Discussiones Mathematicae
Graph Theory 44 (2024) 363–381
https://doi.org/10.7151/dmgt.2450

EDGE DEGREE CONDITIONS FOR DOMINATING AND SPANNING CLOSED TRAILS

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Abstract

Edge degree conditions have been studied since the 1980s, mostly with regard to hamiltonicity of line graphs and the equivalent existence of dominating closed trails in their root graphs, as well as the stronger property of being supereulerian, i.e., admitting a spanning closed trail. For a graph G, let $\overline{\sigma}_2(G) = \min\{d(u) + d(v) \mid uv \in E(G)\}$. Chen *et al.* conjectured that a 3-edge-connected graph G with sufficientl large order n and $\overline{\sigma}_2(G) > \frac{n}{9} - 2$ is either supereulerian or contractible to the Petersen graph. We show that the conjecture is true when $\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1)$. Furthermore, we show that for an essentially k-edge-connected graph G with sufficiently large order n, the following statements hold.

(i) If $k = 2$ and $\overline{\sigma}_2(G) \geq 2(\lfloor n/8 \rfloor - 1)$, then either $L(G)$ is hamiltonian or G can be contracted to one of a set of six graphs which are not supereulerian;

(ii) If $k = 3$ and $\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1)$, then either $L(G)$ is hamiltonian or G can be contracted to the Petersen graph.

Keywords: hamiltonicity, supereulerian, degree sum, line graph.

2020 Mathematics Subject Classification: 05C07, 05C38, 05C45.

1. Introduction

We follow Bondy and Murty [2] for undefined terms and notation, and consider finite and loopless graphs only, but we allow multiple edges. Whenever we allow multiple edges in the sequel, we will indicate this by using the term *multigraph*; if we use the term graph, we will always assume the graph under consideration is simple. A graph G is *hamiltonian* if it has a *Hamilton cycle* (i.e., a spanning cycle). For a vertex x of a graph G, we denote by $N_G(x)$ the *neighborhood* of x in G, i.e., the set of vertices adjacent to x in G, and by $d_G(x) = |N_G(x)|$ (or simply $d(x)$) the *degree* of x in G. For multigraphs these concepts need to be adjusted in the obvious way. To distinguish vertex sets with different degrees, we use $D_i(G) = \{v \in V(G) \mid d(v) = i\}$, and we let $D(G) = D_1(G) \cup D_2(G)$. The *circumference* of G, denoted by $c(G)$, is the length of a longest cycle of G. The *girth* of G, denoted by $g(G)$, is the length of a shortest cycle of G. A graph is called *claw-free* if it has no induced subgraph isomorphic to $K_{1,3}$. The *line graph* of a graph G will be denoted by $L(G)$. As in [2], the *matching number*, the *connectivity* and the *edge-connectivity* of G are denoted by $\alpha'(G)$, $\kappa(G)$ and $\kappa'(G)$, respectively.

An edge cut X of G is *essential* if $G - X$ has at least two nontrivial components, i.e., containing at least one edge. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G is connected and does not have an essential edge cut X with $|X| < k$. Note that a graph G is essentially k-edge-connected if and only if its line graph $L(G)$ is k-connected or complete.

For a fixed function $f(n)$ of the order n of a graph G, degree conditions like $\delta(G) \geq f(n)$ are usually referred to as Dirac-type degree conditions. Similarly, with $\sigma_2(G) = \min\{d(u)+d(v)|\ uv \notin E(G)\}$, conditions of the form $\sigma_2(G) \geq f(n)$ are commonly called Ore-type degree conditions. Here we focus on so-called edge degree conditions, i.e., conditions of the type $\overline{\sigma}_2(G) \geq f(n)$, where $\overline{\sigma}_2(G) = \min\{d(u) + d(v)|\ uv \in E(G)\}$.

Degree conditions of the above three types are well-known in the area of hamiltonian graph theory as the classic approach to establishing sufficient conditions for guaranteeing hamiltonian properties. We refer the interested reader to the surveys [13, 14] for a wealth of results and references in this field, but here we only repeat the basic results involving the three above parameters. Let G be a graph of order $n \geq 3$. In [12], Dirac proved that $\delta(G) \geq \frac{n}{2}$ is a sufficient condition for G to be hamiltonian. As a generalization of Dirac's result, in [16], Ore proved that $\sigma_2(G) \geq n$ guarantees the hamiltonicity of G. In [3], Brualdi and Shanny proved that a graph G of order $n \geq 4$ with $\overline{\sigma}_2(G) \geq n$ has a hamiltonian line graph $L(G)$. Motivated by the hamiltonicity results on claw-free graph obtained in [8, 19], we focus on sufficient conditions for the hamiltonicity of the line graph $L(G)$ of a graph G involving $\overline{\sigma}_2(G)$.

Regarding substructures in a graph G that are crucial for investigating the existence of Hamilton cycles in $L(G)$, we introduce the following terminology. A closed trail Ψ is called a *spanning closed trail* (SCT) in G if $V(G) = V(\Psi)$, and is called a *dominating closed trail* (DCT) if $E(G - V(\Psi)) = \emptyset$. A graph is *supereulerian* if it contains an SCT. The family of supereulerian graphs is denoted by $\mathcal{SL}$. In order to limit the length of this paper, we refrain from repeating the details of the techniques that are nowadays more or less standard for this field. We refer the reader to [18] for a detailed explanation of the general approaches that are used in our proofs, like Catlin's reduction method of collapsible graphs, the core of a graph, Veldman's reduction, and so on.

All results on hamiltonicity of line graphs are based on the following close relationship between DCTs in graphs and the hamiltonicity of their line graphs.

Theorem 1 (Harary and Nash-Willians [15]). *Let G be a graph with at least three edges. Then $L(G)$ is hamiltonian if and only if G has a* DCT.

The following was conjectured by Benhocine *et al.* in [1], and proved by Veldman in [20].

Theorem 2 (Veldman [20]). *Let G be an essentially 2-edge-connected graph of order n such that $\overline{\sigma}_2(G) > \frac{2n}{5} - 2$. If n is sufficiently large, then $L(G)$ is hamiltonian.*

In [20], Veldman also obtained the following related result, showing that the lower bound in the above result can be improved, but only by allowing a class of exceptional graphs.

Theorem 3 (Veldman [20]). *Let G be an essentially 2-edge-connected graph of order n such that $\overline{\sigma}_2(G) > 2(\lfloor \frac{n}{7} \rfloor - 1)$. If n is sufficiently large, then either $L(G)$ is hamiltonian or G is contractible to a $K_{2,3}$ such that all vertices of degree 2 in $K_{2,3}$ are contracted vertices.*

We use $\theta(i, j, k)$ to denote the graph that is obtained from the multigraph consisting of two vertices and three multiple (parallel) edges by subdividing the three edges i, j, and k times, respectively. For example, $\theta(1, 1, 1) \cong K_{2,3}$. We also define the following two classes of graphs, referring to Figure 1.

