of a graph  $G$ , denoted by  $G^k$ , is the graph on the same vertex set as  $G$  such that  $uv$  is an edge if and only if the distance between  $u$  and  $v$  is at most  $k$  in  $G$ . An edge  $uv \in E(G)$  is *subdivided* by deleting the edge  $uv$ , then adding a new vertex  $x$  and two new edges  $ux$  and  $xv$ . Let  $K_n$ ,  $C_n$  and  $P_n$  denote the complete graph, the cycle and the path, all of order  $n$ , respectively; and let  $S_n$  denote the star of order  $n + 1$ . For any positive integer  $n$ , let  $[n]$  be the set of positive integers not exceeding  $n$ . For notation and terminology not defined here, we refer the reader to [31].

For a positive integer  $k$ , a subset  $D \subseteq V(G)$  is a  $k$ -*dominating set* of the graph  $G$  if  $|N_G(v) \cap D| \geq k$  for every  $v \in V(G) \setminus D$ . The  $k$ -*domination number* of  $G$ , denoted by  $\gamma_k(G)$ , is the minimum cardinality among the  $k$ -dominating sets of  $G$ . Note that the 1-domination number,  $\gamma_1(G)$ , is the classical domination number  $\gamma(G)$ .

A graph  $G$  is called  $F$ -*free* if it does not contain any induced subgraph isomorphic to  $F$ . More generally, let  $\mathcal{F}$  be a (finite or infinite) class of graphs, then  $G$  is  $\mathcal{F}$ -free if it is  $F$ -free for all  $F \in \mathcal{F}$ . On the other hand, let  $G^D$  denote a graph  $G$  with a specified subset  $D \subseteq V(G)$ . Then,  $F^{D'}$  is an (induced) subgraph of  $G^D$  if  $F$  is an (induced) subgraph of  $G$  and  $D' = V(F) \cap D$ . We say that  $F_1^{D_1}$

is isomorphic to  $F_2^{D_2}$  if there is an edge-preserving bijection between  $V(F_1)$  and  $V(F_2)$  which maps  $D_1$  onto  $D_2$ . Analogously, we may define the  $F^{D'}$ -freeness of  $G^D$  and forbidden (induced) subgraph characterization with a specified vertex subset  $D$ .

## 1.2. Preliminary results

The concept of  $k$ -domination in graphs was introduced by Fink and Jacobson [15, 16] and it has been studied extensively by many researchers (see for example [5–8, 10, 13, 14, 18, 19, 26, 30, 32]). For more details, we refer the reader to the books on domination by Haynes, Hedetniemi and Slater [23, 24] and to the survey on  $k$ -domination and  $k$ -independence by Chellali *et al.* [9].

Fink and Jacobson [15] established the following basic theorem.

**Theorem 1** [15]. *For any graph  $G$  with  $\Delta(G) \geq k \geq 2$ ,  $\gamma_k(G) \geq \gamma(G) + k - 2$ .*

Although it is proved that the above inequality is sharp for every  $k \geq 2$ , the characterization of graphs attaining the equality is still open, even for the case when  $k = 2$ . The corresponding characterization problem was studied in [18, 20, 21], while similar problems involving different domination-type graph and hypergraph invariants were considered for example in [3, 4, 22, 26, 29].

In this paper, we study  $(\gamma, \gamma_2)$ -graphs, that is, graphs for which Theorem 1 holds with equality if  $k = 2$ . Note that  $G$  is a  $(\gamma, \gamma_2)$ -graph, that is  $\gamma_2(G) = \gamma(G)$ , if and only if every component of  $G$  is a  $(\gamma, \gamma_2)$ -graph. Thus, we only deal with connected graphs in the rest of the paper.

Hansberg and Volkmann [21] characterized the cactus graphs (i.e., graphs in which no two cycles share an edge) which are  $(\gamma, \gamma_2)$ -graphs and they also gave some general properties of the graphs attaining the equality. In 2016, the claw-free (i.e.,  $S_3$ -free)  $(\gamma, \gamma_2)$ -graphs and the line graphs which are  $(\gamma, \gamma_2)$ -graphs were characterized by Hansberg *et al.* [20]. We will refer to the following basic lemmas proved in these papers.

**Lemma 1** [21]. *If  $G$  is a connected nontrivial graph with  $\gamma_2(G) = \gamma(G)$ , then  $\delta(G) \geq 2$ .*

**Lemma 2** [20]. *Let  $D$  be a minimum 2-dominating set of a graph  $G$ . If  $\gamma_2(G) = \gamma(G)$ , then  $D$  is independent.*

**Lemma 3** [20]. *Let  $G$  be a connected nontrivial graph with  $\gamma_2(G) = \gamma(G)$  and let  $D$  be a minimum 2-dominating set of  $G$ . Then, for each vertex  $u' \in V \setminus D$  and  $u, v \in D \cap N(u')$ , there is a vertex  $v' \in V \setminus D$  such that  $u, u', v$  and  $v'$  induce a  $C_4$ .*

We strengthen Lemma 3 by proving the following statement.

**Lemma 4.** *Let  $G$  be a connected nontrivial graph with  $\gamma_2(G) = \gamma(G)$  and let  $D$  be a minimum 2-dominating set of  $G$ . For every pair  $u, v \in D$ , if  $N_G(u) \cap N_G(v) \neq \emptyset$ , then there exists a nonadjacent pair  $u', v' \in V \setminus D$  such that  $N_G(u') \cap D = N_G(v') \cap D = \{u, v\}$ .*

**Proof.** For every vertex  $x \in N_G(u) \cap N_G(v)$ , there is a vertex  $y$  different from  $x$  such that  $N_G(y) \cap D = \{u, v\}$  and  $xy \notin E(G)$ , since otherwise  $(D \setminus \{u, v\}) \cup \{x\}$  would be a dominating set of  $G$ , a contradiction. This proves that we have at least two non-adjacent vertices  $u'$  and  $v'$  with the property  $N_G(u') \cap D = N_G(v') \cap D = \{u, v\}$ . ■

The following simple proposition demonstrates that  $(\gamma, \gamma_2)$ -graphs form a rich class and it indicates the possible difficulties in a general characterization.

**Proposition 5.** *There is no forbidden (induced) subgraph for the graphs satisfying the equality of domination number and 2-domination number.*

**Proof.** Consider an arbitrary graph  $F$  and a four-cycle  $C_4$ , which is vertex-disjoint to  $F$ . Let  $u$  and  $v$  be two non-adjacent vertices of  $C_4$ . Construct the graph  $G_F$  by joining each vertex of  $F$  to both  $u$  and  $v$ . Since, for any  $F$ , the graph  $G_F$  contains  $F$  as an induced subgraph and it satisfies the equality  $\gamma_2(G_F) = \gamma(G_F) = 2$ , there is no forbidden induced subgraph for  $(\gamma, \gamma_2)$ -graphs. ■

As a consequence of Lemmas 1–4, we will prove that all  $(\gamma, \gamma_2)$ -graphs belong to the following graph class  $\mathcal{G}$  that we define together with its subclasses  $\mathcal{G}_1$  and  $\mathcal{G}_2$ .

**Definition 1.** Given an arbitrary simple graph  $F$  with vertex set  $V(F) = D = \{v_1, \dots, v_d\}$ , a graph  $G$  belongs to the class  $\mathcal{G}(F)$  if  $G$  can be obtained from  $F$  by the following rules.

- (i) Define a pair of vertices  $X_{i,j} = \{x_{i,j}^1, x_{i,j}^2\}$  for every edge  $v_i v_j$  of  $F$ , and further, let  $Y$  be an arbitrary (possibly empty) set of vertices, such that  $D$ ,  $Y$  and all the pairs  $X_{i,j}$  are mutually disjoint sets of vertices. Define  $V(G) = D \cup X \cup Y$ , where  $X = \bigcup_{v_i v_j \in E(F)} X_{i,j}$ .
- (ii) The edges between  $D$  and  $X \cup Y$  are defined such that  $N_G(x_{i,j}^s) \cap D = \{v_i, v_j\}$  for every vertex  $x_{i,j}^s \in X$ , and the set  $N_G(u) \cap D$  contains at least two vertices and induces a complete subgraph in  $F$  for any  $u \in Y$ . The induced subgraph  $G[D]$  cannot contain edges.
- (iii) The edges inside  $X \cup Y$  can be chosen arbitrarily, but each  $X_{i,j}$  must remain independent.

