

THE ACHROMATIC NUMBER OF THE CARTESIAN PRODUCT OF K_6 AND K_q

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Abstract

Let G be a graph and C a finite set of colours. A vertex colouring $f : V(G) \rightarrow C$ is complete if for any pair of distinct colours $c_1, c_2 \in C$ one can find an edge $\{v_1, v_2\} \in E(G)$ such that $f(v_i) = c_i$, $i = 1, 2$. The achromatic number of G is defined to be the maximum number $\text{achr}(G)$ of colours in a proper complete vertex colouring of G . In the paper $\text{achr}(K_6 \square K_q)$ is determined for any integer q such that either $8 \leq q \leq 40$ or $q \geq 42$ is even.

Keywords: complete vertex colouring, achromatic number, Cartesian product, complete graph.

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1. INTRODUCTION

Let G be a finite simple graph and C a finite set of colours. A vertex colouring $f : V(G) \rightarrow C$ is *complete* provided that for any pair $\{c_1, c_2\} \in \binom{C}{2}$ (of distinct colours of C) there exists an edge $\{v_1, v_2\}$ (usually shortened to v_1v_2) of G such that $f(v_i) = c_i$, $i = 1, 2$. The *achromatic number* of G , in symbols $\text{achr}(G)$, is the maximum cardinality of the colour set in a proper complete vertex colouring of G . The achromatic number was introduced in Harary, Hedetniemi, and Prins [3], where among other things the following interpolation result was proved.

Theorem 1. *If G is a graph, and an integer k satisfies $\chi(G) \leq k \leq \text{achr}(G)$, there exists a proper complete vertex colouring of G using k colours.*

In the present paper the achromatic number of $K_6 \square K_q$, the Cartesian product of K_6 and K_q (the notation following Imrich and Klavžar [8] is adopted), is determined for all q satisfying either $8 \leq q \leq 40$ or $q \geq 42$ and $q \equiv 0 \pmod{2}$.

This is the third in a series of three papers, in which the problem of finding $\text{achr}(K_6 \square K_q)$ is completely solved. Some historical remarks concerning the achromatic number, a motivation of the problem and basic facts on proper complete colourings of Cartesian products of two complete graphs are available in the paper Horňák [4], where $\text{achr}(K_6 \square K_q)$ has been determined for odd $q \geq 41$. Maybe a bit surprisingly, proving that $\text{achr}(K_6 \square K_7) = 18$ has required quite a long analysis contained in the paper Horňák [5].

For $m, n \in \mathbb{Z}$ we work with *integer intervals* defined by

$$[m, n] = \{z \in \mathbb{Z} : m \leq z \leq n\}, \quad [m, \infty) = \{z \in \mathbb{Z} : m \leq z\}.$$

If $p, q \in [1, \infty)$ and $V(K_r) = [1, r]$, $r = p, q$, then $V(K_p \square K_q) = [1, p] \times [1, q]$, and $E(K_p \square K_q)$ consists of edges $(i_1, j_1)(i_2, j_2)$, where $i_1, i_2 \in [1, p]$ and $j_1, j_2 \in [1, q]$ satisfy either $i_1 = i_2$ and $j_1 \neq j_2$ or $i_1 \neq i_2$ and $j_1 = j_2$.

Let $\mathcal{M}(p, q, C)$ denote the set of $p \times q$ matrices M with entries from C such that all lines (rows and columns) of M have pairwise distinct entries, and any pair $\{\alpha, \beta\} \in \binom{C}{2}$ is *good* in M , which means that there is a line of M containing both α and β ; the pair $\{\alpha, \beta\}$ is either *row-based* or *column-based* (in M) depending on whether the involved line is a row or a column. In other words, the number of lines witnessing the fact that the pair $\{\alpha, \beta\}$ is good, is positive, and it may happen that the pair $\{\alpha, \beta\}$ is simultaneously row-based and column-based as well. For a matrix M we denote by $(M)_{i,j}$ the entry of M appearing in the i th row and the j th column.

Proposition 2 [4]. *If $p, q \in [1, \infty)$ and C is a finite set, then the following statements are equivalent.*

- (1) *There is a proper complete vertex colouring of $K_p \square K_q$ using as colours elements of C .*
- (2) $\mathcal{M}(p, q, C) \neq \emptyset$.

The implication (2) \Rightarrow (1) of Proposition 2 is based on a straightforward observation that if $M \in \mathcal{M}(p, q, C)$, then the vertex colouring f_M of $K_p \square K_q$ defined by $f_M(i, j) = (M)_{i,j}$ is proper and complete as well.

Proposition 3 [4]. *If $p, q \in [1, \infty)$, C, D are finite sets, $M \in \mathcal{M}(p, q, C)$, mappings $\rho : [1, p] \rightarrow [1, p]$, $\sigma : [1, q] \rightarrow [1, q]$, $\pi : C \rightarrow D$ are bijections, and $M_{\rho, \sigma}$, M_π are $p \times q$ matrices defined by $(M_{\rho, \sigma})_{i,j} = (M)_{\rho(i), \sigma(j)}$ and $(M_\pi)_{i,j} = \pi((M)_{i,j})$, then $M_{\rho, \sigma} \in \mathcal{M}(p, q, C)$ and $M_\pi \in \mathcal{M}(p, q, D)$.*

Let $M \in \mathcal{M}(p, q, C)$. The *frequency* of a colour $\gamma \in C$ is the number $\text{frq}(\gamma)$ of times γ appears in M , while $\text{frq}(M)$, the *frequency* of M , is the minimum of frequencies of colours in C . A colour of frequency l is an l -colour, C_l is the set of l -colours and $c_l = |C_l|$. C_{l+} is the set of colours of frequency at least

l and $c_{l+} = |C_{l+}|$. For the (complete) colouring f_M mentioned above denote $V_\gamma = f_M^{-1}(\gamma) \subseteq [1, p] \times [1, q]$, and let $N(V_\gamma)$ be the neighbourhood of V_γ (the union of neighborhoods of vertices in V_γ). The *excess* of γ is defined to be the maximum number $\text{exc}(\gamma)$ of vertices in a set $S \subseteq N(V_\gamma)$ such that the restriction of f_M formed by uncolouring the vertices of S is still complete with respect to pairs of colours containing γ . The *excess* of M is the minimum $\text{exc}(M)$ of excesses of colours in C .

We denote by $\mathbb{R}(i)$ the set of colours in the i th row of M and by $\mathbb{C}(j)$ the set of colours in the j th column of M . Further, let

$$\begin{aligned} \mathbb{R}_l(i) &= C_l \cap \mathbb{R}(i), & r_l(i) &= |\mathbb{R}_l(i)|, \\ \mathbb{C}_l(j) &= C_l \cap \mathbb{C}(j), & c_l(j) &= |\mathbb{C}_l(j)|, \end{aligned}$$

so that $\mathbb{R}_l(i)$ and $\mathbb{C}_l(j)$ is the set of l -colours appearing in the row i and those appearing in the column j , respectively. For $i, j, k \in [1, p]$ let

$$\begin{aligned} \mathbb{R}(i, j) &= C_2 \cap \mathbb{R}(i) \cap \mathbb{R}(j), & r(i, j) &= |\mathbb{R}(i, j)|, \\ \mathbb{R}(i, j, k) &= C_3 \cap \mathbb{R}(i) \cap \mathbb{R}(j) \cap \mathbb{R}(k), & r(i, j, k) &= |\mathbb{R}(i, j, k)|, \end{aligned}$$

and for $m, n \in [1, q]$ let

$$\mathbb{C}(m, n) = C_2 \cap \mathbb{C}(m) \cap \mathbb{C}(n), \quad c(m, n) = |\mathbb{C}(m, n)|.$$

Thus $\mathbb{R}(i, j)$ and $\mathbb{C}(m, n)$ are respectively the sets of colours which appear exactly in rows i and j or columns m and n , while $\mathbb{R}(i, j, k)$ is the set of colours which appear exactly in the rows i, j and k . For $\gamma \in C$ define

$$\mathbb{R}(\gamma) = \{i \in [1, p] : \gamma \in \mathbb{R}(i)\}$$

as the set of numbers of rows containing the colour γ . To avoid a possible confusion coming from the double usage of $\mathbb{R}(\cdot)$ in $\mathbb{R}(\gamma)$ and $\mathbb{R}(i)$ note that whenever $\mathbb{R}(\cdot)$ is used with a Greek alphabet letter argument, then the argument points to a colour in C , and not to a row of the matrix M .

If $S \subseteq [1, p] \times [1, q]$, we say that a colour $\gamma \in C$ *occupies a position in S* (appears in S or simply *is in S*) if there exists $(i, j) \in S$ with $(M)_{i,j} = \gamma$. For a nonempty set of colours $A \subseteq C$, the *set of columns covered by A* is

$$\text{Cov}(A) = \{j \in [1, q] : \mathbb{C}(j) \cap A \neq \emptyset\},$$

i.e., the set of (numbers of) columns containing a colour of A . We put $\text{cov}(A) = |\text{Cov}(A)|$, and for $A = \{\alpha\}, \{\alpha, \beta\}, \{\alpha, \beta, \gamma\}$ we use a simplified notation $\text{Cov}(\alpha)$, $\text{Cov}(\alpha, \beta)$, $\text{Cov}(\alpha, \beta, \gamma)$ and $\text{cov}(\alpha)$, $\text{cov}(\alpha, \beta)$, $\text{cov}(\alpha, \beta, \gamma)$ instead of $\text{Cov}(A)$ and $\text{cov}(A)$.

2. MATRIX CONSTRUCTIONS

Proposition 4. $\text{achr}(K_6 \square K_4) \geq 12$, $\text{achr}(K_6 \square K_6) \geq 18$ and $\text{achr}(K_6 \square K_8) \geq 21$.

Proof. Let M_q be the $6 \times q$ matrix below, $q = 4, 6, 8$, where \bar{n} stands for $10 + n$ and $\bar{\bar{n}}$ for $20 + n$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & \bar{0} & \bar{1} & \bar{2} \\ 2 & 1 & 4 & 3 \\ 7 & 8 & 5 & 6 \\ \bar{2} & \bar{1} & \bar{0} & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & \bar{0} & \bar{1} & \bar{2} \\ \bar{3} & \bar{4} & \bar{5} & \bar{6} & \bar{7} & \bar{8} \\ 2 & 1 & 7 & \bar{2} & \bar{5} & \bar{0} \\ \bar{1} & \bar{8} & 4 & 3 & 7 & \bar{4} \\ \bar{6} & 9 & 8 & \bar{3} & 6 & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \bar{6} & \bar{7} & \bar{8} \\ 6 & 7 & 8 & 9 & \bar{0} & \bar{8} & \bar{6} & \bar{7} \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} & \bar{7} & \bar{8} & \bar{6} \\ 4 & 8 & 7 & 1 & 9 & \bar{5} & \bar{0} & \bar{1} \\ \bar{3} & 5 & \bar{1} & \bar{1} & 2 & 9 & \bar{9} & \bar{0} \\ \bar{0} & \bar{4} & \bar{0} & \bar{2} & 6 & 3 & \bar{1} & 9 \end{pmatrix}$$

One can check easily that $M_4 \in \mathcal{M}(6, 4, [1, 12])$, $M_6 \in \mathcal{M}(6, 6, [1, 18])$ and $M_8 \in \mathcal{M}(6, 8, [1, 21])$. Therefore, we are done by Proposition 2. \blacksquare

For $r \in [3, 9]$ consider r -element colour sets

$$V_r = \{v_r(k) : k \in [1, r]\}, \quad W_r = \{w_r(k) : k \in [1, r]\}$$

such that V_r , W_r and $[1, 18]$ are pairwise disjoint. Further, let N_r be the $6 \times r$ matrix below.

