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CHORDED k-PANCYCLIC AND WEAKLY k-PANCYCLIC GRAPHS

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Abstract

As natural relaxations of pancyclic graphs, we say a graph G is k-pancyclic if G contains cycles of each length from k to $|V(G)|$ and G is *weakly pancyclic* if it contains cycles of all lengths from the girth to the circumference of G, while G is *weakly k-pancyclic* if it contains cycles of all lengths from k to the circumference of G. A cycle C is chorded if there is an edge between two vertices of the cycle that is not an edge of the cycle. Combining these ideas, a graph is *chorded pancyclic* if it contains chorded cycles of each length from 4 to the circumference of the graph, while G is *chorded k-pancyclic* if there is a chorded cycle of each length from k to $|V(G)|$. Further, G is *chorded weakly k-pancyclic* if there is a chorded cycle of each length from k to the circumference of the graph. We consider conditions for graphs to be chorded weakly k-pancyclic and chorded k-pancyclic.

Keywords: cycle, chord, pancyclic, weakly pancyclic.

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1. Introduction

The study of cycles has a long and diverse history. Many different properties have been developed concerning cycles. For example, early on Bondy [2] studied one of the most important of these; pancyclicity. We say a graph G is *pancyclic* if G contains a cycle of each length from three to the order of G and G is *k-pancyclic* if it contains cycles of all lengths from k to the order of the graph. Natural relaxations of pancyclic graphs have also been developed. In his thesis, Brandt [3] introduced one such variation of pancyclic graphs. A graph is *weakly pancyclic* if it contains cycles of all lengths from the girth to the circumference of the graph. Further, a graph is *weakly k-pancyclic* if it contains cycles of all lengths from k to the circumference (see for example [5]).

Another, more recent cycle variation is that of chorded cycles. We say an edge between two vertices of a cycle is a *chord* if it is not an edge of the cycle. We say cycle C is a *chorded cycle* if the vertices of C induce at least one chord. Pósa [13] asked what conditions imply a graph contains a chorded cycle. This question has seen considerable interest lately (see for example [7–9]).

In this paper we consider a merging of the ideas we have discussed. We say a graph is *chorded k-pancyclic* if it contains chorded cycles of all lengths from k to $|V(G)|$ (see for example [10]). Further, G is *chorded weakly k-pancyclic* if G contains chorded cycles of each length from k to the circumference of the graph. Note that we did not say chorded cycles existed from the girth on up, since the smallest chorded cycle contains a smaller cycle.

We consider only simple graphs in this paper. We use the standard notation of $V(G)$, $E(G)$, and $\delta(G)$ for the vertex set, edge set, and minimum degree of the graph G. Let $K_{a,b}$ denote the complete bipartite graph with parts of order a and b. Let C_k denote the cycle of order k and P_k denote the path of order k. Let $N_H(x)$ denote the set of neighbors of the vertex x in the graph (or subgraph) H and let $< S >$ denote the graph induced by the vertex set S. Given an orientation of some path or cycle, we denote by x^+ and x^- the successor and predecessor of the vertex x following the given orientation. Further, let $x^{+2} = (x^+)^+$ and similarly, let $x^{-2} = (x^-)^-$, etc. Similarly, $N_C^+(x)$ denotes the set of successors of the neighbors of x on the cycle C following the given orientation. Let $d(u, v)$ denote the distance in the graph between vertices u and v. Given a subgraph or vertex subset S let $G - S$ be the graph obtained by removing S from G. The girth is the length of the shortest cycle and the circumference is the length of a longest cycle. For terms not defined here see [11].

In his thesis, Brandt [3] showed the following result.

Theorem 1. *Let $G \neq C_5$ be a nonbipartite triangle-free graph of order n. If $\delta(G) > n/3$, then G is weakly pancyclic with girth 4 and circumference $\min\{2(n-\alpha(G)), n\}$.*

In [4] it is shown that Theorem 1 is best possible.

Brandt, Faudree, and Goddard [5] provided another result on weak pancyclic graphs, removing the triangle free condition of the previous result.

Theorem 2. *Every nonbipartite graph G of order n with minimum degree $\delta(G) \geq (n+2)/3$ is weakly pancyclic with girth 3 or 4.*

This result is almost best possible. The graph formed from K_{m+1} and $K_{m,m}$ $(m \geq 3)$ by identifying a vertex from each has order $n = 3m$ and minimum degree $m = \frac{n}{3}$, but contains no odd cycle of length more than $m+1$, while having all even cycles up to $2m$.

We extend each of these last two results as follows.

Theorem 3. *Let G be a nonbipartite triangle-free graph of order $n \geq 13$. If $\delta(G) \geq \frac{n+1}{3}$, then G is chorded weakly 6-pancyclic with circumference* $\min\{2(n - \alpha(G)), n\}$.

