Discussiones Mathematicae
Graph Theory 44 (2024) 411–433
https://doi.org/10.7151/dmgt.2451

THE ACHROMATIC NUMBER OF THE CARTESIAN PRODUCT OF K_6 AND K_q

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Abstract

Let G be a graph and C a finite set of colours. A vertex colouring $f : V(G) \to C$ is complete if for any pair of distinct colours $c_1, c_2 \in C$ one can find an edge $\{v_1, v_2\} \in E(G)$ such that $f(v_i) = c_i$, $i = 1, 2$. The achromatic number of G is defined to be the maximum number $\mathrm{achr}(G)$ of colours in a proper complete vertex colouring of G. In the paper $\mathrm{achr}(K_6 \square K_q)$ is determined for any integer q such that either $8 \leq q \leq 40$ or $q \geq 42$ is even.

Keywords: complete vertex colouring, achromatic number, Cartesian product, complete graph.

2020 Mathematics Subject Classification: 05C15.

1. Introduction

Let G be a finite simple graph and C a finite set of colours. A vertex colouring $f : V(G) \to C$ is *complete* provided that for any pair $\{c_1, c_2\} \in \binom{C}{2}$ (of distinct colours of C) there exists an edge $\{v_1, v_2\}$ (usually shortened to v_1v_2) of G such that $f(v_i) = c_i$, $i = 1, 2$. The *achromatic number* of G, in symbols $\mathrm{achr}(G)$, is the maximum cardinality of the colour set in a proper complete vertex colouring of G. The achromatic number was introduced in Harary, Hedetniemi, and Prins [3], where among other things the following interpolation result was proved.

Theorem 1. *If G is a graph, and an integer k satisfies $\chi(G) \leq k \leq \mathrm{achr}(G)$, there exists a proper complete vertex colouring of G using k colours.*

In the present paper the achromatic number of $K_6 \square K_q$, the Cartesian product of K_6 and K_q (the notation following Imrich and Klavžar [8] is adopted), is determined for all q satisfying either $8 \leq q \leq 40$ or $q \geq 42$ and $q \equiv 0 \pmod 2$.

This is the third in a series of three papers, in which the problem of finding $\mathrm{achr}(K_6\Box K_q)$ is completely solved. Some historical remarks concerning the achromatic number, a motivation of the problem and basic facts on proper complete colourings of Cartesian products of two complete graphs are available in the paper Horňák [4], where $\mathrm{achr}(K_6\Box K_q)$ has been determined for odd $q \geq 41$. Maybe a bit surprisingly, proving that $\mathrm{achr}(K_6\Box K_7) = 18$ has required quite a long analysis contained in the paper Horňák [5].

For $m, n \in \mathbb{Z}$ we work with *integer intervals* defined by

$$[m, n] = \{z \in \mathbb{Z} : m \leq z \leq n\}, \qquad [m, \infty) = \{z \in \mathbb{Z} : m \leq z\}.$$

If $p, q \in [1, \infty)$ and $V(K_r) = [1, r]$, $r = p, q$, then $V(K_p\Box K_q) = [1, p] \times [1, q]$, and $E(K_p\Box K_q)$ consists of edges $(i_1, j_1)(i_2, j_2)$, where $i_1, i_2 \in [1, p]$ and $j_1, j_2 \in [1, q]$ satisfy either $i_1 = i_2$ and $j_1 \neq j_2$ or $i_1 \neq i_2$ and $j_1 = j_2$.

Let $\mathcal{M}(p, q, C)$ denote the set of $p \times q$ matrices M with entries from C such that all lines (rows and columns) of M have pairwise distinct entries, and any pair $\{\alpha, \beta\} \in \binom{C}{2}$ is *good in* M, which means that there is a line of M containing both α and β; the pair $\{\alpha, \beta\}$ is either *row-based* or *column-based* (in M) depending on whether the involved line is a row or a column. In other words, the number of lines witnessing the fact that the pair $\{\alpha, \beta\}$ is good, is positive, and it may happen that the pair $\{\alpha, \beta\}$ is simultaneously row-based and column-based as well. For a matrix M we denote by $(M)_{i,j}$ the entry of M appearing in the ith row and the jth column.

Proposition 2 [4]**.** *If $p, q \in [1, \infty)$ and C is a finite set, then the following statements are equivalent.*

(1) *There is a proper complete vertex colouring of $K_p\Box K_q$ using as colours elements of C.*

(2) *$\mathcal{M}(p, q, C) \neq \emptyset$.*

The implication (2) $\Rightarrow$ (1) of Proposition 2 is based on a straightforward observation that if $M \in \mathcal{M}(p, q, C)$, then the vertex colouring f_M of $K_p\Box K_q$ defined by $f_M(i, j) = (M)_{i,j}$ is proper and complete as well.

Proposition 3 [4]**.** *If $p, q \in [1, \infty)$, C, D are finite sets, $M \in \mathcal{M}(p, q, C)$, mappings $\rho : [1, p] \to [1, p]$, $\sigma : [1, q] \to [1, q]$, $\pi : C \to D$ are bijections, and $M_{\rho,\sigma}$, M_π are $p \times q$ matrices defined by $(M_{\rho,\sigma})_{i,j} = (M)_{\rho(i),\sigma(j)}$ and $(M_\pi)_{i,j} = \pi((M)_{i,j})$, then $M_{\rho,\sigma} \in \mathcal{M}(p, q, C)$ and $M_\pi \in \mathcal{M}(p, q, D)$.*

Let $M \in \mathcal{M}(p, q, C)$. The *frequency* of a colour $\gamma \in C$ is the number $\mathrm{frq}(\gamma)$ of times γ appears in M, while $\mathrm{frq}(M)$, the *frequency* of M, is the minimum of frequencies of colours in C. A colour of frequency l is an *l-colour*, C_l is the set of l-colours and $c_l = |C_l|$. C_{l+} is the set of colours of frequency at least

l and $c_{l+} = |C_{l+}|$. For the (complete) colouring f_M mentioned above denote $V_\gamma = f_M^{-1}(\gamma) \subseteq [1,p] \times [1,q]$, and let $N(V_\gamma)$ be the neighbourhood of V_γ (the union of neighborhoods of vertices in V_γ). The *excess* of γ is defined to be the maximum number $\mathrm{exc}(\gamma)$ of vertices in a set $S \subseteq N(V_\gamma)$ such that the restriction of f_M formed by uncolouring the vertices of S is still complete with respect to pairs of colours containing γ. The *excess* of M is the minimum $\mathrm{exc}(M)$ of excesses of colours in C.

We denote by $\mathbb{R}(i)$ the set of colours in the ith row of M and by $\mathbb{C}(j)$ the set of colours in the jth column of M. Further, let

$$\mathbb{R}_l(i) = C_l \cap \mathbb{R}(i), \qquad r_l(i) = |\mathbb{R}_l(i)|,$$
$$\mathbb{C}_l(j) = C_l \cap \mathbb{C}(j), \qquad c_l(j) = |\mathbb{C}_l(j)|,$$

so that $\mathbb{R}_l(i)$ and $\mathbb{C}_l(j)$ is the set of l-colours appearing in the row i and those appearing in the column j, respectively. For $i,j,k \in [1,p]$ let

$$\mathbb{R}(i,j) = C_2 \cap \mathbb{R}(i) \cap \mathbb{R}(j), \qquad r(i,j) = |\mathbb{R}(i,j)|,$$
$$\mathbb{R}(i,j,k) = C_3 \cap \mathbb{R}(i) \cap \mathbb{R}(j) \cap \mathbb{R}(k), \qquad r(i,j,k) = |\mathbb{R}(i,j,k)|,$$

and for $m,n \in [1,q]$ let

$$\mathbb{C}(m,n) = C_2 \cap \mathbb{C}(m) \cap \mathbb{C}(n), \qquad c(m,n) = |\mathbb{C}(m,n)|.$$

Thus $\mathbb{R}(i,j)$ and $\mathbb{C}(m,n)$ are respectively the sets of colours which appear exactly in rows i and j or columns m and n, while $\mathbb{R}(i,j,k)$ is the set of colours which appear exactly in the rows i,j and k. For $\gamma \in C$ define

$$\mathbb{R}(\gamma) = \{i \in [1,p] : \gamma \in \mathbb{R}(i)\}$$

as the set of numbers of rows containing the colour γ. To avoid a possible confusion coming from the double usage of $\mathbb{R}(\cdot)$ in $\mathbb{R}(\gamma)$ and $\mathbb{R}(i)$ note that whenever $\mathbb{R}(\cdot)$ is used with a Greek alphabet letter argument, then the argument points to a colour in C, and not to a row of the matrix M.

If $S \subseteq [1,p] \times [1,q]$, we say that a colour $\gamma \in C$ *occupies a position in* S (*appears in* S or simply *is in* S) if there exists $(i,j) \in S$ with $(M)_{i,j} = \gamma$. For a nonempty set of colours $A \subseteq C$, the *set of columns covered by* A is

$$\mathrm{Cov}(A) = \{j \in [1,q] : \mathbb{C}(j) \cap A \neq \emptyset\},$$

i.e., the set of (numbers of) columns containing a colour of A. We put $\mathrm{cov}(A) = |\mathrm{Cov}(A)|$, and for $A = \{\alpha\}, \{\alpha,\beta\}, \{\alpha,\beta,\gamma\}$ we use a simplified notation $\mathrm{Cov}(\alpha)$, $\mathrm{Cov}(\alpha,\beta)$, $\mathrm{Cov}(\alpha,\beta,\gamma)$ and $\mathrm{cov}(\alpha)$, $\mathrm{cov}(\alpha,\beta)$, $\mathrm{cov}(\alpha,\beta,\gamma)$ instead of $\mathrm{Cov}(A)$ and $\mathrm{cov}(A)$.

2. Matrix Constructions

Proposition 4. $\operatorname{achr}(K_6 \square K_4) \geq 12$, $\operatorname{achr}(K_6 \square K_6) \geq 18$ *and* $\operatorname{achr}(K_6 \square K_8) \geq 21$.

Proof. Let M_q be the $6 \times q$ matrix below, $q = 4, 6, 8$, where $\bar{n}$ stands for $10 + n$ and $\bar{\bar{n}}$ for $20 + n$.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & \bar{0} & \bar{1} & \bar{2} \\ 2 & 1 & 4 & 3 \\ 7 & 8 & 5 & 6 \\ \bar{2} & \bar{1} & \bar{0} & 9 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 7 & 8 & 9 & \bar{0} & \bar{1} & \bar{2} \\ \bar{3} & \bar{4} & \bar{5} & \bar{6} & \bar{7} & \bar{8} \\ 2 & 1 & \bar{7} & \bar{2} & \bar{5} & \bar{0} \\ \bar{1} & \bar{8} & 4 & 3 & 7 & \bar{4} \\ \bar{6} & 9 & 8 & \bar{3} & 6 & 5 \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & \bar{6} & \bar{7} & \bar{8} \\ 6 & 7 & 8 & 9 & \bar{0} & \bar{8} & \bar{6} & \bar{7} \\ \bar{1} & \bar{2} & \bar{3} & \bar{4} & \bar{5} & \bar{7} & \bar{8} & \bar{6} \\ 4 & 8 & 7 & 1 & \bar{9} & \bar{5} & \bar{\bar{0}} & \bar{\bar{1}} \\ \bar{3} & 5 & \bar{1} & \bar{\bar{1}} & 2 & 9 & \bar{9} & \bar{\bar{0}} \\ \bar{0} & \bar{4} & \bar{\bar{0}} & \bar{2} & 6 & 3 & \bar{\bar{1}} & \bar{9} \end{pmatrix}$$

One can check easily that $M_4 \in \mathcal{M}(6, 4, [1, 12])$, $M_6 \in \mathcal{M}(6, 6, [1, 18])$ and $M_8 \in \mathcal{M}(6, 8, [1, 21])$. Therefore, we are done by Proposition 2. ■

For $r \in [3, 9]$ consider r-element colour sets

$$V_r = \{v_r(k) : k \in [1, r]\},\ W_r = \{w_r(k) : k \in [1, r]\}$$

such that V_r, W_r and $[1, 18]$ are pairwise disjoint. Further, let N_r be the $6 \times r$ matrix below.

