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ON s-HAMILTONIAN-CONNECTED LINE GRAPHS

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Abstract

For an integer $s \geq 0$, G is s-hamiltonian-connected if for any vertex subset $S \subseteq V(G)$ with $|S| \leq s$, $G - S$ is hamiltonian-connected. Thomassen in 1984 conjectured that every 4-connected line graph is hamiltonian (see [*Reflections on graph theory*, J. Graph Theory 10 (1986) 309–324]), and Kužel

and Xiong in 2004 conjectured that every 4-connected line graph is hamiltonian-connected (see [Z. Ryjáček and P. Vrána, *Line graphs of multigraphs and Hamilton-connectedness of claw-free graphs*, J. Graph Theory 66 (2011) 152–173]). In this paper we prove the following.
(i) For $s \geq 3$, every $(s+4)$-connected line graph is s-hamiltonian-connected.
(ii) For $s \geq 0$, every $(s+4)$-connected line graph of a claw-free graph is s-hamiltonian-connected.

Keywords: line graph, claw-free graph, s-hamiltonian-connected, collapsible graphs, reductions.

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1. Introduction

Graphs considered here are finite and loopless. Unless otherwise noted, we follow [1] for notation and terms. As in [1], $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of a graph G, respectively. A graph is *nontrivial* if it contains edges. An edge cut X is *essential* if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut X with $|X| < k$. For a connected graph G, let $ess'(G) = \max\{k : G \text{ is essentially } k\text{-edge-connected}\}$, and for an integer $i \geq 0$, let $D_i(G) = \{u \in V(G) : d_G(u) = i\}$. Throughout this paper, for an integer $n \geq 2$, C_n denotes a cycle on n vertices (called an n-cycle), W_n denotes the graph obtained from an n-cycle by adding a new vertex and connecting it to every vertex of the n-cycle. If $S \subseteq V(G)$ or $S \subseteq E(G)$, then $G[S]$ is the subgraph induced in G by S. We use $H \subseteq G$ to denote the fact that H is a subgraph of G. For $H \subseteq G$, $x \in V(G)$, $A \subseteq V(G)$, $X \subseteq E(G)$, and $Y \subseteq E(G) - E(H)$, define $E_G(x) = \{e : e \text{ is incident to } x\}$, $N_H(x) = N_G(x) \cap V(H)$, $d_H(x) = |N_H(x)|$, $G - A = G[V(G) - A]$, $G - X = G[E(G) - X]$, and $H + Y = G[E(H) \cup Y]$. When $A = \{v\}$ and $X = \{e\}$, we use $G - v$ for $G - \{v\}$ and $G - e$ for $G - \{e\}$.

Let $O(G)$ denote the set of odd degree vertices of G. A graph G is *eulerian* if $O(G) = \emptyset$ and G is connected. A graph G is *supereulerian* if G has a spanning eulerian subgraph. A graph G is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. From the definition of a line graph, if $L(G)$ is not a complete graph, then $L(G)$ is k-connected if and only if G is essentially k-edge-connected. The following are several fascinating conjectures in the literature.

Conjecture 1. (i) (Thomassen [20]) *Every 4-connected line graph is hamiltonian.*

(ii) (Matthews and Sumner [16]) *Every 4-connected claw-free graph is hamiltonian.*

(iii) (Kužel and Xiong [11]) *Every* 4*-connected line graph is hamiltonian-connected.*

(iv) (Ryjáček and Vrána [17]) *Every* 4*-connected claw-free graph is hamiltonian-connected.*

Ryjáček and Vrána in [17] indicated that the statements in Conjecture 1 are mutually equivalent. There have been many studies on these conjectures in the literature. Among them are the following.

Theorem 2 (Zhan [21])**.** *Every* 7*-connected line graph is hamiltonian-connected.*

Theorem 3 (Kaiser and Vrána [9])**.** *Every* 5*-connected line graph with minimum degree at least* 6 *is hamiltonian.*

Theorem 4 (Kriesell [10])**.** *Every* 4*-connected line graph of a claw-free graph is hamiltonian-connected.*

For an integer $s \geq 0$, a graph G is *s-hamiltonian* (or *s-hamiltonian-connected*, respectively) if for any vertex subset $S \subseteq V(G)$ with $|S| \leq s$, $G-S$ is hamiltonian (or hamiltonian-connected, respectively). It is routine to observe that every s-hamiltonian graph is $(s+2)$-connected, and every s-hamiltonian-connected graph is $(s+3)$-connected. The converse, on the other hand, is not true, as $K_{m,m+1}$ is m-connected but nonhamiltonian.

Theorem 5 (Kaiser, Ryjáček, and Vrána [8])**.** *Every* 5*-connected claw-free graph with minimum degree* 6 *is* 1*-hamiltonian-connected.*

Theorem 6. *Let s be an integer.*

(i) (Theorem 1.4 of [12]) *For $s \geq 3$, every $(s+4)$-connected line graph is $(s-1)$-hamiltonian-connected.*

(ii) (Theorem 1.3 of [13]) *For $s \geq 5$, every $(s+2)$-connected line graph is s-hamiltonian.*

(iii) (Theorem 1.6 of [15])) *For $s \geq 0$, every $(s+2)$-connected line graph of a claw-free graph is s-hamiltonian.*

(iv) (Theorem 1.6 of [15])) *Every* 4*-connected line graph of a claw-free graph is* 1*-hamiltonian-connected.*

Motivated by Conjecture 1 as well as the results in [5, 12] and [13], the following conjecture was proposed.

Conjecture 7 [15]**.** *Let s be an integer.*

(i) *For $s \geq 2$, a line graph is s-hamiltonian if and only if it is $(s+2)$-connected.*

(ii) *For $s \geq 2$, a claw-free graph is s-hamiltonian if and only if it is $(s+2)$-connected.*

(iii) *For $s \geq 1$, a line graph is s-hamiltonian-connected if and only if it is $(s+3)$-connected.*

(iv) *For $s \geq 1$, a claw-free graph is s-hamiltonian-connected if and only if it is $(s+3)$-connected.*

In [18], Ryjáček and Vrána showed that when $s = 1$, Conjecture 7(iii) is equivalent to Conjecture 1(i). The main results in this paper are presented below.

Theorem 8. *For $s \geq 3$, every $(s+4)$-connected line graph is s-hamiltonian-connected.*

Theorem 9. *For $s \geq 0$, every $(s+4)$-connected line graph of a claw-free graph is s-hamiltonian-connected.*

Catlin's reduction method will be refreshed in Section 2, together with other useful tools developed in this paper for our proofs of the main results. The proof of Theorem 8 is presented in Section 3, and the proof of Theorem 9 is presented in Section 4. We would like to point out that some of the mechanisms developed in [15] will be utilized in the proof arguments of Theorem 9, as shown in Section 4.