Theorem 4. *Every nonbipartite graph G of order $n \geq 13$ with minimum degree $\delta(G) \geq (n+2)/3$ is chorded weakly 6-pancyclic.*

Theorem 3 is best possible in the sense that as G is triangle-free, it contains no chorded 4 or 5-cycles. We will prove Theorems 3 and 4 in Section 2.

Our second goal concerns the following. A well-known result of Chvátal and Erdős relates connectivity ($\kappa(G)$) and independence number ($\alpha(G)$) to cycle length.

Theorem 5 (Chvátal, Erdős [6])**.** *If G is a graph of order $n \geq 3$ such that $\alpha(G) \leq \kappa(G)$, then G is hamiltonian, that is, it contains a spanning cycle.*

Amar *et al.* [1] conjectured that if $\alpha(G) \leq \kappa(G)$ and G is not bipartite, then G has cycles of every length from 4 to $|V(G)|$. Lou [12] considered this conjecture and proved the following.

Theorem 6. *Let G be a triangle-free graph of order $n \geq 4$ with $\alpha(G) \leq \kappa(G)$. Then G is 4-pancyclic or $G = K_{\frac{n}{2},\frac{n}{2}}$, or $G = C_5$.*

Our goal is to extend Lou's Theorem as follows.

Theorem 7. *Let G be a triangle-free graph of order $n \geq 13$ with $\alpha(G) \leq \kappa(G)$. Then G is chorded weakly 8-pancyclic, or $G = K_{\frac{n}{2},\frac{n}{2}}$.*

Note that since G is triangle-free, there cannot be a chorded C_4 or C_5 in G. In Section 3 we will prove Theorem 7 and provide examples to show there may not be chorded 6 and 7-cycles in such graphs. Thus, in general, this result is best possible.

2. Proofs of Theorems 3 and 4

In this section we prove Theorems 3 and 4. In order to do so, we begin with several general lemmas that will apply in both proofs.

Lemma 8. *Let G be a graph of order $n \geq 12$ with $\delta(G) \geq \frac{n+1}{3}$. If H is a subgraph of G of order $6+t$ $(0 \leq t \leq 5)$ and x, y, z are vertices of H such that $d = \deg_H(x) + deg_H(y) + deg_H(z) \leq 6+t$, and*

$$N_{G-H}(x) \cap N_{G-H}(y) = \emptyset = N_{G-H}(x) \cap N_{G-H}(z),$$

then $|N_{G-H}(y) \cap N_{G-H}(z)| \geq 1$.

Proof. Since $\delta(G) \geq \frac{n+1}{3}$, we see that $3\delta(G) - d \geq n-5-t$. But from the neighborhood intersection conditions, since $|V(G-H)| = n-6-t$, it then follows that $|N_{G-H}(y) \cap N_{G-H}(z)| \geq 1$. ■

Lemma 9. *If G has order $n \geq 12$ and $\delta(G) \geq \frac{n+1}{3}$, then G contains a chorded 6-cycle.*

Proof. By Theorem 1 we know G contains 6-cycles. Suppose that G satisfies the conditions of the Theorem and further, suppose the result fails to hold. Let $C : v_1, v_2, v_3, \ldots, v_6, v_1$ be a chordless 6-cycle in G and let $H = C$.

Case 1. Assume that no two consecutive vertices of C have a common neighbor in $G - C$.

Consider the vertices v_1, v_2, v_3. By our assumption and Lemma 8, we see that there exists a vertex x with $x \in N_{G-H}(v_1) \cap N_{G-H}(v_3)$. Let $H_1 =< V(C) \cup \{x\} >$ and now consider v_2, v_3, v_4. Again by Lemma 8, we can select a vertex y with $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_4)$. But then, the cycle $v_1, x, v_3, v_4, y, v_2, v_1$ is a 6-cycle with chord v_2v_3.

Case 2. Assume that there are two consecutive vertices of C with at least one neighbor in $G - H$.

Without loss of generality, we may assume that $x \in N_{G-H}(v_1) \cap N_{G-H}(v_2)$. Let $H_1 =< V(C) \cup \{x\} >$ and consider x, v_2, v_5. If there exists a vertex y with $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$ then $v_1, x, v_2, y, v_5, v_6, v_1$ is a 6-cycle with chord v_1v_2. Similarly, if $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_5)$ then $v_1, v_2, x, y, v_5, v_6, v_1$ is a 6-cycle with chord xv_1. If both these fail to hold, then by Lemma 8, we conclude instead that $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_2)$ and let $H_2 =< V(H_1) \cup \{y\} >$.