$$\begin{pmatrix} v_3(1) & v_3(2) & v_3(3) \\ v_3(2) & v_3(3) & v_3(1) \\ v_3(3) & v_3(1) & v_3(2) \\ w_3(1) & w_3(2) & w_3(3) \\ w_3(2) & w_3(3) & w_3(1) \\ w_3(3) & w_3(1) & w_3(2) \end{pmatrix} \quad \begin{pmatrix} v_6(1) & v_6(2) & v_6(3) & v_6(4) & v_6(5) & v_6(6) \\ v_6(2) & v_6(3) & v_6(4) & v_6(5) & v_6(6) & v_6(1) \\ v_6(3) & v_6(4) & v_6(5) & v_6(6) & v_6(1) & v_6(2) \\ w_6(1) & w_6(2) & w_6(3) & w_6(4) & w_6(5) & w_6(6) \\ w_6(3) & w_6(4) & w_6(5) & w_6(6) & w_6(1) & w_6(2) \\ w_6(5) & w_6(6) & w_6(1) & w_6(2) & w_6(3) & w_6(4) \end{pmatrix}$$

$$\begin{pmatrix} v_4(1) & v_4(2) & v_4(3) & v_4(4) \\ v_4(2) & v_4(3) & v_4(4) & v_4(1) \\ v_4(3) & v_4(4) & v_4(1) & v_4(2) \\ w_4(1) & w_4(2) & w_4(3) & w_4(4) \\ w_4(2) & w_4(3) & w_4(4) & w_4(1) \\ w_4(3) & w_4(4) & w_4(1) & w_4(2) \end{pmatrix} \quad \begin{pmatrix} v_5(1) & v_5(2) & v_5(3) & v_5(4) & v_5(5) \\ v_5(2) & v_5(3) & v_5(4) & v_5(5) & v_5(1) \\ v_5(3) & v_5(4) & v_5(5) & v_5(1) & v_5(2) \\ w_5(1) & w_5(2) & w_5(3) & w_5(4) & w_5(5) \\ w_5(3) & w_5(4) & w_5(5) & w_5(1) & w_5(2) \\ w_5(5) & w_5(1) & w_5(2) & w_5(3) & w_5(4) \end{pmatrix}$$

$$\begin{pmatrix} v_7(1) & v_7(2) & v_7(3) & v_7(4) & v_7(5) & v_7(6) & v_7(7) \\ v_7(2) & v_7(3) & v_7(4) & v_7(5) & v_7(6) & v_7(7) & v_7(1) \\ v_7(3) & v_7(4) & v_7(5) & v_7(6) & v_7(7) & v_7(1) & v_7(2) \\ w_7(1) & w_7(2) & w_7(3) & w_7(4) & w_7(5) & w_7(6) & w_7(7) \\ w_7(3) & w_7(4) & w_7(5) & w_7(6) & w_7(7) & w_7(1) & w_7(2) \\ w_7(5) & w_7(6) & w_7(7) & w_7(1) & w_7(2) & w_7(3) & w_7(4) \end{pmatrix}$$

$$\begin{pmatrix} v_8(1) & v_8(2) & v_8(3) & v_8(4) & v_8(5) & v_8(6) & v_8(7) & v_8(8) \\ v_8(2) & v_8(3) & v_8(4) & v_8(5) & v_8(6) & v_8(7) & v_8(8) & v_8(1) \\ v_8(3) & v_8(4) & v_8(5) & v_8(6) & v_8(7) & v_8(8) & v_8(1) & v_8(2) \\ w_8(1) & w_8(2) & w_8(3) & w_8(4) & w_8(5) & w_8(6) & w_8(7) & w_8(8) \\ w_8(4) & w_8(5) & w_8(6) & w_8(7) & w_8(8) & w_8(1) & w_8(2) & w_8(3) \\ w_8(7) & w_8(8) & w_8(1) & w_8(2) & w_8(3) & w_8(4) & w_8(5) & w_8(6) \end{pmatrix}$$

$$\begin{pmatrix} v_9(1) & v_9(2) & v_9(3) & v_9(4) & v_9(5) & v_9(6) & v_9(7) & v_9(8) & v_9(9) \\ v_9(2) & v_9(3) & v_9(4) & v_9(5) & v_9(6) & v_9(7) & v_9(8) & v_9(9) & v_9(1) \\ v_9(3) & v_9(4) & v_9(5) & v_9(6) & v_9(7) & v_9(8) & v_9(9) & v_9(1) & v_9(2) \\ w_9(1) & w_9(2) & w_9(3) & w_9(4) & w_9(5) & w_9(6) & w_9(7) & w_9(8) & w_9(9) \\ w_9(4) & w_9(5) & w_9(6) & w_9(7) & w_9(8) & w_9(9) & w_9(1) & w_9(2) & w_9(3) \\ w_9(7) & w_9(8) & w_9(9) & w_9(1) & w_9(2) & w_9(3) & w_9(4) & w_9(5) & w_9(6) \end{pmatrix}$$

Lemma 5. *If* $r \in [3, 9]$, *then* $N_r \in \mathcal{M}(6, r, V_r \cup W_r)$.

Proof. By inspection of the matrix N_r. ∎

Proposition 6. *If* $q \in [9, 15]$, *then* $\operatorname{achr}(K_6 \Box K_q) \geq 2q + 6$.

Proof. The block matrix $M_q = (M_6\, N_{q-6})$ belongs to $\mathcal{M}(6, q, C)$ with $C = [1, 18] \cup V_{q-6} \cup W_{q-6}$. To see it first realise that since the colourings f_{M_6} and $f_{N_{q-6}}$ are proper (Proposition 4, Lemma 5), and $[1, 18] \cap (V_{q-6} \cup W_{q-6}) = \emptyset$, the colouring f_{M_q} is proper, too.

Next, we have to show that each pair $\{\alpha, \beta\} \in \binom{C}{2}$ is good in M_q. The colourings f_{M_6} and $f_{N_{q-6}}$ are complete, hence it suffices to restrict our attention to $\alpha \in [1, 18]$ and $\beta \in V_{q-6} \cup W_{q-6}$. In such a case $|\mathbb{R}(\alpha) \cap [1, 3]| = 1 = |\mathbb{R}(\alpha) \cap [4, 6]|$ and $\mathbb{R}(\beta) \in \{[1, 3], [4, 6]\}$ so that $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) \neq \emptyset$, and the pair $\{\alpha, \beta\}$ is row-based.

So, Proposition 2 yields $\operatorname{achr}(K_6 \Box K_q) \geq |C| = 18 + 2(q - 6) = 2q + 6$. ∎

For $l = 0, 1, 2, 3$, let $r_l \in [3, 9]$, and let $N^l_{r_l}$ be the $6 \times r_l$ matrix obtained from N_{r_l} in such a way that $v_{r_l}(k)$ is replaced with $v^l_{r_l}(k)$ and $w_{r_l}(k)$ is replaced with $w^l_{r_l}(k)$ for each $k \in [1, r_l]$; here we suppose that, given a fixed quadruple (r_0, r_1, r_2, r_3), the sets $[1, 12]$ and

$$V^l_{r_l} = \left\{ v^l_{r_l}(k) : k \in [1, r_l] \right\}, \; W^l_{r_l} = \left\{ w^l_{r_l}(k) : k \in [1, r_l] \right\}, \; l = 0, 1, 2, 3,$$

are pairwise disjoint. Further, let $\tilde{N}^l_{r_l}$ be the $6 \times r_l$ matrix obtained from $N^l_{r_l}$ by interchanging its rows l and $l + 3$, $l = 1, 2, 3$.

Proposition 7. *If* $q \in [16, 40]$, *then* $\operatorname{achr}(K_6 \Box K_q) \geq 2q + 4$.

Proof. Since $4 \cdot 3 = 12 \le q - 4 \le 36 = 4 \cdot 9$, there are integers $r_l \in [3, 9]$, $l = 0, 1, 2, 3$, such that $\sum_{l=0}^{3} r_l = q - 4$. Let us show that the block matrix $M_q = (M_4\, N_{r_0}^0\, \tilde{N}_{r_1}^1\, \tilde{N}_{r_2}^2\, \tilde{N}_{r_3}^3)$ belongs to $\mathcal{M}(6, q, C)$ with $C = [1, 12] \cup \bigcup_{l=0}^{3} (V_{r_l}^l \cup W_{r_l}^l)$.

By Lemma 5 and Proposition 3 we have $N_{r_l} \in \mathcal{M}(6, r_l, V_{r_l} \cup W_{r_l})$ and $N_{r_l}^l \in \mathcal{M}(6, r_l, V_{r_l}^l \cup W_{r_l}^l)$, $l = 0, 1, 2, 3$, as well as $\tilde{N}_{r_l}^l \in \mathcal{M}(6, r_l, V_{r_l}^l \cup W_{r_l}^l)$, $l = 1, 2, 3$. The colouring f_M with $M \in \{M_4, N_{r_0}^0\, \tilde{N}_{r_1}^1\, \tilde{N}_{r_2}^2\, \tilde{N}_{r_3}^3\}$ is proper, and the sets $[1, 12]$, $V_{r_l}^l \cup W_{r_l}^l$, $l = 0, 1, 2, 3$, are pairwise disjoint, hence the colouring f_{M_q} is proper.

Now consider a pair $\{\alpha, \beta\} \in \binom{C}{2}$. If both α, β are either in $[1, 12]$ or in $V_{r_l}^l \cup W_{r_l}^l$ with $l \in [0, 3]$, then the pair $\{\alpha, \beta\}$ is good in M_q, because the colourings f_{M_4} and f_M with $M \in \{N_{r_0}^0, \tilde{N}_{r_1}^1, \tilde{N}_{r_2}^2, \tilde{N}_{r_3}^3\}$ are complete (Propositions 3, 4, Lemma 5).

In all remaining cases we show that $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) \neq \emptyset$, which means that the pair $\{\alpha, \beta\}$ is row-based.

If $\alpha \in [1, 12]$ and $\beta \in \bigcup_{l=0}^{3} (V_{r_l}^l \cup W_{r_l}^l)$, then $\mathbb{R}(\alpha) \in \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$ and $\mathbb{R}(\beta) \in \{\{1, 2, 3\}, \{1, 2, 6\}, \{1, 3, 5\}, \{1, 5, 6\}, \{2, 3, 4\}, \{2, 4, 6\}, \{3, 4, 5\}, \{4, 5, 6\}\}$ so that $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) \neq \emptyset$ follows immediately.

If $\alpha \in V_{r_0}^0$ and $\beta \in \bigcup_{l=1}^{3} V_{r_l}^l$, then $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = [1, 3] \setminus \{l\}$; similarly, with $\alpha \in W_{r_0}^0$ and $\beta \in \bigcup_{l=1}^{3} W_{r_l}^l$ we have $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = [4, 6] \setminus \{l\}$.

If $\{i, j, k\} = [1, 3]$, $\alpha \in V_{r_i}^i$ and $\beta \in V_{r_j}^j$, then $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = \{k\}$; the same conclusion holds provided that $\{i, j, k\} = [4, 6]$, $\alpha \in W_{r_i}^i$ and $\beta \in W_{r_j}^j$.

If there is $l \in [1, 3]$ such that either $\alpha \in V_{r_0}^0$ and $\beta \in W_{r_l}^l$ or $\alpha \in W_{r_0}^0$ and $\beta \in V_{r_l}^l$, then $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = \{l\}$.

Finally, $\alpha \in V_{r_i}^i$ with $i \in [1, 3]$ and $\beta \in W_{r_j}^j$ with $j \in [1, 3] \setminus \{i\}$ leads to $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = \{i + 3\}$.

Thus the colouring f_{M_q} is complete and $M_q \in \mathcal{M}(6, q, C)$. Since $|C| = 12 + \sum_{l=0}^{3} 2r_l = 2q + 4$, by Proposition 2 we get $\operatorname{achr}(K_6 \square K_q) \ge 2q + 4$. ∎

For a given $s \in [2, \infty)$ consider colour sets $U_s = \{u_k : k \in [1, s]\}$ with $U \in \{X, Y, Z, T\}$ such that the sets $[1, 12]$, X_s, Y_s, Z_s and T_s are pairwise disjoint.

Proposition 8. *If* $q \in [42, \infty)$ *and* $q \equiv 0 \pmod 2$, *then* $\operatorname{achr}(K_6 \square K_q) \ge 2q + 4$.

Proof. Let $s = \frac{q-4}{2}$, and let M_q be the $6 \times q$ matrix below. We show that $M_q \in \mathcal{M}(6, q, C)$ for $C = [1, 12] \cup X_s \cup Y_s \cup Z_s \cup T_s$. Obviously, since $s \ge 19 \ge 2$, the colouring f_{M_q} is proper.

$$\begin{pmatrix}
1 & 2 & 3 & 4 & x_1 & x_2 & \cdots & x_{s-1} & x_s & y_1 & y_2 & \cdots & y_{s-1} & y_s \\
5 & 6 & 7 & 8 & x_s & x_1 & \cdots & x_{s-2} & x_{s-1} & z_1 & z_2 & \cdots & z_{s-1} & z_s \\
9 & 10 & 11 & 12 & t_1 & t_2 & \cdots & t_{s-1} & t_s & x_1 & x_2 & \cdots & x_{s-1} & x_s \\
2 & 1 & 4 & 3 & z_1 & z_2 & \cdots & z_{s-1} & z_s & t_1 & t_2 & \cdots & t_{s-1} & t_s \\
7 & 8 & 5 & 6 & t_s & t_1 & \cdots & t_{s-2} & t_{s-1} & y_s & y_1 & \cdots & y_{s-2} & y_{s-1} \\
12 & 11 & 10 & 9 & y_1 & y_2 & \cdots & y_{s-1} & y_s & z_s & z_1 & \cdots & z_{s-2} & z_{s-1}
\end{pmatrix}$$

Notice that M_q has a submatrix M_4 (formed by the first four columns of M_q). The colouring f_{M_4} is complete (Proposition 4), hence a pair $\{\alpha, \beta\} \in \binom{C}{2}$ is good in M_q provided that $\alpha, \beta \in [1, 12]$. So, it remains to consider pairs $\{\alpha, \beta\}$ with $\alpha \in C$ and $\beta \in X_s \cup Y_s \cup Z_s \cup T_s$. Realise that $\mathbb{R}(\alpha) \in \mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathbb{R}(\beta) \in \mathcal{R}_2$, where $\mathcal{R}_1 = \{\{1,4\}, \{2,5\}, \{3,6\}\}$ and $\mathcal{R}_2 = \{\{1,2,3\}, \{1,5,6\}, \{2,4,6\}, \{3,4,5\}\}$. As $R \cap R_2 \neq \emptyset$ for any $R \in \mathcal{R}_1 \cup \mathcal{R}_2$ and any $R_2 \in \mathcal{R}_2$, the pair $\{\alpha, \beta\}$ is row-based.

Thus, by Proposition 2 we see that $\operatorname{achr}(K_6 \square K_q) \geq 4s + 12 = 2q + 4$. ■

3. Some Basic Facts Concerning Matrices in $\mathcal{M}(p, q, C)$

In this section we first reproduce those facts from [4] that are necessary for our paper.