2. Preliminaries

We view a trail of G as a vertex-edge alternating sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ such that all the e_i's are distinct and for each $i = 1, 2, \ldots, k$, e_i is incident with both v_{i-1} and v_i. The vertices in $v_1, v_2, \ldots, v_{k-1}$ are *internal vertices* of the trail. For edges $e', e'' \in E(G)$, an (e', e'')-trail of G is a trail T of G whose first edge is e' and whose last edge is e''. An *internally dominating (e', e'')-trail* of G is an (e', e'')-trail T of G such that every edge of G is incident with an internal vertex of T, and a *spanning (e', e'')-trail* of G is an internally dominating (e', e'')-trail T of G such that $V(T) = V(G)$. Harary and Nash-Williams [6] first showed the relationship between eulerian subgraphs in G and hamiltonicity in $L(G)$. Theorem 10(ii) below is observed in [14].

Theorem 10. *Let G be a graph with $|E(G)| \geq 3$. Each of the following holds.*

(i) (Harary and Nash-Williams [6]) *$L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

(ii) ([14]) *$L(G)$ is hamiltonian-connected if and only if for any pair of edges $e', e'' \in E(G)$, G has an internally dominating (e', e'')-trail.*

Theorem 11. *Let G be a connected graph with at least three edges and $s > 0$ an integer. The line graph $L(G)$ is s-hamiltonian-connected if and only if $G - S$ has an internally dominating (e', e'')-trail for any $S \subset E(G)$ with $|S| \leq s$, and for any pair of edges $e', e'' \in E(G - S)$.*

We say that an edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by the path of length 2 is called *subdividing* e. For a graph G and edges $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e', and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e''. Then $V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}$.

Proposition 12. *For a graph G and edges $e', e'' \in E(G)$, if $G(e', e'')$ has a dominating* (*spanning, respectively*) *$(v(e'), v(e''))$-trail, then G has an internally dominating* (*spanning, respectively*) *(e', e'')-trail.*

Let $X \subseteq E(G)$ be an edge subset of G. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. If v_K is the vertex in G/H onto which the connected subgraph K is contracted, then K is called the *preimage* of v_K, and denoted by $PI(v_K)$. In [2] Catlin defined collapsible graphs. Given an even subset R of $V(G)$, a subgraph Γ of G is called an *R-subgraph* if $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is *collapsible* if for any even subset R of $V(G)$, G has an R-subgraph. In particular, K_1 is collapsible. Catlin [2] showed that for any graph G, one can obtain the *reduction* G' of G by contracting all maximal collapsible subgraphs of G. A graph G' is *reduced* if G' has no nontrivial collapsible subgraphs. A vertex x in G' is *c-nontrivial* (or *c-trivial*) if $|V(PI(x))| \geq 2$ (or $|V(PI(x))| = 1$). By definition, every collapsible graph is supereulerian. We summarize some results on Catlin's reduction method and other related facts below.

Theorem 13. *Let G be a graph and let H be a collapsible connected subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*

(i) (Catlin, Theorem 3 of [2]) *G is collapsible if and only if G/H is collapsible. Therefore, G is collapsible if and only if the reduction of G is K_1.*

(ii) (Catlin, implied by definition and Theorem 3 of [2]) *C_2, C_3 are collapsible, and when $n \geq 4$, for any $e \in E(W_n)$, $W_n(e)$ is collapsible.*

(iii) (Theorem 2.3(iii) of [14]) *If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*

(iv) (Theorem 2.3(iv) of [14]) *For vertices $u, v \in V(G/H) - \{v_H\}$, if G/H has a spanning (u, v)-trail, then G has a spanning (u, v)-trail.*

(v) (Theorem 3.3 of [14]) *Let G be a 3-edge-connected graph. If every 3-edge-cut X has at least one edge in a 2-cycle or 3-cycle of G, then, for any two edges $e', e'' \in E(G)$, $G(e', e'')$ is collapsible.*

Let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. Let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following theorem summarizes results related to $F(G)$ and supereulerian properties.

Theorem 14. *Let G be a connected graph and let G' be the reduction of G. Then each of the following holds.*

(i) (Jaeger [7]) *If $F(G) = 0$, then G is collapsible.*

(ii) (Catlin [2]) *If $F(G) \leq 1$, then $G' \in \{K_1, K_2\}$. Therefore, G is supereulerian if and only if $G' \neq K_2$.*

(iii) (Catlin *et al.* [3]) *If $F(G) \leq 2$, then $G' \in \{K_1, K_2, K_{2,t}\}$ for some integer $t \geq 1$. Therefore, G is supereulerian if and only if $G' \notin \{K_2, K_{2,t}\}$ for some odd integer t.*

(iv) (Theorem 1.1 of [4]) *Let $k \geq 1$ be an integer. Then $\kappa'(G) \geq 2k$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.*

Lemma 15 ([15])**.** *Assume that $K = v_1v_2v_3v_1$ is a triangle in a connected graph G with $d_G(v_1) = 3$. Also assume that $N_G(v_1) = \{v_2, v_3, x\}$ and $e \in \{v_1v_2, v_2v_3\}$. Let w be the new vertex in G/K on which K is contracted, and let $u(\neq w) \in V(G/K)$. Let T be a spanning (u, w)-trail in G/K. Then each of the following holds.*

(i) *For $e = v_1v_2$, $G(e)$ has a dominating $(u, v(e))$-trail T_1 such that $V(G(e)) - V(T_1) \subseteq \{v_1\}$.*

(ii) *For $e = v_2v_3$, if $xv_1 \notin E(T)$, then $G(e)$ has a spanning $(u, v(e))$-trail T_2.*

Lemma 16 ([15])**.** *Let $s \geq 3$ be an integer and G be a graph with $\kappa'(G) \geq 3$ and $ess'(G) \geq s + 2$. If $v \in D_3(G)$, then $\kappa'(G - v) \geq 3$ and $ess'(G - v) \geq s + 1$.*

3. Proof of Theorem 8.

Let $s \geq 3$ be an integer, and let G be a connected, essentially s-edge-connected graph such that $L(G)$ is not a complete graph. Then for any edge $vx \in E(G)$ with $d_G(v) \in \{1, 2\}$, we have $d_G(x) \geq s + 2 - d_G(v)$. Following [19], the core of the graph G, written as G_0, is obtained by the following two operations repeatedly.

Operation 1. Delete each vertex of degree 1.

Operation 2. For each vertex y of degree 2 with $E_G(y) = \{xy, yz\}$, contract exactly one edge in $E_G(y)$. This amounts to deleting vertex y in G with $d_G(y) = 2$ and replacing xy and yz with a new edge xz.

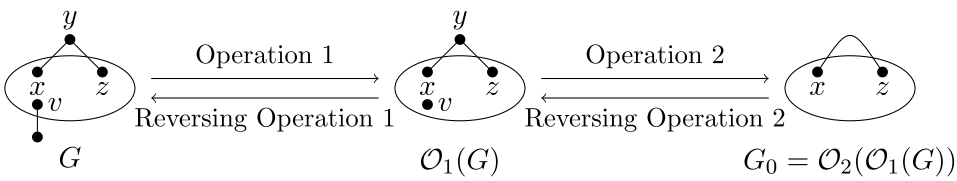

Figure 1. The core graph.