Now consider v_6, x, y. If there exists a vertex $z \in N_{G-H_2}(v_6) \cap N_{G-H_2}(x)$ then $v_1, v_2, y, x, z, v_6, v_1$ is a 6-cycle with chord xv_2. If instead $z \in N_{G-H_2}(y) \cap N_{G-H_2}(v_6)$ then $v_1, v_2, x, y, z, v_6, v_1$ is a 6-cycle with chord xv_1. If both of these fail to hold, we conclude from Lemma 8 that $z \in N_{G-H_2}(x) \cap N_{G-H_2}(y)$ and we let $H_3 =< V(H_2) \cup \{z\} >$.

Now consider v_1, v_3, z. If there exists a vertex $w \in N_{G-H_3}(v_1) \cap N_{G-H_3}(v_3)$ we then have a 6-cycle $v_1, w, v_3, v_2, y, x, v_1$ with chord xv_2. But, if instead $w \in N_{G-H_3}(z) \cap N_{G-H_3}(v_3)$ then $v_2, y, x, z, w, v_3, v_2$ is a 6-cycle with chord xv_2. Finally, if both of these fail to hold, then by Lemma 8, $w \in N_{G-H_3}(v_1) \cap N_{G-H_3}(z)$, then $v_1, w, z, x, y, v_2, v_1$ is a 6-cycle with chord xv_2, completing the proof. ■

Lemma 10. *If G has order $n \geq 12$ and $\delta(G) \geq \frac{n+1}{3}$, then G contains a chorded 7-cycle.*

Proof. By Theorem 1 we know G contains a 7-cycle. Let G be as stated, and suppose the result fails to hold. Let $C : v_1, v_2, v_3, \ldots, v_7, v_1$ be a chordless 7-cycle in G and let $R = G - C$ and $H = C$. We now consider the following cases.

Case 1. Suppose that no two consecutive vertices of C have a common neighbor in R.

Consider v_1, v_2, v_3. By our assumption and Lemma 8 we see that there exists a vertex $x \in N_R(v_1) \cap N_R(v_3)$. Let $H_1 =< H \cup \{x\} >$. Now consider v_2, v_5, v_6. If there exists a vertex $w \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$, then $v_1, v_2, w, v_5, v_4, v_3, x, v_1$ is a 7-cycle with chord v_2v_3. If instead $w \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_6)$, then $v_1, v_7, v_6, w, v_2, v_3, x, v_1$ is a 7-cycle with chord v_1v_2. However, by our assumption and Lemma 8, one of these two facts must hold.

Case 2. Suppose there are two consecutive vertices on C with a common neighbor in R.

Without loss of generality let $x \in N_R(v_1) \cap N_R(v_2)$, set $H_1 =< C \cup \{x\} >$ and consider v_2, v_5, x. If there exists $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$, then

$$v_1, x, v_2, y, v_5, v_6, v_7, v_1$$

is a 7-cycle with chord v_1v_2. If instead $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_5)$, then $v_1, v_2, x, y, v_5, v_6, v_7, v_1$ is a 7-cycle with chord xv_1. If both of these fail to happen, then by Lemma 8 there exists $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(x)$. Let $H_2 =< H_1 \cup \{y\} >$.

Now consider x, y, v_6. If there exists a vertex $z \in N_{G-H_2}(v_6) \cap N_{G-H_2}(x)$, then $v_1, v_2, y, x, z, v_6, v_7, v_1$ is a 7-cycle with chord xv_1. If instead, $z \in N_{G-H_2}(y) \cap N_{G-H_2}(v_6)$, then $v_1, v_2, x, y, z, v_6, v_7, v_1$ is a 7-cycle with chord xv_1. Otherwise, by Lemma 8, there is a vertex $z \in N_{G-H_2}(x) \cap N_{G-H_2}(y)$. We now consider v_3, v_7, z, with $H_3 =< H_2 \cup \{z\} >$.

If there exists a vertex w such that $w \in N_{G-H_3}(z) \cap N_{G-H_3}(v_7)$, then $v_1, v_2, y, x, z, w, v_7, v_1$ is a 7-cycle with chord xv_1. If instead $w \in N_{G-H_3}(z) \cap N_{G-H_3}(v_3)$, then $v_2, v_1, x, y, z, w, v_3, v_2$ is a 7-cycle with chord yv_2. Otherwise, by Lemma 8, there us a vertex $w \in N_{G-H_3}(v_3) \cap N_{G-H_3}(v_7)$ and then

$$v_1, x, y, v_2, v_3, w, v_7, v_1$$

is a 7-cycle with chord v_1v_2. This completes the proof of the lemma. ■

Lemma 11. *Let G have order $n \geq 12$ and $\delta(G) \geq \frac{n+1}{3}$. Then G contains a chorded 8-cycle.*

Proof. By Theorem 1 we know that G contains an 8-cycle. Suppose all 8-cycles are chordless and consider the 8-cycle $C : v_1, v_2, v_3, \ldots, v_8, v_1$ and let $H = C$. We now consider two cases.