$$\begin{pmatrix} v_3(1) & v_3(2) & v_3(3) \\ v_3(2) & v_3(3) & v_3(1) \\ v_3(3) & v_3(1) & v_3(2) \\ w_3(1) & w_3(2) & w_3(3) \\ w_3(2) & w_3(3) & w_3(1) \\ w_3(3) & w_3(1) & w_3(2) \end{pmatrix} \quad \begin{pmatrix} v_6(1) & v_6(2) & v_6(3) & v_6(4) & v_6(5) & v_6(6) \\ v_6(2) & v_6(3) & v_6(4) & v_6(5) & v_6(6) & v_6(1) \\ v_6(3) & v_6(4) & v_6(5) & v_6(6) & v_6(1) & v_6(2) \\ w_6(1) & w_6(2) & w_6(3) & w_6(4) & w_6(5) & w_6(6) \\ w_6(3) & w_6(4) & w_6(5) & w_6(6) & w_6(1) & w_6(2) \\ w_6(5) & w_6(6) & w_6(1) & w_6(2) & w_6(3) & w_6(4) \end{pmatrix}$$

$$\begin{pmatrix} v_4(1) & v_4(2) & v_4(3) & v_4(4) \\ v_4(2) & v_4(3) & v_4(4) & v_4(1) \\ v_4(3) & v_4(4) & v_4(1) & v_4(2) \\ w_4(1) & w_4(2) & w_4(3) & w_4(4) \\ w_4(2) & w_4(3) & w_4(4) & w_4(1) \\ w_4(3) & w_4(4) & w_4(1) & w_4(2) \end{pmatrix} \quad \begin{pmatrix} v_5(1) & v_5(2) & v_5(3) & v_5(4) & v_5(5) \\ v_5(2) & v_5(3) & v_5(4) & v_5(5) & v_5(1) \\ v_5(3) & v_5(4) & v_5(5) & v_5(1) & v_5(2) \\ w_5(1) & w_5(2) & w_5(3) & w_5(4) & w_5(5) \\ w_5(3) & w_5(4) & w_5(5) & w_5(1) & w_5(2) \\ w_5(5) & w_5(1) & w_5(2) & w_5(3) & w_5(4) \end{pmatrix}$$

$$\begin{pmatrix} v_7(1) & v_7(2) & v_7(3) & v_7(4) & v_7(5) & v_7(6) & v_7(7) \\ v_7(2) & v_7(3) & v_7(4) & v_7(5) & v_7(6) & v_7(7) & v_7(1) \\ v_7(3) & v_7(4) & v_7(5) & v_7(6) & v_7(7) & v_7(1) & v_7(2) \\ w_7(1) & w_7(2) & w_7(3) & w_7(4) & w_7(5) & w_7(6) & w_7(7) \\ w_7(3) & w_7(4) & w_7(5) & w_7(6) & w_7(7) & w_7(1) & w_7(2) \\ w_7(5) & w_7(6) & w_7(7) & w_7(1) & w_7(2) & w_7(3) & w_7(4) \end{pmatrix}$$

$$\begin{pmatrix} v_8(1) & v_8(2) & v_8(3) & v_8(4) & v_8(5) & v_8(6) & v_8(7) & v_8(8) \\ v_8(2) & v_8(3) & v_8(4) & v_8(5) & v_8(6) & v_8(7) & v_8(8) & v_8(1) \\ v_8(3) & v_8(4) & v_8(5) & v_8(6) & v_8(7) & v_8(8) & v_8(1) & v_8(2) \\ w_8(1) & w_8(2) & w_8(3) & w_8(4) & w_8(5) & w_8(6) & w_8(7) & w_8(8) \\ w_8(4) & w_8(5) & w_8(6) & w_8(7) & w_8(8) & w_8(1) & w_8(2) & w_8(3) \\ w_8(7) & w_8(8) & w_8(1) & w_8(2) & w_8(3) & w_8(4) & w_8(5) & w_8(6) \end{pmatrix}$$

$$\begin{pmatrix} v_9(1) & v_9(2) & v_9(3) & v_9(4) & v_9(5) & v_9(6) & v_9(7) & v_9(8) & v_9(9) \\ v_9(2) & v_9(3) & v_9(4) & v_9(5) & v_9(6) & v_9(7) & v_9(8) & v_9(9) & v_9(1) \\ v_9(3) & v_9(4) & v_9(5) & v_9(6) & v_9(7) & v_9(8) & v_9(9) & v_9(1) & v_9(2) \\ w_9(1) & w_9(2) & w_9(3) & w_9(4) & w_9(5) & w_9(6) & w_9(7) & w_9(8) & w_9(9) \\ w_9(4) & w_9(5) & w_9(6) & w_9(7) & w_9(8) & w_9(9) & w_9(1) & w_9(2) & w_9(3) \\ w_9(7) & w_9(8) & w_9(9) & w_9(1) & w_9(2) & w_9(3) & w_9(4) & w_9(5) & w_9(6) \end{pmatrix}$$

Lemma 5. *If $r \in [3, 9]$, then $N_r \in \mathcal{M}(6, r, V_r \cup W_r)$.*

Proof. By inspection of the matrix N_r . ■

Proposition 6. *If $q \in [9, 15]$, then $\text{achr}(K_6 \square K_q) \geq 2q + 6$.*

Proof. The block matrix $M_q = (M_6 N_{q-6})$ belongs to $\mathcal{M}(6, q, C)$ with $C = [1, 18] \cup V_{q-6} \cup W_{q-6}$. To see it first realise that since the colourings f_{M_6} and $f_{N_{q-6}}$ are proper (Proposition 4, Lemma 5), and $[1, 18] \cap (V_{q-6} \cup W_{q-6}) = \emptyset$, the colouring f_{M_q} is proper, too.

Next, we have to show that each pair $\{\alpha, \beta\} \in \binom{C}{2}$ is good in M_q . The colourings f_{M_6} and $f_{N_{q-6}}$ are complete, hence it suffices to restrict our attention to $\alpha \in [1, 18]$ and $\beta \in V_{q-6} \cup W_{q-6}$. In such a case $|\mathbb{R}(\alpha) \cap [1, 3]| = 1 = |\mathbb{R}(\alpha) \cap [4, 6]|$ and $\mathbb{R}(\beta) \in \{[1, 3], [4, 6]\}$ so that $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) \neq \emptyset$, and the pair $\{\alpha, \beta\}$ is row-based.

So, Proposition 2 yields $\text{achr}(K_6 \square K_q) \geq |C| = 18 + 2(q - 6) = 2q + 6$. ■

For $l = 0, 1, 2, 3$, let $r_l \in [3, 9]$, and let $N_{r_l}^l$ be the $6 \times r_l$ matrix obtained from N_{r_l} in such a way that $v_{r_l}(k)$ is replaced with $v_{r_l}^l(k)$ and $w_{r_l}(k)$ is replaced with $w_{r_l}^l(k)$ for each $k \in [1, r_l]$; here we suppose that, given a fixed quadruple (r_0, r_1, r_2, r_3) , the sets $[1, 12]$ and

$$V_{r_l}^l = \{v_{r_l}^l(k) : k \in [1, r_l]\}, \quad W_{r_l}^l = \{w_{r_l}^l(k) : k \in [1, r_l]\}, \quad l = 0, 1, 2, 3,$$

are pairwise disjoint. Further, let $\tilde{N}_{r_l}^l$ be the $6 \times r_l$ matrix obtained from $N_{r_l}^l$ by interchanging its rows l and $l + 3$, $l = 1, 2, 3$.

Proposition 7. *If $q \in [16, 40]$, then $\text{achr}(K_6 \square K_q) \geq 2q + 4$.*

Proof. Since $4 \cdot 3 = 12 \leq q - 4 \leq 36 = 4 \cdot 9$, there are integers $r_l \in [3, 9]$, $l = 0, 1, 2, 3$, such that $\sum_{l=0}^3 r_l = q - 4$. Let us show that the block matrix $M_q = (M_4 N_{r_0}^0 \tilde{N}_{r_1}^1 \tilde{N}_{r_2}^2 \tilde{N}_{r_3}^3)$ belongs to $\mathcal{M}(6, q, C)$ with $C = [1, 12] \cup \bigcup_{l=0}^3 (V_{r_l}^l \cup W_{r_l}^l)$.

By Lemma 5 and Proposition 3 we have $N_{r_l} \in \mathcal{M}(6, r_l, V_{r_l} \cup W_{r_l})$ and $N_{r_l}^l \in \mathcal{M}(6, r_l, V_{r_l}^l \cup W_{r_l}^l)$, $l = 0, 1, 2, 3$, as well as $\tilde{N}_{r_l}^l \in \mathcal{M}(6, r_l, V_{r_l}^l \cup W_{r_l}^l)$, $l = 1, 2, 3$. The colouring f_M with $M \in \{M_4, N_{r_0}^0, \tilde{N}_{r_1}^1, \tilde{N}_{r_2}^2, \tilde{N}_{r_3}^3\}$ is proper, and the sets $[1, 12]$, $V_{r_l}^l \cup W_{r_l}^l$, $l = 0, 1, 2, 3$, are pairwise disjoint, hence the colouring f_{M_q} is proper.

Now consider a pair $\{\alpha, \beta\} \in \binom{C}{2}$. If both α, β are either in $[1, 12]$ or in $V_{r_l}^l \cup W_{r_l}^l$ with $l \in [0, 3]$, then the pair $\{\alpha, \beta\}$ is good in M_q , because the colourings f_{M_4} and f_M with $M \in \{N_{r_0}^0, \tilde{N}_{r_1}^1, \tilde{N}_{r_2}^2, \tilde{N}_{r_3}^3\}$ are complete (Propositions 3, 4, Lemma 5).

In all remaining cases we show that $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) \neq \emptyset$, which means that the pair $\{\alpha, \beta\}$ is row-based.

If $\alpha \in [1, 12]$ and $\beta \in \bigcup_{l=0}^3 (V_{r_l}^l \cup W_{r_l}^l)$, then $\mathbb{R}(\alpha) \in \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$ and $\mathbb{R}(\beta) \in \{\{1, 2, 3\}, \{1, 2, 6\}, \{1, 3, 5\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 4, 6\}, \{3, 4, 5\}, \{4, 5, 6\}\}$ so that $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) \neq \emptyset$ follows immediately.

If $\alpha \in V_{r_0}^0$ and $\beta \in \bigcup_{l=1}^3 V_{r_l}^l$, then $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = [1, 3] \setminus \{l\}$; similarly, with $\alpha \in W_{r_0}^0$ and $\beta \in \bigcup_{l=1}^3 W_{r_l}^l$ we have $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = [4, 6] \setminus \{l\}$.

If $\{i, j, k\} = [1, 3]$, $\alpha \in V_{r_i}^i$ and $\beta \in V_{r_j}^j$, then $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = \{k\}$; the same conclusion holds provided that $\{i, j, k\} = [4, 6]$, $\alpha \in W_{r_i}^i$ and $\beta \in W_{r_j}^j$.

If there is $l \in [1, 3]$ such that either $\alpha \in V_{r_0}^0$ and $\beta \in W_{r_l}^l$ or $\alpha \in W_{r_0}^0$ and $\beta \in V_{r_l}^l$, then $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = \{l\}$.

Finally, $\alpha \in V_{r_i}^i$ with $i \in [1, 3]$ and $\beta \in W_{r_j}^j$ with $j \in [1, 3] \setminus \{i\}$ leads to $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = \{i + 3\}$.

Thus the colouring f_{M_q} is complete and $M_q \in \mathcal{M}(6, q, C)$. Since $|C| = 12 + \sum_{l=0}^3 2r_l = 2q + 4$, by Proposition 2 we get $\text{achr}(K_6 \square K_q) \geq 2q + 4$. ■

For a given $s \in [2, \infty)$ consider colour sets $U_s = \{u_k : k \in [1, s]\}$ with $U \in \{X, Y, Z, T\}$ such that the sets $[1, 12]$, X_s, Y_s, Z_s and T_s are pairwise disjoint.

Proposition 8. *If $q \in [42, \infty)$ and $q \equiv 0 \pmod{2}$, then $\text{achr}(K_6 \square K_q) \geq 2q + 4$.*

Proof. Let $s = \frac{q-4}{2}$, and let M_q be the $6 \times q$ matrix below. We show that $M_q \in \mathcal{M}(6, q, C)$ for $C = [1, 12] \cup X_s \cup Y_s \cup Z_s \cup T_s$. Obviously, since $s \geq 19 \geq 2$, the colouring f_{M_q} is proper.

$$\begin{pmatrix} 1 & 2 & 3 & 4 & x_1 & x_2 & \cdots & x_{s-1} & x_s & y_1 & y_2 & \cdots & y_{s-1} & y_s \\ 5 & 6 & 7 & 8 & x_s & x_1 & \cdots & x_{s-2} & x_{s-1} & z_1 & z_2 & \cdots & z_{s-1} & z_s \\ 9 & 10 & 11 & 12 & t_1 & t_2 & \cdots & t_{s-1} & t_s & x_1 & x_2 & \cdots & x_{s-1} & x_s \\ 2 & 1 & 4 & 3 & z_1 & z_2 & \cdots & z_{s-1} & z_s & t_1 & t_2 & \cdots & t_{s-1} & t_s \\ 7 & 8 & 5 & 6 & t_s & t_1 & \cdots & t_{s-2} & t_{s-1} & y_s & y_1 & \cdots & y_{s-2} & y_{s-1} \\ 12 & 11 & 10 & 9 & y_1 & y_2 & \cdots & y_{s-1} & y_s & z_s & z_1 & \cdots & z_{s-2} & z_{s-1} \end{pmatrix}$$

Notice that M_q has a submatrix M_4 (formed by the first four columns of M_q). The colouring f_{M_4} is complete (Proposition 4), hence a pair $\{\alpha, \beta\} \in \binom{C}{2}$ is good in M_q provided that $\alpha, \beta \in [1, 12]$. So, it remains to consider pairs $\{\alpha, \beta\}$ with $\alpha \in C$ and $\beta \in X_s \cup Y_s \cup Z_s \cup T_s$. Realise that $\mathbb{R}(\alpha) \in \mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathbb{R}(\beta) \in \mathcal{R}_2$, where $\mathcal{R}_1 = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$ and $\mathcal{R}_2 = \{\{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}\}$. As $R \cap R_2 \neq \emptyset$ for any $R \in \mathcal{R}_1 \cup \mathcal{R}_2$ and any $R_2 \in \mathcal{R}_2$, the pair $\{\alpha, \beta\}$ is row-based.

Thus, by Proposition 2 we see that $\text{achr}(K_6 \square K_q) \geq 4s + 12 = 2q + 4$. ■

3. SOME BASIC FACTS CONCERNING MATRICES IN $\mathcal{M}(p, q, C)$

In this section we first reproduce those facts from [4] that are necessary for our paper.

Lemma 9 [4]. *If $p, q \in [1, \infty)$, C is a finite set, $M \in \mathcal{M}(p, q, C)$ and $\gamma \in C$, then the following hold.*

1. $\text{frq}(\gamma) \leq \min(p, q)$.
2. $\text{frq}(\gamma) = l$ implies $\text{exc}(\gamma) = l(p + q - l - 1) - (|C| - 1) \geq 0$.
3. $\text{frq}(M) = l$ implies $|C| \leq \lfloor \frac{pq}{l} \rfloor$.