Lemma 9 [4]**.** *If $p, q \in [1, \infty)$, C is a finite set, $M \in \mathcal{M}(p, q, C)$ and $\gamma \in C$, then the following hold.*

1. $\operatorname{frq}(\gamma) \leq \min(p, q)$.
2. $\operatorname{frq}(\gamma) = l$ *implies* $\operatorname{exc}(\gamma) = l(p + q - l - 1) - (|C| - 1) \geq 0$.
3. $\operatorname{frq}(M) = l$ *implies* $|C| \leq \left\lfloor \frac{pq}{l} \right\rfloor$.

Lemma 10 [4]**.** *If $p, q \in [1, \infty)$, C is a finite set and $M \in \mathcal{M}(p, q, C)$, then* $\operatorname{exc}(M) = \operatorname{exc}(\gamma)$, *where $\gamma \in C$ satisfies* $\operatorname{frq}(\gamma) = \operatorname{frq}(M)$.

Lemma 11 [4]**.** *If $q \in [7, \infty)$, $s \in [0, 7]$, C is a set of cardinality $2q + s$ and $M \in \mathcal{M}(6, q, C)$, then the following hold.*

1. $c_1 = 0$.
2. $c_l = 0$ *for* $l \in [7, \infty)$.
3. $c_2 \geq 3s$.
4. $c_{3+} \leq 2q - 2s$.
5. $\sum_{i=3}^{6} ic_i \leq 6q - 6s$.
6. $\operatorname{frq}(M) = 2$.
7. $\operatorname{exc}(M) = 7 - s$.
8. $c_{4+} \leq c_2 - 3s$.
9. $\{i, k\} \in \binom{[1,6]}{2}$ *implies* $r(i, k) \leq 8 - s$.

Lemma 12. *If $q \in [7, \infty)$, then* $\operatorname{achr}(K_6 \square K_q) \leq 2q + 6$.

Proof. If our lemma is false, according to Theorem 1 and Proposition 2 there is a $(2q + 7)$-element set C and $M \in \mathcal{M}(6, q, C)$. By Lemma 11.3 and 11.7 then $c_2 \geq 3 \cdot 7 = 21$ and $\operatorname{exc}(M) = 0$. Further, by Proposition 3 we may suppose without loss of generality that $r_2(1) \geq r_2(i)$ for $i \in [2, 6]$, which implies $6r_2(1) \geq$

$\sum_{i=1}^{6} r_2(i) = 2c_2$ and $r_2(1) \geq \left\lceil \frac{2c_2}{6} \right\rceil \geq 7$; we shall use on similar occasions (w) to indicate that it is Proposition 3, which is behind the fact that the generality is not lost. As a consequence there is $i \in [2,6]$ such that $r(1,i) \geq \left\lceil \frac{r_2(1)}{5} \right\rceil \geq 2$. With $\gamma \in \mathbb{R}(1,i)$ then each colour of $\mathbb{R}(1,i) \setminus \{\gamma\}$ contributes one to the excess of γ, hence we have $0 = \mathrm{exc}(M) = \mathrm{exc}(\gamma) \geq r(1,i) - 1 \geq 1$, a contradiction. ■

4. Solution

It turns out that the matrix constructions given by Propositions 4 and 6–8 are optimum from the point of view of $\mathrm{achr}(K_6 \square K_q)$. The optimality was already known for $q = 4$ (Horňák and Puntigán [7]) and $q = 6$ (Bouchet [1]), while the rest of the present paper is devoted to the analysis of remaining q's.

Theorem 13. $\mathrm{achr}(K_6 \square K_8) = 21$.

Proof. By Proposition 4 and Lemma 12 we know that $21 \leq \mathrm{achr}(K_6 \square K_8) \leq 22$. Suppose that $\mathrm{achr}(K_6 \square K_8) = 22$; because of Proposition 2 there is a 22-element set of colours C and a matrix $M \in \mathcal{M}(6,8,C)$. With $q = 8$ and $s = 6$ Lemma 11 yields $c_2 \geq 18$, $c_{3+} \leq 4$, $\mathrm{frq}(M) = 2$, $\mathrm{exc}(M) = 1$, $c_{4+} \leq c_2 - 18$ and $r(i,k) \leq 2$ for $\{i,k\} \in \binom{[1,6]}{2}$. We are going to strengthen step by step the requirements on M to finally reach a conclusion that M cannot exist at all.

Claim 14. *If $i \in [1,6]$ and $j \in [1,8]$, then $|\mathbb{R}_2(i) \cap \mathbb{C}(j)| \leq 2$.*

Proof. Suppose that $|C'_2| \geq 3$ for $C'_2 = \mathbb{R}_2(i) \cap \mathbb{C}(j)$. If $\alpha = (M)_{i,j} \in C_2$, then each colour of $C'_2 \setminus \{\alpha\}$ contributes one to the excess of α so that, by Lemma 10, $1 = \mathrm{exc}(M) = \mathrm{exc}(\alpha) \geq |C'_2 \setminus \{\alpha\}| \geq 2$, a contradiction. On the other hand, if $\alpha \in C_{3+}$, then for any $\beta \in C'_2$ we have $1 = \mathrm{exc}(M) = \mathrm{exc}(\beta) \geq |C'_2 \setminus \{\beta\}| \geq 2$, a contradiction again. □

Claim 15. *If $\{i,k\} \in \binom{[1,6]}{2}$ and $\alpha, \beta \in \mathbb{R}_2(i,k)$, $\alpha \neq \beta$, then $\mathrm{cov}(\alpha,\beta) = 2$, so that $\{\alpha,\beta\} \subseteq \mathbb{C}(j)$ for both $j \in \mathrm{Cov}(\alpha,\beta)$.*

Proof. Suppose (w) $i = 1$, $k = 2$ and $(M)_{1,1} = (M)_{2,2} = \alpha$. If $\mathrm{cov}(\alpha,\beta) = 4$, (w) $(M)_{1,3} = (M)_{2,4} = \beta$. Denote $A = \mathbb{R}(1) \cup \mathbb{R}(2)$. From $\mathrm{exc}(\alpha) = \mathrm{exc}(\beta) = 1$ it is clear that $|A| = 14$, $|C \setminus A| = 8$, and that, for both $l \in [1,2]$, each colour of $C \setminus A$ occupies a position in the set $[3,6] \times [2l-1, 2l]$ (the colouring f_M is complete). Since $|(C \setminus A) \cap C_2| = |C \setminus A| - |(C \setminus A) \cap C_{3+}| \geq 8 - c_{3+} \geq 4$, there is a colour $\gamma \in (C \setminus A) \cap C_2$. The neighbourhood of the 2-element vertex set $f_M^{-1}(\gamma)$ contains ten vertices belonging to $[3,6] \times [1,4]$, all coloured with seven colours of $(C \setminus A) \setminus \{\gamma\}$. As a consequence we obtain $1 = \mathrm{exc}(M) = \mathrm{exc}(\gamma) \geq 10 - 7 = 3$, a contradiction.

If $\mathrm{cov}(\alpha,\beta)=3$, (w) $(M)_{1,2}=(M)_{2,3}=\beta$. With $B=\mathbb{R}(1)\cup\mathbb{R}(2)\cup\mathbb{C}(2)$ then $\mathrm{exc}(\alpha)=1$ implies $|B|=18$ and $|C\setminus B|=4$. Each colour $\gamma\in C\setminus B$ belongs to $\mathbb{C}(1)\cap\mathbb{C}(3)$ (both pairs $\{\gamma,\alpha\}$ and $\{\gamma,\beta\}$ are good in M) and satisfies $\mathrm{exc}(\gamma)\ge|(C\setminus B)\setminus\{\gamma\}|=3$, hence $\gamma\notin C_2$ and $\gamma\in C_{3+}$. Consequently, $c_{3+}\ge 4$, $c_{3+}=4$, $C\setminus B=C_{3+}$, $B=C_2$, $\delta=(M)_{1,3}\in C_2$, and the second copy of δ appears in $[3,6]\times[4,8]$ so that $\mathrm{exc}(\delta)\ge 2$ (if $\delta=(M)_{m,n}$ with $(m,n)\in[3,6]\times[4,8]$, then both β and $(M)_{m,1}\in C_{3+}$ contribute to the excess of δ), a contradiction. □

If $\mathrm{Cov}(\alpha,\beta)=\{j,l\}$ for the 2-colours α,β of Claim 15, then (w) $\alpha=(M)_{i,j}=(M)_{k,l}$ and $\beta=(M)_{i,l}=(M)_{k,j}$; we say that the set of 2-colours $\{\alpha,\beta\}$ forms an $\mathbb{X}$-*configuration* (in M): both copies of a colour $\gamma\in\{\alpha,\beta\}$ are diagonal to each other in the "rectangle" of the matrix M with corners $(i,j),(i,l),(k,j),(k,l)$, and this fact will be in the sequel for simplicity denoted by $\{\alpha,\beta\}\to\mathbb{X}$.

Claim 16. *If* $i\in[1,6]$, *then* $r_2(i)=6$ *and* $r_3(i)=2$.

Proof. Let (w) $r_2(1)\ge r_2(i)$ for each $i\in[2,6]$ and $r(1,i)\ge r(1,i+1)$ for each $i\in[2,5]$. Suppose that $r_2(1)\ge 7$. Then $2\ge r(1,2)\ge\left\lceil\frac{r_2(1)}{5}\right\rceil=2$ so that $r(1,2)=2$. Moreover, $r(1,6)\le\left\lfloor\frac{r_2(1)}{5}\right\rfloor=1$, and there is $p\in[2,5]$ such that $r(1,p)=2$ and $r(1,p+1)\le 1$ (which implies $r(1,i)\le 1$ for any $i\in[p+1,6]$). As $r_2(6)=8-r_{3+}(6)\ge 8-c_{3+}\ge 4$, we have $|\mathbb{R}_2(6)\setminus\mathbb{R}_2(1)|=|\mathbb{R}_2(6)\setminus\mathbb{R}(1,6)|=r_2(6)-r(1,6)=[8-r_{3+}(6)]-r(1,6)\ge 4-1=3$, and there exists $\alpha\in\mathbb{R}_2(6)\setminus\mathbb{R}_2(1)$.

Consider a colour $\beta\in\mathbb{R}(i,k)$, where $1<i<k$. When counting the number of pairs $\{\beta,\gamma\}$ with $\gamma\in\mathbb{R}_2(1)$, that are good in M because of the copy of β in the mth row of M, $m\in\{i,k\}$, we see that $r(1,m)$ of them are row-based, and, by Claim 14, at most two of them are column-based. There is $i\in[2,5]$ such that $\alpha\in\mathbb{R}(i,6)$. Proceeding as above we obtain that the number of pairs $\{\alpha,\gamma\}$ with $\gamma\in\mathbb{R}_2(1)$, that are good in M, is at most $\rho=[r(1,i)+2]+[r(1,6)+2]\le 6+r(1,6)$. Observe that we cannot have $r(1,6)\le r_2(1)-7$, because then $\rho\le 6+[r_2(1)-7]<r_2(1)$, a contradiction. Therefore, $1\ge r(1,6)\ge r_2(1)-6\ge 1$, $r(1,6)=1$ and $r_2(1)=7$. Now $p\le 3$, because $p\ge 4$ would mean $7=r_2(1)=\sum_{i=2}^{6}r(1,i)\ge 2(p-1)+(6-p)=p+4\ge 8$, a contradiction. Consequently, any colour $\delta\in C_2\setminus\mathbb{R}(1)$ needs a copy in a row $k\in[2,p]$; to see it realise that, under the assumption $\delta\in\mathbb{R}(a,b)$ with $a,b\in[p+1,6]$, the number of pairs $\{\delta,\gamma\}$, $\gamma\in\mathbb{R}_2(1)$, that are good in M, is at most $[r(1,a)+2]+[r(1,b)+2]\le 2\cdot 1+4<r_2(1)$, which is impossible. Thus $18\le c_2\le 7+\sum_{i=2}^{p}[r_2(i)-r(1,i)]\le 7+(p-1)(7-2)=5p+2\le 17$, a contradiction.

Since the assumption $r_2(1)\ge 7$ was false, we have $36\ge\sum_{i=1}^{6}r_2(i)=2c_2\ge 36$. Therefore, $r_2(i)=6$ for each $i\in[1,6]$, $c_2=18$, $c_{4+}=0$, $C=C_2\cup C_3$, and the proof follows. □

By Claim 16, $c_2 = \frac{6 \cdot 6}{2} = 18$ and $c_3 = \frac{6 \cdot 2}{3} = 4$; moreover, for every $i \in [1,6]$ there is (at least one) $p_i \in [1,6] \setminus \{i\}$ such that $r(i, p_i) = \lceil \frac{6}{5} \rceil = 2$. Then, by Claim 15, $\mathbb{R}(i, p_i) \to \mathbb{X}$ for $i \in [1,6]$. Let $\tilde{C_2} = \{\alpha, \beta, \gamma, \delta\} \subseteq C_2$ be such that $\{\alpha, \beta\} \to \mathbb{X}$ and $\{\gamma, \delta\} \to \mathbb{X}$, where $\{\alpha, \beta\} \neq \{\gamma, \delta\}$ (which immediately yields $\{\alpha, \beta\} \cap \{\gamma, \delta\} = \emptyset$). We have $\operatorname{cov}(\tilde{C_2}) \in [3,4]$, since with $\operatorname{cov}(\tilde{C_2}) = 2$ each of β, γ, δ contributes one to the excess of α so that $\operatorname{exc}(\alpha) \geq 3$, a contradiction.