Case 1. Suppose no two consecutive vertices on C have a common neighbor in $G - H$.

Consider v_1, v_2, v_3. Then, by our assumption and by Lemma 8 there exists a vertex $x \in N_{G-H}(v_1) \cap N_{G-H}(v_3)$. Let $H_1 =< H \cup \{x\} >$. Similarly, there exists a vertex y with $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_4)$. Let $H_2 =< H_1 \cup \{y\} >$.

Next consider v_8, v_1, v_2. Again, by our assumption and Lemma 8, there exists a vertex z such that $z \in N_{G-H_2}(v_2) \cap N_{G-H_2}(v_8)$. Then, $v_1, x, v_3, v_4, y, v_2, z, v_8, v_1$ is an 8-cycle with chord v_1v_2.

Case 2. Suppose there is a pair of consecutive vertices on C with a common neighbor in $G - H$.

Without loss of generality, let $x \in N_{G-H}(v_1) \cap N_{G-H}(v_2)$ and $H_1 =< H \cup \{x\} >$. Now consider v_2, x, v_5. If there exists y with $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_5)$ then, $v_1, v_2, x, y, v_5, v_6, v_7, v_8, v_1$ is an 8-cycle with chord xv_1. If instead $y \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$ then, $v_1, x, v_2, y, v_5, v_6, v_7, v_8, v_1$ is an 8-cycle with chord v_1v_2. If both these cases fail to hold, then by Lemma 8 there exists y with $y \in N_{G-H_1}(x) \cap N_{G-H_1}(v_2)$.

Let $H_2 =< H_1 \cup \{y\} >$ and now consider x, y, v_6. If there exists $w \in N_{G-H_2}(x) \cap N_{G-H_2}(v_6)$, then $v_1, v_2, y, x, w, v_6, v_7, v_8, v_1$ is an 8-cycle with chord xv_1. If instead $w \in N_{G-H_2}(y) \cap N_{G-H_2}(v_6)$, then $v_1, v_2, x, y, w, v_6, v_7, v_8, v_1$ is an 8-cycle with chord xv_1. If both of these cases fail to hold, then again by Lemma 8, there exists $w \in N_{G-H_2}(x) \cap N_{G-H_2}(y)$.

Now let $H_3 =< H_2 \cup \{w\} >$ and consider v_7, w, v_4. If there exists $z \in N_{G-H_3}(v_7) \cap N_{G-H_3}(v_4)$, then $v_1, x, v_2, v_3, v_4, z, v_7, v_8, v_1$ is an 8-cycle with chord v_1v_2. If instead $z \in N_{G-H_3}(w) \cap N_{G-H_3}(v_7)$, then $v_1, v_2, y, x, w, z, v_7, v_8, v_1$ is an 8-cycle with chord xv_1. If both the previous cases fail to hold, then by Lemma 8 there exists $z \in N_{G-H_3}(v_4) \cap N_{G-H_3}(w)$, in which case $v_2, v_1, x, y, w, z, v_4, v_3, v_2$ is an 8-cycle with chord xv_2. This completes the proof of the lemma. ■

Lemma 12. *Let G be a graph of order $n \geq 13$ with $\delta(G) \geq \frac{n+1}{3}$. Then G contains chorded cycles of each length from 9 to the circumference of the graph.*

Proof. By Theorem 1, G contains cycles of each length from 9 to the circumference of G. Let G be as stated and suppose G has no chorded k-cycle for some $k \geq 9$. Let $C = C_k : v_1, v_2, \ldots, v_k$ be such a cycle in G. Further, let $H = G - C_k$. We consider the following cases.

Case 1. Suppose, no two consecutive vertices of C_k have a common neighbor off C.

By our assumption and Lemma 8, for any three consecutive vertices on C_k, v_i, v_{i+1}, v_{i+2}, $N_{G-H}(v_i) \cap N_{G-H}(v_{i+2}) \neq \emptyset$. Let $w \in N_{G-H}(v_1) \cap N_{G-H}(v_3)$ and let $H_1 =< H \cup \{w\} >$. If $N_{G-H_1}(v_2) \cap N_{G-H_1}(v_6) \neq \emptyset$, then take $w_2 \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_6)$ and note that

$$v_1, w, v_3, v_2, w_2, v_6, v_7, \ldots, v_k, v_1$$

is a k-cycle with chord v_1v_2. Thus, we may assume $N_{G-H_1}(v_2) \cap N_{G-H_1}(v_6) = \emptyset$ and by symmetry $N_{G-H_1}(v_2) \cap N_{G-H_1}(v_{k-2}) = \emptyset$.