Let $\mathcal{O}_1(G)$ denote the graph obtained from G by applying Operation 1 to each vertex of degree 1, and $\mathcal{O}_2(G)$ the graph obtained from G by applying Operation 2 to each vertex of degree 2. Thus $G_0 = \mathcal{O}_2(\mathcal{O}_1(G))$. As shown in [19], we observe that G_0 is well-defined, and is 3-edge-connected and essentially s-edge-connected. By the definitions of these operations, any trail in G is contracted to a trail in G_0. Conversely, for any trail T' in G_0, there is a trail T in G such that T' is the contraction image of T. We call T a *lift* of T', or say that T' can be lifted to T.

In the rest of this section, we assume that G is a connected, essentially s-edge-connected graph, where $s \geq 6$ is an integer, and let $X \subseteq E(G)$ with $|X| \leq 3$. Let $H = G_0 - (E(G_0) \cap X)$. If H is not connected, then H contains an isolated vertex v with $E_G(v) = X = E_{G_0}(v)$ and $|X| = 3$, and $H - v$ is essentially s-edge-connected. If H is connected, then H is essentially $(s-3)$-edge-connected since G is essentially s-edge-connected. Let $G_X = \begin{cases} H, & \text{if } H \text{ is connected,} \\ H - v, & \text{if } H \text{ is not connected.} \end{cases}$
Then we have

(1) $\quad G_X$ is essentially $(s-3)$-connected.

Let $(G_X)_0$ be the core of G_X. Then

(2) $\quad (G_X)_0$ is 3-edge-connected and essentially $(s-3)$-edge-connected.

Theorem 17 (Theorem 4.1 of [13]). *Let G be an essentially 7-edge-connected graph. If $X \subseteq E(G)$ with $|X| \leq 3$, then $\tau((G_X)_0) \geq 2$.*

Lemma 18. *Let G be an essentially 7-edge-connected graph. Let $X \subseteq E(G)$ be a subset with $|X| \leq 3$ and $\{e_1, e_2\} \subseteq E(G) - X$. Then $G - X$ has an internally dominating (e_1, e_2)-trail.*

Proof. Let G_0 be the core of G. Notice that G is essentially 7-edge-connected. By (2),

(3) $\quad (G_X)_0$ is 3-edge-connected and essentially 4-edge-connected.

Claim 1. *Let $e = xy \in E(G)$. We assume that $d_{G_X}(y) \geq d_{G_X}(x)$ if $e \in E(G_0)$ but $e \notin E((G_X)_0)$; otherwise, we assume that $d_G(y) \geq d_G(x)$. Then $y \in V((G_X)_0)$. Therefore, $d_{(G_X)_0}(y) \geq 3$.*

Proof. Notice that there are three possibilities for the location of e: $e \in E((G_X)_0)$, $e \notin E(G_0)$, or $e \in E(G_0)$ and $e \notin E((G_X)_0)$. If $e \in E((G_X)_0)$, then both x and y are in $V((G_X)_0)$.

If $e \notin E(G_0)$, then since $d_G(y) \geq d_G(x)$, $d_G(x) \in \{1, 2\}$. As G is essentially 7-edge-connected, $d_G(x) + d_G(y) \geq 9$, and so $d_G(y) \geq 9 - d_G(x)$. Therefore, $d_{G_0}(y) \geq 7$ and $d_{G_X}(y) \geq 7 - 3 = 4$. This implies that $y \in V((G_X)_0)$.

If $e \in E(G_0)$ and $e \notin E((G_X)_0)$, then since $d_{G_X}(y) \geq d_{G_X}(x)$, we have $d_{G_X}(x) \in \{1, 2\}$. By (1), G_X is essentially 4-edge-connected. Then $d_{G_X}(x) + d_{G_X}(y) \geq 6$. Thus, $d_{G_X}(y) \geq 6 - d_{G_X}(x) \geq 4$. So $y \in V((G_X)_0)$. Claim 1 holds. □

For $i = 1, 2$, denote $e_i = x_i y_i$ in such a way that if $e_i \in E(G_0)$ but $e_i \notin E((G_X)_0)$, then the labeling of x_i and y_i satisfies $d_{G_X}(y_i) \geq d_{G_X}(x_i)$; otherwise we label x_i and y_i so that $d_G(y_i) \geq d_G(x_i)$. Let

$$Q = \begin{cases} (G_X)_0(e_1, e_2), & \text{if } e_1, e_2 \in E((G_X)_0), \\ (G_X)_0(e_i), & \text{if } \{e_1, e_2\} \cap E((G_X)_0) = \{e_i\}, \\ (G_X)_0, & \text{otherwise} \end{cases}$$

and

$$v_i = \begin{cases} v(e_i), & \text{if } e_i \in E((G_X)_0), \\ y_i, & \text{otherwise.} \end{cases}$$

By Theorem 17, $\tau((G_X)_0) \geq 2$ and so $F(Q) \leq 2$. By Theorem 14(iii) and (3), Q is collapsible. By Theorem 13(iii),

$$\text{(4)} \qquad Q \text{ has a spanning } (v_1, v_2)\text{-trail } T_1.$$

Let T_2 be the lift of T_1 in G_X and let T_3 be the lift of T_2 in $(G - X)(e_1, e_2)$. Let T be a trail obtained from T_3 by replacing v_i by e_i. Then T is an (e_1, e_2)-trail of $G - X$. Let $T = w_1 f_1 w_2 f_2 \cdots f_k w_k$, where $f_1 = e_1$ and $f_k = e_2$, and let $\mathcal{I} = \{w_2, w_3, \ldots, w_{k-1}\}$. Then $V(T_1) - \{v(e_1), v(e_2)\} \subseteq \mathcal{I}$. To show that T is an internally dominating (e_1, e_2)-trail in $G - X$, it suffices to show that every edge $e = xy$ of $G - X$ is incident with an internal vertex of T, i.e., either $x \in \mathcal{I}$ or $y \in \mathcal{I}$.

We assume that $d_{G_X}(y) \geq d_{G_X}(x)$ if $e \in E(G_0)$ but $e \notin (G_X)_0$; otherwise, we assume that $d_G(y) \geq d_G(x)$. By Claim 1, $y \in V((G_X)_0)$. By (4), $y \in V(T_1)$. As $d_{(G_X)_0}(y) \geq 3$, $y \notin \{v(e_1), v(e_2)\}$ and so $y \in V(T_1) - \{v(e_1), v(e_2)\} \subseteq \mathcal{I}$. ■

Lemma 19. *Every 7-connected line graph is 3-hamiltonian-connected.*

Proof. Lemma 19 follows from Lemma 18 and Theorem 11. ■

***Proof of Theorem* 8.** By Lemma 19, Theorem 8 holds when $s = 3$. We assume that $s \geq 4$ and that Theorem 8 holds for smaller values of s. Let G be a graph with $\kappa(L(G)) \geq s+4$. For any $S \subseteq V(L(G))$ with $|S| \leq s$, pick $v_{e_0} \in S$. Assume that the edge in G corresponding to v_{e_0} in $L(G)$ is e_0. Let $G^* = G - e_0$. Since $\kappa(L(G)) \geq s+4$, $\kappa(L(G^*)) = \kappa(L(G) - v_{e_0}) \geq s+3$. It follows by induction that as $L(G^*)$ is $(s+3)$-connected, $L(G^*)$ is $(s-1)$-hamiltonian-connected, and so $L(G) - S = L(G^*) - (S - \{v_{e_0}\})$ must be hamiltonian-connected. It follows by definition that $L(G)$ is s-hamiltonian-connected. ∎

4. Graphs with Property $\mathcal{K}(s)$ and Proof of Theorem 9

Throughout this section, we assume that $s \geq 0$ is an integer. Following [15], we shall introduce a property of graphs which will play an important role in our arguments.