Thus (w) $\operatorname{cov}(\alpha, \beta) = [1,2]$ and $\operatorname{cov}(\gamma, \delta) = [l, l+1]$, where $l \in [2,3]$. Then $\mathbb{C}(1) \cup \mathbb{C}(2) \subseteq C_2$ (because of $r_3(1) = r_3(2) = 2$ and $\operatorname{exc}(\alpha) = 1$, colours of C_3 occupy four positions in $\mathbb{R}(1) \cup \mathbb{R}(2)$ and neither position in $\mathbb{C}(1) \cup \mathbb{C}(2)$), and, similarly, $\mathbb{C}(l) \cup \mathbb{C}(l+1) \subseteq C_2$. So, all 3-colours appear exclusively in $7 - l$ columns of M numbered from $l+2$ to 8. There is $j \in [l+2, 8]$ such that $c_3(j) \geq \left\lceil \frac{3c_3}{7-l} \right\rceil \geq \lceil \frac{12}{5} \rceil = 3$, while $c_2(j) \geq 6 - c_3 = 2$. If $\varepsilon \in \mathbb{C}_2(j) \cap \mathbb{R}(m,n)$, then 3-colours occupy at least $r_3(m) + r_3(n) + [c_3(j) - 1] = c_3(j) + 3 \geq 6$ positions in $N(V_\varepsilon)$ (since $\varepsilon \in \{(M)_{m,j}, (M)_{n,j}\}$, at least $c_3(j) - 1$ positions in $[1,6] \times \{j\}$ occupied by 3-colours are positions that are not in $\{m,n\}) \times [1,8]$), hence $\operatorname{exc}(\varepsilon) \geq 6 - c_3 = 2$, a final contradiction for the proof of Theorem 13. ■

Theorem 17. *If* $q \in [9,15]$, *then* $\operatorname{achr}(K_6 \square K_q) = 2q + 6$.

Proof. See Proposition 6 and Lemma 12. ■

Theorem 18. *If either* $q \in [42, \infty)$ *and* $q \equiv 0 \pmod 2$ *or* $q \in [16, 40]$, *then* $\operatorname{achr}(K_6 \square K_q) = 2q + 4$.

Proof. We proceed by the way of contradiction. Since $\operatorname{achr}(K_6 \square K_q) \geq 2q + 4$ (Propositions 7 and 8), the assumption $\operatorname{achr}(K_6 \square K_q) \geq 2q + 5$ by Theorem 1 and Proposition 2 means that there is a colour set C with $|C| = 2q + 5 = 2q + s$ and a matrix $M \in \mathcal{M}(6, q, C)$. By Lemma 11 then $C = \bigcup_{l=2}^{6} C_l$, $c_2 \geq 15$, $c_{3+} \leq 2q - 10$, $\sum_{i=3}^{6} i c_i \leq 6q - 30$, $\operatorname{frq}(M) = 2 = \operatorname{exc}(M)$, $c_{4+} \leq c_2 - 15$, and $\{i,k\} \in \binom{[1,6]}{2}$ implies $r(i,k) \leq 3$. A contradiction is reached first for $q \geq 19$, then for $q \in [17,18]$, and finally for $q = 16$.

Let G be an auxiliary graph G associated with M, in which $V(G) = [1,6]$ and $\{i,k\} \in E(G)$ if and only if $r(i,k) \geq 1$.

Claim 19. $\Delta(G) \leq 3$.

Proof. In [4] it has been proved that $\Delta(G) \geq 4$ together with $\operatorname{achr}(K_6 \square K_q) = 2q + s$ implies $q \leq 40 - 5s = 15$, a contradiction. □

In [4] one can find also proofs of the following two claims.

Claim 20. *If* $\{i,j,k,l,m,n\} = [1,6]$ *and* $r(i,j,k) \geq 1$, *then* $r(l,m,n) \leq 9$.

Claim 21. *If* $\{i,j,k,l,m,n\} = [1,6]$, $r(i,l) \geq 1$, $r(j,l) \geq 1$ *and* $r(k,l) \geq 1$, *then* $r(l,m,n) \geq q + 3s - 24 = q - 9$.

Claim 22. *If* $\{i,j,k,l,m,n\} = [1,6]$*,* $r(i,l) \geq 1$*,* $r(j,l) \geq 1$ *and* $r(k,l) \geq 1$*,* $\{a,b\} \in \binom{[1,6]}{2}$ *and* $r(a,b) \geq 1$*, then* $|\{i,j,k\} \cap \{a,b\}| = 1 = |\{l,m,n\} \cap \{a,b\}|$*.*

Proof. The assumptions of Claim 22 imply that $r(l,m,n) \geq q-9 \geq 7$ (see Claim 21). Consider a colour $\alpha \in \mathbb{R}(a,b)$.

If $\{a,b\} \subseteq \{i,j,k\}$, the number of pairs $\{\alpha,\beta\}$, $\beta \in \mathbb{R}(l,m,n)$, that are good in M (and necessarily column-based), is at most $3\operatorname{cov}(\alpha) = 6 < r(l,m,n)$, a contradiction.

On the other hand, with $\{a,b\} \subseteq \{l,m,n\}$ each colour of $\mathbb{R}(l,m,n)$ contributes one to the excess of α so that $2 = \operatorname{exc}(M) = \operatorname{exc}(\alpha) \geq r(l,m,n) \geq 7$, a contradiction again.

Therefore, $2 = |\{i,j,k,l,m,n\} \cap \{a,b\}| = |\{i,j,k\} \cap \{a,b\}| + |\{l,m,n\} \cap \{a,b\}| \leq 1+1 = 2$, and then $|\{i,j,k\} \cap \{a,b\}| = 1 = |\{l,m,n\} \cap \{a,b\}|$. □

Claim 23. *If* $\{i,k\} \in \binom{[1,6]}{2}$*, then* $r(i,k) \leq 2$*.*

Proof. Let (w) $i=1$, $k=2$, $\operatorname{Cov}(\mathbb{R}(1,2)) = [1,n]$, and assume (for a proof by contradiction) that $r(1,2) = 3$ (see Lemma 11.9), which implies $n \in [3,6]$.

We are going to show that $A = C \setminus (\mathbb{R}(1) \cup \mathbb{R}(2)) \subseteq C_{3+}$. First observe that each colour $\alpha \in A$ occupies at least two positions in $S_n = [3,6] \times [1,n]$ (all pairs $\{\alpha,\beta\}$, $\beta \in \mathbb{R}(1,2)$, are good in M), hence $|A| \leq \left\lfloor \frac{4n}{2} \right\rfloor = 2n$. Moreover, from $|\mathbb{R}(1) \cap \mathbb{R}(2)| \geq r(1,2)$ we get $|\mathbb{R}(1) \cup \mathbb{R}(2)| \leq 2q-3$. Consequently, $2q+5 = |C| = |A| + |\mathbb{R}(1) \cup \mathbb{R}(2)| \leq |A| + (2q-3)$ leads to $8 \leq |A| \leq 2n$ and $n \in [4,6]$.

If $n=4$, then $|A| = 8$, and any colour of A occupies exactly two positions in S_4. Suppose there is a colour $\alpha \in A \cap C_2$. If a vertex $(i,j) \in S_4$ belongs to $N(V_\alpha)$, then $(M)_{i,j} \in A \setminus \{\alpha\}$, hence $2 = \operatorname{exc}(M) = \operatorname{exc}(\alpha) \geq 10 - |A \setminus \{\alpha\}| = 3$ (the set $N(V_\alpha)$ has 10 vertices in S_4), a contradiction. Therefore, $A \subseteq C_{3+}$.

If $n=5$, there is $j \in [1,5]$ such that $|\mathbb{C}(j) \cap \mathbb{R}(1,2)| = 2$ and $|\mathbb{C}(l) \cap \mathbb{R}(1,2)| = 1$ for $l \in [1,5] \setminus \{j\}$. Then $A = A_2 \cup A_3 \cup A_4$, where A_l consists of colours of A occupying l positions in S_5. With $a_l = |A_l|$ we obtain $a_2 \leq 4$ (if $\alpha \in A_2 \setminus \mathbb{C}(j)$, at least one of three pairs $\{\alpha,\beta\}$ with $\beta \in \mathbb{R}(1,2)$ is not good in M, a contradiction), $a_3 + a_4 = |A| - a_2 \geq 8 - 4 = 4$, $16 + a_3 + a_4 \leq 16 + a_3 + 2a_4 \leq 2(a_2 + a_3 + a_4) + a_3 + 2a_4 = \sum_{l=2}^{4} l a_l = 4 \cdot 5 = 20$, $a_3 + a_4 \leq 20 - 16 = 4$, $a_3 + a_4 = 4$, all six above expressions are 20, which implies $a_4 = 0$, $a_3 = 4 = a_2$, and then all positions in S_5 are occupied by colours of $A_2 \cup A_3$. If $\operatorname{Cov}(\mathbb{R}(1,2) \setminus \mathbb{C}(j)) = \{s,t\} \subseteq [1,5] \setminus \{j\}$, then $A_2 \subseteq \mathbb{C}(j) \cup \mathbb{C}(s) \cup \mathbb{C}(t)$. For the set B of colours in $C_2 \setminus A_2$ that are not in $[1,2] \times [1,5]$ we have $|B| \geq c_2 - [(2|[1,5]| - 3) + a_2] \geq 15 - 11 = 4$. However, the number of pairs $\{\alpha,\beta\}$ with $\alpha \in A_2$ and $\beta \in B$, that are good in M, is at most three (if $\beta \in \mathbb{R}(u)$, $u \in [3,6]$, only $(M)_{u,j}, (M)_{u,s}$ and $(M)_{u,t}$ are available as α), a contradiction.

If $n=6$, the frequency of each colour in $\alpha \in A$ is at least three, since all pairs $\{\alpha,\beta\}$ with $\beta \in \mathbb{R}(1,2)$ are column-based, and at most one of them satisfies the implication $\alpha \in \mathbb{C}(j) \Rightarrow \beta \in \mathbb{C}(j)$.

Thus $A \subseteq C_{3+}$ and $C_2 \subseteq \mathbb{R}(1) \cup \mathbb{R}(2)$. For $l \in [1,6]$ let

$$K(l) = \{m \in [1,6] \setminus \{l\} : r(l,m) \geq 1\},$$

so that $\Delta(G) \leq 3$ (Claim 19) implies $|K(l)| \leq 3$. Further, for $l, m \in [1,6]$, $l \neq m$, let $r_2^-(l,m) = |\mathbb{R}_2(l) \setminus \mathbb{R}_2(m)|$ be the number of 2-colours which occur in row l but not row m. Observe that if $m \in K(l)$, then $r_2^-(l,m) = \sum_{p \in K(l) \setminus \{m\}} r(l,p) \leq 2 \cdot 3 = 6$.

The inclusion $C_2 \subseteq \mathbb{R}(1) \cup \mathbb{R}(2)$ implies $c_2 = r_2^-(1,2) + r_2^-(2,1) + r(1,2)$. Then $15 \leq c_2 = r_2^-(1,2) + r_2^-(2,1) + r(1,2) \leq 2 \cdot 6 + 3 = 15$, hence $c_2 = 15$, $r_2^-(1,2) = r_2^-(2,1) = 6$, $r(1,2) = 3$, $r_2(1) = r_2^-(1,2) + r(1,2) = 9 = r_2^-(2,1) + r(2,1) = r_2(2)$, $|K(l)| = 3$ and $r(l,p) = 3$ for each $p \in K(l)$, $l = 1, 2$.

The last two facts imply in particular that $|K(1) \setminus \{2\}| = 2$ and that $r_1(m) = 3$ for both $m \in K(1) \setminus \{2\}$. This means that we can repeat the entire reasoning from the start of the proof of Claim 23 with the pair $(1,m)$ instead of the pair $(1,2)$. Among other things we obtain $r_2(m) = 9$ for both $m \in K(1) \setminus \{2\} \subseteq [3,6]$. Together with $r_2(1) = r_2(2) = 9$ we then have

$$15 = c_2 = \frac{1}{2} \sum_{m=1}^{6} r_2(m) \geq \frac{4 \cdot 9}{2} = 18,$$

a contradiction. □

Claim 24. *If $i \in [1,6]$, then $r_2(i) \leq 6$.*

Proof. Since $\Delta(G) \leq 3$, the claim is a direct consequence of Claim 23. □

Claim 25. *The following statements are true.*

1. $\Delta(G) = 3$.
2. *G is a subgraph of $K_{3,3}$.*
3. $c_2 \leq 18$.

Proof. 1. The assumption $\Delta(G) \leq 2$ would mean, by Claim 23, $r_2(i) \leq 2 \cdot 2 = 4$ for $i \in [1,6]$ and $15 \leq c_2 = \frac{1}{2} \sum_{i=1}^{6} r_2(i) \leq \frac{6 \cdot 4}{2} = 12$, a contradiction.

2. From Claims 25.1 and 22 it follows that there is a partition $\{I, K\}$ of $[1,6]$ satisfying $|I| = |K| = 3$ such that $r(i,k) \geq 1$ with $\{i,k\} \in \binom{[1,6]}{2}$ implies $|\{i,k\} \cap I| = 1 = |\{i,k\} \cap K|$. Thus, G is a subgraph of $K_{3,3}$ with the bipartition $\{I, K\}$.

3. Finally, by Claim 23, $c_2 = \sum_{i \in I} \sum_{k \in K} r(i,k) \leq 9 \cdot 2 = 18$. □

Henceforth we suppose (w) that the bipartition of the graph $K_{3,3}$ from Claim 25.2 is $\{[1,3], [4,6]\}$, which leads to

$$C_2 = \bigcup_{i=1}^{3} \bigcup_{k=4}^{6} \mathbb{R}(i,k).$$

Note that this assumption somehow restricts the meaning of (w) in the subsequent analysis, namely the bijection $\rho : [1,6] \to [1,6]$ in Proposition 3 should satisfy $\rho([1,3]) \in \{[1,3],[4,6]\}$.