If $N_{G-H_1}(v_3) \cap N_{G-H_1}(v_7) \neq \emptyset$, then let $w_3 \in N_{G-H_1}(v_3) \cap N_{G-H_1}(v_7)$. Now, by our assumptions there exists $w_k \in N_{G-H_1}(v_k) \cap N_{G-H_1}(v_2)$. Now by Lemma 8, we have that

$$v_2, v_1, w, v_3, w_3, v_7, v_8, \ldots, v_k, w_k, v_2$$

is a k-cycle with chord v_2v_3. Note that if any pair of vertices v_i, v_{i+4} for $i = 2, 3, \ldots, k-3$ share a common neighbor off C, then we can always find a chorded k-cycle in a similar fashion. So we may assume this never happens.

Then, in particular, considering v_2, v_5, v_6, we know by our assumptions there exists a vertex $x \in N_{G-H_1}(v_2) \cap N_{G-H_1}(v_5)$, and let $H_2 =< H_1 \cup \{x\} >$. Similarly, considering v_5, v_8, v_9, we know there exists a vertex $y \in N_{G-H_2}(x_5) \cap N_{G-H_2}(v_8)$. Now $v_1, w, v_3, v_2, x, v_5, y, v_8, v_9, \ldots, v_k, v_1$ is a k-cycle with chord v_1v_2, completing this case.

Case 2. Suppose two consecutive vertices of $C = H$ do have a common neighbor in $G - H$.

Without loss of generality, say $w \in N_{G-H}(v_1) \cap N_{G-H}(v_2)$ and let $H_1 =< H \cup \{w\} >$. Then if any pair v_i, v_{i+3} for $i = 2, 3, \ldots, k-3$ satisfies $N_{G-H_1}(v_i) \cap N_{G-H_1}(v_{i+3}) \neq \emptyset$ with a vertex $x \in N_{G-H_1}(v_i) \cap N_{G-H_1}(v_{i+3})$, there exists a k-cycle $v_1, w, v_2, v_3, \ldots, v_i, x, v_{i+3}, \ldots, v_k, v_1$ with chord v_1v_2.

Thus, assume no such pair exists. Then, in particular, considering v_2, v_5, v_8 we see that there exists a vertex, say w_2, such that $w_2 \in N_{G-H}(v_2) \cap N_{G-H}(v_8)$, and considering the triple v_3, v_6, v_9, there must exists a vertex $w_3 \in N_{G-H}(v_3) \cap N_{G-H}(v_9)$ and considering v_4, v_7, v_{10} (here v_{10} may be v_1) we have a vertex $w_4 \in N_{G-H}(v_4) \cap N_{G-H}(v_{10})$. Then the cycle $v_1, v_2, w_2, v_8, v_9, w_3, v_3, v_4, w_4, v_{10}, v_{11}, \ldots, v_1$ (note again that it is possible that $v_1 = v_{10}$) is a k-cycle with chord v_2v_3. This completes the proof. ■

Note that Case 2 may require at least 13 vertices, hence the condition that $n \geq 13$. As this lemma is used in Theorems 3 and 4, the condition that $n \geq 13$ must be assumed in each result.

We are now ready to prove Theorem 3.

***Proof of Theorem* 3.** By Theorem 1, G is weakly pancyclic with girth 3 or 4. Let G be a graph of order $n \geq 13$ with $\delta(G) \geq \frac{n+1}{3}$. Then by Lemmas 9, 10, 11, and 12 we see that G contains chorded cycles of length 6 up to the circumference of G. ■

As G is triangle-free, there can be no chorded 4 or 5-cycles, thus the result is best possible.

Example for Theorem 3. We construct a graph G as follows. Begin with a copy of $C_4 = v_1, v_2, v_3, v_4, v_1$. Blowup each of the vertices v_1 and v_3 into sets of $\frac{n-2}{3}$ independent vertices and blowup the vertices v_2 and v_4 into sets of $\frac{n-2}{6}$ independent vertices. For any edge of C_4 insert all edges between the corresponding sets. Finally, insert two new vertices x and y that are themselves adjacent and join x to all vertices in the blowup of v_1 and y to all vertices in the blowup of v_3 (see Figure 1). Note that $\delta(G) = \frac{n+1}{3}$. Further, it is easy to see that G is chorded weakly 6-pancyclic.

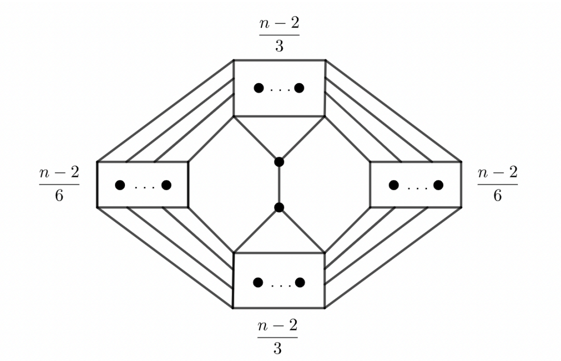

Figure 1. Sharpness example for Theorem 3.