Lemma 10 [4]. *If $p, q \in [1, \infty)$, C is a finite set and $M \in \mathcal{M}(p, q, C)$, then $\text{exc}(M) = \text{exc}(\gamma)$, where $\gamma \in C$ satisfies $\text{frq}(\gamma) = \text{frq}(M)$.*

Lemma 11 [4]. *If $q \in [7, \infty)$, $s \in [0, 7]$, C is a set of cardinality $2q + s$ and $M \in \mathcal{M}(6, q, C)$, then the following hold.*

1. $c_1 = 0$.
2. $c_l = 0$ for $l \in [7, \infty)$.
3. $c_2 \geq 3s$.
4. $c_{3+} \leq 2q - 2s$.
5. $\sum_{i=3}^6 ic_i \leq 6q - 6s$.
6. $\text{frq}(M) = 2$.
7. $\text{exc}(M) = 7 - s$.
8. $c_{4+} \leq c_2 - 3s$.
9. $\{i, k\} \in \binom{[1, 6]}{2}$ implies $r(i, k) \leq 8 - s$.

Lemma 12. *If $q \in [7, \infty)$, then $\text{achr}(K_6 \square K_q) \leq 2q + 6$.*

Proof. If our lemma is false, according to Theorem 1 and Proposition 2 there is a $(2q + 7)$ -element set C and $M \in \mathcal{M}(6, q, C)$. By Lemma 11.3 and 11.7 then $c_2 \geq 3 \cdot 7 = 21$ and $\text{exc}(M) = 0$. Further, by Proposition 3 we may suppose without loss of generality that $r_2(1) \geq r_2(i)$ for $i \in [2, 6]$, which implies $6r_2(1) \geq$

$\sum_{i=1}^6 r_2(i) = 2c_2$ and $r_2(1) \geq \lceil \frac{2c_2}{6} \rceil \geq 7$; we shall use on similar occasions (w) to indicate that it is Proposition 3, which is behind the fact that the generality is not lost. As a consequence there is $i \in [2, 6]$ such that $r(1, i) \geq \lceil \frac{r_2(1)}{5} \rceil \geq 2$. With $\gamma \in \mathbb{R}(1, i)$ then each colour of $\mathbb{R}(1, i) \setminus \{\gamma\}$ contributes one to the excess of γ , hence we have $0 = \text{exc}(M) = \text{exc}(\gamma) \geq r(1, i) - 1 \geq 1$, a contradiction. ■

4. SOLUTION

It turns out that the matrix constructions given by Propositions 4 and 6–8 are optimum from the point of view of $\text{achr}(K_6 \square K_q)$. The optimality was already known for $q = 4$ (Horňák and Puntigán [7]) and $q = 6$ (Bouchet [1]), while the rest of the present paper is devoted to the analysis of remaining q 's.

Theorem 13. $\text{achr}(K_6 \square K_8) = 21$.

Proof. By Proposition 4 and Lemma 12 we know that $21 \leq \text{achr}(K_6 \square K_8) \leq 22$. Suppose that $\text{achr}(K_6 \square K_8) = 22$; because of Proposition 2 there is a 22-element set of colours C and a matrix $M \in \mathcal{M}(6, 8, C)$. With $q = 8$ and $s = 6$ Lemma 11 yields $c_2 \geq 18$, $c_{3+} \leq 4$, $\text{frq}(M) = 2$, $\text{exc}(M) = 1$, $c_{4+} \leq c_2 - 18$ and $r(i, k) \leq 2$ for $\{i, k\} \in \binom{[1, 6]}{2}$. We are going to strengthen step by step the requirements on M to finally reach a conclusion that M cannot exist at all.

Claim 14. If $i \in [1, 6]$ and $j \in [1, 8]$, then $|\mathbb{R}_2(i) \cap \mathbb{C}(j)| \leq 2$.

Proof. Suppose that $|C'_2| \geq 3$ for $C'_2 = \mathbb{R}_2(i) \cap \mathbb{C}(j)$. If $\alpha = (M)_{i,j} \in C_2$, then each colour of $C'_2 \setminus \{\alpha\}$ contributes one to the excess of α so that, by Lemma 10, $1 = \text{exc}(M) = \text{exc}(\alpha) \geq |C'_2 \setminus \{\alpha\}| \geq 2$, a contradiction. On the other hand, if $\alpha \in C_{3+}$, then for any $\beta \in C'_2$ we have $1 = \text{exc}(M) = \text{exc}(\beta) \geq |C'_2 \setminus \{\beta\}| \geq 2$, a contradiction again. □

Claim 15. If $\{i, k\} \in \binom{[1, 6]}{2}$ and $\alpha, \beta \in \mathbb{R}_2(i, k)$, $\alpha \neq \beta$, then $\text{cov}(\alpha, \beta) = 2$, so that $\{\alpha, \beta\} \subseteq \mathbb{C}(j)$ for both $j \in \text{Cov}(\alpha, \beta)$.

Proof. Suppose (w) $i = 1$, $k = 2$ and $(M)_{1,1} = (M)_{2,2} = \alpha$. If $\text{cov}(\alpha, \beta) = 4$, (w) $(M)_{1,3} = (M)_{2,4} = \beta$. Denote $A = \mathbb{R}(1) \cup \mathbb{R}(2)$. From $\text{exc}(\alpha) = \text{exc}(\beta) = 1$ it is clear that $|A| = 14$, $|C \setminus A| = 8$, and that, for both $l \in [1, 2]$, each colour of $C \setminus A$ occupies a position in the set $[3, 6] \times [2l - 1, 2l]$ (the colouring f_M is complete). Since $|(C \setminus A) \cap C_2| = |C \setminus A| - |(C \setminus A) \cap C_{3+}| \geq 8 - c_{3+} \geq 4$, there is a colour $\gamma \in (C \setminus A) \cap C_2$. The neighbourhood of the 2-element vertex set $f_M^{-1}(\gamma)$ contains ten vertices belonging to $[3, 6] \times [1, 4]$, all coloured with seven colours of $(C \setminus A) \setminus \{\gamma\}$. As a consequence we obtain $1 = \text{exc}(M) = \text{exc}(\gamma) \geq 10 - 7 = 3$, a contradiction.

If $\text{cov}(\alpha, \beta) = 3$, (w) $(M)_{1,2} = (M)_{2,3} = \beta$. With $B = \mathbb{R}(1) \cup \mathbb{R}(2) \cup \mathbb{C}(2)$ then $\text{exc}(\alpha) = 1$ implies $|B| = 18$ and $|C \setminus B| = 4$. Each colour $\gamma \in C \setminus B$ belongs to $\mathbb{C}(1) \cap \mathbb{C}(3)$ (both pairs $\{\gamma, \alpha\}$ and $\{\gamma, \beta\}$ are good in M) and satisfies $\text{exc}(\gamma) \geq |(C \setminus B) \setminus \{\gamma\}| = 3$, hence $\gamma \notin C_2$ and $\gamma \in C_{3+}$. Consequently, $c_{3+} \geq 4$, $c_{3+} = 4$, $C \setminus B = C_{3+}$, $B = C_2$, $\delta = (M)_{1,3} \in C_2$, and the second copy of δ appears in $[3, 6] \times [4, 8]$ so that $\text{exc}(\delta) \geq 2$ (if $\delta = (M)_{m,n}$ with $(m, n) \in [3, 6] \times [4, 8]$, then both β and $(M)_{m,1} \in C_{3+}$ contribute to the excess of δ), a contradiction. \square

If $\text{Cov}(\alpha, \beta) = \{j, l\}$ for the 2-colours α, β of Claim 15, then (w) $\alpha = (M)_{i,j} = (M)_{k,l}$ and $\beta = (M)_{i,l} = (M)_{k,j}$; we say that the set of 2-colours $\{\alpha, \beta\}$ forms an \mathbb{X} -configuration (in M): both copies of a colour $\gamma \in \{\alpha, \beta\}$ are diagonal to each other in the “rectangle” of the matrix M with corners $(i, j), (i, l), (k, j), (k, l)$, and this fact will be in the sequel for simplicity denoted by $\{\alpha, \beta\} \rightarrow \mathbb{X}$.

Claim 16. *If $i \in [1, 6]$, then $r_2(i) = 6$ and $r_3(i) = 2$.*

Proof. Let (w) $r_2(1) \geq r_2(i)$ for each $i \in [2, 6]$ and $r(1, i) \geq r(1, i+1)$ for each $i \in [2, 5]$. Suppose that $r_2(1) \geq 7$. Then $2 \geq r(1, 2) \geq \left\lceil \frac{r_2(1)}{5} \right\rceil = 2$ so that $r(1, 2) = 2$. Moreover, $r(1, 6) \leq \left\lfloor \frac{r_2(1)}{5} \right\rfloor = 1$, and there is $p \in [2, 5]$ such that $r(1, p) = 2$ and $r(1, p+1) \leq 1$ (which implies $r(1, i) \leq 1$ for any $i \in [p+1, 6]$). As $r_2(6) = 8 - r_{3+}(6) \geq 8 - c_{3+} \geq 4$, we have $|\mathbb{R}_2(6) \setminus \mathbb{R}_2(1)| = |\mathbb{R}_2(6) \setminus \mathbb{R}(1, 6)| = r_2(6) - r(1, 6) = [8 - r_{3+}(6)] - r(1, 6) \geq 4 - 1 = 3$, and there exists $\alpha \in \mathbb{R}_2(6) \setminus \mathbb{R}_2(1)$.

Consider a colour $\beta \in \mathbb{R}(i, k)$, where $1 < i < k$. When counting the number of pairs $\{\beta, \gamma\}$ with $\gamma \in \mathbb{R}_2(1)$, that are good in M because of the copy of β in the m th row of M , $m \in \{i, k\}$, we see that $r(1, m)$ of them are row-based, and, by Claim 14, at most two of them are column-based. There is $i \in [2, 5]$ such that $\alpha \in \mathbb{R}(i, 6)$. Proceeding as above we obtain that the number of pairs $\{\alpha, \gamma\}$ with $\gamma \in \mathbb{R}_2(1)$, that are good in M , is at most $\rho = [r(1, i) + 2] + [r(1, 6) + 2] \leq 6 + r(1, 6)$. Observe that we cannot have $r(1, 6) \leq r_2(1) - 7$, because then $\rho \leq 6 + [r_2(1) - 7] < r_2(1)$, a contradiction. Therefore, $1 \geq r(1, 6) \geq r_2(1) - 6 \geq 1$, $r(1, 6) = 1$ and $r_2(1) = 7$. Now $p \leq 3$, because $p \geq 4$ would mean $7 = r_2(1) = \sum_{i=2}^6 r(1, i) \geq 2(p-1) + (6-p) = p+4 \geq 8$, a contradiction. Consequently, any colour $\delta \in C_2 \setminus \mathbb{R}(1)$ needs a copy in a row $k \in [2, p]$; to see it realise that, under the assumption $\delta \in \mathbb{R}(a, b)$ with $a, b \in [p+1, 6]$, the number of pairs $\{\delta, \gamma\}$, $\gamma \in \mathbb{R}_2(1)$, that are good in M , is at most $[r(1, a) + 2] + [r(1, b) + 2] \leq 2 \cdot 1 + 4 < r_2(1)$, which is impossible. Thus $18 \leq c_2 \leq 7 + \sum_{i=2}^p [r_2(i) - r(1, i)] \leq 7 + (p-1)(7-2) = 5p+2 \leq 17$, a contradiction.

Since the assumption $r_2(1) \geq 7$ was false, we have $36 \geq \sum_{i=1}^6 r_2(i) = 2c_2 \geq 36$. Therefore, $r_2(i) = 6$ for each $i \in [1, 6]$, $c_2 = 18$, $c_{4+} = 0$, $C = C_2 \cup C_3$, and the proof follows. \square

By Claim 16, $c_2 = \frac{6 \cdot 6}{2} = 18$ and $c_3 = \frac{6 \cdot 2}{3} = 4$; moreover, for every $i \in [1, 6]$ there is (at least one) $p_i \in [1, 6] \setminus \{i\}$ such that $r(i, p_i) = \lceil \frac{6}{5} \rceil = 2$. Then, by Claim 15, $\mathbb{R}(i, p_i) \rightarrow \mathbb{X}$ for $i \in [1, 6]$. Let $\tilde{C}_2 = \{\alpha, \beta, \gamma, \delta\} \subseteq C_2$ be such that $\{\alpha, \beta\} \rightarrow \mathbb{X}$ and $\{\gamma, \delta\} \rightarrow \mathbb{X}$, where $\{\alpha, \beta\} \neq \{\gamma, \delta\}$ (which immediately yields $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$). We have $\text{cov}(\tilde{C}_2) \in [3, 4]$, since with $\text{cov}(\tilde{C}_2) = 2$ each of β, γ, δ contributes one to the excess of α so that $\text{exc}(\alpha) \geq 3$, a contradiction.