([15]) Let $\mathcal{K}$ denote the graph family such that a (connected) graph G is in $\mathcal{K}$ if and only if G satisfies each of the following.

(KS1) For any $w \in D_3(G)$, the subgraph induced by $N_G(w)$ contains at least one edge.

(KS2) Let $w \in N_G(x_1) \cap N_G(x_2)$, where $x_1, x_2 \in D_3(G)$ and $x_1x_2 \notin E(G)$. If $N_G(w) = \{x_1, x_2, v\}$, then either $vx_1 \notin E(G)$ or $vx_2 \notin E(G)$.

(KS3) Let $w_1, w_2 \in N_G(x_1) \cap N_G(x_2)$, where $x_1, x_2 \in D_3(G)$ and $x_1x_2 \notin E(G)$. If $w_1w_2 \in E(G)$, then $N_G(w_1) \cup N_G(w_2) \subseteq N_G(x_1) \cup N_G(x_2) \cup \{x_1, x_2\}$.

By definition, $K_4 \in \mathcal{K}$ and every claw-free graph satisfies (KS1) and (KS3). Since (KS2) is violated for the graphs W_4 and W_5, we have $W_4, W_5 \notin \mathcal{K}$. For an integer $s \geq 0$, a graph G is said to have *Property* $\mathcal{K}(s)$ if G is in $\mathcal{K} - \{K_4\}$ and satisfies both $\kappa'(G) \geq 3$ and $ess'(G) \geq s+4$.

Lemma 20 [15]**.** *If the graph G has Property $\mathcal{K}(s)$, then there is a set $\triangle(G)$ of edge-disjoint triangles in G such that $D_3(G) \subseteq V(L)$, where L is the subgraph induced by $\bigcup_{K \in \triangle(G)} E(K)$, and $D_3(G) \cap V(K) \neq \emptyset$ for each $K \in \triangle(G)$.*

Let G have Property $\mathcal{K}(s)$ and $v \in D_3(G)$. By Lemma 20, there is a triangle in $\triangle(G)$ that contains v. We denote this triangle by $\triangle_v$. Thus, for $v, u \in D_3(G)$, we have either $E(\triangle_v) = E(\triangle_u)$ or $E(\triangle_v) \cap E(\triangle_u) = \emptyset$. Fix a given subset $E' \subseteq E(G)$. Define $\triangle'(G) = \{\triangle_v \in \triangle(G) : v \in D_3(G) \text{ and } E(\triangle_v) \cap E' = \emptyset\}$ and $\triangle^*(G) = \triangle(G) - \triangle'(G)$. Then $\triangle(G) = \triangle'(G)$ if $E' \cap E(\triangle(G)) = \emptyset$. Let $G_1 = G/\triangle(G)$ and $G_1^* = G/\triangle'(G)$ be the graphs obtained from G by contracting the edges in $\triangle(G)$ and $\triangle'(G)$, respectively. Thus if $E' \cap E(\triangle(G)) = \emptyset$, then $G_1 = G_1^*$. We call G_1 a $\triangle$-contraction of G and G_1^* a $\triangle$-contraction of G with respect to E'. Since G is 3-edge-connected and essentially $(s+4)$-edge-connected, we have

(5) $\kappa'(G_1) \geq 4$ and $ess'(G_1) \geq s+4$, and $\kappa'(G_1^*) \geq 3$ and $ess'(G_1^*) \geq s+4$.

By Theorem 14(iv), for any $X \subseteq E(G_1)$ with $|X| \leq 2$, $\tau(G_1 - X) \geq 2$, and so $F(G_1 - X) = 0$. Let t be the number of different triangles in $\triangle^*(G)$ and let $\triangle^*(G) = \{\triangle_{v_1}, \ldots, \triangle_{v_t}\}$ with $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$. Then $\{v_1, \ldots, v_t\} \subseteq D_3(G_1^*)$ and $E(\triangle_{v_i}) \cap E' \neq \emptyset$ for $i = 1, \ldots, t$ (Figure 2). Since G_1^* is 3-edge-connected and essentially 4-edge-connected, we have either $d_{G_1^*}(u_i) \geq 4$ or $d_{G_1^*}(w_i) \geq 4$. Without loss of generality, we assume that

$$d_{G_1^*}(w_i) \geq 4. \tag{6}$$

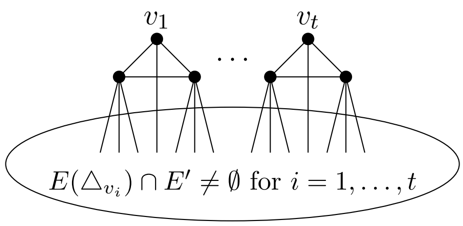

Figure 2. $G_1^* = G_1/\triangle'(G)$.

Lemma 21. (i) *If $t = 0$, then $G_1^* = G_1$.*

(ii) *If $t = 1$, then for any edge $e \in E(G_1^*)$, $\tau(G_1^* - e) \geq 2$.*

(iii) *If $s = 0$ and $t = 2$, then $\tau(G_1^*) \geq 2$.*

(iv) *If $s \geq 1$ and $t = 2$, then for any $e \in E(G_1^*)$, $\tau(G_1^* - e) \geq 2$.*

Proof. If $t = 0$, then $\triangle^*(G) = \emptyset$. Thus $G_1^* = G_1$. If $t = 1$, then $\triangle^*(G) = \{\triangle_{v_1}\}$ with $V(\triangle_{v_1}) = \{v_1, u_1, w_1\}$. By (6), $d_{G_1^*}(w_1) \geq 4$. Let Q_1 be the graph obtained from G_1^* by adding the new edge v_1u_1. Actually this new edge and the edge v_1u_1 in the triangle $\triangle_{v_1}$ are parallel. We denote this new edge by $(v_1u_1)'$. Then Q_1 is 4-edge-connected. Thus, for any edge $e \in E(G_1^*)$, $\tau(G_1^* - e) = \tau(Q_1 - \{e, (v_1u_1)'\})$ ≥ 2.

If $t = 2$, then $\triangle^*(G) = \{\triangle_{v_1}, \triangle_{v_2}\}$ with $V(\triangle_{v_i}) = \{v_i, u_i, w_i\} (i = 1, 2)$. By (6), $d_{G_1^*}(w_i) \geq 4$. If $s = 0$, then we set Q_2 to be the graph obtained from G_1^* by adding the new edges v_1v_2 and u_1u_2. Thus, Q_2 is 4-edge-connected. So $\tau(G_1^*) = \tau(Q_2 - \{v_1v_2, u_1u_2\}) \geq 2$. If $s \geq 1$, then G_1^* is essentially 5-edge-connected. Thus, for $x \in \{u_1, w_1, u_2, w_2\}$, $d_{G_1^*}(x) \geq 4$. Let Q_3 be the graph obtained from G_1^* by adding the new edge v_1v_2. Then Q_3 is 4-edge-connected. So for any edge $e \in E(G_1^*)$, $\tau(G_1^* - e) = \tau(Q_3 - \{v_1v_2, e\}) \geq 2$. ∎

The next lemma will be used in the proof of Theorem 9. For any edge subset X of G with $|X| = s$, to prove that $L(G)$ is s-hamiltonian connected, it suffices to prove that for any two edges $e_1, e_2 \in G - X$, $G - X$ has an internally dominating (e_1, e_2)-trail. By Theorem 8, we only need to consider $s \in \{0, 1, 2\}$.