***Proof of Theorem* 4.** By Theorem 2, G is weakly pancyclic with girth 3 or 4. Again by Lemmas 9, 10, 11, and 12, we see that G has chorded cycles of each length from 6 to the circumference of G. ■

3. Proof of Theorem 7

The following from [12] will be useful.

Lemma 13. *If G is a triangle-free graph of order $n \geq 4$ and C is a cycle in G, then for every vertex $v \in G - C$, the set $N_C^+(v)$ is non-empty and $N_C^+(v)$ is not an independent set, hence $|N_C^+(v)| \geq 2$.*

The next lemma has appeared in numerous papers, thus we attribute it to folklore.

Lemma 14. *Let C be a cycle in a graph G and $v \in V(G-C)$. If there is an edge in $N_C^+(v)$, then G contains a cycle D with $V(D) = V(C) \cup \{v\}$.*

We now state, in more detail, what Lou [12] proved.

Theorem 15. *If G ($G \neq K_{m,m}$ or C_5) is a triangle-free graph with $\alpha(G) \leq \kappa(G)$, then*

1. *G is k-regular and*
2. *$k = \alpha(G) = \kappa(G) = \kappa$ and G is κ-regular*
3. *G has diameter 2, and*
4. *G contains cycles of length 4 up to $|V(G)|$.*

What Lou proved actually puts some real restrictions on graphs G that satisfy the conditions of being triangle-free with $\alpha(G) = \kappa(G)$. The most severe is a bound on the order of G.

Lemma 16. *If G is a triangle-free graph of order n with $\alpha(G) = \kappa(G) = k$, then $n \leq k^2 + 1$.*

Proof. Let G be as stated above. Select any vertex v. Then v has exactly k mutually nonadjacent neighbors and each of these vertices may have at most $k-1$ distinct new neighbors. If there are any other vertices, say x, then $d(v,x) > 2$ and there is no way to create a path to v that would be of length at most 2. Thus, no such x exists and so $n \leq 1 + k + k(k-1) = k^2 + 1$. ■

This lemma provides another simple observation that if $\alpha(G) = 2$, then G is either C_5 or C_4, and if $\alpha(G) = 3$, then $n \leq 10$. Thus, from now on we need only consider $\alpha(G) \geq 4$.

The graphs in Figures 2 and 3 show that the conditions of being triangle-free with $\alpha(G) = \kappa(G)$ are not enough to guarantee that 6 and 7-cycles are chorded.

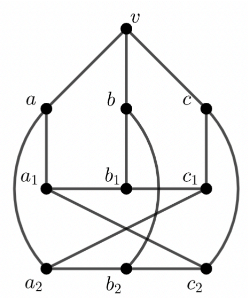

Figure 2. Here $\alpha(G) = 3$.

We now present our proof of Theorem 7, the extension of Lou's Theorem, which utilizes an expansion of the ideas in his approach.

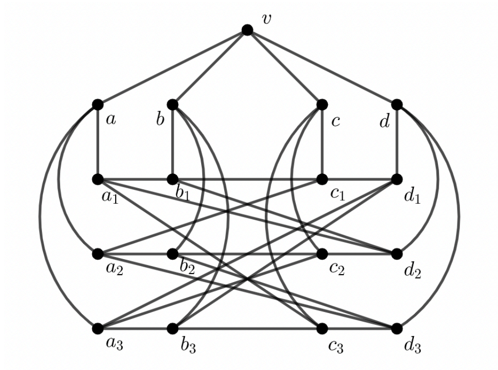

Figure 3. Here $\alpha(G) = 4$.

Proof. Suppose the result fails to hold. Then by Theorem 6, there must exist an integer k with $4 \leq k \leq |V(G)| - 2$, such that G contains a C_k but no chorded C_{k+2}. We next show that each of the following structures (see Figure 4) on a C_k, actually provides a chorded C_{k+2}.

To see this for structure (I), consider the $(k+2)$-cycle $a, u, v, a^+, a^{+2}, \ldots, a$ with chord aa^+.

For structure (III), consider the $(k+2)$-cycle $a, u, b, b^-, \ldots, a^+, v, b^+, b^{+2}, \ldots, a$ with chord aa^+.

For structure (IV), consider the $(k+2)$-cycle $a, u, v, b, b^-, \ldots a^+, b^+, b^{+2}, \ldots, a$ with chord aa^+.

In order to handle structure II we first need to develop several facts.