Claim 26. *There is at most one pair $(i,k) \in [1,3] \times [4,6]$ with $r(i,k) = 0$.*

Proof. If $|\{(i,k) \in [1,3] \times [4,6] : r(i,k) = 0\}| \geq 2$, Claim 23 yields $15 \leq c_2 \leq 7 \cdot 2 = 14$, a contradiction. $\square$

Claim 27. *If $(i,j,k) \in \{(1,2,3),(4,5,6)\}$, then $7 \leq q-9 \leq r(i,j,k) \leq 9$.*

Proof. From Claim 26 it immediately follows that $\max(\deg_G(p) : p \in [1,3]) = 3 = \max(\deg_G(p) : p \in [4,6])$. So, by Claims 21 and 20, $q - 9 \leq r(i,j,k) \leq 9$. $\square$

Use for an edge $\{i,k\}$ of the graph $K_{3,3}$ with bipartition $\{[1,3],[4,6]\}$ the label $r(i,k) \in [0,2]$ (see Claim 23). A colour $\alpha \in C_2$ *corresponds* to an edge $\{i,k\} \in E(K_{3,3})$ if $\alpha \in \mathbb{R}(i,k)$, and α *corresponds* to a set $E \subseteq E(K_{3,3})$ if there is $e \in E$ such that α corresponds to e. We denote by $\mathrm{Col}(E)$ the set of colours corresponding to E. Colours $\alpha, \beta \in C_2$, $\alpha \neq \beta$, are *column-related* (in M) provided that $\mathbb{R}(\alpha) \cap \mathbb{R}(\beta) = \emptyset$ (and, consequently, $|\mathbb{R}(\alpha) \cup \mathbb{R}(\beta)| = 4$); then the pair $\{\alpha,\beta\}$ is not row-based, and hence it is column-based (thus if 2-colours α,β are column-related, then the pair $\{\alpha,\beta\}$ is column-based, but not necessarily vice versa). Evidently, if $\{\gamma_j : j \in [1,l]\}$ is a set of pairwise column-related 2-colours, where $\gamma_j \in \mathbb{R}(i_j,k_j)$, then $r(i_j,k_j) \geq 1$, $\{i_j,k_j\} \in E(K_{3,3})$, $|\bigcup_{j\in[1,l]}\{i_j,k_j\}| = 2l$, $\{\{i_j,k_j\} : j \in [1,l]\}$ is a matching in $K_{3,3}$, and so $l \leq 3$. For a matching $\mathcal{M}$ in $K_{3,3}$ we denote by $\mathrm{wt}(\mathcal{M})$ the *weight* of $\mathcal{M}$, i.e., the sum of labels of edges of $\mathcal{M}$.

Claim 28. *If $\mathcal{M}^1$ is a perfect matching in $K_{3,3}$, then there are perfect matchings $\mathcal{M}^2$ and $\mathcal{M}^3$ in $K_{3,3}$ such that $\{\mathcal{M}^1,\mathcal{M}^2,\mathcal{M}^3\}$ is a partition of $E(K_{3,3})$ and $\mathrm{wt}(\mathcal{M}^2) \geq \mathrm{wt}(\mathcal{M}^3)$; moreover, $\bigcup_{s=1}^3 \mathrm{Col}(\mathcal{M}^s) = C_2$ and $\sum_{s=1}^3 \mathrm{wt}(\mathcal{M}^s) = c_2$.*

Proof. The set $E(K_{3,3}) \setminus \mathcal{M}^1$ induces a 6-vertex cycle C in $K_{3,3}$. Then $E(C)$ has a partition $\{\mathcal{M}^2,\mathcal{M}^3\}$ into perfect matchings of $K_{3,3}$ so that $\{\mathcal{M}^1,\mathcal{M}^2,\mathcal{M}^3\}$ is a partition of $E(K_{3,3})$; without loss of generality we may suppose $\mathrm{wt}(\mathcal{M}^2) \geq \mathrm{wt}(\mathcal{M}^3)$. Notice that each colour of C_2 is in exactly one of the sets $\mathrm{Col}(\mathcal{M}^s) \subseteq C_2$, $s = 1,2,3$, hence $C_2 = \bigcup_{s=1}^3 \mathrm{Col}(\mathcal{M}^s)$ and $c_2 = \sum_{i=1}^3 \sum_{k=4}^6 r(i,k) = \sum_{s=1}^3 \mathrm{wt}(\mathcal{M}^s)$. $\square$

Let A be a nonempty subset of C_2. We say that A is of *type* $t_1^{a_1} \cdots t_p^{a_p}$ if $|A| \geq t_1 \geq \cdots \geq t_p \geq 1$, every column of M contains either 0 or exactly t_s colours of A for some $s \in [1,p]$, and a_s is the number of columns of M that contain exactly t_s colours of A. Note that $\sum_{s=1}^{p} a_s t_s = 2|A|$. Clearly, the type of A is unique.

Claim 29. *Under the assumptions* $\{i,j,k\} = \{1,2,3\}$, $\{l,m,n\} = \{4,5,6\}$, $\alpha \in \mathbb{R}(i,l)$, $\beta \in \mathbb{R}(j,m)$ *and* $\gamma \in \mathbb{R}(k,n)$, *the following statements are true.*

1. *If the set* $\{\alpha,\beta,\gamma\}$ *is of the type* $t_1^{a_1}\cdots t_p^{a_p}$, *then* $\sum_{s=1}^{p} a_s \binom{t_s}{2} \geq 3$.
2. *If* $\operatorname{cov}(\alpha,\beta,\gamma) \leq 3$, *then* $\operatorname{cov}(\alpha,\beta,\gamma) = 3$, *each colour of* $C_2 \setminus \{\alpha,\beta,\gamma\}$ *appears exactly once in the set* $[1,6] \times \operatorname{Cov}(\alpha,\beta,\gamma)$, *and* $\bigcup_{s\in\operatorname{Cov}(\alpha,\beta,\gamma)} \mathbb{C}(s) = C_2$.
3. *The type of the set* $\{\alpha,\beta,\gamma\}$ *is either* $3^1 1^3$ *or* 2^3.

Proof. 1. The colours α,β,γ are pairwise column-related, hence each of the pairs $\{\alpha,\beta\}$, $\{\alpha,\gamma\}$ and $\{\beta,\gamma\}$ is column-based; colours of such a pair share a column whose number is in $\operatorname{Cov}(\alpha,\beta,\gamma)$. A column of M containing exactly t_s colours of $\{\alpha,\beta,\gamma\}$ hosts exactly $\binom{t_s}{2}$ from among the above pairs, and so $\sum_{s=1}^{p} a_s \binom{t_s}{2} \geq 3$.

2. In the set $[1,6] \times \operatorname{Cov}(\alpha,\beta,\gamma)$ there are at most eighteen positions of which six are occupied by α,β,γ. Since $c_2 \geq 15$, there are at least twelve other colours in C_2, and these must all occupy one of the remaining positions, otherwise for a colour $\delta \in C_2 \setminus \{\alpha,\beta,\gamma\}$, that is out of $[1,6] \times \operatorname{Cov}(\alpha,\beta,\gamma)$, the number of $\varepsilon \in \{\alpha,\beta,\gamma\}$ such that the pair $\{\delta,\varepsilon\}$ is good in M, is only 2 (such pairs are row-based). Thus $\operatorname{cov}(\alpha,\beta,\gamma) = 3$, $c_2 = 15$, each colour of $C_2 \setminus \{\alpha,\beta,\gamma\}$ occupies exactly one position in $[1,6] \times \operatorname{Cov}(\alpha,\beta,\gamma)$, and the set of colours, that appear in $[1,6] \times \operatorname{Cov}(\alpha,\beta,\gamma)$, is equal to C_2, i.e., $\bigcup_{s\in\operatorname{Cov}(\alpha,\beta,\gamma)} \mathbb{C}(s) = C_2$.

3. Possible types of the set $\{\alpha,\beta,\gamma\}$ (that must satisfy Claim 29.1) are 3^2, $3^1 2^1 1^1$, $3^1 1^3$ and 2^3. However, the type 3^2 has to be excluded since $\operatorname{cov}(\alpha,\beta,\gamma) = 3$ by Claim 29.2.

Suppose the set $\{\alpha,\beta,\gamma\}$ is of the type $3^1 2^1 1^1$. Observe that if $b \in \operatorname{Cov}(\alpha,\beta,\gamma)$ satisfies $\{\alpha,\beta,\gamma\} \cap \mathbb{C}(b) = \{\varepsilon\}$, there is (a unique) $a \in [1,6]$ such that $\{\alpha,\beta,\gamma\} \cap \mathbb{R}(a) = \{\varepsilon\}$ and $\zeta = (M)_{a,b} \neq \varepsilon$. By Claim 29.2 then $\zeta \in C_2$, and the second copy of ζ is in $[1,6] \times ([1,q] \setminus \operatorname{Cov}(\alpha,\beta,\gamma))$; so, the number of pairs $\{\zeta,\eta\}$ with $\eta \in \{\alpha,\beta,\gamma\} \setminus \{\varepsilon\}$, that are good in M (and necessarily row-based), is one, while $|\{\alpha,\beta,\gamma\} \setminus \{\varepsilon\}| = 2$, a contradiction. □

Claim 30. *If* $\{i,j,k\} = \{1,2,3\}$, $\{l,m,n\} = \{4,5,6\}$, $\mathbb{R}(i,l) = \{\alpha_1,\alpha_2\}$, $\mathbb{R}(j,m) = \{\beta_1,\beta_2\}$, $\gamma_1 \in \mathbb{R}(k,n)$ *and* $a,b \in [1,2]$, *then the set* $\{\alpha_a,\beta_b,\gamma_1\}$ *is of the type* $3^1 1^3$.

Proof. If the claim is false, then, by Claim 29.3, (w) $\{\alpha_1,\beta_1,\gamma_1\}$ is of the type 2^3, $\operatorname{Cov}(\alpha_1,\beta_1,\gamma_1) = [1,3]$, and, by Claim 29.2, α_2 occupies exactly one position in $[1,6] \times [1,3]$. Clearly, α_2 appears in the column of M containing both β_1 and γ_1 (the pair $\{\beta_1,\gamma_1\}$ is column-based), for otherwise $\{\alpha_2,\beta_1,\gamma_1\}$ would be of the type $2^2 1^2$, which is impossible by Claim 29.3; so, by the same claim, $\{\alpha_2,\beta_1,\gamma_1\}$ is of the type $3^1 1^3$, (w) $\operatorname{Cov}(\alpha_2,\beta_1,\gamma_1) = [1,4]$.

Proceeding similarly as above we see that β_2 appears in the column containing α_1, γ_1, and $\{\alpha_1,\beta_2,\gamma_1\}$ is of the type $3^1 1^3$, so that the pair $\{\alpha_2,\beta_2\}$ can be good in M only if $\{\alpha_2,\beta_2\} \subseteq \mathbb{C}(4)$. Consequently, $\{\alpha_2,\beta_2,\gamma_1\}$ is of the type 2^3. If $\{\alpha_1,\beta_1\} \subseteq \mathbb{C}(b)$, $b \in [1,3]$, then $\mathrm{Cov}(\alpha_2,\beta_2,\gamma_1) = [1,4] \setminus \{b\}$, so that, by Claim 29.2, $\mathbb{C}(b), \mathbb{C}(4) \subseteq C_2$ and $\mathbb{C}(b,4) = \mathbb{C}(b) \setminus \{\alpha_1,\beta_1\} = \mathbb{C}(4) \setminus \{\alpha_2,\beta_2\}$. In such a case any colour $\delta \in \mathbb{C}(b,4) \subseteq C_2$ satisfies $2 = \mathrm{exc}(\delta) \ge |\mathbb{C}(b,4) \setminus \{\delta\}| = 3$, a contradiction. □

From Claim 30 we see that if $\mathcal{M}$ is a perfect matching in $K_{3,3}$ with $\mathrm{wt}(\mathcal{M}) \ge 5$ and colours $\alpha,\beta,\gamma \in \mathrm{Col}(\mathcal{M})$ are pairwise column-related, then the set $\{\alpha,\beta,\gamma\}$ is of the type $3^1 1^3$.

Claim 31. *Under the assumptions* $\{i,j,k\} = [1,3]$, $\{l,m,n\} = [4,6]$, $\mathbb{R}(i,l) = \{\alpha_1,\alpha_2\}$, $\mathbb{R}(j,m) = \{\beta_1,\beta_2\}$, $\mathbb{R}(k,n) \in \{\{\gamma_1\},\{\gamma_1,\gamma_2\}\}$ *and* $C_2^1 = \mathbb{R}(i,l) \cup \mathbb{R}(j,m) \cup \mathbb{R}(k,n)$, *the following statements are true.*

1. *There is* $a \in [1,q]$ *such that* $C_2^1 \subseteq \mathbb{C}(a)$.
2. *If colours* $\delta,\varepsilon \in C_2^1$, $\delta \ne \varepsilon$, *are column-related, then* $\mathrm{cov}(\delta,\varepsilon) = 3$.
3. *If* $\delta \in \{\alpha,\beta\}$, $C_2^1 \subseteq \mathbb{C}(a)$ *and* $\mathrm{Cov}(\delta_1,\delta_2) = \{a,b,d\}$, *then* $|\mathbb{C}(b) \cap \mathbb{C}(d)| = 3$ *and* $\mathbb{C}(b,d) \ne \emptyset$.
4. $\{\alpha_1,\alpha_2\} \to \mathbb{X}$ *and* $\{\beta_1,\beta_2\} \to \mathbb{X}$.
5. *If* $\mathbb{R}(k,n) = \{\gamma_1,\gamma_2\}$, *then* $\{\gamma_1,\gamma_2\} \to \mathbb{X}$.
6. $\mathrm{cov}(C_2^1) = 4$.
7. *If* $\delta \in C_2 \setminus C_2^1$, *then* δ *is in* $[1,6] \times \mathrm{Cov}(C_2^1)$.