Lemma 22. *Let X be an edge subset of G, and $s = |X| \in \{0, 1, 2\}$. If G satisfies Property $\mathcal{K}(s)$, then for any two edges $e_1, e_2 \in G - X$, $G - X$ has an internally dominating (e_1, e_2)-trail T such that $V(G) - V(T) \subseteq \bigcup_{i=3}^{s+3} D_i(G)$.*

Proof. Let $E' = X \cup \{e_1, e_2\}$ and let G_1 be a $\triangle$-contraction of G and G_1^* be a $\triangle$-contraction of G with respect to E'. If $\triangle^*(G) \neq \emptyset$, then we assume that $\triangle^*(G) = \{\triangle_{v_1}, \ldots, \triangle_{v_t}\}$ with $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$ $(i = 1, \ldots, t)$. Thus $E(\triangle_{v_i}) \cap E' \neq \emptyset$ and $t \in \{0, 1, \ldots, 2 + s\}$. By (5), we have

$$D_i(G_1^*) \subseteq D_i(G) \text{ for } i = 3, \ldots, s + 3. \tag{7}$$

Since a triangle is collapsible, to prove Lemma 22, by Proposition 12 and Theorem 13(iv), it suffices to prove that $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that

$$V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \bigcup_{i=3}^{s+3} D_i(G_1^*). \tag{8}$$

Claim 1. *If $s = 0$, then for any two edges e', e'' in G_1^*, $G_1^*(e', e'')$ is collapsible. Therefore, $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T.*

Proof. Since $s = 0$, we have $X = \emptyset$ and $t \in \{0, 1, 2\}$. Thus $G_1^* - X = G_1^*$. By (5), G_1^* is 3-edge-connected and essentially 4-edge-connected. Therefore, $G_1^*(e', e'')$ is 2-edge-connected. Let G' be the reduction of $G_1^*(e', e'')$. By Lemma 21(i)–(iii), $\tau(G_1^*) \geq 2$. Thus $F(G_1^* - \{e', e''\}) \leq 2$. As $F(G_1^* - \{e', e''\}) = F(G_1^*(e', e''))$, we have $F(G_1^*(e', e'')) \leq 2$. Since $G_1^*(e', e'')$ has only two vertices of degree two and since G_1^* is essentially 4-edge-connected, G' has at most two vertices of degree two. By Theorem 14(iii), $G' = K_1$. So $G_1^*(e', e'')$ is collapsible. By Theorem 13(iii), $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T. Claim 1 holds. □

Claim 2. *Assume that $s = 1$. Let $X = \{f\}$.*

(i) *If $t = 0$, then $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T.*

(ii) *If $t \in \{1, 2, 3\}$, then $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $|V((G_1^* - X)(e_1, e_2)) - V(T)| \leq 1$ and $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq D_3(G_1^*)$. Furthermore, if $t \in \{1, 2\}$ and $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v\}$, then $E_G(v) = \{f, e_1, e_2\}$.*

Proof. As $s = 1$, G_1^* is 3-edge-connected and essentially 5-edge-connected. Thus, $(G_1^* - f)(e_1, e_2)$ is 2-edge-connected and $|D_2((G_1^* - f)(e_1, e_2))| \leq 3$. If $t = 0$, then $G_1^* = G_1$ and so $F((G_1^* - f)(e_1, e_2)) \leq 1$. By Theorem 14(ii), $(G_1^* - f)(e_1, e_2)$ is collapsible. So $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T.

If $t \in \{1, 2\}$, then by Lemma 21(ii) and Lemma 21(iv), $F((G_1^* - f)(e_1, e_2)) \leq 2$. Let G' be the reduction of $(G_1^* - f)(e_1, e_2)$. Since $|D_2((G_1^* - f)(e_1, e_2))| \leq 3$

and since G_1^* is essentially 5-edge-connected, we have $|D_2(G')| \leq 4$. By Theorem 14(iii), $G' \in \{K_1, K_{2,2}, K_{2,3}, K_{2,4}\}$. If $G' = K_1$, by Theorem 13(iii), $(G_1^* - f)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' \in \{K_{2,3}, K_{2,4}\}$, then $v(e_1), v(e_2) \in D_2(G')$. By Theorem 13(iv), $(G_1^* - f)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' = K_{2,2}$, then $v(e_1), v(e_2) \in D_2(G')$ such that $v(e_1), v(e_2)$ are not adjacent. Let $V(G') - \{v(e_1), v(e_2)\} = \{v, v'\}$. Then either $PI(v)$ or $PI(v')$ is trivial. Without loss of generality, we assume that $PI(v)$ is trivial. So f is incident to v, and $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $V((G_1^* - f)(e_1, e_2)) - V(T) = \{v\}$, where $E_G(v) = \{e_1, e_2, f\}$.

If $t = 3$, then $f \in X'$ and $\triangle^*(G) = \{\triangle_{v_1}, \triangle_{v_2}, \triangle_{v_3}\}$, where for $1 \leq i \leq 3$, $v_i \in D_3(G_1^*)$. Without loss of generality, we assume that $f \in E(\triangle_{v_1})$, $e_1 \in E(\triangle_{v_2})$ and $e_2 \in E(\triangle_{v_3})$. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 4-edge-connected. By Claim 1, $(G_1^* - v_1)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T'. Let $T = \begin{cases} (T' - f) + \{v_1u_1, v_1w_1\}, & \text{if } f \in E(T') \\ T', & \text{otherwise} \end{cases}$. Then T is a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ such that $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \{v_1\} \subseteq D_3(G_1^*)$. Claim 2 holds. □

Claim 3. *If $s = 2$, then $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \bigcup_{i=3}^{5} D_i(G_1^*)$.*

Proof. Since $s = 2$, G_1^* is 3-edge-connected and essentially 6-edge-connected. Let $X = \{f_1, f_2\}$. Then $G_1^* - X$ is connected and essentially 4-edge-connected. So $(G_1^* - X)(e_1, e_2)$ is connected. As $s = 2$, we have $t \in \{0, 1, 2, 3, 4\}$.