Claim 1. *Every vertex off the k-cycle $C_k : x_1, x_2, \ldots, x_k, x_1$ $(k \geq 6)$ has at least one adjacency on C_k.*

Proof. Suppose there is a vertex $v \notin V(C_k)$ such that v has no adjacencies on C_k. Then since G is triangle-free, $E(N(v)) = \emptyset$. However, for any $x_i \in V(C_k)$, $N(v) \cup \{x_i\}$ is a set of cardinality $\kappa(G) + 1$. If this set is independent, a contradiction arises to Theorem 15. Thus, every vertex on C_k is adjacent to at least one vertex in $N(v)$. Without loss of generality, say $x_1v_1 \in E(G)$ for some $v_1 \in N(v)$. Now $d(v, x_3) > 2$. Either there exists $v_3 \in N(v)$ such that $v_3 \neq v_1$ with $v_3x_3 \in E(G)$ or $x_3v_1 \in E(G)$.

First suppose latter happens, then $d(v, x_4) > 2$. Since G is triangle-free, $v_1x_4 \notin E(G)$, which implies there exists $v_4 \in N(v)$ such that $v_4x_4 \in E(G)$. Next note that $d(v, x_2) > 2$. If $x_2v_4 \in E(G)$ we obtain structure III, and hence a chorded $(k+2)$-cycle exists in G, a contradiction. Further, to avoid a triangle, $x_2v_1 \notin E(G)$, so there exists $v_2 \in N(v)$ ($v_2 \neq v_1, v_4$) such that $v_2x_2 \in E(G)$. Note that we can extend C_k to a $(k+2)$-cycle

$$C^* = x_2, v_2, v, v_4, x_4, x_5, \ldots, x_1, x_2.$$

Now $d(x_2, x_5) > 2$. Further, $x_2x_5 \notin E(G)$ as that would provide a chord for C^*. Also $x_1x_5 \notin E(G)$ for the same reason, and $x_3x_5 \notin E(G)$ since G is triangle-free. Thus, there exists some $w \notin V(C_k)$ such that $x_2w, x_5w \in E(G)$. Now $x_1, v_1, v, v_2, x_2, w, x_5, x_6, \ldots, x_k, x_1$ is a $(k+2)$-cycle with chord x_1x_2, a contra-

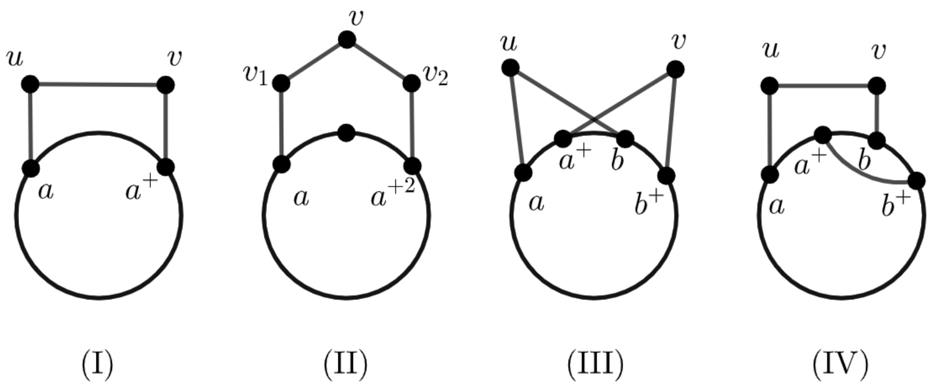

Figure 4. Four structures producing chorded $(k+2)$-cycles.

Now consider the former case, that is, that there exists $v_3 \notin V(C_k)$ such that $v_3 \neq v_1$ and $v_3v, v_3x_3 \in E(G)$. Since $d(v, x_2) > 2$ and $v_1x_2, v_3, x_2 \notin E(G)$, there exists a $v_2 \in N(v)$ with $v_2x_2 \in E(G)$. Again there is a $(k+2)$-cycle

$$C' : x_1, v_1, v, v_3, x_3, x_4, \ldots, x_k, x_1.$$

If there are any chords in C_k not involving x_2, then C' is chorded, a contradiction. Now $d(x_2, x_5) > 2$. If there exists $w \in N_{G-C_k}(x_2) - \{x_1\}$ such that $wx_5 \in E(G)$, then

$$x_1, v_1, v, v_2, x_2, w, x_5, x_6, \ldots, x_k, x_1$$

is a $(k+2)$-cycle with chord x_1x_2. So we may assume $x_2x_5 \in E(G)$. Next note that $d(x_3, x_k) > 2$. If $v_3x_k \in E(G)$, then again C' is chorded with chord v_3x_k. So we may assume there exists a vertex $w \in N_{G-C_k}(x_3)$ with $wx_k \in E(G)$. Now we see that

$$x_3, v_3, v, v_2, x_2, x_5, x_6, \ldots x_k, w, x_3$$

is a $(k+2)$-cycle with chord x_2x_3, a contradiction completing this case and the proof of the claim. □

Claim 2. *If structure* II *exists in* G*, then a chorded* $(k+2)$*-cycle exists in* G.