Thus (w) $\text{cov}(\alpha, \beta) = [1, 2]$ and $\text{cov}(\gamma, \delta) = [l, l + 1]$, where $l \in [2, 3]$. Then $\mathbb{C}(1) \cup \mathbb{C}(2) \subseteq C_2$ (because of $r_3(1) = r_3(2) = 2$ and $\text{exc}(\alpha) = 1$, colours of C_3 occupy four positions in $\mathbb{R}(1) \cup \mathbb{R}(2)$ and neither position in $\mathbb{C}(1) \cup \mathbb{C}(2)$), and, similarly, $\mathbb{C}(l) \cup \mathbb{C}(l + 1) \subseteq C_2$. So, all 3-colours appear exclusively in $7 - l$ columns of M numbered from $l + 2$ to 8. There is $j \in [l + 2, 8]$ such that $c_3(j) \geq \lceil \frac{3c_3}{7-l} \rceil \geq \lceil \frac{12}{5} \rceil = 3$, while $c_2(j) \geq 6 - c_3 = 2$. If $\varepsilon \in \mathbb{C}_2(j) \cap \mathbb{R}(m, n)$, then 3-colours occupy at least $r_3(m) + r_3(n) + [c_3(j) - 1] = c_3(j) + 3 \geq 6$ positions in $N(V_\varepsilon)$ (since $\varepsilon \in \{(M)_{m,j}, (M)_{n,j}\}$, at least $c_3(j) - 1$ positions in $[1, 6] \times \{j\}$ occupied by 3-colours are positions that are not in $\{m, n\} \times [1, 8]$), hence $\text{exc}(\varepsilon) \geq 6 - c_3 = 2$, a final contradiction for the proof of Theorem 13. ■

Theorem 17. *If $q \in [9, 15]$, then $\text{achr}(K_6 \square K_q) = 2q + 6$.*

Proof. See Proposition 6 and Lemma 12. ■

Theorem 18. *If either $q \in [42, \infty)$ and $q \equiv 0 \pmod{2}$ or $q \in [16, 40]$, then $\text{achr}(K_6 \square K_q) = 2q + 4$.*

Proof. We proceed by the way of contradiction. Since $\text{achr}(K_6 \square K_q) \geq 2q + 4$ (Propositions 7 and 8), the assumption $\text{achr}(K_6 \square K_q) \geq 2q + 5$ by Theorem 1 and Proposition 2 means that there is a colour set C with $|C| = 2q + 5 = 2q + s$ and a matrix $M \in \mathcal{M}(6, q, C)$. By Lemma 11 then $C = \bigcup_{l=2}^6 C_l$, $c_2 \geq 15$, $c_{3+} \leq 2q - 10$, $\sum_{i=3}^6 ic_i \leq 6q - 30$, $\text{frq}(M) = 2 = \text{exc}(M)$, $c_{4+} \leq c_2 - 15$, and $\{i, k\} \in \binom{[1,6]}{2}$ implies $r(i, k) \leq 3$. A contradiction is reached first for $q \geq 19$, then for $q \in [17, 18]$, and finally for $q = 16$.

Let G be an auxiliary graph G associated with M , in which $V(G) = [1, 6]$ and $\{i, k\} \in E(G)$ if and only if $r(i, k) \geq 1$.

Claim 19. $\Delta(G) \leq 3$.

Proof. In [4] it has been proved that $\Delta(G) \geq 4$ together with $\text{achr}(K_6 \square K_q) = 2q + s$ implies $q \leq 40 - 5s = 15$, a contradiction. □

In [4] one can find also proofs of the following two claims.

Claim 20. *If $\{i, j, k, l, m, n\} = [1, 6]$ and $r(i, j, k) \geq 1$, then $r(l, m, n) \leq 9$.*

Claim 21. *If $\{i, j, k, l, m, n\} = [1, 6]$, $r(i, l) \geq 1$, $r(j, l) \geq 1$ and $r(k, l) \geq 1$, then $r(l, m, n) \geq q + 3s - 24 = q - 9$.*

Claim 22. *If $\{i, j, k, l, m, n\} = [1, 6]$, $r(i, l) \geq 1$, $r(j, l) \geq 1$ and $r(k, l) \geq 1$, $\{a, b\} \in \binom{[1, 6]}{2}$ and $r(a, b) \geq 1$, then $|\{i, j, k\} \cap \{a, b\}| = 1 = |\{l, m, n\} \cap \{a, b\}|$.*

Proof. The assumptions of Claim 22 imply that $r(l, m, n) \geq q - 9 \geq 7$ (see Claim 21). Consider a colour $\alpha \in \mathbb{R}(a, b)$.

If $\{a, b\} \subseteq \{i, j, k\}$, the number of pairs $\{\alpha, \beta\}$, $\beta \in \mathbb{R}(l, m, n)$, that are good in M (and necessarily column-based), is at most $3 \text{cov}(\alpha) = 6 < r(l, m, n)$, a contradiction.

On the other hand, with $\{a, b\} \subseteq \{l, m, n\}$ each colour of $\mathbb{R}(l, m, n)$ contributes one to the excess of α so that $2 = \text{exc}(M) = \text{exc}(\alpha) \geq r(l, m, n) \geq 7$, a contradiction again.

Therefore, $2 = |\{i, j, k, l, m, n\} \cap \{a, b\}| = |\{i, j, k\} \cap \{a, b\}| + |\{l, m, n\} \cap \{a, b\}| \leq 1 + 1 = 2$, and then $|\{i, j, k\} \cap \{a, b\}| = 1 = |\{l, m, n\} \cap \{a, b\}|$. \square

Claim 23. *If $\{i, k\} \in \binom{[1, 6]}{2}$, then $r(i, k) \leq 2$.*

Proof. Let (w) $i = 1$, $k = 2$, $\text{Cov}(\mathbb{R}(1, 2)) = [1, n]$, and assume (for a proof by contradiction) that $r(1, 2) = 3$ (see Lemma 11.9), which implies $n \in [3, 6]$.

We are going to show that $A = C \setminus (\mathbb{R}(1) \cup \mathbb{R}(2)) \subseteq C_{3+}$. First observe that each colour $\alpha \in A$ occupies at least two positions in $S_n = [3, 6] \times [1, n]$ (all pairs $\{\alpha, \beta\}$, $\beta \in \mathbb{R}(1, 2)$, are good in M), hence $|A| \leq \lfloor \frac{4n}{2} \rfloor = 2n$. Moreover, from $|\mathbb{R}(1) \cap \mathbb{R}(2)| \geq r(1, 2)$ we get $|\mathbb{R}(1) \cup \mathbb{R}(2)| \leq 2q - 3$. Consequently, $2q + 5 = |C| = |A| + |\mathbb{R}(1) \cup \mathbb{R}(2)| \leq |A| + (2q - 3)$ leads to $8 \leq |A| \leq 2n$ and $n \in [4, 6]$.

If $n = 4$, then $|A| = 8$, and any colour of A occupies exactly two positions in S_4 . Suppose there is a colour $\alpha \in A \cap C_2$. If a vertex $(i, j) \in S_4$ belongs to $N(V_\alpha)$, then $(M)_{i,j} \in A \setminus \{\alpha\}$, hence $2 = \text{exc}(M) = \text{exc}(\alpha) \geq 10 - |A \setminus \{\alpha\}| = 3$ (the set $N(V_\alpha)$ has 10 vertices in S_4), a contradiction. Therefore, $A \subseteq C_{3+}$.

If $n = 5$, there is $j \in [1, 5]$ such that $|\mathbb{C}(j) \cap \mathbb{R}(1, 2)| = 2$ and $|\mathbb{C}(l) \cap \mathbb{R}(1, 2)| = 1$ for $l \in [1, 5] \setminus \{j\}$. Then $A = A_2 \cup A_3 \cup A_4$, where A_l consists of colours of A occupying l positions in S_5 . With $a_l = |A_l|$ we obtain $a_2 \leq 4$ (if $\alpha \in A_2 \setminus \mathbb{C}(j)$, at least one of three pairs $\{\alpha, \beta\}$ with $\beta \in \mathbb{R}(1, 2)$ is not good in M , a contradiction), $a_3 + a_4 = |A| - a_2 \geq 8 - 4 = 4$, $16 + a_3 + a_4 \leq 16 + a_3 + 2a_4 \leq 2(a_2 + a_3 + a_4) + a_3 + 2a_4 = \sum_{l=2}^4 la_l = 4 \cdot 5 = 20$, $a_3 + a_4 \leq 20 - 16 = 4$, $a_3 + a_4 = 4$, all six above expressions are 20, which implies $a_4 = 0$, $a_3 = 4 = a_2$, and then all positions in S_5 are occupied by colours of $A_2 \cup A_3$. If $\text{Cov}(\mathbb{R}(1, 2) \setminus \mathbb{C}(j)) = \{s, t\} \subseteq [1, 5] \setminus \{j\}$, then $A_2 \subseteq \mathbb{C}(j) \cup \mathbb{C}(s) \cup \mathbb{C}(t)$. For the set B of colours in $C_2 \setminus A_2$ that are not in $[1, 2] \times [1, 5]$ we have $|B| \geq c_2 - [(2|[1, 5]| - 3) + a_2] \geq 15 - 11 = 4$. However, the number of pairs $\{\alpha, \beta\}$ with $\alpha \in A_2$ and $\beta \in B$, that are good in M , is at most three (if $\beta \in \mathbb{R}(u)$, $u \in [3, 6]$, only $(M)_{u,j}$, $(M)_{u,s}$ and $(M)_{u,t}$ are available as α), a contradiction.

If $n = 6$, the frequency of each colour in $\alpha \in A$ is at least three, since all pairs $\{\alpha, \beta\}$ with $\beta \in \mathbb{R}(1, 2)$ are column-based, and at most one of them satisfies the implication $\alpha \in \mathbb{C}(j) \Rightarrow \beta \in \mathbb{C}(j)$.

Thus $A \subseteq C_{3+}$ and $C_2 \subseteq \mathbb{R}(1) \cup \mathbb{R}(2)$. For $l \in [1, 6]$ let

$$K(l) = \{m \in [1, 6] \setminus \{l\} : r(l, m) \geq 1\},$$

so that $\Delta(G) \leq 3$ (Claim 19) implies $|K(l)| \leq 3$. Further, for $l, m \in [1, 6]$, $l \neq m$, let $r_2^-(l, m) = |\mathbb{R}_2(l) \setminus \mathbb{R}_2(m)|$ be the number of 2-colours which occur in row l but not row m . Observe that if $m \in K(l)$, then $r_2^-(l, m) = \sum_{p \in K(l) \setminus \{m\}} r(l, p) \leq 2 \cdot 3 = 6$.

The inclusion $C_2 \subseteq \mathbb{R}(1) \cup \mathbb{R}(2)$ implies $c_2 = r_2^-(1, 2) + r_2^-(2, 1) + r(1, 2)$. Then $15 \leq c_2 = r_2^-(1, 2) + r_2^-(2, 1) + r(1, 2) \leq 2 \cdot 6 + 3 = 15$, hence $c_2 = 15$, $r_2^-(1, 2) = r_2^-(2, 1) = 6$, $r(1, 2) = 3$, $r_2(1) = r_2^-(1, 2) + r(1, 2) = 9 = r_2^-(2, 1) + r(2, 1) = r_2(2)$, $|K(l)| = 3$ and $r(l, p) = 3$ for each $p \in K(l)$, $l = 1, 2$.

The last two facts imply in particular that $|K(1) \setminus \{2\}| = 2$ and that $r_1(m) = 3$ for both $m \in K(1) \setminus \{2\}$. This means that we can repeat the entire reasoning from the start of the proof of Claim 23 with the pair $(1, m)$ instead of the pair $(1, 2)$. Among other things we obtain $r_2(m) = 9$ for both $m \in K(1) \setminus \{2\} \subseteq [3, 6]$. Together with $r_2(1) = r_2(2) = 9$ we then have

$$15 = c_2 = \frac{1}{2} \sum_{m=1}^6 r_2(m) \geq \frac{4 \cdot 9}{2} = 18,$$

a contradiction. □

Claim 24. *If $i \in [1, 6]$, then $r_2(i) \leq 6$.*

Proof. Since $\Delta(G) \leq 3$, the claim is a direct consequence of Claim 23. □

Claim 25. *The following statements are true.*

1. $\Delta(G) = 3$.
2. G is a subgraph of $K_{3,3}$.
3. $c_2 \leq 18$.

Proof. 1. The assumption $\Delta(G) \leq 2$ would mean, by Claim 23, $r_2(i) \leq 2 \cdot 2 = 4$ for $i \in [1, 6]$ and $15 \leq c_2 = \frac{1}{2} \sum_{i=1}^6 r_2(i) \leq \frac{6 \cdot 4}{2} = 12$, a contradiction.

2. From Claims 25.1 and 22 it follows that there is a partition $\{I, K\}$ of $[1, 6]$ satisfying $|I| = |K| = 3$ such that $r(i, k) \geq 1$ with $\{i, k\} \in \binom{[1, 6]}{2}$ implies $|\{i, k\} \cap I| = 1 = |\{i, k\} \cap K|$. Thus, G is a subgraph of $K_{3,3}$ with the bipartition $\{I, K\}$.

3. Finally, by Claim 23, $c_2 = \sum_{i \in I} \sum_{k \in K} r(i, k) \leq 9 \cdot 2 = 18$. □

Henceforth we suppose (w) that the bipartition of the graph $K_{3,3}$ from Claim 25.2 is $\{[1, 3], [4, 6]\}$, which leads to

$$C_2 = \bigcup_{i=1}^3 \bigcup_{k=4}^6 \mathbb{R}(i, k).$$

Note that this assumption somehow restricts the meaning of (w) in the subsequent analysis, namely the bijection $\rho : [1, 6] \rightarrow [1, 6]$ in Proposition 3 should satisfy $\rho([1, 3]) \in \{[1, 3], [4, 6]\}$.