Moreover,  $G$  belongs to  $\mathcal{G}_1(F)$  if  $|N_G(y) \cap D| = 2$  for each  $y \in Y$ ; and  $G$  belongs to  $\mathcal{G}_2(F)$  if  $Y = \emptyset$ . The graph classes  $\mathcal{G}$ ,  $\mathcal{G}_1$ ,  $\mathcal{G}_2$  contain those graphs  $G$  for which

there exists a graph  $F$  such that  $G$  belongs to  $\mathcal{G}(F)$ ,  $\mathcal{G}_1(F)$ ,  $\mathcal{G}_2(F)$ , respectively (see Figure 1 for an example).

We will prove in Section 2 that if a graph  $G \in \mathcal{G}_1$  is obtained by starting with the underlying graph  $F$ , then  $D = V(F)$  is a minimum 2-dominating set of  $G$ . The vertices in  $X$  make sure that the necessary condition from Lemma 4 is satisfied.

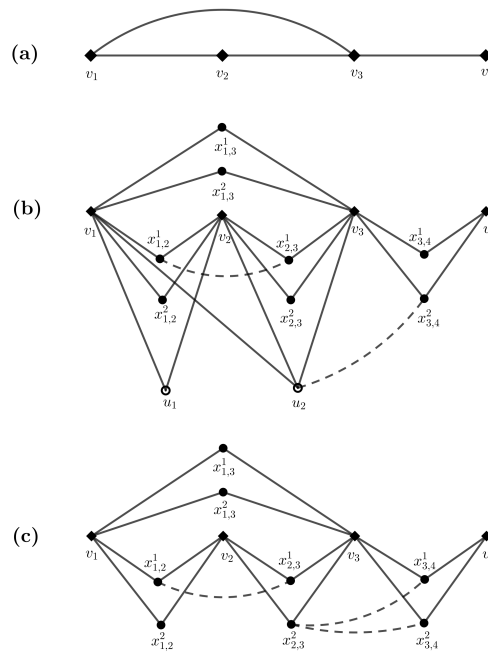


Figure 1. (a) Graph  $F$  which is the underlying graph of  $G$  and  $G'$ . (b) A graph  $G$  from  $\mathcal{G}(F)$ . (c) A graph  $G'$  from  $\mathcal{G}_2(F)$ .

For  $G \in \mathcal{G}(F)$  with the fixed partition  $V(G) = D \cup X \cup Y$  as per above definition, a vertex  $v$  is a *D-vertex* (or *original vertex*) if  $v \in D$ ;  $v$  is a *subdivision vertex* if  $v \in X$ ; and  $v$  is a *supplementary vertex* if  $v \in Y$ . The edges inside  $G[X \cup Y]$  are called *supplementary edges*, and  $F$  is said to be the *underlying graph* of  $G$ . In Section 5, we will show that the underlying graph is not necessarily unique by presenting a  $(\gamma, \gamma_2)$ -graph having two non-isomorphic underlying graphs. Note that the construction in the proof of Proposition 5 always belongs to the class  $\mathcal{G}_1$ . Hence, Proposition 5 remains true under the condition  $G \in \mathcal{G}_1$ . This motivates us to focus on the smaller class  $\mathcal{G}_2$ .

In Figure 1, two different graphs obtained from the same underlying graph  $F$  are illustrated, namely  $G \in \mathcal{G}(F)$  and  $G' \in \mathcal{G}_2(F)$ . The supplementary edges

are shown by the dashed lines. It is worth to note that  $Y = \{u_1, u_2\}$  in Figure 1(a) and  $Y = \emptyset$  in Figure 1(c).

Alternatively, we may define the graph class  $\mathcal{G}_2(F)$  in the following constructive way. Let  $F$  be a simple graph with vertex set  $V(F)$  and edge set  $E(F)$ . Consider the *double subdivision graph*  $F^*$  obtained by substituting each edge  $v_i v_j$  by two parallel edges and subdividing each edge once by adding the vertices  $x_{i,j}^1$  and  $x_{i,j}^2$ . Let  $X_{i,j} = \{x_{i,j}^1, x_{i,j}^2\}$  and define the set of subdivision vertices  $X = \bigcup_{v_i v_j \in E(F)} X_{i,j}$ . The graph class  $\mathcal{G}_2(F)$  consists of the graphs obtained by adding some (maybe zero) supplementary edges between subdivision vertices of  $F^*$  such that each  $X_{i,j}$  remains independent (see Figure 2 for an example).

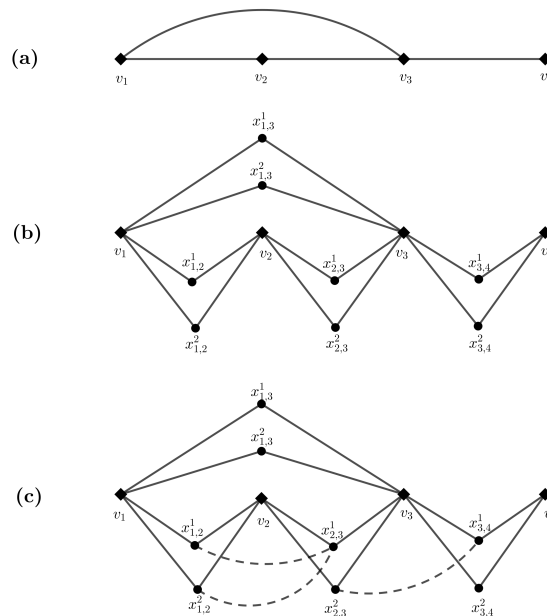


Figure 2. (a) The underlying graph  $F$ . (b) The double subdivision graph  $F^*$ . (c) The graph  $G \in \mathcal{G}_2(F)$  obtained by adding three supplementary edges between subdivision vertices of  $F^*$ .

**Proposition 6.** *If  $G$  is a graph with  $\gamma_2(G) = \gamma(G)$ , then  $G \in \mathcal{G}$ .*

**Proof.** Assuming  $\gamma_2(G) = \gamma(G)$ , choose a minimum 2-dominating set  $D$  of  $G$  and define the graph  $F = G^2[D]$ ; i.e., we take the 2<sup>nd</sup> power of  $G$  (as defined in the Introduction) and then consider the subgraph induced by  $D$  in  $G^2$ . We first note that, by Lemma 2,  $D$  is independent in  $G$ . Since  $D$  is a 2-dominating set, every  $u \in V(G) \setminus D$  has at least two neighbors in  $D$  and, by the definition

of  $F$ , the set  $N_G(u) \cap D$  induces a complete subgraph in  $F$ . By Lemma 4, for every edge  $v_i v_j$  of  $F$ , there exist at least two different and non-adjacent vertices  $u, u' \in V(G) \setminus D$  such that  $N_G(u) \cap D = N_G(u') \cap D = \{v_i, v_j\}$ . If we select such a pair and define  $X_{i,j} = \{u, u'\}$  for every  $v_i v_j \in E(F)$ , and let  $Y = V(G) \setminus (D \cup X)$ , then  $G$  can be obtained from the underlying graph  $F$  with the vertex partition  $V(G) = D \cup X \cup Y$ , proving that  $G \in \mathcal{G}(F)$ . ■

In a follow-up paper of the present work [12], we studied the analogous problem for each  $k \geq 3$ . There we gave a characterization for connected bipartite graphs satisfying  $\gamma_k(G) = \gamma(G) + k - 2$  and  $\Delta(G) \geq k$ . This result is based on the notion of the  $k$ -uniform “underlying hypergraph” that corresponds to the underlying graph, as defined here, if  $k = 2$ .

### 1.3. Structure of the paper

In Section 2, we define the class  $\mathcal{H}$  of those graphs which are contained in  $\mathcal{G}_2$  with an underlying graph of girth at least 5 and we give a characterization for  $(\gamma, \gamma_2)$ -graphs over  $\mathcal{H}$ . Then, in Section 3, we discuss algorithmic complexity questions. First, we prove that the recognition problem of  $(\gamma, \gamma_2)$ -graphs is NP-hard on  $\mathcal{G}_1$  (even if a minimum 2-dominating set is given together with the problem instance). Then, on the positive side, we show that there is a polynomial-time algorithm which recognizes  $(\gamma, \gamma_2)$ -graphs over the class  $\mathcal{H}$  if the instance is given together with the minimum 2-dominating set  $D = V(F)$ . The algorithm is based on our characterization theorem and Edmond’s Blossom Algorithm. In Section 4, we consider the hereditary version of the property and characterize  $(\gamma, \gamma_2)$ -perfect graphs. As a direct consequence, we get that  $(\gamma, \gamma_2)$ -perfect graphs are easy to recognize. In the concluding section, we put remarks on the underlying graphs and discuss some open problems.