- $\mathcal{G}_1 = \{K_{2,3}, K_{2,5}, W_3^*, \theta(1,1,2), \theta(1,1,3), \theta(1,2,2)\}$ and
- $\mathcal{G}_2 = \{J(2,2), J(2,3), K_{2,5}^*, K_{2,5}^{**}, C(6,2), C(6,2)', C(6,4), C(6,4)', \theta(1,1,4), \theta(1,1,4)', \theta(1,1,4)'', \theta(1,1,4)''', \theta(2,2,2), \theta(1,2,3), \theta(1,2,3)', W_3^{**}\}$.

As already mentioned in [20], Theorem 3 is best possible in the sense that there exist infinitely many essentially 2-edge-connected graphs G with $\overline{\sigma}_2(G) = 2(\lfloor \frac{n}{7} \rfloor - 1)$ such that $L(G)$ is nonhamiltonian and G is not contractible to $K_{2,3}$. Examples of such graphs can be found among the graphs contractible to $K_{2,5}$

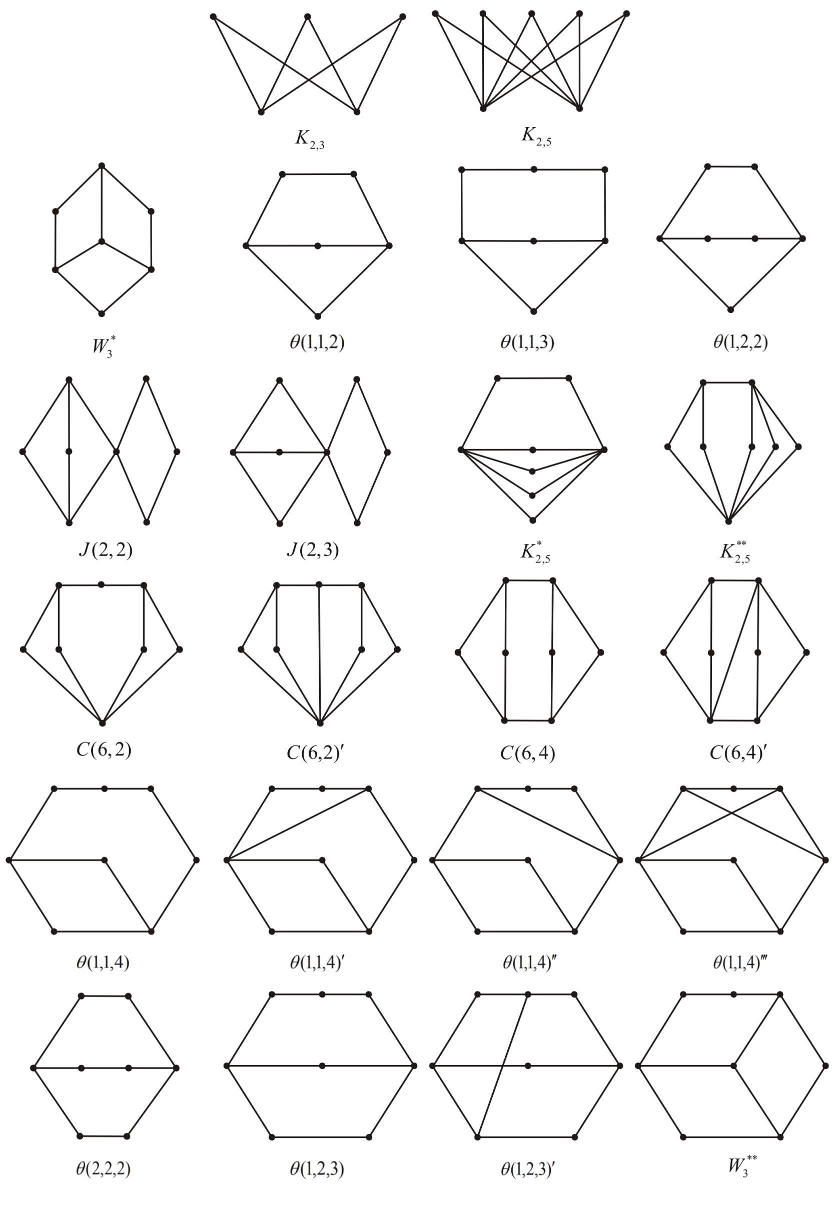

Figure 1. The graphs in $\mathcal{G}_1 \cup \mathcal{G}_2$.

or the 3-cube minus a vertex (the graph W_3^*). The following result confirms this, and shows that we can lower the bound on $\overline{\sigma}_2(G)$ in Theorem 3 slightly by excluding these two classes of exceptional graphs.

Theorem 4 (Tian and Xiong [19]). *Let G be an essentially 2-edge-connected graph of order n such that $\overline{\sigma}_2(G) \geq 2(\lfloor \frac{n}{7} \rfloor - 1)$. If n is sufficiently large, then either $L(G)$ is hamiltonian or G is contractible to a $K_{2,3}$, a $K_{2,5}$, or a W_3^* such that all vertices of degree 2 in $K_{2,3}$, $K_{2,5}$, and W_3^* are contracted vertices.*

The next result shows that for 3-edge-connected graphs, the lower bound in Theorem 4 can be improved considerably, even with a stronger conclusion. Here, the Petersen graph is the graph $P(10)$ depicted in Figure 2.

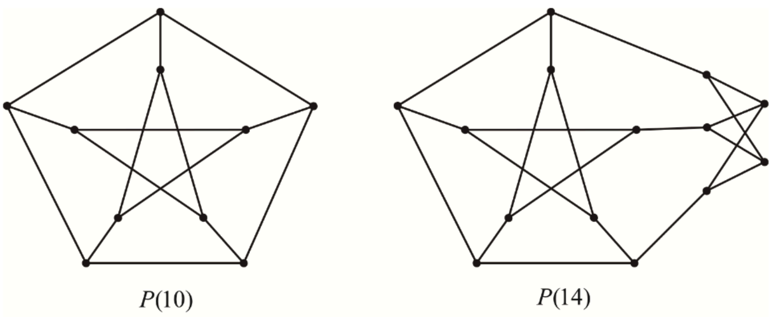

Figure 2. The Petersen graph $P(10)$, and the graph $P(14)$.

Theorem 5 (Chen and Lai [9]). *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq \frac{n}{5} - 2. \tag{1}$$

If n is sufficiently large, then either G is supereulerian or G can be contracted to the Petersen graph.