Proof. Consider the perfect matching $\mathcal{M}^1 = \{\{i,l\},\{j,m\},\{k,n\}\}$. From the assumptions of Claim 31 we get $5 \le \mathrm{wt}(\mathcal{M}^1) \le 6$.

1. By Claim 30 we know that (among others) all of the following sets are of the type $3^1 1^3$: $\{\alpha_1,\beta_1,\gamma_1\}$, $\{\alpha_2,\beta_1,\gamma_1\}$, $\{\alpha_1,\beta_2,\gamma_1\}$ and $\{\alpha_1,\beta_1,\gamma_2\}$ (provided that $\gamma_2 \in \mathbb{R}(k,n)$). Then there is $a \in [1,q]$ with $\{\alpha_1,\beta_1,\gamma_1\} \subseteq \mathbb{C}(a)$. Now $|\{\alpha_2,\beta_1,\gamma_1\} \cap \mathbb{C}(a)| \ge 2 > 1$, hence $|\{\alpha_2,\beta_1,\gamma_1\} \cap \mathbb{C}(a)| = 3$ and $\alpha_2 \in \mathbb{C}(a)$. A similar reasoning shows that $\beta_2 \in \mathbb{C}(a)$ as well as $\gamma_2 \in \mathbb{C}(a)$ (under the assumption $\gamma_2 \in \mathbb{R}(k,n)$).

Before proceeding further let us mention that, by Claim 31.1, if $\mathcal{M}$ is a perfect matching in $K_{3,3}$ with $\mathrm{wt}(\mathcal{M}) \ge 5$, then all colours of $\mathrm{Col}(\mathcal{M})$ occur in (exactly) one of columns of M.

2. There are $s,t,u \in \{1,2\}$ such that $\{\delta,\varepsilon\} \subseteq \{\alpha_s,\beta_t,\gamma_u\}$. Therefore, the statement is a direct consequence of Claim 31.1 and the fact that, by Claim 30, the set $\{\alpha_s,\beta_t,\gamma_u\}$ is of the type $3^1 1^3$.

3. If $\delta = \alpha$, $\mathrm{Cov}(\alpha_1,\alpha_2) = \{a,b,d\}$ and $\varepsilon \in C' = C \setminus (\mathbb{R}(i) \cup \mathbb{R}(l) \cup \mathbb{C}(a))$, then, as both pairs $\{\varepsilon,\alpha_1\}$ and $\{\varepsilon,\alpha_2\}$ are good in M, we get $\varepsilon \in \mathbb{C}(b) \cap \mathbb{C}(d)$. Consequently, $|\mathbb{C}(b) \cap \mathbb{C}(d)| \ge |C'| \ge 2q+5-(2q-2+4) = 3$; further, by Claim 24,

$|(\mathbb{R}(i)\cup\mathbb{R}(l)\cup\mathbb{C}(a))\cap C_2|\le 2\cdot 6-2+4=14<c_2$, and so C' contains a 2-colour ζ. Having in mind that $\zeta\in\mathbb{C}(b,d)$ and $2=\operatorname{exc}(\zeta)\ge|(\mathbb{C}(b)\cap\mathbb{C}(d))\setminus\{\zeta\}|\ge|C'\setminus\{\zeta\}|\ge 2$, we obtain $|\mathbb{C}(b)\cap\mathbb{C}(d)|=3$.

An analogous reasoning applies in the case $\delta=\beta$ and $\operatorname{Cov}(\beta_1,\beta_2)=\{a,b,d\}$.

4. By Claim 31.1 there is $a\in[1,q]$ with $C_2^1\subseteq\mathbb{C}(a)$, hence $2\le\operatorname{cov}(\alpha_1,\alpha_2)\le 3$. Suppose that $\operatorname{cov}(\alpha_1,\alpha_2)=3$ (which means that $\{\alpha_1,\alpha_2\}\to\mathbb{X}$ is not true) and $\operatorname{Cov}(\alpha_1,\alpha_2)=\{a,b,d\}$. By Claim 31.3 then $|\mathbb{C}(b)\cap\mathbb{C}(d)|=3$, and there is (a 2-colour) $\varepsilon\in\mathbb{C}(b,d)$.

By Claim 28 there exists a partition $\{\mathcal{M}^1,\mathcal{M}^2,\mathcal{M}^3\}$ of $E(K_{3,3})$ into perfect matchings in $K_{3,3}$, $\bigcup_{s=1}^3\operatorname{Col}(\mathcal{M}^s)=C_2$ and $\operatorname{wt}(\mathcal{M}^2)\ge\operatorname{wt}(\mathcal{M}^3)$. Let $C_2^s=\operatorname{Col}(\mathcal{M}^s)$, $s=2,3$, so that $\{C_2^1,C_2^2,C_2^3\}$ is a partition of C_2; then there is $p\in[2,3]$ with $\varepsilon\in C_2^p$. Note that $\{\mathcal{M}^2,\mathcal{M}^3\}=\{\mathcal{M}^p,\mathcal{M}^{5-p}\}$ and $\{C_2^2,C_2^3\}=\{C_2^p,C_2^{5-p}\}$.

Let us show that $\operatorname{wt}(\mathcal{M}^p)\le 4$. Indeed, with $\operatorname{wt}(\mathcal{M}^p)=|\operatorname{Col}(\mathcal{M}^p)|\ge 5$, by Claim 31.1 all colours of $C_2^p=\operatorname{Col}(\mathcal{M}^p)$ occur in one of columns of M, say in the column e. Since $\varepsilon\in C_2^p$, we have necessarily $e\in\{b,d\}$. The column e contains exactly one of colours α_1,α_2 of C_2^1 and all colours of C_2^p (thus 2-colours only), which implies $\mathbb{C}(b)\cap\mathbb{C}(d)=\mathbb{C}(b)\cap\mathbb{C}(d)\cap\mathbb{C}(e)\subseteq\mathbb{C}(e)\subseteq C_2$, $\mathbb{C}(b,d)=C_2\cap\mathbb{C}(b)\cap\mathbb{C}(d)=\mathbb{C}(b)\cap\mathbb{C}(d)$ and $|\mathbb{C}(b,d)|=3$. From among three colours of $\mathbb{C}(b)\cap\mathbb{C}(d)\subseteq\mathbb{C}(e)$ at least two appear in one of two "halves" of the column e (the "upper half" and the "lower half"); more precisely, there is $I\in\{[1,3],[4,6]\}$ such that the colours of $\mathbb{C}(b,d)$ occupy at least two positions in $I\times\{e\}$. Let ζ,η be distinct colours of $\mathbb{C}(b,d)$ occupying two positions in $I\times\{e\}$. Then the remaining copies of ζ,η occupy two positions in $([1,6]\setminus I)\times\{f\}$, where $\{b,d\}=\{e,f\}$. Thus $|\mathbb{R}(\zeta)\cup\mathbb{R}(\eta)|=4$, and so the colours ζ,η are column-related. The colours ζ and η correspond to e_1 and e_2, repectively, with $e_1,e_2\in\mathcal{M}^p$, $e_1\ne e_2$. If $\mathcal{M}^p=\{e_1,e_2,e_3\}$, from Claim 23 we know that the label of the edge e_3 is either 1 or 2; let ϑ be a colour of C_2^p that corresponds to e_3. The colours ζ,η,ϑ are pairwise column-related. Therefore, by Claim 30, the set $\{\zeta,\eta,\vartheta\}$ is of the type 3^11^3. This, however, is contradicted by the following two facts: $|\{\zeta,\eta,\vartheta\}\cap\mathbb{C}(e)|=3$ and $|\{\zeta,\eta,\vartheta\}\cap\mathbb{C}(f)|\ge|\{\zeta,\eta\}|=2>1$.

Since $\operatorname{wt}(\mathcal{M}^1)\in[5,6]$ and Claim 28 yields $\sum_{s=1}^3\operatorname{wt}(\mathcal{M}^s)=c_2\ge 15$, we have $\operatorname{wt}(\mathcal{M}^{5-p})=c_2-\operatorname{wt}(\mathcal{M}^1)-\operatorname{wt}(\mathcal{M}^p)\ge 15-6-4=5$. By Claim 31.1 all colours of $C_2^{5-p}=\operatorname{Col}(\mathcal{M}^{5-p})$ occur in one of columns of M, say in the column e. From $C_2^1\cap C_2^{5-p}=\emptyset$ it follows that $e\ne a$. Further, note that $\mathbb{C}(b)$ contains $\varepsilon\in C_2^p$ as well as one of colours $\alpha_1,\alpha_2\in C_2^1$, and the same is true for $\mathbb{C}(d)$; as a consequence of $|C_2^{5-p}|=\operatorname{wt}(\mathcal{M}^{5-p})\ge 5$ then $e\notin\{b,d\}$. Consider $e_1,e_2\in\mathcal{M}^{5-p}$ with $i\in e_1$ and $l\in e_2$. Having in mind that $\{i,l\}\in\mathcal{M}^1$, we obtain $e_1\ne e_2$. If $\mathcal{M}^{5-p}=\{e_1,e_2,e_3\}$, then the edge e_3 is labelled with either 1 or 2. Observe that there is a colour $\zeta\in C_2^{5-p}$ corresponding to e_3, which does not belong to $\mathbb{C}(a)$. This is clear if $\operatorname{wt}(\mathcal{M}^1)=6$, when $\mathbb{C}(a)\cap\operatorname{Col}(\mathcal{M}^{5-p})=\operatorname{Col}(\mathcal{M}^1)\cap\operatorname{Col}(\mathcal{M}^{5-p})=\emptyset$. On the other hand, with $\operatorname{wt}(\mathcal{M}^1)=5$ we have $\operatorname{wt}(\mathcal{M}^{5-p})=6$, the edge e_3 is labelled with

2, and for the choice of ζ there are two possibilities, at least one of which satisfies the above requirement: $\mathbb{C}(a)$ contains at most one colour of $\mathrm{Col}(\mathcal{M}^{5-p})$. Now it is clear that no pair $\{\zeta, \alpha_s\}$ with $s \in [1,2]$ is row-based (because $\mathbb{R}(\alpha_1) = \mathbb{R}(\alpha_2) = \{i,l\}$ and $\mathbb{R}(\zeta) \cap \{i,l\} = \emptyset$). As a consequence both pairs $\{\zeta, \alpha_1\}, \{\zeta, \alpha_2\}$ are necessarily column-based. However, from $e \in \mathrm{Cov}(\zeta)$, $e \notin \mathrm{Cov}(\alpha_1, \alpha_2) = \{a, b, d\}$ and $a \notin \mathrm{Cov}(\zeta)$ it follows that $\mathrm{Cov}(\zeta) \cap \mathrm{Cov}(\alpha_1, \alpha_2) = \mathrm{Cov}(\zeta) \cap \{b, d\}$ and $|\mathrm{Cov}(\zeta) \cap \{b, d\}| \leq 1$; since $|\mathbb{C}(b) \cap \{\alpha_1, \alpha_2\}| = 1 = |\mathbb{C}(d) \cap \{\alpha_1, \alpha_2\}|$, the number of pairs $\{\zeta, \alpha_1\}$, $\{\zeta, \alpha_2\}$, that are good in M, is at most 1, a contradiction.

Thus $\mathrm{cov}(\alpha_1, \alpha_2) = 2$ and $\{\alpha_1, \alpha_2\} \to \mathbb{X}$.

The assumption $\mathrm{cov}(\beta_1, \beta_2) = 3$ leads to a contradiction similarly as above, hence $\mathrm{cov}(\beta_1, \beta_2) = 2$ and $\{\beta_1, \beta_2\} \to \mathbb{X}$.

5. Use Claim 31.4 with $\{\gamma_1, \gamma_2\}$ in the role of $\{\alpha_1, \alpha_2\}$.

6. This claim is a consequence of Claim 31.2, Claim 31.4 and Claim 31.5, where the last one applies only if $\mathbb{R}(k, n) = \{\gamma_1, \gamma_2\}$.

7. If $\delta \in C_2 \setminus C_2^1$ occupies only (two) positions in $[1,6] \times ([1,q] \setminus \mathrm{Cov}(C_2^1))$, then the number of pairs $\{\delta, \varepsilon\}$ with $\varepsilon \in \{\alpha_1, \beta_1, \gamma_1\}$, that are good in M (and necessarily row-based), is $2 < 3 = |\{\alpha_1, \beta_1, \gamma_1\}|$, a contradiction. □

Let $C_3^* = C_3 \setminus (\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6))$, $c_3^* = |C_3^*|$ and $\rho_3 = r(1,2,3) + r(4,5,6)$ so that $c_3 = \rho_3 + c_3^*$ and $\rho_3 \leq 18$ (see Claim 27).

Claim 32. *If* $\rho_3 \geq 15$*, then* $\mathrm{cov}(\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6)) \leq 9$.

Proof. Suppose $r(1,2,3) \geq r(4,5,6)$ and observe that if $j \in \mathrm{Cov}(\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6))$, then $\mathbb{R}(1,2,3) \cap \mathbb{C}(j) \neq \emptyset$ and $\mathbb{R}(4,5,6) \cap \mathbb{C}(j) \neq \emptyset$ as well. Indeed, if $\delta \in \mathbb{R}(1,2,3) \cap \mathbb{C}(j)$ and $\mathbb{R}(4,5,6) \cap \mathbb{C}(j) = \emptyset$, then the number of pairs $\{\delta, \varepsilon\}$ with $\varepsilon \in \mathbb{R}(4,5,6)$, that are good in M (they must be column-based), is at most $3|\mathrm{Cov}(\delta) \setminus \{j\}| = 6 < 7 \leq q - 9 \leq r(4,5,6)$ (Claim 27), a contradiction. A similar contradiction can be reached under the assumption $\mathbb{R}(4,5,6) \cap \mathbb{C}(j) \neq \emptyset$ and $\mathbb{R}(1,2,3) \cap \mathbb{C}(j) = \emptyset$. So, $\mathrm{cov}(\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6)) = \mathrm{cov}(\mathbb{R}(1,2,3)) = \mathrm{cov}(\mathbb{R}(4,5,6))$.