## 2. CHARACTERIZATION OF $(\gamma, \gamma_2)$ -GRAPHS OVER $\mathcal{H}$

To formulate the main result of this section, we will refer to the following definitions.

**Definition 2.** Let  $\mathcal{H}$  be the union of those graph classes  $\mathcal{G}_2(F)$  where the underlying graph  $F$  is  $(C_3, C_4)$ -free.

When we consider a graph  $G \in \mathcal{H}$ , we will always assume that a fixed  $(C_3, C_4)$ -free underlying graph  $F$  and a corresponding partition  $V(G) = D \cup X$  are given. In order to indicate this structure, we will use the notation  $G^D$ .

**Definition 3.** For a positive integer  $k \geq 2$ , let  $A_k^{W_k}$  be the graph on the vertex set

$$V(A_k) = \{v, w_1, \dots, w_k, x_1^1, \dots, x_k^1, x_1^2, \dots, x_k^2\}$$

and with the edge set

$$E(A_k) = \{vx_i^1, vx_i^2, w_ix_i^1, w_ix_i^2 : 1 \leq i \leq k\} \cup \{x_i^1x_{i+1}^2 : 1 \leq i \leq k\} \cup \{x_k^1x_1^2\}.$$

The specified vertex set is  $W_k = \{v\} \cup \{w_i : 1 \leq i \leq k\}$  (for illustration see Figure 3).

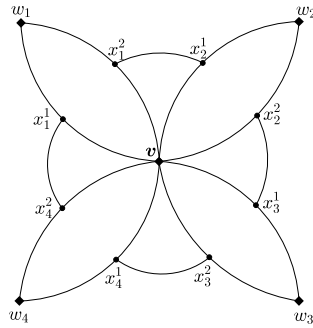


Figure 3. The graph  $A_4$ .

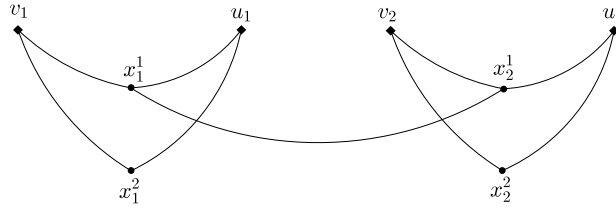


Figure 4. The graph  $B$ .

**Definition 4.** Let  $B^W$  be the graph of order 8 with

$$V(B) = \{v_1, u_1, v_2, u_2, x_1^1, x_1^2, x_2^1, x_2^2\},$$

$$E(B) = \{v_ix_i^1, v_ix_i^2, u_ix_i^1, u_ix_i^2 : 1 \leq i \leq 2\} \cup \{x_1^1x_2^1\}.$$

The specified vertex set is  $W = \{v_1, u_1, v_2, u_2\}$  (for illustration see Figure 4).

Note that  $A_k \in \mathcal{G}_2(S_k)$  and  $B \in \mathcal{G}_2(2K_2)$ .

We first prove a lemma which will be referred to in the proof of our main theorem and also in later sections.



**Lemma 7.** *If  $G^D \in \mathcal{G}_1(F)$ , then  $D$  is a minimum 2-dominating set of  $G$ .*

**Proof.** By definition, every vertex from  $X$  has two neighbors in  $D$ . Thus,  $D$  is a 2-dominating set in  $G$ . Suppose to the contrary that  $D'$  is a 2-dominating set of  $G$  such that  $|D'| < |D|$ . Let  $D_1 = D \cap D'$  and  $D_2 = D \setminus D'$ . Since  $D$  is independent in  $G$ , the vertices in  $D_2$  have to be 2-dominated by the vertices of  $D' \setminus D$ , that is, every vertex in  $D_2$  has at least two neighbors in  $D'$ . Then we have

$$|E[D', D_2]| \geq 2|D_2|.$$

Moreover, by the definition of  $\mathcal{G}_1(F)$ , every vertex in  $D' \setminus D$  has exactly two neighbors in  $D$ , so we have

$$2|D' \setminus D| \geq |E[D', D_2]|.$$

Thus,  $|D' \setminus D| \geq |D_2|$ . Since  $D' = (D' \setminus D) \cup D_1$ , we conclude  $|D'| \geq |D_2| + |D_1| = |D|$ , a contradiction. ■

**Theorem 2.** *Let  $G^D$  be a graph from  $\mathcal{H}$ . Then  $\gamma(G) = \gamma_2(G)$  holds if and only if  $G^D$  contains no subgraph isomorphic to  $B^W$  and no subgraph isomorphic to  $A_k^{W_k}$  for any  $k \geq 2$ .*

**Proof.** Throughout the proof, we assume that  $G \in \mathcal{H}$  and hence there exists a  $(C_3, C_4)$ -free underlying graph  $F$  such that  $G \in \mathcal{G}_2(F)$ . By Lemma 7,  $D = V(F)$  is a minimum 2-dominating set of  $G$ .

First assume that  $G^D$  contains a (not necessarily induced) subgraph which is isomorphic to  $B^W$ . We may assume, without loss of generality, that this subgraph contains the vertices  $S = \{v_1, u_1, v_2, u_2, x_1^1, x_1^2, x_2^1, x_2^2\}$ , the edges correspond to those in Figure 4, and  $S \cap D = \{v_1, u_1, v_2, u_2\}$ . Since  $F$  is  $\{C_3, C_4\}$ -free, the induced subgraph  $F[S \cap D]$  is  $\{C_3, C_4\}$ -free as well. Therefore, as  $|S \cap D| = 4$ ,  $F[S \cap D]$  is a forest. It contains at least two edges, namely  $v_1 u_1$  and  $v_2 u_2$ . Hence,  $F[S \cap D]$  contains a leaf, say  $v_1$ . Consider the set  $D' = (D \setminus S) \cup \{u_1, x_1^1, x_2^2\}$ . Observe that  $D'$  dominates all the vertices in  $D$ ; the vertex  $x_1^1 \in D'$  dominates  $x_2^1$ ; the vertex  $u_1$  dominates  $x_1^2$ . By the choice of  $v_1$  and  $u_1$ ,  $F[\{v_1, v_2, u_2\}]$  contains only the edge  $v_2 u_2$ . Hence, all the subdivision vertices different from  $\{x_1^1, x_1^2, x_2^1, x_2^2\}$  are dominated either by  $D \setminus S$  or  $u_1$ . Hence,  $D'$  is a dominating set in  $G$  and  $|D'| < |D|$ . These imply  $\gamma(G) < \gamma_2(G)$ .

Next assume that  $G^D$  contains a subgraph which is isomorphic to  $A_k^{W_k}$ . We may assume, without loss of generality, that the vertices of this subgraph are named as given in the definition of  $A_k^{W_k}$ . Let  $W = W_k$ . Consider the set  $D' = (D \setminus W) \cup \{x_1^1, \dots, x_k^1\}$ . Observe that  $D'$  dominates all the vertices in  $D$ ; the set  $\{x_1^1, \dots, x_k^1\} \subseteq D'$  dominates all the vertices of the form  $x_i^s$  ( $i \in [k]$ ,  $s \in [2]$ ). Since  $F$  is assumed to be  $C_3$ -free, for any further subdivision vertex  $x_{i,j}^s$  of  $G$ , at least one of its neighbors which is a  $D$ -vertex, namely at least one of  $v_i$  and  $v_j$ ,

is not included in  $W$ . Thus,  $x_{i,j}^s$  is dominated by a vertex in  $D \setminus W$ . We may conclude that  $D'$  is a dominating set in  $G$ . Since  $|W| = k + 1$ , we have  $|D'| < |D|$  from which  $\gamma(G) < \gamma_2(G)$  follows. This finishes the proof of one direction of our theorem.

For the converse, we assume that  $G$  contains no subgraph isomorphic to  $B^W$  and no subgraph isomorphic to  $A_k^{W_k}$  for any  $k \geq 2$ , and then prove that  $\gamma(G) = \gamma_2(G)$ . In particular, having no subgraph isomorphic to  $B^W$  means that every supplementary edge is inside a neighborhood of a  $D$ -vertex and, therefore,  $N[x_{i,j}^s] \subseteq N[v_i] \cup N[v_j]$  holds for each supplementary vertex  $x_{i,j}^s$ . Now, suppose for a contradiction that  $\gamma(G) < \gamma_2(G)$ . Let  $D'$  be a minimum dominating set of  $G$  such that  $|D' \cap D|$  is maximum under this condition. It is clear that  $|D'| = \gamma(G) < \gamma_2(G) = |D|$ .