Claim 3.1. *If $G_1^* - X$ is not 2-edge-connected, then Claim 3 holds.*

Proof. Assume that $G_1^* - X$ is not 2-edge-connected. Let e be a cut edge of $G_1^* - X$, and let H_1 and H_2 be components of $(G_1^* - X) - e$. Then $\{f_1, f_2, e\}$ is a 3-edge cut of G_1^*. As G_1^* is essentially 6-edge-connected, we may assume that $V(H_1) = \{v_1\}$. Then $E_{G_1^*}(v_1) = \{f_1, f_2, e\}$. Thus $t \leq 3$. Consider $G_1^* - v_1$. Then $d_{G_1^* - v_1}(x) \geq 4$ for any $x \in N_{G_1^*}(v_1)$. Since $t \leq 3$, $G_1^* - v_1$ contains at most two vertices of degree 3. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected. Thus $\tau((G_1^* - v) - e_1) \geq 2$. This implies that $F((G_1^* - v) - \{e_1, e_2\}) \leq 1$ and so $F((G_1^* - v_1)(e_1, e_2)) \leq 1$. By Theorem 14(ii), $(G_1^* - v_1)(e_1, e_2)$ is collapsible. Let T be a spanning $(v(e_1), v(e_2))$-trail of $(G_1^* - v_1)(e_1, e_2)$. Then T is a dominating $(v(e_1), v(e_2))$-trail of $(G_1^* - X)(e_1, e_2)$ with $V(G_1^*) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. Claim 3.1 holds. □

By Claim 3.1, we may assume that $G_1^* - X$ is 2-edge-connected. Then $(G_1^* - X)(e_1, e_2)$ is also 2-edge-connected. If $t = 0$, then $f_1, f_2, e_1, e_2 \in E(G_1)$ and $G_1^* = G_1$. So $(G_1 - X)(e_1, e_2)$ has at most three vertices of degree 2. Let G' be the reduction of $(G_1 - X)(e_1, e_2)$. Then $|D_2(G')| \leq 4$. By (5), G_1 is 4-edge-connected.

By Theorem 14(iv), we have $\tau(G_1 - X) \geq 2$ and so $F((G_1 - X)(e_1, e_2)) \leq 2$. Since $(G_1 - X)(e_1, e_2)$ is 2-edge-connected and G_1 is essentially 6-edge-connected, $G' \in \{K_1, K_{2,2}, K_{2,3}\}$. If $G' = K_1$, by Theorem 13(iii), $(G_1 - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' = K_{2,3}$, then $v(e_1), v(e_2) \in D_2(G')$. By Theorem 13(iv), $(G_1 - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' = K_{2,2}$, then $v(e_1), v(e_2) \in D_2(G')$ such that $v(e_1), v(e_2)$ are not adjacent. Let $V(G') - \{v(e_1), v(e_2)\} = \{v, v'\}$. Then either $PI(v)$ or $PI(v')$ is trivial. Without loss of generality, we assume that $PI(v)$ is trivial. So f_1, f_2 are incident to v and $(G_1 - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $V((G_1 - X)(e_1, e_2)) - V(T) = \{v\} \subseteq D_4(G_1)$, where $E_G(v) = \{e_1, e_2, f_1, f_2\}$. Next we just need to consider $t \in \{1, 2, 3, 4\}$.

Claim 3.2. *If $f_1, f_2 \in E(G_1)$, then Claim 3 holds.*

Proof. In this case, $|\{e_1, e_2\} \cap E(G_1)| \leq 1$ and $t \in \{1, 2\}$. Without loss of generality, we assume that $e_2 \notin E(G_1)$. We consider two cases.

Case 1. $t = 1$. Then $\triangle^*(G) = \{\triangle_{v_1}\}$ with $v_1 \in D_3(G_1^*)$ and $V(\triangle_{v_1}) = \{v_1, u_1, w_1\}$, and $e_2 \in E(\triangle_{v_1})$. Let $E_{G_1^*}(v_1) = \{v_1u_1, v_1w_1, v_1z\}$. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected. Since G_1^* is essentially 6-edge-connected, we have $d_{G_1^* - v_1}(y) \geq 4$ for $y \in \{u_1, w_1, z\}$. Thus $G_1^* - v_1$ is 4-edge-connected, and so $\tau((G_1^* - v_1) - \{f_1, f_2\}) \geq 2$.

If $e_2 \in \{v_1u_1, v_1w_1\}$, assume that $e_2 = v_1w_1$. Let $a = \begin{cases} z, & \text{if } e_1 = v_1z, \\ u_1, & \text{if } e_1 = v_1u_1, \\ v(e_1), & \text{otherwise} \end{cases}$ and let $H = \begin{cases} (G_1^* - v_1) - \{f_1, f_2\}, & \text{if } e_1 \in \{v_1z, v_1u_1\} \\ ((G_1^* - v_1) - \{f_1, f_2\})(e_1), & \text{otherwise} \end{cases}$. Then $F(H) \leq 1$ and H is 2-edge-connected. By Theorem 14(ii), H has a spanning (w_1, a)-trail. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$.

If $e_2 \notin \{v_1u_1, v_1w_1\}$, then $e_2 = u_1w_1$. If $e_1 \in E_{G_1^*}(v_1)$, then $e_1 = v_1b$, where $b \in \{w_1, u_1, z\}$. As $F(((G_1^* - v_1) - \{f_1, f_2\})(e_2)) \leq 1$, $((G_1^* - v_1) - \{f_1, f_2\})(e_2)$ has a spanning $(b, v(e_2))$-trail. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. If $e_1 \notin E_{G_1^*}(v_1)$, then as $F(((G_1^* - v_1) - \{f_1, f_2\})(e_1)) \leq 1$, $((G_1^* - v_1) - \{f_1, f_2\})(e_1)$ has a spanning $(w_1, v(e_1))$-trail T'. If $e_2 \notin E(T')$, then this trail T' can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. If $e_2 \in E(T')$, then $T = T' - \{w_1u_1\} + \{v_1u_1, v_1w_1, w_1v(e_2)\}$ is a spanning $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$.

Case 2. $t = 2$. Then $e_1, e_2 \notin E(G_1)$ and $\triangle^*(G) = \{\triangle_{v_1}, \triangle_{v_2}\}$ with $v_i \in D_3(G_1^*)$ and $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$ $(i = 1, 2)$. For $i = 1, 2$, we assume that $e_i \in E(\triangle_{v_i})$, $E_{G_1^*}(v_i) = \{v_iu_i, v_iw_i, v_iz_i\}$, and let x_i be the vertex on which $\triangle_{v_i}$ is contracted in G_1. If $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$ is not connected, then $z_1 = z_2$ and f_1, f_2, x_1z_1, x_2z_2 are incident to z_1 of degree 4. Thus $\{f_1, f_2, v_1z_1, v_2u_2, v_2w_2\}$ is an essential 5-edge cut in G_1^*, a contradiction. So $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$ is connected. Similarly, $G_1 - \{f_1, f_2, x_1z_1\}$ and $G_1 - \{f_1, f_2, x_2z_2\}$ are 2-edge-connected.

Consider $G_1 - \{f_1, f_2, x_1z_1\}$. By (5), G_1 is 4-edge-connected. By Theorem 14(iv), $F(G_1 - \{f_1, f_2, x_1z_1\}) \leq 1$ and $F(G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}) \leq 2$. By Theorem 14(ii), $G_1 - \{f_1, f_2, x_1z_1\}$ is collapsible. Thus $G_1 - \{f_1, f_2, x_1z_1\}$ has a spanning (x_1, x_2)-trail T. If $e_2 \neq u_2w_2$, then by Lemma 15, $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T with $V(G_1^*) - V(T) \subseteq \{v_1, v_2\} \subseteq D_3(G_1^*)$. So we may assume that $e_2 = u_2w_2$. Similarly, we assume that $e_1 = u_1w_1$. By Lemma 15, $\{f_1, f_2\} \cap \{x_1z_1, x_2z_2\} = \emptyset$.