In [10], Chen and Lai improved the lower bound in Theorem 5 even further, and obtained the following result.

Theorem 6 (Chen and Lai [10]). *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq \frac{n}{6} - 2. \tag{2}$$

If n is sufficiently large, then either G is supereulerian or G can be contracted to the Petersen graph.

A natural question in this context is the following. What is the best possible lower bound for the degree sum condition on pairs of adjacent vertices for which all the exceptional graphs can be contracted to the Petersen graph? Due to a construction based on the so-called Blanuša snarks in [10], there is an infinite family of graphs showing that in the following conjecture, if true, the lower bound in (3) would be best possible.

Conjecture 7 (Chen and Lai [10]). *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) > \frac{n}{9} - 2. \tag{3}$$

If n is sufficiently large, then either G is supereulerian or G can be contracted to the Petersen graph.

We define $\mathcal{G}_3 = \{P(10), P(14)\}$, where $P(10)$ and $P(14)$ are the graphs depicted in Figure 2. In the sequel, we use G' to denote the reduction of a graph G, and G'_0 for the reduction of the core G_0 of a graph G. In [6], Chen *et al.* improved the result of Theorem 6 and obtained the following result.

Theorem 8 (Chen *et al.* [6]). *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) > 2\left(\frac{n}{15} - 1\right). \tag{4}$$

If n is sufficiently large, then either $G \in \mathcal{SL}$ or $G' \in \mathcal{G}_3$. Furthermore, if $\overline{\sigma}_2(G) \geq 2(\frac{n}{14} - 1)$ and $G' = P(14)$, then $n = 14s$ and each vertex in $P(14)$ is obtained by contracting a K_s or $K_s - e$ for some $e \in E(K_s)$.

Let $\mathcal{Q}_0(r, k)$ be the family of k-edge-connected triangle-free graphs of order at most r that do not admit an SCT. We will show that $\mathcal{Q}_0(8, 2) = \mathcal{G}_1 \cup \mathcal{G}_2$ (see Theorem 16 in Section 2). In the following, for given integer $p > 0$, we use "$n \gg p$" for "n is sufficiently large relative to p". Our main result reads as follows. As we will see, Theorems 2, 3, 4, 5, 6 and 8 are all special cases of Theorem 9, with $(p, k) \in \{(5, 2), (7, 2), (10, 3), (12, 3), (15, 3)\}$.

Theorem 9. *Let G be an essentially k-edge-connected graph of order n ($k \in \{2, 3\}$), and let $p \geq 2$ be an integer such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1). \tag{5}$$

If $n \gg p$, then either $L(G)$ is hamiltonian or G has no DCT *and $G'_0 \in \mathcal{Q}_0(\max\{p, \frac{3}{2}p - 4\}, k)$.*

We continue with stating some consequences of our main result, and postpone all proofs to later sections. As applications of Theorem 9, the following results are obtained.

Theorem 10. *Let G be an essentially 2-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/8 \rfloor - 1). \tag{6}$$

If n is sufficiently large, then either $L(G)$ is hamiltonian or G is contractible to a graph in $\{K_{2,3}, K_{2,5}, W_3^*, C(6,2)', C(6,4)', \theta(1,1,4)'''\}$.

Theorem 11. *Let G be an essentially 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1). \tag{7}$$

If n is sufficiently large, then either $L(G)$ is hamiltonian or G can be contracted to the Petersen graph.

Theorem 12. *Let G be a 3-edge-connected graph of order n such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1). \tag{8}$$

If n is sufficiently large, then either G is supereulerian or G can be contracted to the Petersen graph.

The remainder of this manuscript is organized as follows. In Section 2, we will present some useful results. In Section 3, we start by presenting and proving some Ore-type analogues of the results in this paper. Section 3.1 contains our proof of Theorem 9, whereas the proofs of Theorems 10, 11 and 12 are given in Section 3.2.

2. Preliminaries

As we mentioned, the first result on edge degrees in this context is due to Brualdi and Shanny [3], but we would like to emphasize that the foundations for all the more recent results in this area were laid by the late Paul Catlin, and subsequently by Catlin and his coworkers. Some of the main results of Catlin *et al.* are presented in the following theorem.

Theorem 13 (Catlin *et al.* [4, 5])**.** *Let G be a connected graph and let G' be the reduction of G.*

(a) *G is collapsible if and only if $G' = K_1$, and $G \in \mathcal{SL}$ if and only if $G' \in \mathcal{SL}$.*

(b) *G has a* DCT *if and only if G' has a* DCT *containing all the contracted vertices of G'.*

(c) *If G is a reduced graph, then G is simple and triangle-free, and $\delta(G) \le 3$. Moreover, any subgraph Γ of G is reduced, and either $\Gamma \in \{K_1, K_2, K_{2,t}$ $(t \ge 2)\}$ or $|E(\Gamma)| \le 2|V(\Gamma)| - 5$.*

We state some facts we need on reduced graphs, as summarized in the following theorem, where the first fact is folklore and easy to prove.

Theorem 14. *Let G be a connected reduced graph of order n. Then each of the following holds.*

(a) *If $G \notin \mathcal{SL}$ and $\kappa'(G) \ge 2$, then $n \ge 5$ and $n = 5$ only if $G = K_{2,3}$.*

(b) (Corollary 4.11 in [11]) *If $n \le 15$ and $\delta(G) \ge 3$, then G is supereulerian if and only if $G \notin \mathcal{G}_3$.*

(c) (Lemma 4.8 in [11]) *If $n \ge 15$, $\kappa'(G) \ge 3$ and $\alpha'(G) \le 7$, then G is supereulerian.*

In [19], Tian and Xiong characterized some small graphs which have no SCT, as follows.

Theorem 15 (Tian and Xiong [19])**.** *Let G be a 2-edge-connected triangle-free graph of order at most 7. Then either G is supereulerian or $G \in \mathcal{G}_1$.*

The following result extends the above result and will be used for our proof of Theorem 10.

Theorem 16. *Let G be a 2-edge-connected triangle-free graph of order at most 8. Then either G is supereulerian or $G \in \mathcal{G}_1 \cup \mathcal{G}_2$.*

Let G be a 2-connected graph, and let $C = v_0 v_1 v_2 \cdots v_{c(G)-1} v_0$ be a longest cycle of G, where the subscripts are taken modulo $c(G)$ throughout. Then any component of $G - V(C)$ has at least two different neighbors on C. Denote by $d_C(v_i, v_j)$ the distance between $v_i, v_j \in V(C)$ (with $v_i \neq v_j$) on C. Obviously, $1 \le d_C(v_i, v_j) \le \left\lfloor \frac{|V(C)|}{2} \right\rfloor$.