We first prove that no pair  $x_{i,j}^1, x_{i,j}^2$  are contained together in  $D'$ . Suppose, to the contrary, that  $\{x_{i,j}^1, x_{i,j}^2\} \subseteq D'$ . Then, since  $N[x_{i,j}^1] \cup N[x_{i,j}^2] \subseteq N[v_i] \cup N[v_j]$ , the set  $D'' = (D' \setminus \{x_{i,j}^1, x_{i,j}^2\}) \cup \{v_i, v_j\}$  would be a dominating set of  $G$ . This contradicts either the minimality of  $|D'|$  or the maximality of  $|D' \cap D|$ .

If we have some edges  $v_i v_j \in E(F)$  such that  $|X_{i,j} \cap D'| = 0$ , then we delete all these  $X_{i,j}$  pairs from  $G$ , delete all the associated edges from  $F$  and obtain  $G'$  and  $F'$ . Note that, by definition,  $G' \in \mathcal{G}_2(F')$  and  $F'$  is still  $(C_3, C_4)$ -free. As  $D'$  contains exactly one vertex from each remaining pair  $X_{i,j}$ , we infer that  $|E(F')| \leq |D'|$ . By Lemma 7,  $\gamma_2(G')$  remains  $|D|$  (we did not delete the possibly arising isolated vertices). We deleted only subdivision vertices not contained in  $D \cup D'$  and  $D'$  contains exactly one vertex from each pair  $X_{i,j}$  corresponding to an edge  $v_i v_j \in E(F')$ . Therefore,

$$(1) \quad |E(F')| \leq |D' \cap V(G')| < |D \cap V(G')|$$

holds and  $D' \cap V(G')$  is a dominating set in  $G'$ . By Lemma 7,  $D \cap V(G')$  remains a minimum 2-dominating set in  $G'$ .

$G'$  might contain several components. By the inequality (1), there is a component, say  $G''$ , such that  $|D' \cap V(G'')| < |D \cap V(G'')| = \gamma_2(G'')$ . It is clear that  $G''$  is not an isolated vertex. Recall that  $N_G[x_{i,j}^s] \subseteq N_G[v_i] \cup N_G[v_j]$  holds for each supplementary vertex  $x_{i,j}^s$  in  $G$  and hence, by construction, the analogous statement remains true in  $G''$ . Thus, the connectivity of the underlying graph  $F''$  of  $G''$  follows from the connectivity of  $G''$ . It also holds that  $V(F'') = D \cap V(G'')$ . Moreover, as  $D' \cap V(G'')$  intersects each pair  $X_{i,j}$  from  $G''$ , we have  $|E(F'')| \leq |D' \cap V(G'')|$ . We may conclude

$$(2) \quad |E(F'')| \leq |D' \cap V(G'')| < |V(F'')|.$$

The underlying graph  $F''$  is therefore a tree and

$$(3) \quad |E(F'')| = |D' \cap V(G'')| = |V(F'')| - 1$$

holds. By the first equality in (3),  $D' \cap D \cap V(G'') = \emptyset$ . Note that  $F''$  is not necessarily an induced subgraph of  $F$  but, as  $F$  is  $C_3$ -free, all the star-subgraphs of  $F''$  are induced stars in  $F$ .

Consider a non-pendant edge  $v_i v_j$  in  $F''$  (if there exists). We know that  $D' \cap V(G'')$  is a dominating set in  $G''$  and it contains exactly one vertex from  $X_{ij}$ . Renaming the vertices if necessary, we may suppose  $x_{i,j}^1 \in D'$ . Then the vertex  $x_{i,j}^2$  must be dominated by a vertex from  $D'$ , which is a neighbor of either  $v_i$  or  $v_j$ . Without loss of generality, assume that  $x_{i,j}^2$  is dominated by a neighbor of  $v_i$ . Let  $S = V(G'') \setminus (N_{G''}(v_j) \setminus X_{ij})$  and consider the induced subgraph  $G''[S]$ . Let  $H$  be the component of the resulting graph, which contains both  $v_i$  and  $v_j$ .

Recall that  $D' \cap V(G'')$  dominates all vertices in  $G''$ . By construction,  $N_{G''}[v_p] \subseteq V(H)$  is true for every vertex  $v_p \neq v_j$  from  $D' \cap V(H)$  and

$$N_{G''}[x_{p,q}^s] \subseteq N_{G''}[v_p] \cup N_{G''}[v_q] \subseteq V(H)$$

holds for every  $x_{p,q}^s \in X \cap V(H)$  if  $p \neq j \neq q$ . The set  $D' \cap V(H)$  therefore dominates all vertices from  $V(H) \setminus N_H[v_j]$ . As  $N_H[v_j] = \{v_j, x_{i,j}^1, x_{i,j}^2\}$ , it can be readily seen that  $D' \cap V(H)$  is a dominating set in  $H$ .

Repeat sequentially this procedure of deleting non-pendant edges in the underlying graph. At the end we obtain a graph  $H_r$  with an underlying graph  $F_r$  such that  $F_r$  is isomorphic to a star graph  $K_{1,m}$ . Then the set  $D_r = V(H_r) \cap D'$  is a dominating set of  $H_r$  and it contains exactly one vertex from each pair  $X_{i,j}$  of subdivision vertices.

We will construct a directed graph  $R$  as follows. We create a vertex  $x_{i,j}$  corresponding to each pair  $X_{i,j} \subset V(H_r)$  of subdivision vertices. Then, we add a directed edge from  $x_{i,j}$  to  $x_{k,\ell}$  in  $R$ , if the vertex in  $X_{i,j} \setminus D_r$  is dominated by the vertex in  $X_{k,\ell} \cap D_r$ . As  $D_r$  has exactly one vertex from each pair  $X_{i,j}$ , the outdegree of each vertex  $x_{i,j} \in V(R)$  is at least one. Thus, there is a directed cycle of order at least  $t \geq 2$ , which corresponds to a subgraph isomorphic to  $A_t^{W_t}$  in  $H_r^{D \cap V(H_r)} \subseteq G^D$ . This contradicts our assumption and finishes the proof of the theorem. ■

### 3. ALGORITHMIC COMPLEXITY

Since there are infinitely many forbidden subgraphs, Theorem 2 does not give directly a polynomial time recognition algorithm for  $(\gamma, \gamma_2)$ -graphs on  $\mathcal{H}$ . However, based on this characterization, we can design a polynomial time algorithm to check whether  $\gamma(G) = \gamma_2(G)$  holds for a general instance  $G^D \in \mathcal{H}$ .

**Theorem 3.** *Let  $G^D \in \mathcal{H}$  be given. It can be decided in polynomial time whether the graph  $G^D$  satisfies the equality  $\gamma(G) = \gamma_2(G)$ .*

**Proof.** By Theorem 2,  $\gamma(G) = \gamma_2(G)$  holds if and only if  $G^D$  contains no subgraph isomorphic to  $B^W$  and no subgraph isomorphic to  $A_k^{W_k}$  for any  $k \geq 2$ .

The algorithm below, first, determines whether  $B^W \subseteq G^D$ . If it holds, then the algorithm halts. It can be readily checked that this part of the algorithm requires polynomial time.

```

Input: A graph  $G^D \in \mathcal{H}$ 
Output: If  $\gamma(G) = \gamma_2(G)$ , then true; else false.
  for each supplementary edge  $uv$  in  $G$ 
    if  $D \cap (N_G(u) \cap N_G(v)) = \emptyset$ , then return false
  for each vertex  $x$  in  $D$ 
     $X \leftarrow N_G(x)$  and  $G' \leftarrow G[X]$ 
     $k = (\deg_G x)/2$ 
    for  $i \leftarrow 1$  to  $k$  do
       $E \leftarrow E(G')$ 
      for  $j \leftarrow 1$  to  $k$  do
        if  $j \neq i$ , then  $E \leftarrow E \cup \{x_j^1 x_j^2\}$ 
       $\mu \leftarrow$  the order of the maximum matching in  $E$ 
      if  $\mu = k$ , then return false
    end-for
  end-for
  return true
end.