Notice that $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$ is connected and $F(G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}) \leq 2$. Let G' be the reduction of $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$. By Theorem 14(iii), $G' \in \{K_1, K_2, K_{2,\ell}\}$. Since G_1 is 4-edge-connected and essentially 6-edge-connected, and since x_1 and x_2 are the vertices on which $\triangle_{v_1}$ and $\triangle_{v_2}$ are contracted in G_1, we have $G' \neq K_{2,\ell}$. If $G' = K_1$, then $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$ has a spanning (x_1, x_2)-trail. By Lemma 15(ii), $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' = K_2$, then we assume that $G' = ab$. Thus either $PI(a)$ or $PI(b)$ is trivial. Without loss of generality, we assume that $PI(a)$ is trivial. Since G_1^* is essentially 6-edge-connected, we have $a \in V(G_1^*)$. As $G_1 - \{f_1, f_2, x_1z_1\}$ and $G_1 - \{f_1, f_2, x_2z_2\}$ are 2-edge-connected, we have $E_{G_1}(a) \cap \{x_1z_1, x_2z_2\} \neq \emptyset$. Without loss of generality, we assume that $a = z_1$. Thus $x_1 \in V(PI(b))$. Since $E_{G_1^*}(a) \cup \{v_1u_1, v_1w_1\} - \{z_1v_1\}$ is essentially edge-cut of G_1^* and since G_1^* is essentially 6-edge-connected, $d_{G_1}(a) \geq 5$. Thus $E_{G_1}(a) = \{ab, z_1x_1, z_2x_2, f_1, f_2\}$, $a = z_1 = z_2$ and $x_1, x_2 \in PI(b)$. Let T be a spanning (x_1, x_2)-trail in $PI(b)$. As $e_1 = u_1w_1$ and $e_2 = u_2w_2$, by Lemma 15(ii), $(G_1^* - X)(e_1, e_2) - a$ has a spanning $(v(e_1), v(e_2))$-trail T'. This trail T' is a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{a\} \subseteq D_5(G_1^*)$. We finish the proof of Claim 3.2. □

By Claim 3.2, we may assume that $f_1 \notin E(G_1)$. In addition, we assume that $\triangle_{v_1} \in \triangle^*(G)$ such that $f_1 \in E(\triangle_{v_1})$. Let $E_{G_1^*}(v_1) = \{v_1u_1, v_1w_1, v_1z_1\}$, where $V(\triangle_{v_1}) = \{v_1, u_1, w_1\}$.

Claim 3.3. *If $E_{G_1^*}(v_1) \cap \{e_1, e_2\} = \emptyset$, then Claim 3 holds.*

Proof. In this case, $e_1, e_2 \in E(G_1^* - v_1)$. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected. If $f_2 \notin E_{G_1^*}(v_1)$, then $f_2 \in E(G_1^* - v_1)$. By Claim 2, $((G_1^* - v_1) - f_2)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T' such that $V(((G_1^* - v_1) - f_2)(e_1, e_2)) - V(T') \subseteq \{y\} \subseteq D_3(G_1^* - v_1)$. Let $T = \begin{cases} (T' - f_1) + \{v_1u_1, v_1w_1\}, & \text{if } f_1 \in E(T') \\ T', & \text{othererwise} \end{cases}$. Then T is a dominating $(v(e_1),$

$v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ such that $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \{v_1, y\} \subseteq D_3(G_1^*)$. If $f_2 \in E_{G_1^*}(v_1)$, then by Claim 2, $((G_1^* - v_1) - u_1w_1)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $V(((G_1^* - v_1) - f_2)(e_1, e_2)) - V(T) \subseteq \{y\} \subseteq D_3(G_1^* - v_1)$. This trail T is also a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ such that $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \{v_1, y\} \subseteq D_3(G_1^*)$. Claim 3.3 holds. $\square$

By Claim 3.3, we assume that $e_1 \in E_{G_1^*}(v_1)$. Let $e_1 = v_1b_1$, where $b_1 \in \{w_1, u_1, z_1\}$. Consider f_2. If $f_2 \in E(G_1)$, then $t \in \{1, 2\}$; if $f_2 \notin E(G_1)$, then we may assume that $f_2 \in E(\triangle_{v_i})$ for some $\triangle_{v_i} \in \triangle^*(G)$. Thus by Claim 3.3, $e_2 \in E_{G_1^*}(v_i)$. So we still have $t \in \{1, 2\}$.

Claim 3.4. *If* $t = 1$*, then Claim* 3 *holds.*

Proof. In this case, $\triangle^*(G) = \{\triangle_{v_1}\}$. Since G_1^* is 3-edge-connected and essentially 6-edge-connected and since v_1 is only the vertex of degree three, $G_1^* - v_1$ is 4-edge-connected. So

$$\tau((G_1^* - v_1) - \{f_2, u_1w_1\}) \geq 2. \tag{9}$$

If $e_2 \in E_{G_1^*}(v_1)$, then we assume that $e_2 = v_1b_2$, where $b_2 \in \{u_1, w_1, z_1\} - \{b_1\}$. By (9), $(G_1^* - v_1) - \{f_2, u_1w_1\}$ is collapsible. Thus $(G_1^* - v_1) - \{f_2, u_1w_1\}$ has a spanning (b_1, b_2)-trail. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. If $e_2 \notin E_{G_1^*}(v_1)$, by (9), $F(((G_1^* - v_1) - \{f_2, u_1w_1\})(e_2)) \leq 1$. By Theorem 14(ii), $((G_1^* - v_1) - \{f_2, u_1w_1\})(e_2)$ is collapsible. Thus $((G_1^* - v_1) - \{f_2, u_1w_1\})(e_2)$ has a spanning $(b_1, v(e_2))$-trail. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. Claim 3.4 holds. $\square$

By Claim 3.4, we assume that $t = 2$. Then $\triangle^*(G) = \{\triangle_{v_1}, \triangle_{v_2}\}$, where $v_i \in D_3(G_1^*)$, $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$ and $\{e_2, f_2\} \cap E(\triangle_{v_2}) \neq \emptyset$. Let $E_{G_1^*}(v_2) = \{v_2u_2, v_2w_2, v_2z_2\}$. Since G_1^* is essentially 6-edge-connected, $d_{G_1^*}(x) \geq 5$ for $x \in \{u_1, w_1, u_2, w_2\}$. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected. Since v_2 is the only vertex of degree 3 in $G_1^* - v_1$, we have $\tau((G_1^* - v_1) - \{f_2\}) \geq 2$, and so $F((G_1^* - v_1) - \{u_1w_1, f_2\}) \leq 1$.