***Proof of Theorem* 16.** Let G be a 2-edge-connected triangle-free graph of order at most 8. If G has an SCT, then we are done. So, in the following, we assume that G has no SCT. If $|V(G)| \le 7$, then by Theorem 15, $G \in \mathcal{G}_1$. So, in the following, we only need to consider the case $|V(G)| = 8$.

Suppose first that $\kappa(G) = 1$. Let $B_1, B_2, \ldots, B_t$ $(t \ge 2)$ be the blocks of G. Since G is triangle-free, $|V(B_i)| \ge 4$. Note that B_i and B_j $(i \neq j)$ have at most one vertex in common. Then $t = 2$; otherwise, $8 = |V(G)| \ge 3 \times 4 - 2 = 10$, a contradiction. Without loss of generality, we may assume that $|V(B_1)| = 4$ and

$|V(B_2)| = 5$. Since G is triangle-free, $B_1 = C_4$. Then $B_2 \notin \mathcal{SL}$; otherwise, G has an SCT, a contradiction. Now, by Theorem 14(a), $B_2 = K_{2,3}$. Note that $|V(B_1) \cap V(B_2)| = 1$. Then G is isomorphic to the graph $J(2,2)$ or $J(2,3)$.

In the following, we suppose that G is 2-connected. Since G is triangle-free and since $G \notin \mathcal{SL}$, $4 \leq c(G) \leq 7$.

Observation 1. *By deleting all chords of C from G, the resulting* 2*-connected graph G^τ is a spanning subgraph of G. Obviously, C is also a longest cycle of G^τ. Then G^τ has no* SCT*; otherwise, G has an* SCT*, a contradiction.*

Note that by adding the deleted chords of C to G^τ one by one, by our assumptions, at each step we obtain a spanning subgraph of G which has no SCT, or we derive at a contradiction. Obviously, if $4 \leq c(G) \leq 5$, then C has no chord. We distinguish the cases that $c(G) = 4, 5, 6$, and 7.

Case 1. $c(G) = 4$. Then $G - V(C) = 4K_1$, or $K_2 \cup 2K_1$, or $P_3 \cup K_1$, or $2K_2$, or P_4, or $K_{1,3}$, or C_4. Suppose that $G - V(C) = K_2 \cup 2K_1$, or $P_3 \cup K_1$, or $2K_2$, or P_4, or $K_{1,3}$, or C_4. Since G is 2-connected and triangle-free, there exists a path $x_1 \cdots x_k$ ($k = 2$ or 3 or 4) in $G - V(C)$ with $v_i \in N_G(x_1) \cap V(C)$ and $v_j \in N_G(x_k) \cap V(C)$ ($v_i \neq v_j$). But now we can find a cycle containing the vertices $x_1, \ldots, x_k$ with length more than 4, a contradiction. Then $G - V(C) = 4K_1$. Since G is 2-connected and triangle-free, and since $c(G) = 4$, $G = K_{2,6}$. Obviously, $K_{2,6}$ has an SCT, a contradiction.

Case 2. $c(G) = 5$. Then $G - V(C) = 3K_1$, or $K_2 \cup K_1$, or P_3. Suppose that $G - V(C) = K_2 \cup K_1$ or P_3. Since G is 2-connected and triangle-free, there exists a path $x_1 \cdots x_k$ ($k = 2$ or 3) in $G - V(C)$ with $v_i \in N_G(x_1) \cap V(C)$ and $v_j \in N_G(x_k) \cap V(C)$ ($v_i \neq v_j$). But now we can find a cycle containing the vertices $x_1, \ldots, x_k$ with length more than 5, a contradiction. Thus, $G - V(C) = 3K_1$. Let $V(G) \setminus V(C) = \{x_1, x_2, x_3\}$. Since G is 2-connected and triangle-free, $d_G(x_i) = 2$ ($i = 1, 2, 3$). Without loss of generality, we may assume that $N_G(x_1) = \{v_i, v_{i+2}\}$. Then, by symmetry and since $c(G) = 5$, we can assume that either $N_G(x_j) = \{v_i, v_{i+2}\}$ or $N_G(x_j) = \{v_i, v_{i+3}\}$ ($j = 2, 3$). Then G is isomorphic to the graph $K^*_{2,5}$ or $K^{**}_{2,5}$.

Case 3. $c(G) = 6$. By Observation 1, without loss of generality, we first assume that C is an induced cycle of G, namely $G = G^\tau$. Then $G - V(C) = 2K_1$ or K_2. Let $V(G) \setminus V(C) = \{x_1, x_2\}$. We distinguish the following two subcases.

Subcase 3.1. $G - V(C) = 2K_1$. Since G is 2-connected and triangle-free, $2 \leq d_G(x_i) \leq 3$ ($i = 1, 2$). Suppose that $d_G(x_1) = 3$ (it is similar for $d_G(x_2) = 3$). Without loss of generality, we may assume that $N_G(x_1) = \{v_i, v_{i+2}, v_{i+4}\}$. By $2 \leq d_G(x_2) \leq 3$, $2 \leq |N_G(x_2) \cap V(C)| \leq 3$. Then, it is easy to check that G has an SCT, a contradiction. Therefore, $d_G(x_1) = d_G(x_2) = 2$. Without loss of generality, we may assume that either $N_G(x_1) = \{v_i, v_{i+2}\}$ or $N_G(x_1) = \{v_i, v_{i+3}\}$.

Suppose first that $N_G(x_1) = \{v_i, v_{i+2}\}$. If $v_i \in N_G(x_2)$ (by symmetry, it is similar for $v_{i+2} \in N_G(x_2)$), then $v_{i+1}, v_{i+2}, v_{i+3}, v_{i+5} \notin N_G(x_2)$; otherwise, either G has a triangle or G has an SCT, a contradiction. Then by $d_G(x_2) = 2$, $N_G(x_2) = \{v_i, v_{i+4}\}$. Then G has a spanning subgraph isomorphic to the graph $C(6,2)$.

If $v_{i+1} \in N_G(x_2)$, then $v_i, v_{i+2}, v_{i+3}, v_{i+4}, v_{i+5} \notin N_G(x_2)$; otherwise, either G has a triangle or G has a cycle of length more than 6, a contradiction. Then $d_G(x_2) = 1$, contrary to $d_G(x_2) = 2$. Therefore, in the following, we may assume that $v_i, v_{i+1}, v_{i+2} \notin N_G(x_2)$. Since G is 2-connected and triangle-free, and by $d_G(x_2) = 2$, $N_G(x_2) = \{v_{i+3}, v_{i+5}\}$. Then G has a spanning subgraph isomorphic to the graph $C(6,4)$.