```

Then, in the next steps of the algorithm, the existence of subgraphs isomorphic to  $A_\ell^{W_\ell}$  is tested. In order to find such a subgraph (if it exists), the algorithm searches for an appropriate matching in  $G[N_G(v_i)]$  for every vertex  $v_i$  from  $D$ . Since a subgraph  $A_\ell^{W_\ell}$  does not necessarily contain all the neighbors of  $v_i$ , it is not enough to check the existence of a perfect matching in  $G[N_G(v_i)]$ . Instead, we define the edge set  $E_i = \{x_{i,j}^1 x_{i,j}^2 : v_j \in N_F(v_i)\}$ . Let  $G_i^*$  be the graph  $G[N_G(v_i)]$  extended by the edges from  $E_i$ . Clearly,  $G_i^*$  contains a perfect matching which is  $E_i$ . On the other hand,  $G_i^*$  contains a perfect matching different from  $E_i$  if and only if  $G[N_G(v_i)]$  has a subgraph isomorphic to  $A_\ell^{W_\ell}$ . Hence, the algorithm checks all possible  $G_i^* - e$  graphs, where  $e \in E_i$ , and if any of them has a perfect matching, then there exists a subgraph isomorphic to  $A_\ell^{W_\ell}$ .

In order to find a maximum matching in  $G_i^* - e$ , we can use Edmond's Blossom Algorithm [11], which was improved by Micali and Vazirani in [28] to run in time  $O(\sqrt{nm})$  for any graph of order  $n$  and size  $m$ . The procedure will be repeated  $(\deg_G(x)/2) = \deg_F(x)$  times for every vertex  $x \in D$ , that is,  $\sum_{v \in V(F)} \deg(v) = 2|E(F)|$ , in total. Thus, the second part of the algorithm requires polynomial-time. This finishes the proof. ■

We now show that the same problem is NP-hard even on the graph class  $\mathcal{G}_1^D$ .

**Theorem 4.** *Consider every graph  $G \in \mathcal{G}_1$  together with a specified set  $D$  such that  $G^2[D] \cong F$  and  $G \in \mathcal{G}_1(F)$ . Then, it is NP-complete to decide whether the inequality  $\gamma(G) < \gamma_2(G)$  holds for a general instance  $G \in \mathcal{G}_1$ .*

**Proof.** By Lemma 7, we have  $\gamma_2(G) = |D|$  and it can be checked in polynomial time whether a given set  $D'$  with  $|D'| < |D|$  is a dominating set of  $G$ . Thus, the decision problem belongs to NP.

In order to prove the NP-hardness, we present a polynomial-time reduction from the well-known 3-SAT problem, which is proved to be NP-complete [17].

Let  $X = \{x_1, x_2, \dots, x_k\}$  be a set of Boolean variables. A truth assignment for  $X$  is a function  $\varphi : X \rightarrow \{t, f\}$ . If  $\varphi(x_i) = t$  holds, then the variable  $x_i$  is called *true*; else if  $\varphi(x_i) = f$  holds, then  $x_i$  is called *false*. If  $x_i$  is a variable in  $X$ , then  $x_i$  and  $\neg x_i$  are literals over  $X$ . The literal  $x_i$  is true under  $\varphi$  if and only if the variable  $x_i$  is true under  $\varphi$ ; the literal  $\neg x_i$  is true if and only if the variable  $x_i$  is false. A clause over  $X$  is a set of three literals over  $X$ , represents the disjunction of those literals and it is satisfied by a truth assignment if and only if at least one of its members is true under that assignment. A collection  $\mathcal{C}$  of clauses over  $X$  is *satisfiable* if and only if there exists some truth assignment for  $X$  that satisfies all the clauses in  $\mathcal{C}$ . Such a truth assignment is called a *satisfying truth assignment* for  $\mathcal{C}$ . The 3-SAT problem is specified as follows.

### 3-SATISFIABILITY (3-SAT) PROBLEM

**Instance:** A collection  $\mathcal{C} = \{C_1, C_2, \dots, C_\ell\}$  of clauses over a finite set  $X$  of variables such that  $|C_j| = 3$ , for  $1 \leq j \leq \ell$ .

**Question:** Is there a truth assignment for  $X$  that satisfies all the clauses in  $\mathcal{C}$ ?

Let  $\mathcal{C}$  be a 3-SAT instance with clauses  $C_1, C_2, \dots, C_\ell$  over the Boolean variables  $X = \{x_1, x_2, \dots, x_k\}$ . We may assume that for every three variables  $x_{i_1}, x_{i_2}, x_{i_3}$  there exists a clause  $C_j$ , where  $j \in [\ell]$ , such that  $C_j$  does not contain any of the variables  $x_{i_1}, x_{i_2}, x_{i_3}$  (neither in positive form, nor in negative form). Otherwise, the problem could be reduced to at most eight (separated) 2-SAT problems, which are solvable in polynomial time.

We now construct a graph  $G \in \mathcal{G}_1(F)$ , where  $F \cong S_{k+1}$ , such that the given instance  $\mathcal{C}$  of 3-SAT problem is satisfiable if and only if  $\gamma(G) < \gamma_2(G)$ . The construction is as follows.

For every variable  $x_i$ , we create three vertices  $\{x_i^t, x_i^f, v_i\}$  and then we add the edges  $x_i^t v_i$  and  $x_i^f v_i$ . For every clause  $C_j \in \mathcal{C}$ , we create a vertex  $c_j$ , and if  $x_i$  is a literal in  $C_j$ , then  $x_i^t c_j \in E(G)$ ; if  $\neg x_i$  is a literal in  $C_j$ , then  $x_i^f c_j \in E(G)$ . Moreover, we add a vertex  $c^*$  and the edges  $c^* x_i^t$  and  $c^* x_i^f$  for every  $i \in [k]$ . We

also add a vertex  $v_{k+1}$  and the edge set  $\{c_i v_{k+1} : 1 \leq i \leq \ell\} \cup \{c^* v_{k+1}\}$ . Finally, we add a new vertex  $v_0$ , which is adjacent to every vertex in  $V(G) \setminus \{v_1, v_2, \dots, v_{k+1}\}$  (for an illustration of the construction see Figure 5). The order of  $G$  is obviously  $3k + \ell + 3$  and this construction can be done in polynomial time. Note that  $G \in \mathcal{G}_1(F)$ , where  $F$  is a star with center  $v_0$  and leaves  $v_1, \dots, v_{k+1}$ . Thus, we have  $\gamma_2(G) = k + 2$ , by Lemma 7.

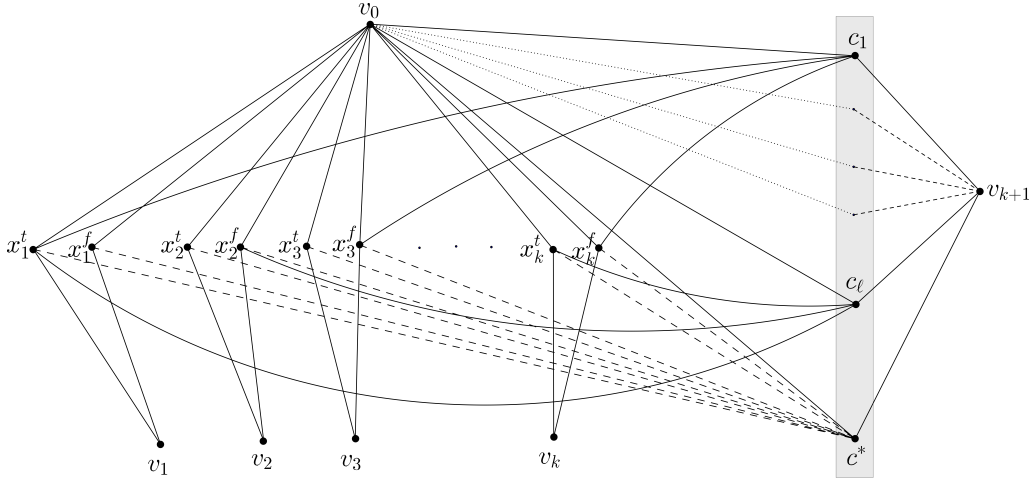


Figure 5. An illustration of the construction for 3-SAT reduction. The clauses  $C_1$  and  $C_\ell$  corresponding to the vertices  $c_1$  and  $c_\ell$ , respectively, are  $C_1 = (x_1 \vee \neg x_3 \vee \neg x_k)$  and  $C_\ell = (x_1 \vee \neg x_2 \vee x_k)$ .

We now prove that  $\mathcal{C}$  is satisfiable if and only if  $\gamma(G) < \gamma_2(G)$ . First, consider a truth assignment  $\varphi : x_i \rightarrow \{t, f\}$  which satisfies  $\mathcal{C}$ . Let  $D_1 = \bigcup_{i \in [k]} \{x_i^t : \varphi(x_i) = t\}$  and let  $D_2 = \bigcup_{i \in [k]} \{x_i^f : \varphi(x_i) = f\}$ . Consider the set  $D' = D_1 \cup D_2 \cup \{c^*\}$ . It can be readily checked that  $D'$  is a dominating set of cardinality  $k + 1$ . Hence,  $\gamma(G) < \gamma_2(G)$  follows.

