Discussiones Mathematicae
Graph Theory 44 (2024) 297–315
https://doi.org/10.7151/dmgt.2448

ON s-HAMILTONIAN-CONNECTED LINE GRAPHS

Xiaoling Ma

College of Mathematics and System Sciences
Xinjiang University, Urumqi, Xinjiang 830046, China
e-mail: mxlmath@sina.com

Hong-Jian Lai

Department of Mathematics
West Virginia University, Morgantown, WV 26506, USA
e-mail: hjlai@math.wvu.edu

Mingquan Zhan

Department of Mathematics
Millersville University of Pennsylvania, Millersville, PA 17551, USA
e-mail: Mingquan.Zhan@millersville.edu

Taoye Zhang

Department of Mathematics
Penn State Worthington Scranton, Dunmore, PA 18512, USA
e-mail: taoyezhang@gmail.com

and

Ju Zhou

Department of Mathematics
Kutztown University of Pennsylvania, Kutztown, PA 19530, USA
e-mail: Zhou@kutztown.edu

Abstract

For an integer $s \geq 0$, G is s-hamiltonian-connected if for any vertex subset $S \subseteq V(G)$ with $|S| \leq s$, $G - S$ is hamiltonian-connected. Thomassen in 1984 conjectured that every 4-connected line graph is hamiltonian (see [*Reflections on graph theory*, J. Graph Theory 10 (1986) 309–324]), and Kužel

and Xiong in 2004 conjectured that every 4-connected line graph is hamiltonian-connected (see [Z. Ryjáček and P. Vrána, *Line graphs of multigraphs and Hamilton-connectedness of claw-free graphs*, J. Graph Theory 66 (2011) 152–173]). In this paper we prove the following.
(i) For $s \geq 3$, every $(s+4)$-connected line graph is s-hamiltonian-connected.
(ii) For $s \geq 0$, every $(s+4)$-connected line graph of a claw-free graph is s-hamiltonian-connected.

Keywords: line graph, claw-free graph, s-hamiltonian-connected, collapsible graphs, reductions.

2020 Mathematics Subject Classification: 05C75, 05C45.

1. Introduction

Graphs considered here are finite and loopless. Unless otherwise noted, we follow [1] for notation and terms. As in [1], $\kappa(G)$ and $\kappa'(G)$ denote the connectivity and the edge-connectivity of a graph G, respectively. A graph is *nontrivial* if it contains edges. An edge cut X is *essential* if $G - X$ has at least two nontrivial components. For an integer $k > 0$, a graph G is *essentially k-edge-connected* if G does not have an essential edge cut X with $|X| < k$. For a connected graph G, let $ess'(G) = \max\{k : G \text{ is essentially } k\text{-edge-connected}\}$, and for an integer $i \geq 0$, let $D_i(G) = \{u \in V(G) : d_G(u) = i\}$. Throughout this paper, for an integer $n \geq 2$, C_n denotes a cycle on n vertices (called an n-cycle), W_n denotes the graph obtained from an n-cycle by adding a new vertex and connecting it to every vertex of the n-cycle. If $S \subseteq V(G)$ or $S \subseteq E(G)$, then $G[S]$ is the subgraph induced in G by S. We use $H \subseteq G$ to denote the fact that H is a subgraph of G. For $H \subseteq G$, $x \in V(G)$, $A \subseteq V(G)$, $X \subseteq E(G)$, and $Y \subseteq E(G) - E(H)$, define $E_G(x) = \{e : e \text{ is incident to } x\}$, $N_H(x) = N_G(x) \cap V(H)$, $d_H(x) = |N_H(x)|$, $G - A = G[V(G) - A]$, $G - X = G[E(G) - X]$, and $H + Y = G[E(H) \cup Y]$. When $A = \{v\}$ and $X = \{e\}$, we use $G - v$ for $G - \{v\}$ and $G - e$ for $G - \{e\}$.

Let $O(G)$ denote the set of odd degree vertices of G. A graph G is *eulerian* if $O(G) = \emptyset$ and G is connected. A graph G is *supereulerian* if G has a spanning eulerian subgraph. A graph G is *claw-free* if it does not contain $K_{1,3}$ as an induced subgraph. The *line graph* of a graph G, denoted by $L(G)$, has $E(G)$ as its vertex set, where two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G are adjacent. From the definition of a line graph, if $L(G)$ is not a complete graph, then $L(G)$ is k-connected if and only if G is essentially k-edge-connected. The following are several fascinating conjectures in the literature.

Conjecture 1. (i) (Thomassen [20]) *Every 4-connected line graph is hamiltonian.*

(ii) (Matthews and Sumner [16]) *Every 4-connected claw-free graph is hamiltonian.*

(iii) (Kužel and Xiong [11]) *Every* 4*-connected line graph is hamiltonian-connected.*

(iv) (Ryjáček and Vrána [17]) *Every* 4*-connected claw-free graph is hamiltonian-connected.*

Ryjáček and Vrána in [17] indicated that the statements in Conjecture 1 are mutually equivalent. There have been many studies on these conjectures in the literature. Among them are the following.