Consider $G_1^* - v_1$. Then $(G_1^* - v_1) - \{u_1w_1, f_2\}$ is essentially 3-edge-connected. If $(G_1^* - v_1) - \{u_1w_1, f_2\}$ has a cut edge f', then we assume that H_1 and H_2 are components of $(G_1^* - v_1) - \{u_1w_1, f_2, f'\}$. Since G_1^* is essentially 6-edge-connected, we have either H_1 or H_2 is trivial. Without loss of generality, we assume that $V(H_1) = \{u_1\}$. Then $E_{G_1^*}(u_1) = \{u_1v_1, u_1w_1, f_2, f'\}$ and $(E_{G_1^*}(u_1) \cup E_{G_1^*}(v_1)) - \{u_1v_1\}$ is an essential 5-edge cut in G_1^*, a contradiction. So $(G_1^* - v_1) - \{u_1w_1, f_2\}$ is 2-edge-connected.

If $e_2 \in E_{G_1^*}(v_1) \cup \{u_1w_1\}$, then $f_2 \in E(\triangle_{v_2})$. We assume that e_2 is incident to b_3, where $b_3 \in \{u_1, w_1, z_1\}$. By Theorem 14(ii), $(G_1^* - v_1) - \{u_1w_1, f_2\}$ is collapsible. Thus $(G_1^* - v_1) - \{u_1w_1, f_2\}$ has a spanning (b_1, b_3)-trail T'. Thus $T = v(e_1)T'v(e_2)$ is a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ such that $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. So we may assume that $e_2 \notin E_{G_1^*}(v_1) \cup \{u_1w_1\}$.

As $\tau((G_1^*-v_1)-\{f_2\}) \geq 2$, $F(((G_1^*-v_1)-\{u_1w_1, f_2\})(e_2)) \leq 2$. Let G' be the reduction of $((G_1^* - v_1) - \{u_1w_1, f_2\})(e_2)$. By Theorem 14(iii), $G' \in \{K_1, K_{2,\ell}\}$ $(\ell \geq 3)$. Since $d_{G_1^*-v_1}(u_1) \geq 4$ and $d_{G_1^*-v_1}(w_1) \geq 4$ and since $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected, $G' = K_1$. Notice that $e_1 = v_1b_1$. Then $((G_1^* - v_1) - \{u_1w_1, f_2\})(e_2)$ has a spanning $(b_1, v(e_2))$-trail T. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. We finish the proof of Claim 3.

□

We finish the proof of Lemma 22. ■

We need one more notation. Let $e = xy \in E(W_5)$ with $x, y \in D_3(W_5)$ and let H be a graph and $e' = x'y' \in E(H)$. Define a new graph $H \oplus W_5$ to be a graph obtained from the disjoint union of $H - e'$ and W_5 by identifying x and x' to form a new vertex, also called x, and by identifying y and y' to form a new vertex, also called y.

Lemma 23 [15]**.** *Suppose that $s \geq 0$ and that G is a claw-free graph such that $\kappa(L(G)) \geq s+4$. Let G_0 be the core of G and let $w_1, w_2, w_3 \in D_3(G_0)$ be vertices with $N_{G_0}(w_2) = \{w_1, w_3, v\}$. If $vw_1, vw_3 \in E(G_0)$, then each of the following holds.*

(i) $s = 0$.

(ii) *Either $G = G_0 \in \{K_4, W_4, W_5\}$, or there exists a subgraph H of G with $\kappa'(H) \geq 3$ and $ess'(H) \geq 4$ such that $G_0 = H \oplus W_5$ (see Figure 3).*

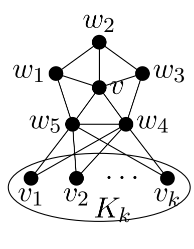

Figure 3. $K_{k+2} \oplus W_5$ in Lemma 23.

***Proof of Theorem* 9.** Let X be any edge subset of G with $|X| = s$. To prove that $L(G)$ is s-hamiltonian connected, it suffices to prove that for any two edges

$e_1, e_2 \in G-X$, $G-X$ has an internally dominating (e_1, e_2)-trail. By Theorem 8, we assume that $s \in \{0, 1, 2\}$. Let G_0 be the core of G. Then it suffices to assume that $X \cup \{e_1, e_2\} \subseteq E(G_0)$, and to show $G_0 - X$ has an internally dominating (e_1, e_2)-trail T with $V(G_0) - V(T) \subseteq \bigcup_{i=3}^{s+3} D_i(G_0)$. By contradiction, we assume that G is a counterexample to Theorem 9 with $|V(G_0)|$ minimized. Then there exist edges $X \cup \{e_1, e_2\} \subseteq E(G_0)$ such that $(G_0 - X)(e_1, e_2)$ does not have a dominating $(v(e_1), v(e_2))$-trail T with

$$V((G_0 - X)(e_1, e_2)) - V(T) \subseteq \bigcup_{i=3}^{s+3} D_i(G_0). \tag{10}$$

By (10) and Theorem 13(iii), we assume that $G_0 \notin \{K_4, W_4, W_5\}$ and G_0^* is not collapsible. By Lemma 22, G_0 does not have Property $\mathcal{K}(s)$. As G_0 is claw-free, (KS2) is violated. Thus there exist $w_1, w_2, w_3 \in D_3(G_0)$ with $N_{G_0}(w_2) = \{w_1, w_3, v\}$ and $vw_1, vw_3 \in E(G_0)$. By Lemma 23, we have $s = 0$ and $G_0 = H \oplus W_5$ for a subgraph H of G_0 with $\kappa'(H) \geq 3$ and $ess'(H) \geq 4$. Assume that $V(W_5) = \{v, w_1, \ldots, w_5\}$ with $w_4w_5 \in E(H) \cap E(W_5)$, as depicted in Figure 3. As H is claw-free, every 3-edge-cut of H has at least one edge in a 3-cycle. By Theorem 13(v), for any two edges $e', e'' \in E(H)$, $H(e', e'')$ is collapsible. Thus H and $H(e')$ are collapsible.

If $\{e_1, e_2\} \cap E(W_5) = \emptyset$, then by the minimality of G_0, $H(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T_1 with $V(H(e_1, e_2)) - V(T_1) \subseteq D_3(H)$. Thus the subgraph induced by $E(T_1) \cup \{vw_5, vw_4, w_5w_1, w_1w_2, w_2w_3, w_3w_4\}$ is a dominating $(v(e_1), v(e_2))$-trail in G_0^*, contrary to (10). If $e_1, e_2 \in E(W_5)$, then by inspection, $W_5(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T_2 that contains either w_4 or w_5. As H is collapsible, H has a spanning eulerian subgraph T_3. Thus $T_4 = G_0^*[(E(T_2) - E(T_3)) \cup (E(T_3) - E(T_2))]$ is a dominating $(v(e_1), v(e_2))$-trail in G_0^* with $V(G_0^*) - V(T_4) \subseteq D_3(G_0)$, contrary to (10). Thus we assume that $e_1 \in E(H) - E(W_5)$ and $e_2 \in E(W_5) - E(H)$. By Theorem 13(ii), $W_5(e_2)$ is collapsible. By Theorem 13(v), $H(e_1)$ is collapsible. Thus G_0^* is collapsible, a contradiction. ■

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