Conversely, assume that  $\gamma(G) < \gamma_2(G)$  and consider a minimum dominating set  $D'$  of cardinality at most  $k + 1$ . In order to dominate  $v_i$ , the set  $D'$  contains at least one vertex from the set  $\{x_i^t, x_i^f, v_i\}$ , for each  $i \in [k]$ . Similarly, to dominate  $v_{k+1}$ , the set  $D'$  contains at least one vertex from the set  $\{c_1, c_2, \dots, c_\ell, c^*, v_{k+1}\}$ . Since  $|D'| \leq k + 1$ , we have  $|D' \cap \{x_i^t, x_i^f, v_i\}| = 1$  for every  $i \in [k]$ . Moreover,  $|D' \cap \{c_1, c_2, \dots, c_\ell, c^*, v_{k+1}\}| = 1$  and  $v_0 \notin D'$ .

Suppose that  $v_{k+1} \in D'$ . In order to dominate the vertices  $x_i^t$  and  $x_i^f$ , the set  $D'$  contains the vertex  $v_i$  for all  $i \in [k]$ . Hence,  $N_G(v_0) \cap D' = \emptyset$ . From the discussion above, we know that  $v_0 \notin D'$ . Thus,  $v_0$  is not dominated by a vertex from  $D'$ , a contradiction.

Suppose that  $c_j \in D'$  for some  $j \in [\ell]$ . Let  $C_j$  be the corresponding clause

containing the variables  $x_{i_1}, x_{i_2}, x_{i_3}$ . Consider any variable  $x_s \in X \setminus \{x_{i_1}, x_{i_2}, x_{i_3}\}$ . Since  $|D' \cap \{x_i^t, x_i^f, v_i\}| = 1$  for each  $i \in [k]$ ,  $D'$  contains  $v_s$  in order to dominate both of the vertices  $x_s^t$  and  $x_s^f$ . By our assumption, there exists a clause  $C_q$  not containing the variables  $x_{i_1}, x_{i_2}, x_{i_3}$  neither in positive nor in negative form. Thus,  $c_q$  is not dominated by a vertex from  $D'$ , a contradiction.

Since  $|D' \cap \{c_1, c_2, \dots, c_\ell, c^*\}| = 1$ , the only remaining case is  $c^* \in D'$ . Under this assumption, every vertex  $c_i$  must be dominated by the vertices corresponding to the literals in  $C_i$ . Thus, the truth assignment

$$\varphi(x_i) = \begin{cases} t, & \text{if } x_i^t \in D', \\ f, & \text{if } x_i^f \in D' \text{ or if } v_i \in D' \end{cases}$$

satisfies  $\mathcal{C}$ . This finishes the proof.  $\blacksquare$

Theorem 4 implies that it is coNP-complete to decide whether the equality  $\gamma(G) = \gamma_2(G)$  holds for a general instance  $G$  from  $\mathcal{G}_1$ . On the other hand, we cannot prove that the problem belongs to NP. Instead, we will consider the complexity class  $\Theta_2^p$ , which consists of those problems solvable by a polynomial-time deterministic algorithm using NP-oracle asked for only  $O(\log n)$  times. (For a detailed introduction, please, see [27].)

**Proposition 8.** *The complexity of deciding whether  $\gamma(G) = \gamma_2(G)$  holds for a general instance  $G$  is in the class  $\Theta_2^p$ .*

**Proof.** Using binary search, the parameters  $\gamma(G)$  and  $\gamma_2(G)$  can be determined by asking the NP-oracle  $O(\log n)$  times whether the inequalities  $\gamma(G) \leq k$  and  $\gamma_2(G) \leq k$  hold. Thus, the decision problem belongs to  $\Theta_2^p$ .  $\blacksquare$

Note that in [3], a similar statement was proved for the problem of deciding whether the transversal number  $\tau(\mathcal{H})$  equals the domination number  $\gamma(\mathcal{H})$  for a general instance hypergraph  $\mathcal{H}$ .

#### 4. CHARACTERIZATION OF $(\gamma, \gamma_2)$ -PERFECT GRAPHS

Recently, Alvarado, Dantas, Rautenbach [1, 2] and Henning, Jäger, Rautenbach [25] studied graphs for which the equality between two fixed domination-type invariants hereditarily holds. The analogous problem for transversal and domination numbers of graphs and hypergraphs was considered in [3].

In this section, we characterize  $(\gamma, \gamma_2)$ -perfect graphs, that is, we characterize the graphs for which the equality between the domination and the 2-domination numbers hereditarily holds. By Lemma 1,  $\delta(G) \geq 2$  is a necessary condition for  $\gamma(G) = \gamma_2(G)$ . Hence, we define  $(\gamma, \gamma_2)$ -perfect graphs as follows.

**Definition 5.** Let  $G$  be a graph with  $\delta(G) \geq 2$ . Then  $G$  is a  $(\gamma, \gamma_2)$ -perfect graph if the equality  $\gamma(H) = \gamma_2(H)$  holds for every induced subgraph  $H$  of minimum degree at least two.

Note that a disconnected graph  $G$  is  $(\gamma, \gamma_2)$ -perfect if and only if all of its components are  $(\gamma, \gamma_2)$ -perfect.

In order to formulate the results of this section we will define the following class.

**Definition 6.** Let  $S_k$  be the star with center vertex  $v$  and end vertices  $\{v_1, v_2, \dots, v_k\}$  such that  $k \geq 1$ . Denote the edge  $vv_j \in E(S_k)$  by  $e_j$  for  $j \in [k]$ . Let  $S(i_1, i_2, \dots, i_k)$  be the graph obtained by substituting each edge  $e_j$  of  $S_k$  by  $i_j$  parallel edges  $e_j^1, e_j^2, \dots, e_j^{i_j}$ , where  $i_j \geq 2$ , and then subdividing each edge  $e_j^r$  by adding the vertex  $x_j^r$  for all  $r \in [i_j]$  and all  $j \in [k]$  (see Figure 6). A graph  $G$  belongs to the class  $\mathcal{S}$  if it is isomorphic to  $S(i_1, i_2, \dots, i_k)$  for some  $k \geq 1$ , where  $i_j \geq 2$  for all  $j \in [k]$ .

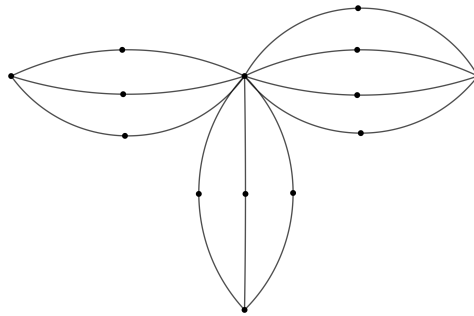


Figure 6. An illustration for the graph  $S(3, 3, 4)$ .

We clearly have  $\mathcal{S} \subseteq \mathcal{G}_1$ , since any  $S(i_1, i_2, \dots, i_k) \in \mathcal{G}_1(F)$ , where  $F \cong S_k$ . On the other hand, if  $G' \in \mathcal{G}(S_k)$ , the underlying graph does not contain a clique of order larger than two and consequently,  $|N(y) \cap D| = 2$  for every supplementary vertex  $y$ . This implies that  $G' \in \mathcal{G}_1(S_k)$ . By the definitions above, we have the following equivalence.

**Proposition 9.** For any graph,  $G \in \mathcal{S}$  holds if and only if  $G \in \mathcal{G}_1(S_k)$  (or, equivalently,  $G \in \mathcal{G}(S_k)$ ) for a non-trivial star  $S_k$  and  $G$  does not contain a supplementary edge.

The main result of this section is a characterization theorem for  $(\gamma, \gamma_2)$ -perfect graphs.



**Theorem 5.**  *$G$  is a connected  $(\gamma, \gamma_2)$ -perfect graph if and only if  $G \in \mathcal{S}$ .*

**Proof.** We first prove that if  $G \in \mathcal{S}$ , then it is  $(\gamma, \gamma_2)$ -perfect graph. By Proposition 9, we know that  $G \in \mathcal{G}_1(F)$ , where  $F \cong S_k$  for  $k \geq 1$ . Then, by Lemma 7,  $\gamma_2(G) = |V(F)| = k + 1$ . Since a minimal 2-dominating set is a dominating set, we have the inequality  $\gamma(G) \leq k + 1$ . In order to prove that  $\gamma(G) = \gamma_2(G)$ , it is enough to show that  $\gamma(G) > k$ . Suppose to the contrary that  $D'$  is a minimum dominating set of  $G$  such that  $|D'| \leq k$ .