Now suppose that $N_G(x_1) = \{v_i, v_{i+3}\}$. By $d_G(x_2) = 2$, $|N_G(x_2) \cap V(C)| = 2$. Then, it is easy to check that either G has an SCT, or G has a triangle, or G has a cycle of length more than 6, a contradiction.

Subcase 3.2. $G - V(C) = K_2$. Without loss of generality, we may assume that $N_G(x_1) \cap V(C) = \{v_i\}$. Since G is 2-connected and triangle-free, and since $c(G) = 6$, $N_G(x_2) \cap V(C) = \{v_{i+3}\}$. Then G has a spanning subgraph isomorphic to $\theta(2,2,2)$.

In both subcases, by Observation 1, joining any two nonadjacent vertices of $C(6,2)$ or $C(6,4)$ or $\theta(2,2,2)$ by an edge (step by step) will result in a triangle, or a $C(6,2)'$, or a $C(6,4)'$, or an SCT in the new graph. Hence, $G \in \{C(6,2), C(6,2)', C(6,4), C(6,4)', \theta(2,2,2)\}$.

Case 4. $c(G) = 7$. By Observation 1, without loss of generality, we first assume that C is an induced cycle of G, namely $G = G^\tau$. Then $G - V(C) = K_1$. Let $V(G) \setminus V(C) = \{x\}$. Since G is 2-connected and triangle-free, $2 \leq d_G(x) \leq 3$.

Suppose that $d_G(x) = 2$. We may assume that $N_G(x) = \{v_i, v_j\}$ ($v_i \neq v_j$). Obviously, $2 \leq d_C(v_i, v_j) \leq 3$. If $d_C(v_i, v_j) = 2$, then, without loss of generality, we may assume that $N_G(x) = \{v_i, v_{i+2}\}$. Then G has a spanning subgraph isomorphic to $\theta(1,1,4)$. If $d_C(v_i, v_j) = 3$, then, without loss of generality, we may assume that $N_G(x) = \{v_i, v_{i+3}\}$. Then G has a spanning subgraph isomorphic to $\theta(1,2,3)$.

Suppose that $d_G(x) = 3$. Without loss of generality, we may assume that $N_G(x) = \{v_i, v_{i+2}, v_{i+4}\}$. Then G has a spanning subgraph isomorphic to W_3^{**}.

By Observation 1, joining any two nonadjacent vertices of $\theta(1,1,4)$ or $\theta(1,2,3)$ or W_3^{**} by an edge (step by step) will result in a triangle, or a graph in $\{\theta(1,1,4)', \theta(1,1,4)'', \theta(1,1,4)''', \theta(1,2,3)', W_3^{**}\}$, or an SCT of the new graph. Hence, G is one of the graphs in $\{\theta(1,1,4), \theta(1,1,4)', \theta(1,1,4)'', \theta(1,1,4)''', \theta(1,2,3), \theta(1,2,3)', W_3^{**}\}$. This completes the proof. ■

Using Theorem 13, Veldman [20] and Shao [17] proved the following.

Theorem 17. *Let G be an essentially k-edge-connected graph with $\overline{\sigma}_2(G) \geq 5$, such that $k \in \{2, 3\}$ and $L(G)$ is not complete. Let G_0' be the reduction of the core G_0 of G. Then each of the following holds.*

(a) *G_0 is well-defined, nontrivial, $\delta(G_0) \geq \kappa'(G_0) \geq k$, and $\kappa'(G_0') \geq \kappa'(G_0) \geq k$.*

(b) (Lemma 5 [20]) *G has a* DCT *if and only if G_0' has a* DCT *containing all the nontrivial vertices.*

In [20], Veldman obtained the following result.

Theorem 18 (Veldman [20])**.** *Let G be a connected graph of order n, and let $p \geq 2$ be an integer such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1). \tag{9}$$

If $n \gg p$, then

$$|V(G'')| \leq \max\left\{p, \frac{3}{2}p - 4\right\}, \tag{10}$$

where G'' is the $D(G)$-reduction of G. Moreover, for $p \leq 7$, (10) holds with equality only if (9) holds with equality.

Using Theorem 18, we can easily deduce the following result.

Theorem 19. *Let G be an essentially k-edge-connected graph of order n (with $k \in \{2, 3\}$), and let $p \geq 2$ be an integer such that*

$$\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1). \tag{11}$$

If $n \gg p$, then exactly one of the following following holds.

(a) *$G_0 \in \mathcal{SL}$;*

(b) *$G_0' \notin \mathcal{SL}$ with $|V(G_0')| \leq \max\left\{p, \frac{3}{2}p - 4\right\}$ and $\kappa'(G_0') \geq k$.*

***Proof of Theorem* 19.** By Theorem 13(a), (a) and (b) of Theorem 19 are mutually exclusive. Suppose that $G_0 \notin \mathcal{SL}$. Then $L(G)$ is not complete; otherwise, $G = K_{1,n-1}$, and so $G_0 \in \mathcal{SL}$, a contradiction. By Theorem 13(a), and since $G_0 \notin \mathcal{SL}$, $G_0' \notin \mathcal{SL}$. Since $\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1)$, if $n \geq 4p$, then $\overline{\sigma}_2(G) \geq 6$, and consequently $D(G)$ is an independent set. Let G'' be the $D(G)$-reduction of G. By Theorem 18, $|V(G'')| \leq \max\{p, \frac{3}{2}p - 4\}$. Since G_0' is a refinement of the $D(G)$-reduction of G, $|V(G_0')| \leq |V(G'')| \leq \max\{p, \frac{3}{2}p - 4\}$. By Theorem 17(a), $\kappa'(G_0') \geq k$. This completes the proof. ∎

3. Supereulerian Graphs and Hamiltonian Line Graphs

Before we continue with the remaining proofs of our results, we mention one other result of Chen [7] and an application of our Theorem 16 in order to obtain an Ore-type analogue of the results in this paper.

Let G be a graph, and let $k \geq 0$ be an integer. If there is a graph G^* such that G can be obtained from G^* by removing at most k edges, then G is said to be *at most k edges short* of being G^*.

Theorem 20 (Chen [7])**.** *Let G be a 2-edge-connected graph with girth $g \in \{3, 4\}$, and let $p \geq 2$ be an integer. If*

$$\sigma_2(G) \geq \frac{2}{g-2}\left(\frac{n}{p} + g - 4\right),$$

and if

$$n \geq 4(g-2)p^2,$$

then exactly one of the following holds.