Proof. Suppose structure II arises and let $C_k = x_1, x_2, , \ldots, x_k$. Without loss of generality suppose that $v_1, v_3 \in N_{G-C}(v)$ and v_1x_1 and v_3x_3 are edges of G. Now, by Claim 1, vertex v must have an adjacency on the cycle C_k. If $vx_2 \in E(G)$, then structure I is formed and we have a chorded $(k+2)$-cycle, a contradiction to our assumption. Thus, assume $vx_2 \notin E(G)$. Further, $vx_1, vx_3 \notin E(G)$, since either edge would create a triangle in G. Thus, $vx_i \in E(G)$ for some i, $4 \le i \le k$. Now this edge is a chord of the $(k+2)$-cycle $x_1, v_1, v, v_3, x_3, x_4, \ldots, x_1$, a contradiction which completes the proof of the claim. □

Next choose a cycle C of length m such that $r = \max_{v \in G-C} |N_C(v)|$ (that is, over all vertices off C, v has the maximum number of adjacencies, and the maximum is taken over all choices of cycles of length m). Take a vertex v from $G - C$ with $|N_C(v)| = r$ and another vertex $u \in V(G-C)$ which is, if possible, adjacent to v. By Lemma 13, the vertex u must have two neighbors y_1 and y_2 on C such that $y_1^+ y_2^+ \in E(G)$. Thus, by Lemma 14, there is an $(m+1)$-cycle D with $V(D) = V(C) \cup \{u\}$. If $r \ge \kappa - 1$, then v has all of its neighbors on D, so again by Lemma 14 there is a cycle on $|V(D) \cup \{v\}| = m+2$ vertices with chord $y_1 y_1^+$, a contradiction.

Now we may assume that $r \le \kappa - 2$. Then $uv \in E(G)$ and v has another neighbor $w \in G-C$. By Lemma 14, w also has two neighbors z_1, z_2 on C such that $z_1^+ z_2^+ \in E(G)$. Since G is triangle-free, in any direction on C, there are at least two vertices between z_1 and z_2, otherwise $< z_1^+, z_2, z_2^+ >= K_3$. Thus, we may assume $z_1 \notin \{y_1^-, y_1, y_1^+\}$. Fix an orientation on D with the path $y_2^+, \ldots, y_1^-, y_1$ and let $S \subset D$ be the set of vertices y of C satisfying $y^- \in N_C(v)$. We wish to show that $N_{G-C}(v) \cup S \cup \{z_1^{+2}\}$ (with respect to the orientation on D) is an independent set with cardinality $\kappa + 1$, the final contradiction.

In order to do this note that on C, the vertex $z_1 \ne {x_1}^{+2}, x_1^{-2}$ or structure II results, a contradiction. Thus, the edges $z_1 z_1^+$ and $z_1^+ z_2^+$ on D are also on C. Similarly, avoiding structure I, we see that there is no edge between any vertex in $N_{G-C}(v)$ and any vertex of S. For the same reason, on D, $z_1^{+2} \notin S$ and, in particular, v is not adjacent to z_1^+ on D. Also, on D, $vz_1^{+2} \notin E(G)$ otherwise,

$$z_1^{+2}, z_1^{+3}, \ldots, z_1, w, v, z_1^{+2}$$

is an $(m+2)$-cycle with chord wz_2. Moreover, z_1^{+2} is not adjacent to a vertex $y \in S$ since otherwise $z_1, w, v, y^-, y^{-2}, \ldots z_1^{+2}, y, y^+, \ldots z_1^-, z_1$ is an $(m+2)$-cycle with chord wz_2, again a contradiction to our assumption. Since an edge in S also creates an $(m+2)$-cycle with chord yy^-, all that remains is to show that z_1^{+2} is not adjacent to any vertex of $N_{G-C}(v)$. For any vertex of $N_{G-C}(v)$ other than w, this follows, as structure II would be formed and hence a chorded $(m+2)$-cycle would exist. But if z_1^{+2} is adjacent to w, we can form a new m cycle C' with w replacing z_1^+. As v is adjacent to w but not to z_1^+, v now has $r+1$ neighbors on C', contradicting our choice of cycle C. Thus, the set $N_{G-C}(v) \cup \{z_1^{+2}\} \cup S$ is independent and has cardinality at least $r+1$, a contradiction completing the proof. ■

4. Conclusion

It is clear from recent work that many conditions implying various cycle properties in a graph can be used to show stronger results concerning cycles with chords. This paper is just one such situation. Broadening our meta-conjecture from [9]:

Almost any condition that implies some cycle property in a graph also implies a chorded cycle property, possibly with some families of exceptional graphs, and small order exceptions.

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