Consider the vertices of  $G$  corresponding to the end vertices of the star  $S_k$ . Let  $\{v_1, v_2, \dots, v_k\} = V(F) \setminus \{v\} \subseteq V(G)$ , where  $v$  is the center of  $F \cong S_k$ . Since  $D'$  is a dominating set,  $|N_G[v_j] \cap D'| \geq 1$  for each  $j \in [k]$ . Note that the closed neighborhoods of any two vertices from the set  $\{v_1, v_2, \dots, v_k\}$  are disjoint. Since  $|D'| \leq k$  by our assumption, we have  $v \notin D'$  and  $|N_G[v_j] \cap D'| = 1$ , for every  $j \in [k]$ . Moreover, as the center  $v$  must also be dominated, there exists some  $j \in [k]$  and  $r \in [i_j]$  such that  $x_j^r \in D'$ . Then,  $v_j \notin D'$  and the vertices in  $(X_j \cup Y_j) \setminus \{x_j^r\}$  are not dominated by  $D'$ , which is a contradiction. Consequently,  $k$  vertices are not enough to dominate all the vertices of  $G$ , that is,  $\gamma(G) \geq k + 1$ . It follows that  $\gamma(G) = \gamma_2(G)$  for any  $G \in \mathcal{S}$ .

Next, suppose that  $H$  is an induced subgraph of  $G$  with minimum degree at least two. If  $H$  does not contain any subdivision vertices, we have  $\delta(H) = 0$ , a contradiction. Thus,  $H$  contains a subdivision vertex. Let  $x_p^q \in V(H)$  for some  $p \in [k]$  and  $q \in [i_p]$ . Since  $\deg_G(x_p^q) = 2$ , then both of the neighbors of  $x_p^q$  must be in  $V(H)$ , i.e.,  $N_G(x_p^q) = \{v, v_p\} \subseteq V(H)$ . Since  $\delta(H) \geq 2$  by the assumption, using an argument similar to the above, we have  $\deg_H(v_p) \geq 2$ . Thus,  $|(X_p \cup Y_p) \setminus \{x_p^q\} \cap V(H)| \geq 1$ . Consequently,  $H \in \mathcal{S}$  and, as it was proved above,  $\gamma(H) = \gamma_2(H)$  holds for every induced subgraph of  $G$  with minimum degree at least two.

To prove the converse, assume that  $G$  is a connected  $(\gamma, \gamma_2)$ -perfect graph. Note that  $\gamma(C_n) = \lceil \frac{n}{3} \rceil$  and  $\gamma_2(C_n) = \lceil \frac{n}{2} \rceil$ , where  $n \geq 3$ . Thus, the  $(\gamma, \gamma_2)$ -perfect graph  $G$  does not contain an induced cycle  $C_n$ , where  $n = 3$  or  $n \geq 5$ .

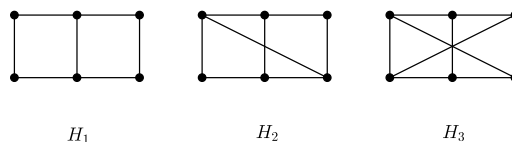


Figure 7. The graphs  $H_1$ ,  $H_2$  and  $H_3$ .

Now, suppose that  $G$  has a non-induced subgraph isomorphic to  $C_r$ , for some  $r \geq 5$ . Since all of its induced cycles are 4-cycles,  $G$  contains at least one of the three graphs  $H_1$ ,  $H_2$  and  $H_3$ , shown in Figure 7, as an induced subgraph. Observe

that  $\gamma(H_i) < \gamma_2(H_i)$  for all  $i \in \{1, 2, 3\}$ . This contradicts our assumption that  $G$  is a  $(\gamma, \gamma_2)$ -perfect graph. Thus,  $G$  does not contain a cycle  $C_r$ , where  $r \neq 4$ .

Since  $G$  is  $(\gamma, \gamma_2)$ -perfect by the assumption, then the equality  $\gamma(G) = \gamma_2(G)$  holds. By Proposition 6, we know that  $G \in \mathcal{G}$ . Thus, if  $D$  is a minimum 2-dominating set of  $G$ , then  $D$  is independent and  $F = G^2[D]$  is the underlying graph of  $G$ .

First, note that  $F$  does not contain a cycle  $C_r$  for  $r \geq 3$ . Otherwise,  $G$  would contain a subgraph isomorphic to  $C_{2r}$ , which is a contradiction. Thus,  $F$  is a forest and  $G \in \mathcal{G}_1(F)$ . Then suppose that  $F$  is not connected. Since  $G$  is connected, there is a supplementary edge  $e = uv$ , where  $u$  and  $v$  are two subdivision vertices of  $G$  such that  $N(u) \cap V(F)$  and  $N(v) \cap V(F)$  are in different components of  $F$ . By the definition of the graph class  $\mathcal{G}_1$ , there are two vertices  $u'$  and  $v'$  such that  $N_G(u) \cap V(F) = N_G(u') \cap V(F)$  and  $N_G(v) \cap V(F) = N_G(v') \cap V(F)$ . Let  $\{x_1, x_2\} = N_G(u) \cap V(F)$  and  $\{x_3, x_4\} = N_G(v) \cap V(F)$ , where the sets  $\{x_1, x_2\}$  and  $\{x_3, x_4\}$  are contained by different components of  $F$ . Consider the set  $A = \{x_1, x_2, x_3, x_4, u, v, u', v'\}$  and the induced subgraph  $G[A]$ . It is easy to check that  $\delta(G[A]) \geq 2$ ,  $\gamma(G[A]) \leq 3$  and  $\gamma_2(G[A]) = 4$ , which is a contradiction. Thus,  $F$  is a tree.

Suppose that  $G$  has a supplementary edge  $e = uv \in E(G)$ , where  $u, v \in V(G) \setminus V(F)$ . Let  $N_G(u) \cap V(F) = \{x_1, x_2\}$  and  $N_G(v) \cap V(F) = \{x_3, x_4\}$ . Note that  $|\{x_1, x_2\} \cap \{x_3, x_4\}| \leq 1$ , otherwise  $G$  would contain a subgraph isomorphic to  $C_3$ . By Lemma 4, there exist two further vertices  $u'$  and  $v'$  satisfying  $N_G(u') \cap V(F) = \{x_1, x_2\}$  and  $N_G(v') \cap V(F) = \{x_3, x_4\}$ . If  $|\{x_1, x_2\} \cap \{x_3, x_4\}| = 1$ , then without loss of generality, assume that  $x_2 = x_3$ . Then, there is a subgraph of  $G$  isomorphic to  $C_3$  induced by the vertices  $u, v$  and  $x_2$ , a contradiction. If  $\{x_1, x_2\} \cap \{x_3, x_4\} = \emptyset$ , then let  $S = \{x_1, x_2, x_3, x_4, u, v, u', v'\}$ . A similar argument applied to the subgraph of  $G$  induced by the vertex set  $S$  yields the inequality  $\gamma(G[S]) \leq 3 < \gamma_2(G[S]) = 4$ . Thus,  $G$  does not have any supplementary edges.

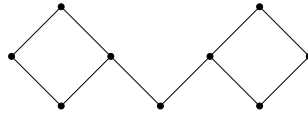


Figure 8. The graph  $H_4$ .

Suppose that  $F$  contains a subgraph isomorphic to  $P_4$ . Since  $G$  does not have a supplementary edge, it contains an induced subgraph isomorphic to  $H_4$  given in Figure 8. Note that  $\delta(H_4) \geq 2$  and  $3 = \gamma(H_4) < \gamma_2(H_4) = 4$ , which contradicts the assumption that  $G$  is  $(\gamma, \gamma_2)$ -perfect. Thus,  $F$  is a star,  $G \in \mathcal{G}_1(F)$ , and  $G$  does not contain supplementary edges. This finishes the proof by Proposition 9. ■

The graph obtained from an edge by attaching two pendant edges to both of

its ends will be called  $T_6$  (for illustration see Figure 9).

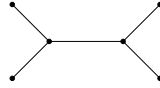


Figure 9. The graph  $T_6$ .