Claim 27 yields $r(1,2,3) \leq 9$. Suppose first that $r(1,2,3) = 9$. In the case $\mathrm{cov}(\mathbb{R}(1,2,3)) \leq 9$ the claim is proved. If $\mathrm{cov}(\mathbb{R}(1,2,3)) \geq 10$, there is $j \in \mathrm{Cov}(\mathbb{R}(1,2,3))$ with $|\mathbb{R}(1,2,3) \cap \mathbb{C}(j)| \leq 2$. In such a case for $\varepsilon \in \mathbb{R}(4,5,6) \cap \mathbb{C}(j)$ the number of pairs $\{\varepsilon, \delta\}$ with $\delta \in \mathbb{R}(1,2,3)$, that are good in M, is at most $2 + 3 + 3 < r(1,2,3)$, a contradiction.

Thus $9 > r(1,2,3) \geq \frac{1}{2}[r(1,2,3) + r(4,5,6)] = \frac{\rho_3}{2} \geq \frac{15}{2} > 7$, $r(1,2,3) = 8$ and $8 \geq r(4,5,6) \geq 7$. If $\mathrm{cov}(\mathbb{R}(1,2,3)) \geq 10$ and $j \in \mathrm{Cov}(\mathbb{R}(1,2,3))$, then necessarily $|\mathbb{R}(1,2,3) \cap \mathbb{C}(j)| \geq 2$, for otherwise $r(1,2,3) \leq 1 + 2 \cdot 3 = 7 < r(1,2,3)$. Consequently, $m = |\{j \in [1,q] : |\mathbb{R}(1,2,3) \cap \mathbb{C}(j)| = 3\}| \leq 4$, because $m \geq 5$ implies that the number of positions in M occupied by colours of $\mathbb{R}(1,2,3)$ is $3m + 2[\mathrm{cov}(\mathbb{R}(1,2,3)) - m] = 2\mathrm{cov}(\mathbb{R}(1,2,3)) + m \geq 20 + 5 = 25 > 24 = 3r(1,2,3)$, a contradiction. For each colour $\zeta \in \mathbb{R}(4,5,6)$ we have $n(\zeta) = |\{j \in [1,q] : |\mathbb{R}(1,2,3) \cap \mathbb{C}(j)| = 3, \zeta \in \mathbb{C}(j)\}| \geq 2$, since $n(\zeta) \leq 1$ leads to $r(1,2,3) \leq 3n(\zeta) + 2[3 - n(\zeta)] = 6 + n(\zeta) \leq 7 < 8 = r(1,2,3)$. Then the number of colours $\zeta \in \mathbb{R}(4,5,6)$, for which all pairs $\{\zeta, \delta\}$ with $\delta \in \mathbb{R}(1,2,3)$ are good in M, is at most $\lfloor \frac{3m}{2} \rfloor \leq \lfloor \frac{3 \cdot 4}{2} \rfloor = 6 < 7 \leq r(4,5,6)$; in other words, there is $\zeta \in \mathbb{R}(4,5,6)$ and $\delta \in \mathbb{R}(1,2,3)$ such that the pair $\{\zeta, \delta\}$ is not good in M, a contradiction.

The case $r(1,2,3) < r(4,5,6)$ can be treated analogously. $\square$

Claim 33. *No perfect matching of $K_{3,3}$ is of weight* 6.

Proof. Suppose that $\mathrm{wt}(\mathcal{M}^1) = 6$, where (w) $\mathcal{M}^1 = \{\{i, 7-i\} : i = 1,2,3\}$, and let $\alpha \in \mathbb{R}(1,6)$, $\beta \in \mathbb{R}(2,5)$, $\gamma \in \mathbb{R}(3,4)$. By Claim 28 there exists a partition $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ of $E(K_{3,3})$ into perfect matchings in $K_{3,3}$, $\bigcup_{s=1}^3 \mathrm{Col}(\mathcal{M}^s) = C_2$, $\sum_{s=1}^3 \mathrm{wt}(\mathcal{M}^s) = c_2 \geq 15$ and $\mathrm{wt}(\mathcal{M}^2) \geq \mathrm{wt}(\mathcal{M}^3)$, hence $\mathrm{wt}(\mathcal{M}^2) + \mathrm{wt}(\mathcal{M}^3) = c_2 - 6 \geq 9$ and $\mathrm{wt}(\mathcal{M}^2) \geq 5$. Let $C_2^s = \mathrm{Col}(\mathcal{M}^s)$, $s = 1,2,3$. Using Claims 31.1 and 31.4–6 we get (w) $C_2^1 = \mathbb{C}(1)$, $\mathrm{Cov}(C_2^1) = [1,4]$ and $C_2^2 \subseteq \mathbb{C}(5)$ (each of columns $2,3,4$ contains two colours of C_2^1). Further, as a consequence of Claim 31.7, and the fact that $C_2^1 = \mathbb{C}(1)$, any colour of C_2 occupies a position in $[1,6] \times (\mathrm{Cov}(C_2^1) \setminus \{1\}) = [1,6] \times [2,4]$.

Now consider an arbitrary colour $\varepsilon \in C_3^*$. There are three distinct integers $a, b, d \in [1,6]$ and a set $I \in \{[1,3],[4,6]\}$ such that $a, b \in I$, $d \in [1,6] \setminus I$ and $\varepsilon \in \mathbb{R}(a,b,d)$. Let us show that the colour ε occupies a position in $[1,6] \times [2,5]$. Suppose this is not true so that every pair $\{\varepsilon, \zeta\}$ with $\zeta \in C_2^1 \cup C_2^2$ is row-based. Then $I = \{a, b, 7-d\}$; otherwise, if $7 - d \in \{a,b\}$, then $\{d, 7-d\} \subseteq \{a, 7-a\} \cup \{b, 7-b\}$, there is $\zeta \in \{\alpha, \beta, \gamma\}$ with $\mathbb{R}(\zeta) \cap (\{a, 7-a\} \cup \{b, 7-b\}) = \emptyset$, which leads to $\mathbb{R}(\varepsilon) \cap \mathbb{R}(\zeta) = \{a,b,d\} \cap \mathbb{R}(\zeta) \subseteq (\{a, 7-a\} \cup \{b, 7-b\}) \cap \mathbb{R}(\zeta) = \emptyset$, and the pair $\{\varepsilon, \zeta\}$ is not good in M, a contradiction. Thus $I = \{a, b, 7-d\}$ and $[1,6] \setminus I = \{d, x, y\}$. The set $\{\{7-d, d\}, \{7-d, x\}, \{7-d, y\}\}$ is a transversal of the collection $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ of pairwise disjoint perfect matchings of $K_{3,3}$, hence there is $z \in \{x, y\}$ with $\{7-d, z\} \in \mathcal{M}^2$ and $\mathbb{R}(7-d, z) \subseteq C_2^2$ (recall that $\{7-d, d\} \in \mathcal{M}^1$). From $\mathrm{wt}(\mathcal{M}^2) \geq 5$ we know, by Claim 23, that $r(7-d, z) \geq 1$; with $\eta \in \mathbb{R}(7-d, z)$ then $\mathbb{R}(\varepsilon) \cap \mathbb{R}(\eta) = \{a,b,d\} \cap \{7-d, z\} = \emptyset$, hence the pair $\{\eta, \varepsilon\}$ is not good in M, a contradiction.

From the above reasoning we see that each colour of $(C_2 \setminus C_2^2) \cup C_3^*$ occupies a position in $[1,6] \times [2,5]$, and each colour of C_2^2 occupies two positions in $[1,6] \times [2,5]$. Therefore, $c_2 + c_3^* + |C_2^2| \leq |[1,6] \times [2,5]| = 6 \cdot 4 = 24$ and $c_2 + c_3^* \leq 24 - |C_2^2| = 24 - \mathrm{wt}(\mathcal{M}^2) \leq 24 - 5 = 19$.

Thus, by Lemma 11.8, $c_3^* + c_{4+} \leq c_3^* + (c_2 - 15) \leq 19 - 15 = 4$. Claim 24 yields $c_2 = \frac{1}{2} \sum_{i=1}^6 r_2(i) \leq \frac{6 \cdot 6}{2} = 18$, hence $2q + 5 = |C| = (c_2 + c_3^*) + \rho_3 + c_{4+} \leq 19 + \rho_3 + (c_2 - 15) \leq 19 + \rho_3 + 3 = 22 + \rho_3$, and $\rho_3 \geq (2q+5) - 22 \geq 15$. Consequently, using Claim 32, $\mathrm{cov}(\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6)) \leq 9$, (w) $\mathrm{Cov}(\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6)) \subseteq [q-8, q]$.

If $\mathrm{wt}(\mathcal{M}^3) \geq 5$, then, by Claim 31.1 applied on C_2^3, (w) $C_2^3 \subseteq \mathbb{C}(6)$, $\mathrm{Cov}(C_2^3) = \{2,3,4,6\}$, $\mathrm{Cov}(C_2) = [1,6]$, and, for any $j \in [7, q-9] \supseteq \{7\}$, $\mathbb{C}(j) \subseteq C_3^* \cup C_{4+}$, so that $6 = |\mathbb{C}(j)| \leq c_3^* + c_{4+} \leq 4$, a contradiction.

So, $\mathrm{wt}(\mathcal{M}^3) \leq 4$, $c_2 = \sum_{s=1}^{3} \mathrm{wt}(\mathcal{M}^s) \leq 6+6+4 = 16$, $37 \leq 2q+5 = c_2 + \rho_3 + c_3^* + c_{4+} \leq 16 + 18 + (c_3^* + c_{4+}) \leq 34 + 4 = 38$ and $q = 16$. Then $37 = |C| = c_2 + c_3^* + \rho_3 + c_{4+} \leq c_2 + c_3^* + \rho_3 + (c_2 - 15) = c_2 + c_3^* + \sum_{s=1}^{3} \mathrm{wt}(\mathcal{M}^s) + \rho_3 - 15 = [c_2 + c_3^* + \mathrm{wt}(\mathcal{M}^2)] + \mathrm{wt}(\mathcal{M}^3) + 6 + \rho_3 - 15 \leq 24 + 4 + \rho_3 - 9 \leq 28 + (18-9) = 37$, $\rho_3 = 18$, $\mathrm{cov}(\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6)) \geq \lceil \frac{18\cdot 3}{6} \rceil = 9$, $\mathrm{cov}(\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6)) = 9$, $\mathrm{Cov}(\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6)) = [8,16]$ and $\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6) = \bigcup_{j=8}^{16} \mathbb{C}(j)$. Let $C' = C \setminus (\mathbb{R}(1,2,3) \cup \mathbb{R}(4,5,6))$, and let M' be the 6×7 submatrix of M formed by the first seven columns of M. Evidently, $M' \in \mathcal{M}(6,7,C')$, which means, by Proposition 2, that $\mathrm{achr}(K_6 \square K_7) \geq |C'| = 19$. This, however, contradicts the result $\mathrm{achr}(K_6 \square K_7) = 18$ proved in [5]. □

Claim 34. *The following statements are true.*

1. $c_2 = 15$.
2. *Each perfect matching of* $K_{3,3}$ *is of weight* 5.
3. $c_{4+} = 0$.
4. *Each edge of* $K_{3,3}$ *is labelled with either* 1 *or* 2.
5. *There are* $I, K \in \{[1,3],[4,6]\}$, $I \neq K$, *and* $k \in K$ *such that for any* $i \in I$ *and any* $l \in K \setminus \{k\}$ *it holds* $r(i,k) = 1$ *and* $r(i,l) = 2$.

Proof. 1. Given a perfect matching $\mathcal{M}^1$ of $K_{3,3}$, by Claim 28 we know that there is a unique partition $\{\mathcal{M}^1, \mathcal{M}^2, \mathcal{M}^3\}$ of $E(K_{3,3})$ into perfect matchings of $K_{3,3}$. By Lemma 11.3 and Claim 33 then $15 \leq c_2 = \sum_{s=1}^{3} \mathrm{wt}(\mathcal{M}^s) \leq \sum_{s=1}^{3} 5 = 15$ and $c_2 = 15$.

2. From the proof of Claim 34.1 we see that $\mathrm{wt}(\mathcal{M}^s) = 5$, $s = 1,2,3$. Thus $\mathrm{wt}(\mathcal{M}) = 5$ for each perfect matching $\mathcal{M}$ of $K_{3,3}$ ($\mathcal{M}$ can be chosen as $\mathcal{M}^1$).

3. By Lemma 11.8 and Claim 34.1 we have $c_{4+} \leq c_2 - 15 = 0$ and $c_{4+} = 0$.

4. No edge of $K_{3,3}$ is labelled with 0, otherwise any perfect matching of $K_{3,3}$ containing such an edge would be of weight at most $2 \cdot 2 = 4$ (Claim 23), which contradicts Claim 34.2.

5. Denote by $l(e)$ the label of an edge $e \in E(K_{3,3})$, and by l_n the number of edges of $K_{3,3}$ labelled with n, $n = 1,2$ (see Claim 34.4); then $l_1 + l_2 = 9$, $15 = c_2 = l_1 + 2l_2 = 9 + l_2$, $l_2 = 6$ and $l_1 = 3$. Let $\{e_1, e_2, e_3\} = \{e \in E(K_{3,3}) : l(e) = 1\}$.