(a) *$G \in \mathcal{SL}$;*

(b) *$G' \notin \mathcal{SL}$ and $|V(G')| \leq p$, where G' is the reduction of G. Further, if $|V(G')| = p$, then $n = (g-2)ps$, for some integer s, and $\delta(G) = \frac{1}{g-2}\big(\frac{n}{p} + g - 4\big)$, and either*

   (i) *$g = 3$, and the preimage H_i of each vertex v_i of G' is at most $\frac{1}{2}d_{G'}(v_i)$ edges short of being K_s, or*

   (ii) *$g = 4$, and the preimage H_i of each vertex v_i of G' is at most $\frac{1}{2}d_{G'}(v_i)$ edges short of being $K_{s,s}$.*

As an application of Theorems 16 and 20, we obtain the following result.

Theorem 21. *Let G be a 2-edge-connected graph with girth $g \in \{3, 4\}$. If*

$$\sigma_2(G) \geq \frac{2}{g-2}\left(\frac{n}{8} + g - 4\right),$$

and if

$$n \geq 256(g-2),$$

then exactly one of the following holds.

(a) *$G \in \mathcal{SL}$;*

(b) *$G' \in \mathcal{G}_1 \cup \mathcal{G}_2$, where G' is the reduction of G. In particular, if $|V(G')| = 8$, then $n = 8(g-2)s$, for some integer s, and either*

(i) $g = 3$, *and the preimage* H_i *of each vertex* v_i *of* G' *is at most* 2 *edges short of being* K_s, *or*

(ii) $g = 4$, *and the preimage* H_i *of each vertex* v_i *of* G' *is at most* 2 *edges short of being* $K_{s,s}$.

***Proof of Theorem* 21.** If G has an SCT, then we are done. In the following, we assume that G has no SCT. By Theorem 20(b), $G' \notin \mathcal{SL}$ and $|V(G')| \leq 8$. By the definition of contraction, $\kappa'(G') \geq \kappa'(G) \geq 2$. By Theorem 13(c), G' is simple and triangle-free. Then by Theorem 16, $G' \in \mathcal{G}_1 \cup \mathcal{G}_2$. So, $d_{G'}(v) \leq 5$ for any $v \in V(G')$. By Theorem 20(b), Theorem 21(b) holds. This completes the proof. ■

3.1. Proof of Theorem 9 and a useful proposition

***Proof of Theorem* 9.** If $L(G)$ is hamiltonian, then we are done. In the following, we assume that $L(G)$ is not hamiltonian, and so $L(G)$ is not complete. Then by Theorem 1, G has no DCT. Since $\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1)$, if $n \geq 4p$, then $\overline{\sigma}_2(G) \geq 6$, and consequently $D(G)$ is an independent set. Let G'' be the $D(G)$-reduction of G. By Theorem 18, $|V(G'')| \leq \max\left\{p, \frac{3}{2}p - 4\right\}$. Let G_0' be the reduction of the core G_0 of G. By Theorem 13(c), G_0' is simple and triangle-free. By Theorem 17, $G_0' \notin \mathcal{SL}$ and $\kappa'(G_0') \geq k$. Since G_0' is a refinement of the $D(G)$-reduction of G, $|V(G_0')| \leq |V(G'')| \leq \max\left\{p, \frac{3}{2}p - 4\right\}$. This completes the proof. ■

Let G_0' be the reduction of the core G_0 of G. For $v \in V(G_0')$, let $\Gamma(v)$ be the preimage of v in G. For convenience, we define the following sets, and we prove a useful proposition that we use in some of the later proofs.

- $S_0 = \{v \in V(G_0') |\ v \text{ is a nontrivial vertex in } G_0'\}$;
- $S_1 = \{v \in S_0 |\ |V(\Gamma(v))| > 1\}$;
- $S_2 = S_0 \setminus S_1$, the set of vertices v with $\Gamma(v) = K_1$ and adjacent to some vertices in $D_2(G)$;
- $V_0 = V(G_0') \setminus S_0$.

Proposition 22. *Let G be an essentially k-edge-connected graph of order n ($k \in \{2,3\}$) with $\overline{\sigma}_2(G) \geq 2(\lfloor n/p \rfloor - 1)$, where $p \geq 2$ is an integer. Let G_0' be the reduction of the core G_0 of G, and suppose $G_0' \notin \mathcal{SL}$. Let S_0, S_1 and V_0 be the sets defined above. If $n \gg p$, then each of the following holds.*

(a) *If $v \in S_1$, then $|V(\Gamma(v))| \geq \left\lfloor \frac{n}{p} \right\rfloor - l + 1$, where $l = \max\left\{p, \frac{3p}{2} - 4\right\}$.*

(b) *$S_1 = S_0$.*

(c) *V_0 is an independent set, and $N_{G_0'}(v) \subseteq S_1$ for any $v \in V_0$.*

(d) $|S_0| \le p$. *Furthermore, if* $|S_0| = p$, *then* $V(G_0') = S_0$.

Proof. As the assumptions of Proposition 22 imply the assumptions of Theorem 19, it follows from Theorem 19 that $|V(G_0')| \le \max\{p, \frac{3p}{2} - 4\}$ and $\kappa'(G_0') \ge k \ge 2$. For convenience, in the following, let $l = \max\{p, \frac{3p}{2} - 4\}$. For $v \in V(G_0')$, let $\Gamma(v)$ be the preimage of v in G. By Theorem 13(c), G_0' is simple and triangle-free. Then

$$d_{G_0'}(v) \le |V(G_0')| - 2 \le l - 2, \text{ for any } v \in V(G_0'). \tag{12}$$

(a) For each $v \in S_1$, by (11) and since $n \gg p$, there exists a vertex $u \in V(\Gamma(v))$ with $d_G(u) \ge \lfloor \frac{n}{p} \rfloor - 1$. Then by (12),

$$|V(\Gamma(v))| \ge |N_G(u) \cap V(\Gamma(v))| \ge d_G(u) - d_{G_0'}(v) \ge \left\lfloor \frac{n}{p} \right\rfloor - l + 1.$$

(b) Suppose that $S_1 \ne S_0$. Let $v \in S_2 = S_0 \setminus S_1$. Then, $d_G(v) = d_{G_0'}(v)$, and v is adjacent to a vertex $u \in D_2(G)$. By (11) and (12),

$$2(\lfloor n/p \rfloor - 1) \le \overline{\sigma}_2(G) \le d_G(v) + d_G(u) = d_{G_0'}(v) + 2 \le l,$$

contrary to the fact that $n \gg p$, and so (b) is proved.