Claim 26. *There is at most one pair $(i, k) \in [1, 3] \times [4, 6]$ with $r(i, k) = 0$.*

Proof. If $|\{(i, k) \in [1, 3] \times [4, 6] : r(i, k) = 0\}| \geq 2$, Claim 23 yields $15 \leq c_2 \leq 7 \cdot 2 = 14$, a contradiction. \square

Claim 27. *If $(i, j, k) \in \{(1, 2, 3), (4, 5, 6)\}$, then $7 \leq q - 9 \leq r(i, j, k) \leq 9$.*

Proof. From Claim 26 it immediately follows that $\max(\deg_G(p) : p \in [1, 3]) = 3 = \max(\deg_G(p) : p \in [4, 6])$. So, by Claims 21 and 20, $q - 9 \leq r(i, j, k) \leq 9$. \square

Use for an edge $\{i, k\}$ of the graph $K_{3,3}$ with bipartition $\{[1, 3], [4, 6]\}$ the label $r(i, k) \in [0, 2]$ (see Claim 23). A colour $\alpha \in C_2$ corresponds to an edge $\{i, k\} \in E(K_{3,3})$ if $\alpha \in \mathbb{R}(i, k)$, and α corresponds to a set $E \subseteq E(K_{3,3})$ if there is $e \in E$ such that α corresponds to e . We denote by $\text{Col}(E)$ the set of colours corresponding to E . Colours $\alpha, \beta \in C_2$, $\alpha \neq \beta$, are *column-related* (in M) provided that $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = \emptyset$ (and, consequently, $|\mathbb{R}(\alpha) \cup \mathbb{R}(\beta)| = 4$); then the pair $\{\alpha, \beta\}$ is not row-based, and hence it is column-based (thus if 2-colours α, β are column-related, then the pair $\{\alpha, \beta\}$ is column-based, but not necessarily vice versa). Evidently, if $\{\gamma_j : j \in [1, l]\}$ is a set of pairwise column-related 2-colours, where $\gamma_j \in \mathbb{R}(i_j, k_j)$, then $r(i_j, k_j) \geq 1$, $\{i_j, k_j\} \in E(K_{3,3})$, $|\bigcup_{j \in [1, l]} \{i_j, k_j\}| = 2l$, $\{\{i_j, k_j\} : j \in [1, l]\}$ is a matching in $K_{3,3}$, and so $l \leq 3$. For a matching \mathcal{M} in $K_{3,3}$ we denote by $\text{wt}(\mathcal{M})$ the *weight* of \mathcal{M} , i.e., the sum of labels of edges of \mathcal{M} .

Claim 28. *If \mathcal{M}^1 is a perfect matching in $K_{3,3}$, then there are perfect matchings \mathcal{M}^2 and \mathcal{M}^3 in $K_{3,3}$ such that $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ is a partition of $E(K_{3,3})$ and $\text{wt}(\mathcal{M}^2) \geq \text{wt}(\mathcal{M}^3)$; moreover, $\bigcup_{s=1}^3 \text{Col}(\mathcal{M}^s) = C_2$ and $\sum_{s=1}^3 \text{wt}(\mathcal{M}^s) = c_2$.*

Proof. The set $E(K_{3,3}) \setminus \mathcal{M}^1$ induces a 6-vertex cycle C in $K_{3,3}$. Then $E(C)$ has a partition $\{\mathcal{M}^2, \mathcal{M}^3\}$ into perfect matchings of $K_{3,3}$ so that $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ is a partition of $E(K_{3,3})$; without loss of generality we may suppose $\text{wt}(\mathcal{M}^2) \geq \text{wt}(\mathcal{M}^3)$. Notice that each colour of C_2 is in exactly one of the sets $\text{Col}(\mathcal{M}^s) \subseteq C_2$, $s = 1, 2, 3$, hence $C_2 = \bigcup_{s=1}^3 \text{Col}(\mathcal{M}^s)$ and $c_2 = \sum_{i=1}^3 \sum_{k=4}^6 r(i, k) = \sum_{s=1}^3 \text{wt}(\mathcal{M}^s)$. \square

Let A be a nonempty subset of C_2 . We say that A is of *type* $t_1^{a_1} \cdots t_p^{a_p}$ if $|A| \geq t_1 \geq \cdots \geq t_p \geq 1$, every column of M contains either 0 or exactly t_s colours of A for some $s \in [1, p]$, and a_s is the number of columns of M that contain

exactly t_s colours of A . Note that $\sum_{s=1}^p a_s t_s = 2|A|$. Clearly, the type of A is unique.

Claim 29. *Under the assumptions $\{i, j, k\} = \{1, 2, 3\}$, $\{l, m, n\} = \{4, 5, 6\}$, $\alpha \in \mathbb{R}(i, l)$, $\beta \in \mathbb{R}(j, m)$ and $\gamma \in \mathbb{R}(k, n)$, the following statements are true.*

1. *If the set $\{\alpha, \beta, \gamma\}$ is of the type $t_1^{a_1} \cdots t_p^{a_p}$, then $\sum_{s=1}^p a_s \binom{t_s}{2} \geq 3$.*
2. *If $\text{cov}(\alpha, \beta, \gamma) \leq 3$, then $\text{cov}(\alpha, \beta, \gamma) = 3$, each colour of $C_2 \setminus \{\alpha, \beta, \gamma\}$ appears exactly once in the set $[1, 6] \times \text{Cov}(\alpha, \beta, \gamma)$, and $\bigcup_{s \in \text{Cov}(\alpha, \beta, \gamma)} \mathbb{C}(s) = C_2$.*
3. *The type of the set $\{\alpha, \beta, \gamma\}$ is either $3^1 1^3$ or 2^3 .*

Proof. 1. The colours α, β, γ are pairwise column-related, hence each of the pairs $\{\alpha, \beta\}$, $\{\alpha, \gamma\}$ and $\{\beta, \gamma\}$ is column-based; colours of such a pair share a column whose number is in $\text{Cov}(\alpha, \beta, \gamma)$. A column of M containing exactly t_s colours of $\{\alpha, \beta, \gamma\}$ hosts exactly $\binom{t_s}{2}$ from among the above pairs, and so $\sum_{s=1}^p a_s \binom{t_s}{2} \geq 3$.

2. In the set $[1, 6] \times \text{Cov}(\alpha, \beta, \gamma)$ there are at most eighteen positions of which six are occupied by α, β, γ . Since $c_2 \geq 15$, there are at least twelve other colours in C_2 , and these must all occupy one of the remaining positions, otherwise for a colour $\delta \in C_2 \setminus \{\alpha, \beta, \gamma\}$, that is out of $[1, 6] \times \text{Cov}(\alpha, \beta, \gamma)$, the number of $\varepsilon \in \{\alpha, \beta, \gamma\}$ such that the pair $\{\delta, \varepsilon\}$ is good in M , is only 2 (such pairs are row-based). Thus $\text{cov}(\alpha, \beta, \gamma) = 3$, $c_2 = 15$, each colour of $C_2 \setminus \{\alpha, \beta, \gamma\}$ occupies exactly one position in $[1, 6] \times \text{Cov}(\alpha, \beta, \gamma)$, and the set of colours, that appear in $[1, 6] \times \text{Cov}(\alpha, \beta, \gamma)$, is equal to C_2 , i.e., $\bigcup_{s \in \text{Cov}(\alpha, \beta, \gamma)} \mathbb{C}(s) = C_2$.

3. Possible types of the set $\{\alpha, \beta, \gamma\}$ (that must satisfy Claim 29.1) are 3^2 , $3^1 2^1 1^1$, $3^1 1^3$ and 2^3 . However, the type 3^2 has to be excluded since $\text{cov}(\alpha, \beta, \gamma) = 3$ by Claim 29.2.

Suppose the set $\{\alpha, \beta, \gamma\}$ is of the type $3^1 2^1 1^1$. Observe that if $b \in \text{Cov}(\alpha, \beta, \gamma)$ satisfies $\{\alpha, \beta, \gamma\} \cap \mathbb{C}(b) = \{\varepsilon\}$, there is (a unique) $a \in [1, 6]$ such that $\{\alpha, \beta, \gamma\} \cap \mathbb{R}(a) = \{\varepsilon\}$ and $\zeta = (M)_{a,b} \neq \varepsilon$. By Claim 29.2 then $\zeta \in C_2$, and the second copy of ζ is in $[1, 6] \times ([1, q] \setminus \text{Cov}(\alpha, \beta, \gamma))$; so, the number of pairs $\{\zeta, \eta\}$ with $\eta \in \{\alpha, \beta, \gamma\} \setminus \{\varepsilon\}$, that are good in M (and necessarily row-based), is one, while $|\{\alpha, \beta, \gamma\} \setminus \{\varepsilon\}| = 2$, a contradiction. \square

Claim 30. *If $\{i, j, k\} = \{1, 2, 3\}$, $\{l, m, n\} = \{4, 5, 6\}$, $\mathbb{R}(i, l) = \{\alpha_1, \alpha_2\}$, $\mathbb{R}(j, m) = \{\beta_1, \beta_2\}$, $\gamma_1 \in \mathbb{R}(k, n)$ and $a, b \in [1, 2]$, then the set $\{\alpha_a, \beta_b, \gamma_1\}$ is of the type $3^1 1^3$.*

Proof. If the claim is false, then, by Claim 29.3, (w) $\{\alpha_1, \beta_1, \gamma_1\}$ is of the type 2^3 , $\text{Cov}(\alpha_1, \beta_1, \gamma_1) = [1, 3]$, and, by Claim 29.2, α_2 occupies exactly one position in $[1, 6] \times [1, 3]$. Clearly, α_2 appears in the column of M containing both β_1 and γ_1 (the pair $\{\beta_1, \gamma_1\}$ is column-based), for otherwise $\{\alpha_2, \beta_1, \gamma_1\}$ would be of the type $2^2 1^2$, which is impossible by Claim 29.3; so, by the same claim, $\{\alpha_2, \beta_1, \gamma_1\}$ is of the type $3^1 1^3$, (w) $\text{Cov}(\alpha_2, \beta_1, \gamma_1) = [1, 4]$.

Proceeding similarly as above we see that β_2 appears in the column containing α_1 , γ_1 , and $\{\alpha_1, \beta_2, \gamma_1\}$ is of the type $3^1 1^3$, so that the pair $\{\alpha_2, \beta_2\}$ can be good in M only if $\{\alpha_2, \beta_2\} \subseteq \mathbb{C}(4)$. Consequently, $\{\alpha_2, \beta_2, \gamma_1\}$ is of the type 2^3 . If $\{\alpha_1, \beta_1\} \subseteq \mathbb{C}(b)$, $b \in [1, 3]$, then $\text{Cov}(\alpha_2, \beta_2, \gamma_1) = [1, 4] \setminus \{b\}$, so that, by Claim 29.2, $\mathbb{C}(b), \mathbb{C}(4) \subseteq C_2$ and $\mathbb{C}(b, 4) = \mathbb{C}(b) \setminus \{\alpha_1, \beta_1\} = \mathbb{C}(4) \setminus \{\alpha_2, \beta_2\}$. In such a case any colour $\delta \in \mathbb{C}(b, 4) \subseteq C_2$ satisfies $2 = \text{exc}(\delta) \geq |\mathbb{C}(b, 4) \setminus \{\delta\}| = 3$, a contradiction. \square

From Claim 30 we see that if \mathcal{M} is a perfect matching in $K_{3,3}$ with $\text{wt}(\mathcal{M}) \geq 5$ and colours $\alpha, \beta, \gamma \in \text{Col}(\mathcal{M})$ are pairwise column-related, then the set $\{\alpha, \beta, \gamma\}$ is of the type $3^1 1^3$.

Claim 31. *Under the assumptions $\{i, j, k\} = [1, 3]$, $\{l, m, n\} = [4, 6]$, $\mathbb{R}(i, l) = \{\alpha_1, \alpha_2\}$, $\mathbb{R}(j, m) = \{\beta_1, \beta_2\}$, $\mathbb{R}(k, n) \in \{\{\gamma_1\}, \{\gamma_1, \gamma_2\}\}$ and $C_2^1 = \mathbb{R}(i, l) \cup \mathbb{R}(j, m) \cup \mathbb{R}(k, n)$, the following statements are true.*

1. *There is $a \in [1, q]$ such that $C_2^1 \subseteq \mathbb{C}(a)$.*
2. *If colours $\delta, \varepsilon \in C_2^1$, $\delta \neq \varepsilon$, are column-related, then $\text{cov}(\delta, \varepsilon) = 3$.*
3. *If $\delta \in \{\alpha, \beta\}$, $C_2^1 \subseteq \mathbb{C}(a)$ and $\text{Cov}(\delta_1, \delta_2) = \{a, b, d\}$, then $|\mathbb{C}(b) \cap \mathbb{C}(d)| = 3$ and $\mathbb{C}(b, d) \neq \emptyset$.*
4. *$\{\alpha_1, \alpha_2\} \rightarrow \mathbb{X}$ and $\{\beta_1, \beta_2\} \rightarrow \mathbb{X}$.*
5. *If $\mathbb{R}(k, n) = \{\gamma_1, \gamma_2\}$, then $\{\gamma_1, \gamma_2\} \rightarrow \mathbb{X}$.*
6. *$\text{cov}(C_2^1) = 4$.*
7. *If $\delta \in C_2 \setminus C_2^1$, then δ is in $[1, 6] \times \text{Cov}(C_2^1)$.*

Proof. Consider the perfect matching $\mathcal{M}^1 = \{\{i, l\}, \{j, m\}, \{k, n\}\}$. From the assumptions of Claim 31 we get $5 \leq \text{wt}(\mathcal{M}^1) \leq 6$.