If $a, b \in [1,3]$, $a \neq b$, then $e_a \cap e_b \neq \emptyset$. To see it suppose that $e_a \cap e_b = \emptyset$, and take $e \in E(K_{3,3}) \setminus \{e_a, e_b\}$ such that $\{e_a, e_b, e\}$ is a perfect matching of $K_{3,3}$. The 6-vertex cycle in $K_{3,3}$ with the edge set $E(K_{3,3}) \setminus \{e_a, e_b, e\}$ has at least five edges labelled with 2, hence one can find in $K_{3,3}$ a perfect matching $\mathcal{M} \subseteq E(K_{3,3}) \setminus \{e_a, e_b, e\}$ with $\mathrm{wt}(\mathcal{M}) = 3 \cdot 2 = 6$, a contradiction.

Thus $e_1 \cap e_2 \neq \emptyset$, $e_1 \cap e_3 \neq \emptyset$ and $e_2 \cap e_3 \neq \emptyset$. Since the subgraph of the bipartite graph $K_{3,3}$ induced by the set of edges $\{e_1, e_2, e_3\}$ is bipartite (and so free of odd cycles), the above three intersections (of 2-element sets) are nonempty only if there is a vertex $k \in [1,6] = V(K_{3,3})$ such that $e_1 \cap e_2 \cap e_3 = \{k\}$. Having in mind that the bipartition of $K_{3,3}$ is $\{[1,3],[4,6]\}$, there are $I, K \in \{[1,3],[4,6]\}$ such that $I \neq K$, $k \in K$, and for any $i \in I$ and any $l \in K \setminus \{k\}$ it holds $r(i,k) = 1$ and $r(i,l) = 2$. □

Based on Claim 34.5 we suppose (w) $I = [1,3]$, $K = [4,6]$ and $k = 6$ so that for any $i \in [1,3]$ and any $l \in [4,5]$ we have $r(i,6) = 1$ and $r(i,l) = 2$. Then $r_2(1) = r_2(2) = r_2(3) = 5$, $r_2(4) = r_2(5) = 6$ and $r_2(6) = 3$. Let $\mathbb{R}(i,6) = \{\alpha_{i,6}\}$, $i = 1,2,3$.

If $\mathcal{M}$ is a perfect matching in $K_{3,3}$, there is $s \in [1,6]$ such that $\mathcal{M} = \mathcal{M}^s$, where

$$\begin{aligned} \mathcal{M}^1 &= \{\{1,6\},\{2,4\},\{3,5\}\}, & \mathcal{M}^2 &= \{\{1,5\},\{2,6\},\{3,4\}\}, \\ \mathcal{M}^3 &= \{\{1,4\},\{2,5\},\{3,6\}\}, & \mathcal{M}^4 &= \{\{1,6\},\{2,5\},\{3,4\}\}, \\ \mathcal{M}^5 &= \{\{1,4\},\{2,6\},\{3,5\}\}, & \mathcal{M}^6 &= \{\{1,5\},\{2,4\},\{3,6\}\}. \end{aligned}$$

We have $|\mathrm{Col}(\mathcal{M}^s)| = 5$ for each $s \in [1,6]$. Applying Claim 31 on five colours of $\mathrm{Col}(\mathcal{M}^s)$ we see that there is $a^s \in [1,q]$ such that $\mathrm{Col}(\mathcal{M}^s) \subseteq \mathbb{C}(a^s)$. If $s,t \in [1,6]$, $s \neq t$, then $|\mathcal{M}^s \cap \mathcal{M}^t| \leq 1$, hence $|\mathrm{Col}(\mathcal{M}^s) \cap \mathrm{Col}(\mathcal{M}^t)| \leq 2$ (Claim 23), and so it is clear that $a^s \neq a^t$. From now on (w)

$$\{a^s : s \in [1,6]\} = [1,6],$$

$$(M)_{i,i} = \alpha_{i,6} = (M)_{6,i+3}, \quad i = 1,2,3.$$

Let us show that $\beta_i = (M)_{i,i+3}$ with $i \in [1,3]$ is not a 2-colour. Indeed, suppose it is. The number of pairs $\{\beta_i, \alpha_{j,6}\}$, $j \in [1,3]$, that are good in M, is 3. However, each copy of β_i provides only one such pair (for the copy $(M)_{i,i+3}$ of β_i it is the pair $\{\beta_i, \alpha_{i,6}\}$, while for the other copy of β_i in one of rows $4,5$ of M it is a pair $\{\beta_i, \alpha_{j,6}\}$ that is column-based), a contradiction. As a consequence of Claims 34.1 and 34.3 then all positions in the set

$$S = \{(1,4),(2,5),(3,6),(6,1),(6,2),(6,3)\}$$

are occupied by 3-colours, and the same is true for the set of positions $[1,6] \times [7,q]$. Moreover, all positions in the set $([1,6] \times [1,6]) \setminus S$ are occupied by 2-colours.

Claim 35. *Each position in the set S is occupied by a colour of C_3^*.*

Proof. If a position $(i, i+3)$ with $i \in [1,3]$ is occupied by a colour $\beta \in \mathbb{R}(1,2,3)$, that copy of β provides no pair $\{\beta, \gamma\}$ with $\gamma \in \mathbb{R}(4,5,6)$ that is good in M.

Claim 27 yields $\min(r(1,2,3), r(4,5,6)) \geq q-9 \geq 7$. However, the number of pairs $\{\beta,\gamma\}$ with $\gamma \in \mathbb{R}(4,5,6)$, that are good in M (and necessarily column-based), is at most $\sum_{l\in \mathrm{Cov}(\beta)\setminus\{i+3\}} |\mathbb{R}(4,5,6)\cap \mathbb{C}(l)| \leq 2\cdot 3 = 6 < r(4,5,6)$, a contradiction.

Similarly, if a position $(6,j)$ with $j \in [1,3]$ is occupied by a colour $\delta \in \mathbb{R}(4,5,6)$, then the number of pairs $\{\delta,\varepsilon\}$ with $\varepsilon \in \mathbb{R}(1,2,3)$, that are good in M, is at most $\sum_{l\in \mathrm{Cov}(\delta)\setminus\{j\}} |\mathbb{R}(1,2,3)\cap\mathbb{C}(l)| \leq 2\cdot 3 = 6 < r(1,2,3)$, a contradiction. □

Claim 36. $C_3^* \subseteq \mathbb{R}(6)$.

Proof. Consider a colour $\beta \in C_3^*$, and let n_i, $i \in \mathbb{R}(\beta)$, denote the number of pairs $\{\beta,\gamma\}$ with $\gamma \in \{\alpha_{1,6}, \alpha_{2,6}, \alpha_{3,6}\}$, that are good in M, and are provided by the copy of β in the row i of M. If $i \in [1,3]$, then $n_i = 1$ (with $\gamma = \alpha_{i,6}$), while $i \in [4,5]$ implies $n_i = 0$, and $i = 6$ yields $n_i = n_6 = 3$. Now, under the assumption $\beta \notin \mathbb{R}(6)$, from the inequalities $1 \leq |\mathbb{R}(\beta)\cap[1,3]| \leq 2$ we obtain $\sum_{i\in\mathbb{R}(\beta)} n_i = |\mathbb{R}(\beta)\cap[1,3]| \leq 2 < 3 = |\{\alpha_{1,6},\alpha_{2,6},\alpha_{3,6}\}|$, a contradiction. □

Claim 37. *$q = 16$, and there is a 3-colour $\beta \in \mathbb{R}(i,m,6)$ with $i \in [1,3]$ and $m \in [4,5]$ that occupies a position in $\{6\}\times[7,16]$.*

Proof. Since $c_{4+} = 0$ (Claim 34.3) and $r_2(6) = 3$, by Claims 36 and 27 we have $q = c_2(6)+c_3(6) = 3+[r(4,5,6)+c_3^*]$ and $c_3^* = q-3-r(4,5,6) \geq q-3-(q-9) = 6$. On the other hand, from Claim 34.1 we get $|C| = 2q+5 = c_2+c_3 = 15+[r(1,2,3)+r(4,5,6)+c_3^*]$ so that $q-3 = r(4,5,6)+c_3^* = 2q-10-r(1,2,3)$, $r(1,2,3) = q-7$, and then Claim 27 yields $9 \geq r(1,2,3) = q-7 \geq 16-7 = 9$, $r(1,2,3) = 9$ and $q = 16$. Using Claim 27 again we obtain $7 = q-9 \leq r(4,5,6) = q-3-c_3^* = 13-c_3^* \leq 13-6 = 7$, $r(4,5,6) = 7$ and $c_3^* = 6$.

The number of positions in $[1,3]\times[1,16]$, that are occupied by colours of C_3^*, is equal to $3\cdot 16 - c_2 - 3r(1,2,3) = 48-15-27 = 6 = c_3^*$, and each colour $\gamma \in C_3^*$ is involved in that counting, since $1 \leq |\mathbb{R}(\gamma)\cap[1,3]| \leq 2$. Therefore, for any $\gamma \in C_3^*$ we get $|\mathbb{R}(\gamma)\cap[1,3]| = 1$ and $|\mathbb{R}(\gamma)\cap[4,6]| = 2$.

Let $\beta \in C_3^*$ occupy a position in $\{6\}\times[7,16]$; the number of such colours is $c_3^*-3 = 3$, because $C_3^* \subseteq \mathbb{R}(6)$ (Claim 36), the positions in $\{6\}\times[1,3]$ are occupied by colours of C_3^* (Claim 35) and $(M)_{6,l} = \alpha_{l-3,6} \in C_2$, $l = 4,5,6$. Then $\mathbb{R}(\beta) = \{i,m,6\}$, where $i \in [1,3]$ and $m \in [4,5]$. □

We are now ready to finish our analysis by showing that for a colour $\beta \in \mathbb{R}(i,m,6)$ of Claim 37 the number of pairs $\{\beta,\gamma\}$ with $\gamma \in C_2$, that are good in M, is less than $c_2 = 15$, which represents a final contradiction proving Theorem 18.

First of all, if β occupies a position in $\{i\}\times[7,16]$, then all pairs $\{\beta,\gamma\}$ with $\gamma \in C_2$, that are good in M, are row-based. The number of such pairs is $r_2(i)+r_2(m)+r_2(6)-[r(i,m)+r(i,6)] = 5+6+3-(2+1) = 11 < 15 = c_2$, a contradiction.

Therefore, $\beta = (M)_{i,i+3}$, and we can find explicitly a colour $\gamma \in C_2$ such that the pair $\{\beta, \gamma\}$ is not good in M. Indeed, in this case $\mathbb{C}(i+3) \cap C_2 = \mathrm{Col}(\mathcal{M}) \supseteq \{\alpha_{i,6}\}$, where the perfect matching $\mathcal{M}$ in $K_{3,3}$ satisfies $\mathcal{M} = \{\{i,6\}, \{j,m\}, \{k,n\}\}$, $\{i,j,k\} = \{1,2,3\}$, $m \in [4,5]$ and $n = 9-m$. Then $\mathbb{C}(i+3) = \{\beta\} \cup \mathbb{R}(i,6) \cup \mathbb{R}(j,m) \cup \mathbb{R}(k,n)$, and so $\mathbb{R}(j,n) \cap \mathbb{C}(i+3) = \emptyset$ (recall that $r(i,6) = 1$ and $r(j,m) = 2 = r(k,n)$); thus, the pair $\{\beta,\gamma\}$ with $\gamma \in \mathbb{R}(j,n)$ is not column-based. Moreover, $\mathbb{R}(\beta) \cap \mathbb{R}(\gamma) = \{i,m,6\} \cap \{j,n\} = \emptyset$, the pair $\{\beta,\gamma\}$ is not row-based, hence it is not good in M, a contradiction. ■

The solution of the problem of determining $\mathrm{achr}(K_6 \Box K_q)$ is now complete. It is summarised in the final theorem of the paper, where

$$\begin{aligned}
J_3 &= [2,3] \cup \{q \in [41,\infty) : q \equiv 1 \,(\mathrm{mod}\ 2)\},\\
J_4 &= \{1,4,7\} \cup [16,40] \cup \{q \in [42,\infty) : q \equiv 0 \,(\mathrm{mod}\ 2)\},\\
J_5 &= \{5,8\},\\
J_6 &= \{6\} \cup [9,15],
\end{aligned}$$

and $J_3 \cup J_4 \cup J_5 \cup J_6 = [1,\infty)$.

Theorem 38. *If $a \in [3,6]$ and $q \in J_a$, then* $\mathrm{achr}(K_6 \Box K_q) = 2q + a$.

Proof. The achromatic number of $K_6 \Box K_q$ was analysed in [7] for $q \leq 4$ (for $q \leq 3$ see also Chiang and Fu [2]), in Horňák and Pčola [6] for $q = 5$, in [1] for $q = 6$, in [5] for $q = 7$, and in [4] for $q \in [41,\infty)$ with $q \equiv 1 \pmod 2$. The remaining statements have been proved in the present paper, see Theorem 13 for $q = 8$, Theorem 17 for $q \in [9,15]$, and Theorem 18 for q satisfying either $q \in [16,40]$ or $q \in [42,\infty)$ together with $q \equiv 0 \pmod 2$. ■

Corollary 39. If $q \in [1,\infty)$, then $2q + 3 \leq \mathrm{achr}(K_6 \Box K_q) \leq 2q + 6$.

Acknowledgements

This work was supported by the Slovak Research and Development Agency under the contract APVV-19-0153 and by the grant VEGA 1/0574/21.

The author is indebted to an anonymous referee for a very careful reading of the paper, which resulted in a substantial improvement of the presentation of the manuscript.

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Received 18 July 2021
Revised 1 March 2022
Accepted 1 March 2022
Available online 12 March 2022