Theorem 2 (Zhan [21])**.** *Every* 7*-connected line graph is hamiltonian-connected.*

Theorem 3 (Kaiser and Vrána [9])**.** *Every* 5*-connected line graph with minimum degree at least* 6 *is hamiltonian.*

Theorem 4 (Kriesell [10])**.** *Every* 4*-connected line graph of a claw-free graph is hamiltonian-connected.*

For an integer $s \geq 0$, a graph G is *s-hamiltonian* (or *s-hamiltonian-connected*, respectively) if for any vertex subset $S \subseteq V(G)$ with $|S| \leq s$, $G - S$ is hamiltonian (or hamiltonian-connected, respectively). It is routine to observe that every s-hamiltonian graph is $(s+2)$-connected, and every s-hamiltonian-connected graph is $(s+3)$-connected. The converse, on the other hand, is not true, as $K_{m,m+1}$ is m-connected but nonhamiltonian.

Theorem 5 (Kaiser, Ryjáček, and Vrána [8])**.** *Every* 5*-connected claw-free graph with minimum degree* 6 *is* 1*-hamiltonian-connected.*

Theorem 6. *Let s be an integer.*

(i) (Theorem 1.4 of [12]) *For $s \geq 3$, every $(s+4)$-connected line graph is $(s-1)$-hamiltonian-connected.*

(ii) (Theorem 1.3 of [13]) *For $s \geq 5$, every $(s+2)$-connected line graph is s-hamiltonian.*

(iii) (Theorem 1.6 of [15])) *For $s \geq 0$, every $(s+2)$-connected line graph of a claw-free graph is s-hamiltonian.*

(iv) (Theorem 1.6 of [15])) *Every* 4*-connected line graph of a claw-free graph is* 1*-hamiltonian-connected.*

Motivated by Conjecture 1 as well as the results in [5, 12] and [13], the following conjecture was proposed.

Conjecture 7 [15]**.** *Let s be an integer.*

(i) *For $s \geq 2$, a line graph is s-hamiltonian if and only if it is $(s+2)$-connected.*

(ii) *For $s \geq 2$, a claw-free graph is s-hamiltonian if and only if it is $(s+2)$-connected.*

(iii) *For $s \geq 1$, a line graph is s-hamiltonian-connected if and only if it is $(s+3)$-connected.*

(iv) *For $s \geq 1$, a claw-free graph is s-hamiltonian-connected if and only if it is $(s+3)$-connected.*

In [18], Ryjáček and Vrána showed that when $s = 1$, Conjecture 7(iii) is equivalent to Conjecture 1(i). The main results in this paper are presented below.

Theorem 8. *For $s \geq 3$, every $(s+4)$-connected line graph is s-hamiltonian-connected.*

Theorem 9. *For $s \geq 0$, every $(s+4)$-connected line graph of a claw-free graph is s-hamiltonian-connected.*

Catlin's reduction method will be refreshed in Section 2, together with other useful tools developed in this paper for our proofs of the main results. The proof of Theorem 8 is presented in Section 3, and the proof of Theorem 9 is presented in Section 4. We would like to point out that some of the mechanisms developed in [15] will be utilized in the proof arguments of Theorem 9, as shown in Section 4.

2. Preliminaries

We view a trail of G as a vertex-edge alternating sequence $v_0, e_1, v_1, e_2, \ldots, e_k, v_k$ such that all the e_i's are distinct and for each $i = 1, 2, \ldots, k$, e_i is incident with both v_{i-1} and v_i. The vertices in $v_1, v_2, \ldots, v_{k-1}$ are *internal vertices* of the trail. For edges $e', e'' \in E(G)$, an (e', e'')-trail of G is a trail T of G whose first edge is e' and whose last edge is e''. An *internally dominating* (e', e'')*-trail* of G is an (e', e'')-trail T of G such that every edge of G is incident with an internal vertex of T, and a *spanning* (e', e'')*-trail* of G is an internally dominating (e', e'')-trail T of G such that $V(T) = V(G)$. Harary and Nash-Williams [6] first showed the relationship between eulerian subgraphs in G and hamiltonicity in $L(G)$. Theorem 10(ii) below is observed in [14].

Theorem 10. *Let G be a graph with $|E(G)| \geq 3$. Each of the following holds.*

(i) (Harary and Nash-Williams [6]) *$L(G)$ is hamiltonian if and only if G has a dominating eulerian subgraph.*

(ii) ([14]) *$L(G)$ is hamiltonian-connected if and only if for any pair of edges $e', e'' \in E(G)$, G has an internally dominating (e', e'')-trail.*

Theorem 11. *Let G be a connected graph with at least three edges and $s > 0$ an integer. The line graph $L(G)$ is s-hamiltonian-connected if and only if $G - S$ has an internally dominating (e', e'')-trail for any $S \subset E(G)$ with $|S| \leq s$, and for any pair of edges $e', e'' \in E(G - S)$.*

We say that an edge $e \in E(G)$ is *subdivided* when it is replaced by a path of length 2 whose internal vertex, denoted by $v(e)$, has degree 2 in the resulting graph. The process of taking an edge e and replacing it by the path of length 2 is called *subdividing* e. For a graph G and edges $e', e'' \in E(G)$, let $G(e')$ denote the graph obtained from G by subdividing e', and let $G(e', e'')$ denote the graph obtained from G by subdividing both e' and e''. Then $V(G(e', e'')) - V(G) = \{v(e'), v(e'')\}$.