1. By Claim 30 we know that (among others) all of the following sets are of the type $3^1 1^3$: $\{\alpha_1, \beta_1, \gamma_1\}$, $\{\alpha_2, \beta_1, \gamma_1\}$, $\{\alpha_1, \beta_2, \gamma_1\}$ and $\{\alpha_1, \beta_1, \gamma_2\}$ (provided that $\gamma_2 \in \mathbb{R}(k, n)$). Then there is $a \in [1, q]$ with $\{\alpha_1, \beta_1, \gamma_1\} \subseteq \mathbb{C}(a)$. Now $|\{\alpha_2, \beta_1, \gamma_1\} \cap \mathbb{C}(a)| \geq 2 > 1$, hence $|\{\alpha_2, \beta_1, \gamma_1\} \cap \mathbb{C}(a)| = 3$ and $\alpha_2 \in \mathbb{C}(a)$. A similar reasoning shows that $\beta_2 \in \mathbb{C}(a)$ as well as $\gamma_2 \in \mathbb{C}(a)$ (under the assumption $\gamma_2 \in \mathbb{R}(k, n)$).

Before proceeding further let us mention that, by Claim 31.1, if \mathcal{M} is a perfect matching in $K_{3,3}$ with $\text{wt}(\mathcal{M}) \geq 5$, then all colours of $\text{Col}(\mathcal{M})$ occur in (exactly) one of columns of M .

2. There are $s, t, u \in \{1, 2\}$ such that $\{\delta, \varepsilon\} \subseteq \{\alpha_s, \beta_t, \gamma_u\}$. Therefore, the statement is a direct consequence of Claim 31.1 and the fact that, by Claim 30, the set $\{\alpha_s, \beta_t, \gamma_u\}$ is of the type $3^1 1^3$.

3. If $\delta = \alpha$, $\text{Cov}(\alpha_1, \alpha_2) = \{a, b, d\}$ and $\varepsilon \in C' = C \setminus (\mathbb{R}(i) \cup \mathbb{R}(l) \cup \mathbb{C}(a))$, then, as both pairs $\{\varepsilon, \alpha_1\}$ and $\{\varepsilon, \alpha_2\}$ are good in M , we get $\varepsilon \in \mathbb{C}(b) \cap \mathbb{C}(d)$. Consequently, $|\mathbb{C}(b) \cap \mathbb{C}(d)| \geq |C'| \geq 2q + 5 - (2q - 2 + 4) = 3$; further, by Claim 24,

$|(\mathbb{R}(i) \cup \mathbb{R}(l) \cup \mathbb{C}(a)) \cap C_2| \leq 2 \cdot 6 - 2 + 4 = 14 < c_2$, and so C' contains a 2-colour ζ . Having in mind that $\zeta \in \mathbb{C}(b, d)$ and $2 = \text{exc}(\zeta) \geq |(\mathbb{C}(b) \cap \mathbb{C}(d)) \setminus \{\zeta\}| \geq |C' \setminus \{\zeta\}| \geq 2$, we obtain $|\mathbb{C}(b) \cap \mathbb{C}(d)| = 3$.

An analogous reasoning applies in the case $\delta = \beta$ and $\text{Cov}(\beta_1, \beta_2) = \{a, b, d\}$.

4. By Claim 31.1 there is $a \in [1, q]$ with $C_2^1 \subseteq \mathbb{C}(a)$, hence $2 \leq \text{cov}(\alpha_1, \alpha_2) \leq 3$. Suppose that $\text{cov}(\alpha_1, \alpha_2) = 3$ (which means that $\{\alpha_1, \alpha_2\} \rightarrow \mathbb{X}$ is not true) and $\text{Cov}(\alpha_1, \alpha_2) = \{a, b, d\}$. By Claim 31.3 then $|\mathbb{C}(b) \cap \mathbb{C}(d)| = 3$, and there is (a 2-colour) $\varepsilon \in \mathbb{C}(b, d)$.

By Claim 28 there exists a partition $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ of $E(K_{3,3})$ into perfect matchings in $K_{3,3}$, $\bigcup_{s=1}^3 \text{Col}(\mathcal{M}^s) = C_2$ and $\text{wt}(\mathcal{M}^2) \geq \text{wt}(\mathcal{M}^3)$. Let $C_2^s = \text{Col}(\mathcal{M}^s)$, $s = 2, 3$, so that $\{C_2^1, C_2^2, C_2^3\}$ is a partition of C_2 ; then there is $p \in [2, 3]$ with $\varepsilon \in C_2^p$. Note that $\{\mathcal{M}^2, \mathcal{M}^3\} = \{\mathcal{M}^p, \mathcal{M}^{5-p}\}$ and $\{C_2^2, C_2^3\} = \{C_2^p, C_2^{5-p}\}$.

Let us show that $\text{wt}(\mathcal{M}^p) \leq 4$. Indeed, with $\text{wt}(\mathcal{M}^p) = |\text{Col}(\mathcal{M}^p)| \geq 5$, by Claim 31.1 all colours of $C_2^p = \text{Col}(\mathcal{M}^p)$ occur in one of columns of M , say in the column e . Since $\varepsilon \in C_2^p$, we have necessarily $e \in \{b, d\}$. The column e contains exactly one of colours α_1, α_2 of C_2^1 and all colours of C_2^p (thus 2-colours only), which implies $\mathbb{C}(b) \cap \mathbb{C}(d) = \mathbb{C}(b) \cap \mathbb{C}(d) \cap \mathbb{C}(e) \subseteq \mathbb{C}(e) \subseteq C_2$, $\mathbb{C}(b, d) = C_2 \cap \mathbb{C}(b) \cap \mathbb{C}(d) = \mathbb{C}(b) \cap \mathbb{C}(d)$ and $|\mathbb{C}(b, d)| = 3$. From among three colours of $\mathbb{C}(b) \cap \mathbb{C}(d) \subseteq \mathbb{C}(e)$ at least two appear in one of two “halves” of the column e (the “upper half” and the “lower half”); more precisely, there is $I \in \{[1, 3], [4, 6]\}$ such that the colours of $\mathbb{C}(b, d)$ occupy at least two positions in $I \times \{e\}$. Let ζ, η be distinct colours of $\mathbb{C}(b, d)$ occupying two positions in $I \times \{e\}$. Then the remaining copies of ζ, η occupy two positions in $([1, 6] \setminus I) \times \{f\}$, where $\{b, d\} = \{e, f\}$. Thus $|\mathbb{R}(\zeta) \cup \mathbb{R}(\eta)| = 4$, and so the colours ζ, η are column-related. The colours ζ and η correspond to e_1 and e_2 , respectively, with $e_1, e_2 \in \mathcal{M}^p$, $e_1 \neq e_2$. If $\mathcal{M}^p = \{e_1, e_2, e_3\}$, from Claim 23 we know that the label of the edge e_3 is either 1 or 2; let ϑ be a colour of C_2^p that corresponds to e_3 . The colours ζ, η, ϑ are pairwise column-related. Therefore, by Claim 30, the set $\{\zeta, \eta, \vartheta\}$ is of the type $3^1 1^3$. This, however, is contradicted by the following two facts: $|\{\zeta, \eta, \vartheta\} \cap \mathbb{C}(e)| = 3$ and $|\{\zeta, \eta, \vartheta\} \cap \mathbb{C}(f)| \geq |\{\zeta, \eta\}| = 2 > 1$.

Since $\text{wt}(\mathcal{M}^1) \in [5, 6]$ and Claim 28 yields $\sum_{s=1}^3 \text{wt}(\mathcal{M}^s) = c_2 \geq 15$, we have $\text{wt}(\mathcal{M}^{5-p}) = c_2 - \text{wt}(\mathcal{M}^1) - \text{wt}(\mathcal{M}^p) \geq 15 - 6 - 4 = 5$. By Claim 31.1 all colours of $C_2^{5-p} = \text{Col}(\mathcal{M}^{5-p})$ occur in one of columns of M , say in the column e . From $C_2^1 \cap C_2^{5-p} = \emptyset$ it follows that $e \neq a$. Further, note that $\mathbb{C}(b)$ contains $\varepsilon \in C_2^p$ as well as one of colours $\alpha_1, \alpha_2 \in C_2^1$, and the same is true for $\mathbb{C}(d)$; as a consequence of $|C_2^{5-p}| = \text{wt}(\mathcal{M}^{5-p}) \geq 5$ then $e \notin \{b, d\}$. Consider $e_1, e_2 \in \mathcal{M}^{5-p}$ with $i \in e_1$ and $l \in e_2$. Having in mind that $\{i, l\} \in \mathcal{M}^1$, we obtain $e_1 \neq e_2$. If $\mathcal{M}^{5-p} = \{e_1, e_2, e_3\}$, then the edge e_3 is labelled with either 1 or 2. Observe that there is a colour $\zeta \in C_2^{5-p}$ corresponding to e_3 , which does not belong to $\mathbb{C}(a)$. This is clear if $\text{wt}(\mathcal{M}^1) = 6$, when $\mathbb{C}(a) \cap \text{Col}(\mathcal{M}^{5-p}) = \text{Col}(\mathcal{M}^1) \cap \text{Col}(\mathcal{M}^{5-p}) = \emptyset$. On the other hand, with $\text{wt}(\mathcal{M}^1) = 5$ we have $\text{wt}(\mathcal{M}^{5-p}) = 6$, the edge e_3 is labelled with

2, and for the choice of ζ there are two possibilities, at least one of which satisfies the above requirement: $\mathbb{C}(a)$ contains at most one colour of $\text{Col}(\mathcal{M}^{5-p})$. Now it is clear that no pair $\{\zeta, \alpha_s\}$ with $s \in [1, 2]$ is row-based (because $\mathbb{R}(\alpha_1) = \mathbb{R}(\alpha_2) = \{i, l\}$ and $\mathbb{R}(\zeta) \cap \{i, l\} = \emptyset$). As a consequence both pairs $\{\zeta, \alpha_1\}, \{\zeta, \alpha_2\}$ are necessarily column-based. However, from $e \in \text{Cov}(\zeta)$, $e \notin \text{Cov}(\alpha_1, \alpha_2) = \{a, b, d\}$ and $a \notin \text{Cov}(\zeta)$ it follows that $\text{Cov}(\zeta) \cap \text{Cov}(\alpha_1, \alpha_2) = \text{Cov}(\zeta) \cap \{b, d\}$ and $|\text{Cov}(\zeta) \cap \{b, d\}| \leq 1$; since $|\mathbb{C}(b) \cap \{\alpha_1, \alpha_2\}| = 1 = |\mathbb{C}(d) \cap \{\alpha_1, \alpha_2\}|$, the number of pairs $\{\zeta, \alpha_1\}, \{\zeta, \alpha_2\}$, that are good in M , is at most 1, a contradiction.

Thus $\text{cov}(\alpha_1, \alpha_2) = 2$ and $\{\alpha_1, \alpha_2\} \rightarrow \mathbb{X}$.

The assumption $\text{cov}(\beta_1, \beta_2) = 3$ leads to a contradiction similarly as above, hence $\text{cov}(\beta_1, \beta_2) = 2$ and $\{\beta_1, \beta_2\} \rightarrow \mathbb{X}$.

5. Use Claim 31.4 with $\{\gamma_1, \gamma_2\}$ in the role of $\{\alpha_1, \alpha_2\}$.

6. This claim is a consequence of Claim 31.2, Claim 31.4 and Claim 31.5, where the last one applies only if $\mathbb{R}(k, n) = \{\gamma_1, \gamma_2\}$.

7. If $\delta \in C_2 \setminus C_2^1$ occupies only (two) positions in $[1, 6] \times ([1, q] \setminus \text{Cov}(C_2^1))$, then the number of pairs $\{\delta, \varepsilon\}$ with $\varepsilon \in \{\alpha_1, \beta_1, \gamma_1\}$, that are good in M (and necessarily row-based), is $2 < 3 = |\{\alpha_1, \beta_1, \gamma_1\}|$, a contradiction. \square

Let $C_3^* = C_3 \setminus (\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6))$, $c_3^* = |C_3^*|$ and $\rho_3 = r(1, 2, 3) + r(4, 5, 6)$ so that $c_3 = \rho_3 + c_3^*$ and $\rho_3 \leq 18$ (see Claim 27).

Claim 32. *If $\rho_3 \geq 15$, then $\text{cov}(\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6)) \leq 9$.*

Proof. Suppose $r(1, 2, 3) \geq r(4, 5, 6)$ and observe that if $j \in \text{Cov}(\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6))$, then $\mathbb{R}(1, 2, 3) \cap \mathbb{C}(j) \neq \emptyset$ and $\mathbb{R}(4, 5, 6) \cap \mathbb{C}(j) \neq \emptyset$ as well. Indeed, if $\delta \in \mathbb{R}(1, 2, 3) \cap \mathbb{C}(j)$ and $\mathbb{R}(4, 5, 6) \cap \mathbb{C}(j) = \emptyset$, then the number of pairs $\{\delta, \varepsilon\}$ with $\varepsilon \in \mathbb{R}(4, 5, 6)$, that are good in M (they must be column-based), is at most $3|\text{Cov}(\delta) \setminus \{j\}| = 6 < 7 \leq q - 9 \leq r(4, 5, 6)$ (Claim 27), a contradiction. A similar contradiction can be reached under the assumption $\mathbb{R}(4, 5, 6) \cap \mathbb{C}(j) \neq \emptyset$ and $\mathbb{R}(1, 2, 3) \cap \mathbb{C}(j) = \emptyset$. So, $\text{cov}(\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6)) = \text{cov}(\mathbb{R}(1, 2, 3)) = \text{cov}(\mathbb{R}(4, 5, 6))$.