(c) Suppose that there are two vertices $v_1, v_2 \in V_0$ such that $v_1v_2 \in E(G_0')$. Since $v_i \in V_0$ $(i = 1, 2)$, $d_G(v_i) = d_{G_0'}(v_i)$. By (11) and (12),

$$2(\lfloor n/p \rfloor - 1) \le \overline{\sigma}_2(G) \le d_G(v_1) + d_G(v_2) \le 2l - 4,$$

contrary to the fact that $n \gg p$, and so (c) is proved.

(d) Suppose that $s = |S_0| > p$. By (b) above, $S_1 = S_0$. Let $S_1 = \{v_1, v_2, \ldots, v_s\}$. Then by (a),

$$s\left(\left\lfloor \frac{n}{p} \right\rfloor - l + 1\right) \le \left| \bigcup_{i=1}^{s} V(\Gamma(v_i)) \right| \le n,$$

a contradiction if $n \gg p$.

Now suppose that $|S_1| = p$ and $V(G_0') \setminus S_1 \ne \emptyset$. Let $v \in V_0 = V(G_0') \setminus S_1$. By (c), we can assume that $N_{G_0'}(v) = \{v_1, v_2, \ldots, v_t\}$ and $N_G(v) = \{w_1, w_2, \ldots, w_t\}$ such that $w_i \in \Gamma(v_i)$ $(1 \le i \le t)$. Note that $D_2(G_0') \subseteq S_1$. Then $d_{G_0'}(v) \ge 3$ and so $t \ge 3$. By (12), $d_G(v) = d_{G_0'}(v) \le l - 2$. Then

$$d_G(w_i) \ge 2\left(\left\lfloor \frac{n}{p} \right\rfloor - 1\right) - d_G(v) \ge 2\left\lfloor \frac{n}{p} \right\rfloor - l. \tag{13}$$

Since G_0' is 2-edge-connected and triangle-free, and since $t \geq 3$, $d_{G_0'}(v_i) \leq l-3$. Then by (13),

$$(14) \quad |V(\Gamma(v_i))| \geq |N_G(w_i) \cap V(\Gamma(v_i))| \geq d_G(w_i) - d_{G_0'}(v_i) \geq 2\left\lfloor \frac{n}{p} \right\rfloor - 2l + 3.$$

By (a) and (14),

$$|V_0| + t\left(2\left\lfloor \frac{n}{p} \right\rfloor - 2l + 3\right) + (p-t)\left(\left\lfloor \frac{n}{p} \right\rfloor - l + 1\right) \leq \left| \bigcup_{u \in V(G_0')} V(\Gamma(u)) \right| = n.$$

Then

$$|V_0| + t + (p+t)\left(\left\lfloor \frac{n}{p} \right\rfloor - l + 1\right) \leq n,$$

a contradiction if $n \gg p$, and so (d) is proved. ■

3.2. Proofs of Theorems 10, 11 and 12

***Proof of Theorem* 10.** This is the special case of Theorem 9 with $p = 8$ and $k = 2$. Suppose that $L(G)$ is not hamiltonian. Because $\overline{\sigma}_2(G) \geq 2(\lfloor n/8 \rfloor - 1)$, if $n \geq 32$, then $\overline{\sigma}_2(G) \geq 6$, and consequently $D(G)$ is an independent set. Let G_0' be the reduction of the core G_0 of G. By Theorem 13(c), G_0' is simple and triangle-free. Then by Theorems 9 and 16, G has no DCT and $G_0' \in \mathcal{G}_1 \cup \mathcal{G}_2$. Note that each of the graphs in the set $\{J(2,2), J(2,3), C(6,2), C(6,4), \theta(1,1,2), \theta(1,1,3), \theta(1,2,2), \theta(1,1,4), \theta(1,1,4)', \theta(1,1,4)'', \theta(2,2,2), \theta(1,2,3), \theta(1,2,3)'\}$ can be contracted to a $K_{2,3}$, each graph in $\{K_{2,5}^*, K_{2,5}^{**}\}$ can be contracted to a $K_{2,5}$, and W_3^{**} can be contracted to a W_3^*. We conclude that G_0' can be contracted to a graph in $\{K_{2,3}, K_{2,5}, W_3^*, C(6,2)', C(6,4)', \theta(1,1,4)'''\}$. This completes the proof. ■

***Proof of Theorem* 11.** This is the special case of Theorem 9 with $p = 15$ and $k = 3$. Suppose that $L(G)$ is not hamiltonian. Because $\overline{\sigma}_2(G) \geq 2(\lfloor n/15 \rfloor - 1)$, if $n \geq 60$, then $\overline{\sigma}_2(G) \geq 6$, and consequently $D(G)$ is an independent set. Let G_0' be the reduction of the core G_0 of G. By Theorem 9, $G_0' \notin \mathcal{SL}$ with $|V(G_0')| \leq 18$ and $\kappa'(G_0') \geq 3$.

By Proposition 22(b), $S_1 = S_0$ and so $V(G_0') = V_0 \cup S_1$. If $|V(G_0')| \leq 15$, then, by Theorem 14(b), $G_0' \in \{P(10), P(14)\}$. Obviously, in this case, G can be contracted to the Petersen graph. In the following, we only need to consider the case that $16 \leq |V(G_0')| \leq 18$.

Let $V_0 = \{v_1, v_2, \ldots, v_t\}$ and $S_1 = \{v_{t+1}, v_{t+2}, \ldots, v_{|V(G_0')|}\}$. Without loss of generality, we can assume that

$$|V(\Gamma(v_{t+1}))| \leq |V(\Gamma(v_{t+2}))| \leq \cdots \leq \left|V\left(\Gamma\left(v_{|V(G_0')|}\right)\right)\right|.$$

Note that $\sum_{i=t+1}^{|V(G_0')|} |V(\Gamma(v_i))| \leq n$. Hence $|V(\Gamma(v_{t+1}))| \leq \frac{n}{|V(G_0')|-t}$.

By Proposition 22(d), $|S_1| \leq 15$. Then, since $|V(G_0')| \geq 16$, $t \geq 1$. By Proposition 22(c), V_0 is an independent set and $\bigcup_{i=1}^t N_{G_0'}(v_i) \subseteq S_1$. Since G_0' is 3-edge-connected,

$$d_{G_0'}(v_i) \geq 3, \text{ for } v_i \in V(G_0'). \tag{15}$$

Let $\{w_1, w_2, \ldots, w_s\}$ be a maximal subset of $\bigcup_{i=1}^t N_G(v_i)$ which satisfies the following two conditions.