**Proposition 10.**  $G \in \mathcal{S}$  if and only if  $G$  is a connected graph with  $\delta(G) \geq 2$  and it contains no subgraph isomorphic to any of  $T_6, P_8$ , or  $C_k$  where  $k \neq 4$ .

**Proof.** If  $G \in \mathcal{S}$ , then it is easy to see that  $G$  is a connected graph with  $\delta(G) \geq 2$  and it does not contain a subgraph isomorphic to  $T_6, P_8$ , or  $C_k$  where  $k \neq 4$ .

Now, assume that  $G$  is a connected graph of minimum degree at least two which does not contain a subgraph isomorphic to  $T_6, P_8$ , or  $C_k$  where  $k \neq 4$ . Note that  $G$  is bipartite. We further have  $\min\{\deg_G(u), \deg_G(v)\} = 2$  for each edge  $uv \in E(G)$ , since  $\delta(G) \geq 2$  and  $G$  does not contain a subgraph isomorphic to  $T_6$  or  $C_3$ .

First, suppose that  $G$  contains an edge  $e = uv \in E(G)$ , which is a bridge. Then  $G - e$  has two components, say  $G_1$  and  $G_2$ . Since  $\delta(G) \geq 2$ , both  $G_1$  and  $G_2$  are non-trivial graphs and may contain at most one vertex, namely either  $u$  or  $v$ , which is of degree 1. Thus, both of the components contain a cycle. These cycles must be vertex-disjoint 4-cycles with a path between them. Hence,  $G$  contains a subgraph isomorphic to  $P_8$  and this contradicts our assumption.

Since  $G$  does not contain a bridge, every edge of  $G$  lies on a 4-cycle. If all the vertices of  $G$  have degree two, then  $G$  is isomorphic to  $C_4$  and  $G \in \mathcal{S}$ . If  $G$  is not isomorphic to  $C_4$ , then every 4-cycle contains a vertex of degree at least three. For a vertex  $v$  of degree two, we define the function  $f(v)$  to denote the vertex opposite to  $v$  in a 4-cycle. Let  $A = \{v \in V(G) : \deg(v) \geq 3 \text{ or } \deg(f(v)) \geq 3\}$ . Consider two vertices  $u, v \in A$ . If  $uv \in E(G)$ , then  $uv$  belongs to a 4-cycle, say  $uvv'u'$ . At least one of  $u$  and  $v$  is of degree two, without loss of generality, say  $\deg(u) = 2$ . Thus,  $u$  belongs only to this 4-cycle. Since  $f(u) = v'$ , by the definition of  $A$ ,  $\deg(v') \geq 3$ . If  $\deg(v) \geq 3$ , then  $vv' \in E(G)$ , we have a contradiction. If  $\deg(v) = 2$ , then  $v \in A$  and  $v$  belongs only to the 4-cycle  $uvv'u'$ . Thus,  $f(v) = u'$ ,  $\deg(u') \geq 3$  and  $u'v' \in E(G)$ , which is a contradiction. Hence,  $A$  is independent. Consider two vertices  $u, v \in V(G) \setminus A$ . If  $uv \in E(G)$ , then at least one of  $f(u)$  or  $f(v)$  is of degree at least three. Then, by the definition of the function  $f$ , we have  $u \in A$  or  $v \in A$ , which is a contradiction. Hence,  $V(G) \setminus A$  is independent.

Consequently,  $(A, V(G) \setminus A)$  is a bipartition of  $V(G)$ . Note that every 4-cycle has exactly two vertices in  $A$ . Hence,  $G^A \in \mathcal{G}_1(F)$  where  $F \cong G^2[A]$ , and there are no supplementary edges. Since  $G$  does not have a subgraph isomorphic to  $C_n$

for  $n \geq 6$ , the underlying graph is a tree. If  $F$  contains a subgraph isomorphic to  $P_4$ , then  $G$  contains a subgraph isomorphic to  $P_8$ , which is a contradiction. Thus,  $F$  is a star, and Proposition 9 implies that  $G \in \mathcal{S}$ . ■

Thus, Proposition 10 allows us to state Theorem 5 in a different form as follows.

**Theorem 6.** *Let  $G$  be a connected graph with  $\delta(G) \geq 2$ . Then  $G$  is a  $(\gamma, \gamma_2)$ -perfect graph if and only if  $G$  contains no subgraph isomorphic to any of  $T_6, P_8$ , or  $C_k$  where  $k \neq 4$ .*

Note that for any  $G \in \mathcal{S}$ , the center of the underlying star can be chosen as a vertex  $v$  of degree  $\Delta(G)$  and then, the subdivision vertices are exactly those contained in  $N_G(v)$ . Therefore, the characterization given in Theorem 5 directly yields a polynomial-time algorithm which recognizes  $(\gamma, \gamma_2)$ -perfect graphs.

## 5. CONCLUDING REMARKS AND OPEN PROBLEMS

In Section 1, we defined the graph class  $\mathcal{G}$  which contains all  $(\gamma, \gamma_2)$ -graphs. Then, in Section 2, we gave a characterization for  $(\gamma, \gamma_2)$ -graphs over a specified subclass  $\mathcal{H}$  of  $\mathcal{G}$ . In the definition of  $\mathcal{H}$  and in the proof of the main theorem, we referred to the properties of the underlying graph. We noted there that the underlying graph is not always unique when a graph  $G$  from  $\mathcal{G}$  is given. In Figure 10, we show a  $(\gamma, \gamma_2)$ -graph having two non-isomorphic underlying graphs. Analogously, one can construct infinitely many graphs with the same property.

In the definition of the class  $\mathcal{H}$ , we forbid 3-cycles and 4-cycles in the underlying graph. The characterization given in Theorem 2 does not hold if 3-cycles are not forbidden in the underlying graph. This is shown by the graph  $A_4^* \in \mathcal{G}_2(F)$  (see Figure 11), where the underlying graph  $F$  is a star supplemented by an edge. One can readily check that even if  $A_4^*$  contains an induced  $A_4^W$  subgraph, it remains a  $(\gamma, \gamma_2)$ -graph as  $\gamma(A_4^*) = \gamma_2(A_4^*) = 5$ . Similarly, it is possible to construct graphs whose underlying graphs are  $C_3$ -free but not  $C_4$ -free such that the statement of Theorem 2 does not remain valid for them. Therefore, the following problems are still open.

**Problem 1.** Characterize  $(\gamma, \gamma_2)$ -graphs over the following graph classes.

1. Over the subclass of  $\mathcal{G}_2$  where the underlying graph does not contain any  $C_4$  subgraphs.
2. Over the subclass of  $\mathcal{G}_2$  where the underlying graph is  $C_3$ -free.
3. Over  $\mathcal{G}_2$ .

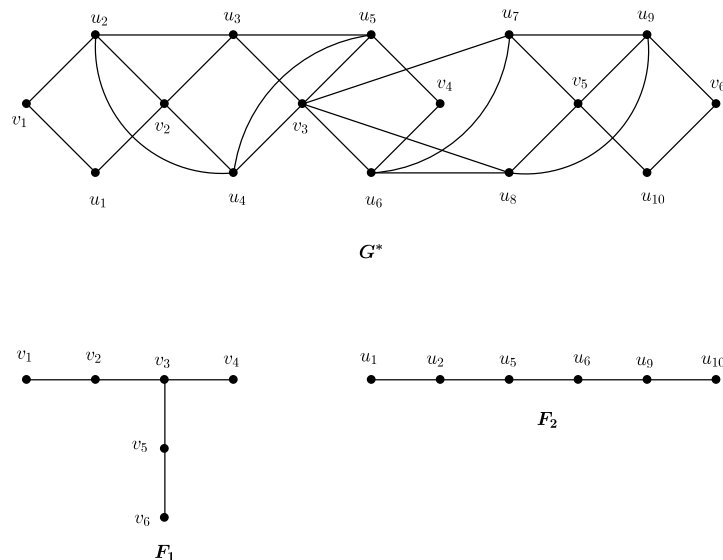


Figure 10.  $G^*$  is a graph with  $\gamma(G^*) = \gamma_2(G^*) = 6$ , which has two non-isomorphic underlying graphs and  $G^* \in \mathcal{H}(F_1) \cap \mathcal{H}(F_2)$ .

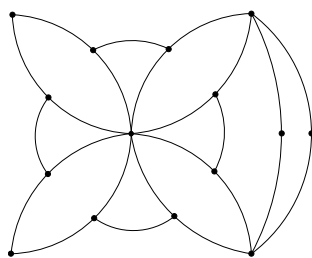


Figure 11. The graph  $A_4^*$ .

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