Claim 27 yields $r(1, 2, 3) \leq 9$. Suppose first that $r(1, 2, 3) = 9$. In the case $\text{cov}(\mathbb{R}(1, 2, 3)) \leq 9$ the claim is proved. If $\text{cov}(\mathbb{R}(1, 2, 3)) \geq 10$, there is $j \in \text{Cov}(\mathbb{R}(1, 2, 3))$ with $|\mathbb{R}(1, 2, 3) \cap \mathbb{C}(j)| \leq 2$. In such a case for $\varepsilon \in \mathbb{R}(4, 5, 6) \cap \mathbb{C}(j)$ the number of pairs $\{\varepsilon, \delta\}$ with $\delta \in \mathbb{R}(1, 2, 3)$, that are good in M , is at most $2 + 3 + 3 < r(1, 2, 3)$, a contradiction.

Thus $9 > r(1, 2, 3) \geq \frac{1}{2}[r(1, 2, 3) + r(4, 5, 6)] = \frac{\rho_3}{2} \geq \frac{15}{2} > 7$, $r(1, 2, 3) = 8$ and $8 \geq r(4, 5, 6) \geq 7$. If $\text{cov}(\mathbb{R}(1, 2, 3)) \geq 10$ and $j \in \text{Cov}(\mathbb{R}(1, 2, 3))$, then necessarily $|\mathbb{R}(1, 2, 3) \cap \mathbb{C}(j)| \geq 2$, for otherwise $r(1, 2, 3) \leq 1 + 2 \cdot 3 = 7 < r(1, 2, 3)$. Consequently, $m = |\{j \in [1, q] : |\mathbb{R}(1, 2, 3) \cap \mathbb{C}(j)| = 3\}| \leq 4$, because $m \geq 5$ implies that the number of positions in M occupied by colours of $\mathbb{R}(1, 2, 3)$ is $3m + 2[\text{cov}(\mathbb{R}(1, 2, 3)) - m] = 2\text{cov}(\mathbb{R}(1, 2, 3)) + m \geq 20 + 5 = 25 > 24 = 3r(1, 2, 3)$,

a contradiction. For each colour $\zeta \in \mathbb{R}(4, 5, 6)$ we have $n(\zeta) = |\{j \in [1, q] : |\mathbb{R}(1, 2, 3) \cap \mathbb{C}(j)| = 3, \zeta \in \mathbb{C}(j)\}| \geq 2$, since $n(\zeta) \leq 1$ leads to $r(1, 2, 3) \leq 3n(\zeta) + 2[3 - n(\zeta)] = 6 + n(\zeta) \leq 7 < 8 = r(1, 2, 3)$. Then the number of colours $\zeta \in \mathbb{R}(4, 5, 6)$, for which all pairs $\{\zeta, \delta\}$ with $\delta \in \mathbb{R}(1, 2, 3)$ are good in M , is at most $\lfloor \frac{3m}{2} \rfloor \leq \lfloor \frac{3 \cdot 4}{2} \rfloor = 6 < 7 \leq r(4, 5, 6)$; in other words, there is $\zeta \in \mathbb{R}(4, 5, 6)$ and $\delta \in \mathbb{R}(1, 2, 3)$ such that the pair $\{\zeta, \delta\}$ is not good in M , a contradiction.

The case $r(1, 2, 3) < r(4, 5, 6)$ can be treated analogously. \square

Claim 33. *No perfect matching of $K_{3,3}$ is of weight 6.*

Proof. Suppose that $\text{wt}(\mathcal{M}^1) = 6$, where (w) $\mathcal{M}^1 = \{\{i, 7-i\} : i = 1, 2, 3\}$, and let $\alpha \in \mathbb{R}(1, 6)$, $\beta \in \mathbb{R}(2, 5)$, $\gamma \in \mathbb{R}(3, 4)$. By Claim 28 there exists a partition $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ of $E(K_{3,3})$ into perfect matchings in $K_{3,3}$, $\bigcup_{s=1}^3 \text{Col}(\mathcal{M}^s) = C_2$, $\sum_{s=1}^3 \text{wt}(\mathcal{M}^s) = c_2 \geq 15$ and $\text{wt}(\mathcal{M}^2) \geq \text{wt}(\mathcal{M}^3)$, hence $\text{wt}(\mathcal{M}^2) + \text{wt}(\mathcal{M}^3) = c_2 - 6 \geq 9$ and $\text{wt}(\mathcal{M}^2) \geq 5$. Let $C_2^s = \text{Col}(\mathcal{M}^s)$, $s = 1, 2, 3$. Using Claims 31.1 and 31.4–6 we get (w) $C_2^1 = \mathbb{C}(1)$, $\text{Cov}(C_2^1) = [1, 4]$ and $C_2^2 \subseteq \mathbb{C}(5)$ (each of columns 2, 3, 4 contains two colours of C_2^1). Further, as a consequence of Claim 31.7, and the fact that $C_2^1 = \mathbb{C}(1)$, any colour of C_2 occupies a position in $[1, 6] \times (\text{Cov}(C_2^1) \setminus \{1\}) = [1, 6] \times [2, 4]$.

Now consider an arbitrary colour $\varepsilon \in C_3^*$. There are three distinct integers $a, b, d \in [1, 6]$ and a set $I \in \{[1, 3], [4, 6]\}$ such that $a, b \in I$, $d \in [1, 6] \setminus I$ and $\varepsilon \in \mathbb{R}(a, b, d)$. Let us show that the colour ε occupies a position in $[1, 6] \times [2, 5]$. Suppose this is not true so that every pair $\{\varepsilon, \zeta\}$ with $\zeta \in C_2^1 \cup C_2^2$ is row-based. Then $I = \{a, b, 7-d\}$; otherwise, if $7-d \in \{a, b\}$, then $\{d, 7-d\} \subseteq \{a, 7-a\} \cup \{b, 7-b\}$, there is $\zeta \in \{\alpha, \beta, \gamma\}$ with $\mathbb{R}(\zeta) \cap (\{a, 7-a\} \cup \{b, 7-b\}) = \emptyset$, which leads to $\mathbb{R}(\varepsilon) \cap \mathbb{R}(\zeta) = \{a, b, d\} \cap \mathbb{R}(\zeta) \subseteq (\{a, 7-a\} \cup \{b, 7-b\}) \cap \mathbb{R}(\zeta) = \emptyset$, and the pair $\{\varepsilon, \zeta\}$ is not good in M , a contradiction. Thus $I = \{a, b, 7-d\}$ and $[1, 6] \setminus I = \{d, x, y\}$. The set $\{\{7-d, d\}, \{7-d, x\}, \{7-d, y\}\}$ is a transversal of the collection $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ of pairwise disjoint perfect matchings of $K_{3,3}$, hence there is $z \in \{x, y\}$ with $\{7-d, z\} \in \mathcal{M}^2$ and $\mathbb{R}(7-d, z) \subseteq C_2^2$ (recall that $\{7-d, d\} \in \mathcal{M}^1$). From $\text{wt}(\mathcal{M}^2) \geq 5$ we know, by Claim 23, that $r(7-d, z) \geq 1$; with $\eta \in \mathbb{R}(7-d, z)$ then $\mathbb{R}(\varepsilon) \cap \mathbb{R}(\eta) = \{a, b, d\} \cap \{7-d, z\} = \emptyset$, hence the pair $\{\eta, \varepsilon\}$ is not good in M , a contradiction.

From the above reasoning we see that each colour of $(C_2 \setminus C_2^2) \cup C_3^*$ occupies a position in $[1, 6] \times [2, 5]$, and each colour of C_2^2 occupies two positions in $[1, 6] \times [2, 5]$. Therefore, $c_2 + c_3^* + |C_2^2| \leq |[1, 6] \times [2, 5]| = 6 \cdot 4 = 24$ and $c_2 + c_3^* \leq 24 - |C_2^2| = 24 - \text{wt}(\mathcal{M}^2) \leq 24 - 5 = 19$.

Thus, by Lemma 11.8, $c_3^* + c_{4+} \leq c_3^* + (c_2 - 15) \leq 19 - 15 = 4$. Claim 24 yields $c_2 = \frac{1}{2} \sum_{i=1}^6 r_2(i) \leq \frac{6 \cdot 6}{2} = 18$, hence $2q + 5 = |C| = (c_2 + c_3^*) + \rho_3 + c_{4+} \leq 19 + \rho_3 + (c_2 - 15) \leq 19 + \rho_3 + 3 = 22 + \rho_3$, and $\rho_3 \geq (2q + 5) - 22 \geq 15$. Consequently, using Claim 32, $\text{cov}(\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6)) \leq 9$, (w) $\text{Cov}(\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6)) \subseteq [q - 8, q]$.

If $\text{wt}(\mathcal{M}^3) \geq 5$, then, by Claim 31.1 applied on C_2^3 , (w) $C_2^3 \subseteq \mathbb{C}(6)$, $\text{Cov}(C_2^3) = \{2, 3, 4, 6\}$, $\text{Cov}(C_2) = [1, 6]$, and, for any $j \in [7, q-9] \supseteq \{7\}$, $\mathbb{C}(j) \subseteq C_3^* \cup C_{4+}$, so that $6 = |\mathbb{C}(j)| \leq c_3^* + c_{4+} \leq 4$, a contradiction.

So, $\text{wt}(\mathcal{M}^3) \leq 4$, $c_2 = \sum_{s=1}^3 \text{wt}(\mathcal{M}^s) \leq 6 + 6 + 4 = 16$, $37 \leq 2q + 5 = c_2 + \rho_3 + c_3^* + c_{4+} \leq 16 + 18 + (c_3^* + c_{4+}) \leq 34 + 4 = 38$ and $q = 16$. Then $37 = |C| = c_2 + c_3^* + \rho_3 + c_{4+} \leq c_2 + c_3^* + \rho_3 + (c_2 - 15) = c_2 + c_3^* + \sum_{s=1}^3 \text{wt}(\mathcal{M}^s) + \rho_3 - 15 = [c_2 + c_3^* + \text{wt}(\mathcal{M}^2)] + \text{wt}(\mathcal{M}^3) + 6 + \rho_3 - 15 \leq 24 + 4 + \rho_3 - 9 \leq 28 + (18 - 9) = 37$, $\rho_3 = 18$, $\text{cov}(\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6)) \geq \lceil \frac{18 \cdot 3}{6} \rceil = 9$, $\text{cov}(\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6)) = 9$, $\text{Cov}(\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6)) = [8, 16]$ and $\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6) = \bigcup_{j=8}^{16} \mathbb{C}(j)$. Let $C' = C \setminus (\mathbb{R}(1, 2, 3) \cup \mathbb{R}(4, 5, 6))$, and let M' be the 6×7 submatrix of M formed by the first seven columns of M . Evidently, $M' \in \mathcal{M}(6, 7, C')$, which means, by Proposition 2, that $\text{achr}(K_6 \square K_7) \geq |C'| = 19$. This, however, contradicts the result $\text{achr}(K_6 \square K_7) = 18$ proved in [5]. \square

Claim 34. *The following statements are true.*

1. $c_2 = 15$.
2. Each perfect matching of $K_{3,3}$ is of weight 5.
3. $c_{4+} = 0$.
4. Each edge of $K_{3,3}$ is labelled with either 1 or 2.
5. There are $I, K \in \{[1, 3], [4, 6]\}$, $I \neq K$, and $k \in K$ such that for any $i \in I$ and any $l \in K \setminus \{k\}$ it holds $r(i, k) = 1$ and $r(i, l) = 2$.

Proof. 1. Given a perfect matching \mathcal{M}^1 of $K_{3,3}$, by Claim 28 we know that there is a unique partition $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ of $E(K_{3,3})$ into perfect matchings of $K_{3,3}$. By Lemma 11.3 and Claim 33 then $15 \leq c_2 = \sum_{s=1}^3 \text{wt}(\mathcal{M}^s) \leq \sum_{s=1}^3 5 = 15$ and $c_2 = 15$.

2. From the proof of Claim 34.1 we see that $\text{wt}(\mathcal{M}^s) = 5$, $s = 1, 2, 3$. Thus $\text{wt}(\mathcal{M}) = 5$ for each perfect matching \mathcal{M} of $K_{3,3}$ (\mathcal{M} can be chosen as \mathcal{M}^1).

3. By Lemma 11.8 and Claim 34.1 we have $c_{4+} \leq c_2 - 15 = 0$ and $c_{4+} = 0$.

4. No edge of $K_{3,3}$ is labelled with 0, otherwise any perfect matching of $K_{3,3}$ containing such an edge would be of weight at most $2 \cdot 2 = 4$ (Claim 23), which contradicts Claim 34.2.

5. Denote by $l(e)$ the label of an edge $e \in E(K_{3,3})$, and by l_n the number of edges of $K_{3,3}$ labelled with n , $n = 1, 2$ (see Claim 34.4); then $l_1 + l_2 = 9$, $15 = c_2 = l_1 + 2l_2 = 9 + l_2$, $l_2 = 6$ and $l_1 = 3$. Let $\{e_1, e_2, e_3\} = \{e \in E(K_{3,3}) : l(e) = 1\}$.