(i) For any pair of vertices $\{w_i, w_j\} \subset \{w_1, w_2, \ldots, w_s\}$ $(w_i \neq w_j)$, there exists a pair of vertices $\{z_i, z_j\} \subset \bigcup_{i=1}^t N_{G_0'}(v_i)$ $(z_i \neq z_j)$ such that $w_i \in \Gamma(z_i), w_j \in \Gamma(z_j)$, and $\Gamma(z_i) \cap \Gamma(z_j) = \emptyset$;

(ii) For each $w_i \in \{w_1, w_2, \ldots, w_s\}$, there is a vertex v_j $(j \leq t)$ that is adjacent to w_i in G.

Note that in this case $\left|\bigcup_{i=1}^t N_{G_0'}(v_i)\right| = s$. Then, since $t \geq 1$ and (15),

$$s \geq 3.$$

Claim 23. $d_{G_0'}(v_i) \leq 15$, *for* $v_i \in V(G_0')$.

Proof. By Theorem 13(c), G_0' is simple and triangle-free. Then, since $\kappa'(G_0') \geq 3$ and $|V(G_0')| \leq 18$, the claim holds immediately. □

By Claim 23,

$$d_G(w_i) \geq 2\left(\left\lfloor\frac{n}{15}\right\rfloor - 1\right) - d_G(v_j) \geq 2\left(\frac{n-14}{15} - 1\right) - 15 = \frac{2n-283}{15}, \tag{16}$$

where v_j $(j \leq t)$ is adjacent to w_i in G.

By (i), for each w_i, there is a vertex $z_i \in \bigcup_{i=1}^t N_{G_0'}(v_i)$ such that $w_i \in V(\Gamma(z_i))$. Hence by (16) and Claim 23,

$$\begin{aligned} |V(\Gamma(z_i))| &\geq |N_G(w_i) \cap V(\Gamma(z_i))| \geq d_G(w_i) - d_{G_0'}(z_i) \\ &\geq \frac{2n-283}{15} - 15 = \frac{2n-508}{15}. \end{aligned} \tag{17}$$

Hence,

$$\left|\bigcup_{i=1}^s V(\Gamma(z_i))\right| = \sum_{i=1}^s |V(\Gamma(z_i))| \geq \frac{s(2n-508)}{15}.$$

Since $\left|\bigcup_{i=1}^s V(\Gamma(z_i))\right| \leq n$ and n is sufficiently large,

$$s \leq 7.$$

Therefore, $3 \leq s \leq 7$.

For $x \in S_1$, by Proposition 22(a),

$$|V(\Gamma(x))| \geq \left\lfloor \frac{n}{15} \right\rfloor - 18 + 1 \geq \frac{n-14}{15} - 17 \geq \frac{n-269}{15}. \tag{18}$$

In particular, if $x \in \bigcup_{i=1}^{t} N_{G_0'}(v_i)$, then $|V(\Gamma(x))| = |V(\Gamma(z_i))|$ for some $z_i \in \bigcup_{i=1}^{t} N_{G_0'}(v_i)$. By (17), $|V(\Gamma(x))| \geq \frac{2n-508}{15}$.

Without loss of generality, we let $V(G_0') = \{v_1, \ldots, v_t, v_{t+1}, \ldots, v_{t+s}, v_{t+s+1}, \ldots, v_{t+s+r}\}$, where

$$\bigcup_{i=1}^{t} N_{G_0'}(v_i) = \{v_{t+1}, \ldots, v_{t+s}\},$$

$S_1 = \{v_{t+1}, \ldots, v_{t+s}, v_{t+s+1}, \ldots, v_{t+s+r}\}$ and $t + s + r = |V(G_0')|$.

By (17) and (18),

$$t + \frac{s(2n-508)}{15} + \frac{r(n-269)}{15} \leq \left| \bigcup_{v_i \in V(G_0')} V(\Gamma(v_i)) \right| = n.$$

Since n is sufficiently large,

$$2s + r \leq 15. \tag{19}$$

Since $s \geq 3$, and by (19), $s + r \leq 12$. Then, since $|V(G_0')| \geq 16$, $t \geq 4$. Let $G^* = G_0'[\{v_1, \ldots, v_t, v_{t+1}, \ldots, v_{t+s}\}]$. By (15), $d_{G^*}(v_i) \geq 3$, for $i \leq t$. By Theorem 13(c), and since $t \geq 4$, $G^* \notin \{K_1, K_2, K_{2,l}\ (l \geq 2)\}$. Then $3t \leq |E(G^*)| \leq 2(t+s) - 5$. So,

$$t \leq 2s - 5. \tag{20}$$

Using $t \geq 4$, (19) and (20), we obtain $5 \leq s \leq 7$.

If $s = 5$ ($s = 6$ or $s = 7$), then, by (19), $r \leq 5$ ($r \leq 3$ or $r \leq 1$, respectively). Note that $\bigcup_{i=1}^{t} N_{G_0'}(v_i) = \{v_{t+1}, \ldots, v_{t+s}\}$ and V_0 is an independent set. Then $\alpha'(G_0') \leq 7$. Using Theorem 14(c), we conclude that G_0' has an SCT, a contradiction. This completes the proof. ■

***Proof of Theorem* 12.** Since G is a 3-edge-connected graph, G is also an essentially 3-edge-connected graph. If G is supereulerian, then we are done. In the following, we assume that $G \notin \mathcal{SL}$. Since G is 3-edge-connected, $D(G) = \emptyset$. Let G_0' be the reduction of the core G_0 of G. Then $G = G_0$ and so $G_0 \notin \mathcal{SL}$. By Theorem 19, $G_0' \notin \mathcal{SL}$, and $|V(G_0')| \leq 18$ and $\kappa'(G_0') \geq 3$. Then, similarly as in the proof of Theorem 11, we conclude that G can be contracted to the Petersen graph. This completes the proof. ■

4. Concluding Remarks

We presented some new results on edge degree conditions of a graph for its line graph to be hamiltonian. In fact, these results imply many known earlier results. In particular, Theorems 2, 3, 4, 5, 6 and 8 are all special cases of Theorem 9, with $(p,k) \in \{(5,2),(7,2),(10,3), (12,3),(15,3)\}$. Moreover, with Theorem 9, we implicitly provide improvements (Theorem 10 and Theorem 11) of Theorem 4 and Theorem 8, since they are special cases of Theorem 9 with $p = 8$ and $k = 2$, and $p = 15$ and $k = 3$, respectively. Furthermore, we note that Theorem 12 slightly improves Theorem 8.

Acknowledgements

This work is supported by the Natural Science Funds of China (Nos. 12131013, 11871099 and 12101126), the China Scholarship Council (No. 201706030019) and the Education Scientific Research Fund for Young and Middle-aged Teachers of Fujian Province (No. JAT190066).

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Received 16 August 2021
Revised 4 February 2022
Accepted 8 February 2022
Available online 26 February 2022