If $a, b \in [1, 3]$, $a \neq b$, then $e_a \cap e_b \neq \emptyset$. To see it suppose that $e_a \cap e_b = \emptyset$, and take $e \in E(K_{3,3}) \setminus \{e_a, e_b\}$ such that $\{e_a, e_b, e\}$ is a perfect matching of $K_{3,3}$. The 6-vertex cycle in $K_{3,3}$ with the edge set $E(K_{3,3}) \setminus \{e_a, e_b, e\}$ has at least five edges labelled with 2, hence one can find in $K_{3,3}$ a perfect matching $\mathcal{M} \subseteq E(K_{3,3}) \setminus \{e_a, e_b, e\}$ with $\text{wt}(\mathcal{M}) = 3 \cdot 2 = 6$, a contradiction.

Thus $e_1 \cap e_2 \neq \emptyset$, $e_1 \cap e_3 \neq \emptyset$ and $e_2 \cap e_3 \neq \emptyset$. Since the subgraph of the bipartite graph $K_{3,3}$ induced by the set of edges $\{e_1, e_2, e_3\}$ is bipartite (and so free of odd cycles), the above three intersections (of 2-element sets) are nonempty only if there is a vertex $k \in [1, 6] = V(K_{3,3})$ such that $e_1 \cap e_2 \cap e_3 = \{k\}$. Having in mind that the bipartition of $K_{3,3}$ is $\{[1, 3], [4, 6]\}$, there are $I, K \in \{[1, 3], [4, 6]\}$ such that $I \neq K$, $k \in K$, and for any $i \in I$ and any $l \in K \setminus \{k\}$ it holds $r(i, k) = 1$ and $r(i, l) = 2$. \square

Based on Claim 34.5 we suppose (w) $I = [1, 3]$, $K = [4, 6]$ and $k = 6$ so that for any $i \in [1, 3]$ and any $l \in [4, 5]$ we have $r(i, 6) = 1$ and $r(i, l) = 2$. Then $r_2(1) = r_2(2) = r_2(3) = 5$, $r_2(4) = r_2(5) = 6$ and $r_2(6) = 3$. Let $\mathbb{R}(i, 6) = \{\alpha_{i,6}\}$, $i = 1, 2, 3$.

If \mathcal{M} is a perfect matching in $K_{3,3}$, there is $s \in [1, 6]$ such that $\mathcal{M} = \mathcal{M}^s$, where

$$\begin{aligned}\mathcal{M}^1 &= \{\{1, 6\}, \{2, 4\}, \{3, 5\}\}, & \mathcal{M}^2 &= \{\{1, 5\}, \{2, 6\}, \{3, 4\}\}, \\ \mathcal{M}^3 &= \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}, & \mathcal{M}^4 &= \{\{1, 6\}, \{2, 5\}, \{3, 4\}\}, \\ \mathcal{M}^5 &= \{\{1, 4\}, \{2, 6\}, \{3, 5\}\}, & \mathcal{M}^6 &= \{\{1, 5\}, \{2, 4\}, \{3, 6\}\}.\end{aligned}$$

We have $|\text{Col}(\mathcal{M}^s)| = 5$ for each $s \in [1, 6]$. Applying Claim 31 on five colours of $\text{Col}(\mathcal{M}^s)$ we see that there is $a^s \in [1, q]$ such that $\text{Col}(\mathcal{M}^s) \subseteq \mathbb{C}(a^s)$. If $s, t \in [1, 6]$, $s \neq t$, then $|\mathcal{M}^s \cap \mathcal{M}^t| \leq 1$, hence $|\text{Col}(\mathcal{M}^s) \cap \text{Col}(\mathcal{M}^t)| \leq 2$ (Claim 23), and so it is clear that $a^s \neq a^t$. From now on (w)

$$\{a^s : s \in [1, 6]\} = [1, 6],$$

$$(M)_{i,i} = \alpha_{i,6} = (M)_{6,i+3}, \quad i = 1, 2, 3.$$

Let us show that $\beta_i = (M)_{i,i+3}$ with $i \in [1, 3]$ is not a 2-colour. Indeed, suppose it is. The number of pairs $\{\beta_i, \alpha_{j,6}\}$, $j \in [1, 3]$, that are good in M , is 3. However, each copy of β_i provides only one such pair (for the copy $(M)_{i,i+3}$ of β_i it is the pair $\{\beta_i, \alpha_{i,6}\}$, while for the other copy of β_i in one of rows 4, 5 of M it is a pair $\{\beta_i, \alpha_{j,6}\}$ that is column-based), a contradiction. As a consequence of Claims 34.1 and 34.3 then all positions in the set

$$S = \{(1, 4), (2, 5), (3, 6), (6, 1), (6, 2), (6, 3)\}$$

are occupied by 3-colours, and the same is true for the set of positions $[1, 6] \times [7, q]$. Moreover, all positions in the set $([1, 6] \times [1, 6]) \setminus S$ are occupied by 2-colours.

Claim 35. *Each position in the set S is occupied by a colour of C_3^* .*

Proof. If a position $(i, i+3)$ with $i \in [1, 3]$ is occupied by a colour $\beta \in \mathbb{R}(1, 2, 3)$, that copy of β provides no pair $\{\beta, \gamma\}$ with $\gamma \in \mathbb{R}(4, 5, 6)$ that is good in M .

Claim 27 yields $\min(r(1, 2, 3), r(4, 5, 6)) \geq q - 9 \geq 7$. However, the number of pairs $\{\beta, \gamma\}$ with $\gamma \in \mathbb{R}(4, 5, 6)$, that are good in M (and necessarily column-based), is at most $\sum_{l \in \text{Cov}(\beta) \setminus \{i+3\}} |\mathbb{R}(4, 5, 6) \cap \mathbb{C}(l)| \leq 2 \cdot 3 = 6 < r(4, 5, 6)$, a contradiction.

Similarly, if a position $(6, j)$ with $j \in [1, 3]$ is occupied by a colour $\delta \in \mathbb{R}(4, 5, 6)$, then the number of pairs $\{\delta, \varepsilon\}$ with $\varepsilon \in \mathbb{R}(1, 2, 3)$, that are good in M , is at most $\sum_{l \in \text{Cov}(\delta) \setminus \{j\}} |\mathbb{R}(1, 2, 3) \cap \mathbb{C}(l)| \leq 2 \cdot 3 = 6 < r(1, 2, 3)$, a contradiction. \square

Claim 36. $C_3^* \subseteq \mathbb{R}(6)$.

Proof. Consider a colour $\beta \in C_3^*$, and let n_i , $i \in \mathbb{R}(\beta)$, denote the number of pairs $\{\beta, \gamma\}$ with $\gamma \in \{\alpha_{1,6}, \alpha_{2,6}, \alpha_{3,6}\}$, that are good in M , and are provided by the copy of β in the row i of M . If $i \in [1, 3]$, then $n_i = 1$ (with $\gamma = \alpha_{i,6}$), while $i \in [4, 5]$ implies $n_i = 0$, and $i = 6$ yields $n_i = n_6 = 3$. Now, under the assumption $\beta \notin \mathbb{R}(6)$, from the inequalities $1 \leq |\mathbb{R}(\beta) \cap [1, 3]| \leq 2$ we obtain $\sum_{i \in \mathbb{R}(\beta)} n_i = |\mathbb{R}(\beta) \cap [1, 3]| \leq 2 < 3 = |\{\alpha_{1,6}, \alpha_{2,6}, \alpha_{3,6}\}|$, a contradiction. \square

Claim 37. $q = 16$, and there is a 3-colour $\beta \in \mathbb{R}(i, m, 6)$ with $i \in [1, 3]$ and $m \in [4, 5]$ that occupies a position in $\{6\} \times [7, 16]$.

Proof. Since $c_{4+} = 0$ (Claim 34.3) and $r_2(6) = 3$, by Claims 36 and 27 we have $q = c_2(6) + c_3(6) = 3 + [r(4, 5, 6) + c_3^*]$ and $c_3^* = q - 3 - r(4, 5, 6) \geq q - 3 - (q - 9) = 6$. On the other hand, from Claim 34.1 we get $|C| = 2q + 5 = c_2 + c_3 = 15 + [r(1, 2, 3) + r(4, 5, 6) + c_3^*]$ so that $q - 3 = r(4, 5, 6) + c_3^* = 2q - 10 - r(1, 2, 3)$, $r(1, 2, 3) = q - 7$, and then Claim 27 yields $9 \geq r(1, 2, 3) = q - 7 \geq 16 - 7 = 9$, $r(1, 2, 3) = 9$ and $q = 16$. Using Claim 27 again we obtain $7 = q - 9 \leq r(4, 5, 6) = q - 3 - c_3^* = 13 - c_3^* \leq 13 - 6 = 7$, $r(4, 5, 6) = 7$ and $c_3^* = 6$.

The number of positions in $[1, 3] \times [1, 16]$, that are occupied by colours of C_3^* , is equal to $3 \cdot 16 - c_2 - 3r(1, 2, 3) = 48 - 15 - 27 = 6 = c_3^*$, and each colour $\gamma \in C_3^*$ is involved in that counting, since $1 \leq |\mathbb{R}(\gamma) \cap [1, 3]| \leq 2$. Therefore, for any $\gamma \in C_3^*$ we get $|\mathbb{R}(\gamma) \cap [1, 3]| = 1$ and $|\mathbb{R}(\gamma) \cap [4, 6]| = 2$.

Let $\beta \in C_3^*$ occupy a position in $\{6\} \times [7, 16]$; the number of such colours is $c_3^* - 3 = 3$, because $C_3^* \subseteq \mathbb{R}(6)$ (Claim 36), the positions in $\{6\} \times [1, 3]$ are occupied by colours of C_3^* (Claim 35) and $(M)_{6,l} = \alpha_{l-3,6} \in C_2$, $l = 4, 5, 6$. Then $\mathbb{R}(\beta) = \{i, m, 6\}$, where $i \in [1, 3]$ and $m \in [4, 5]$. \square

We are now ready to finish our analysis by showing that for a colour $\beta \in \mathbb{R}(i, m, 6)$ of Claim 37 the number of pairs $\{\beta, \gamma\}$ with $\gamma \in C_2$, that are good in M , is less than $c_2 = 15$, which represents a final contradiction proving Theorem 18.

First of all, if β occupies a position in $\{i\} \times [7, 16]$, then all pairs $\{\beta, \gamma\}$ with $\gamma \in C_2$, that are good in M , are row-based. The number of such pairs is $r_2(i) + r_2(m) + r_2(6) - [r(i, m) + r(i, 6)] = 5 + 6 + 3 - (2 + 1) = 11 < 15 = c_2$, a contradiction.

Therefore, $\beta = (M)_{i,i+3}$, and we can find explicitly a colour $\gamma \in C_2$ such that the pair $\{\beta, \gamma\}$ is not good in M . Indeed, in this case $\mathbb{C}(i+3) \cap C_2 = \text{Col}(\mathcal{M}) \supseteq \{\alpha_{i,6}\}$, where the perfect matching \mathcal{M} in $K_{3,3}$ satisfies $\mathcal{M} = \{\{i, 6\}, \{j, m\}, \{k, n\}\}$, $\{i, j, k\} = \{1, 2, 3\}$, $m \in [4, 5]$ and $n = 9 - m$. Then $\mathbb{C}(i+3) = \{\beta\} \cup \mathbb{R}(i, 6) \cup \mathbb{R}(j, m) \cup \mathbb{R}(k, n)$, and so $\mathbb{R}(j, n) \cap \mathbb{C}(i+3) = \emptyset$ (recall that $r(i, 6) = 1$ and $r(j, m) = 2 = r(k, n)$); thus, the pair $\{\beta, \gamma\}$ with $\gamma \in \mathbb{R}(j, n)$ is not column-based. Moreover, $\mathbb{R}(\beta) \cap \mathbb{R}(\gamma) = \{i, m, 6\} \cap \{j, n\} = \emptyset$, the pair $\{\beta, \gamma\}$ is not row-based, hence it is not good in M , a contradiction. ■

The solution of the problem of determining $\text{achr}(K_6 \square K_q)$ is now complete. It is summarised in the final theorem of the paper, where

$$\begin{aligned} J_3 &= [2, 3] \cup \{q \in [41, \infty) : q \equiv 1 \pmod{2}\}, \\ J_4 &= \{1, 4, 7\} \cup [16, 40] \cup \{q \in [42, \infty) : q \equiv 0 \pmod{2}\}, \\ J_5 &= \{5, 8\}, \\ J_6 &= \{6\} \cup [9, 15], \end{aligned}$$

and $J_3 \cup J_4 \cup J_5 \cup J_6 = [1, \infty)$.

Theorem 38. *If $a \in [3, 6]$ and $q \in J_a$, then $\text{achr}(K_6 \square K_q) = 2q + a$.*

Proof. The achromatic number of $K_6 \square K_q$ was analysed in [7] for $q \leq 4$ (for $q \leq 3$ see also Chiang and Fu [2]), in Horňák and Pčola [6] for $q = 5$, in [1] for $q = 6$, in [5] for $q = 7$, and in [4] for $q \in [41, \infty)$ with $q \equiv 1 \pmod{2}$. The remaining statements have been proved in the present paper, see Theorem 13 for $q = 8$, Theorem 17 for $q \in [9, 15]$, and Theorem 18 for q satisfying either $q \in [16, 40]$ or $q \in [42, \infty)$ together with $q \equiv 0 \pmod{2}$. ■

Corollary 39. *If $q \in [1, \infty)$, then $2q + 3 \leq \text{achr}(K_6 \square K_q) \leq 2q + 6$.*

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