Proposition 12. *For a graph G and edges $e', e'' \in E(G)$, if $G(e', e'')$ has a dominating (spanning, respectively) $(v(e'), v(e''))$-trail, then G has an internally dominating (spanning, respectively) (e', e'')-trail.*

Let $X \subseteq E(G)$ be an edge subset of G. The *contraction* G/X is the graph obtained from G by identifying the two ends of each edge in X and then deleting the resulting loops. If H is a subgraph of G, then we write G/H for $G/E(H)$. If v_K is the vertex in G/H onto which the connected subgraph K is contracted, then K is called the *preimage* of v_K, and denoted by $PI(v_K)$. In [2] Catlin defined collapsible graphs. Given an even subset R of $V(G)$, a subgraph Γ of G is called an *R-subgraph* if $O(\Gamma) = R$ and $G - E(\Gamma)$ is connected. A graph G is *collapsible* if for any even subset R of $V(G)$, G has an R-subgraph. In particular, K_1 is collapsible. Catlin [2] showed that for any graph G, one can obtain the *reduction* G' of G by contracting all maximal collapsible subgraphs of G. A graph G' is *reduced* if G' has no nontrivial collapsible subgraphs. A vertex x in G' is *c-nontrivial* (or *c-trivial*) if $|V(PI(x))| \geq 2$ (or $|V(PI(x))| = 1$). By definition, every collapsible graph is supereulerian. We summarize some results on Catlin's reduction method and other related facts below.

Theorem 13. *Let G be a graph and let H be a collapsible connected subgraph of G. Let v_H denote the vertex onto which H is contracted in G/H. Each of the following holds.*

- (i) (Catlin, Theorem 3 of [2]) *G is collapsible if and only if G/H is collapsible. Therefore, G is collapsible if and only if the reduction of G is K_1.*
- (ii) (Catlin, implied by definition and Theorem 3 of [2]) *C_2, C_3 are collapsible, and when $n \geq 4$, for any $e \in E(W_n)$, $W_n(e)$ is collapsible.*
- (iii) (Theorem 2.3(iii) of [14]) *If G is collapsible, then for any pair of vertices $u, v \in V(G)$, G has a spanning (u, v)-trail.*
- (iv) (Theorem 2.3(iv) of [14]) *For vertices $u, v \in V(G/H) - \{v_H\}$, if G/H has a spanning (u, v)-trail, then G has a spanning (u, v)-trail.*
- (v) (Theorem 3.3 of [14]) *Let G be a 3-edge-connected graph. If every 3-edge-cut X has at least one edge in a 2-cycle or 3-cycle of G, then, for any two edges $e', e'' \in E(G)$, $G(e', e'')$ is collapsible.*

Let $\tau(G)$ denote the maximum number of edge-disjoint spanning trees of G. Let $F(G)$ be the minimum number of additional edges that must be added to G so that the resulting graph has two edge-disjoint spanning trees. The following theorem summarizes results related to $F(G)$ and supereulerian properties.

Theorem 14. *Let G be a connected graph and let G' be the reduction of G. Then each of the following holds.*

(i) (Jaeger [7]) *If $F(G) = 0$, then G is collapsible.*

(ii) (Catlin [2]) *If $F(G) \leq 1$, then $G' \in \{K_1, K_2\}$. Therefore, G is supereulerian if and only if $G' \neq K_2$.*

(iii) (Catlin *et al.* [3]) *If $F(G) \leq 2$, then $G' \in \{K_1, K_2, K_{2,t}\}$ for some integer $t \geq 1$. Therefore, G is supereulerian if and only if $G' \notin \{K_2, K_{2,t}\}$ for some odd integer t.*

(iv) (Theorem 1.1 of [4]) *Let $k \geq 1$ be an integer. Then $\kappa'(G) \geq 2k$ if and only if for any edge subset $X \subseteq E(G)$ with $|X| \leq k$, $\tau(G - X) \geq k$.*

Lemma 15 ([15])**.** *Assume that $K = v_1v_2v_3v_1$ is a triangle in a connected graph G with $d_G(v_1) = 3$. Also assume that $N_G(v_1) = \{v_2, v_3, x\}$ and $e \in \{v_1v_2, v_2v_3\}$. Let w be the new vertex in G/K on which K is contracted, and let $u(\neq w) \in V(G/K)$. Let T be a spanning (u, w)-trail in G/K. Then each of the following holds.*

(i) *For $e = v_1v_2$, $G(e)$ has a dominating $(u, v(e))$-trail T_1 such that $V(G(e)) - V(T_1) \subseteq \{v_1\}$.*

(ii) *For $e = v_2v_3$, if $xv_1 \notin E(T)$, then $G(e)$ has a spanning $(u, v(e))$-trail T_2.*

Lemma 16 ([15])**.** *Let $s \geq 3$ be an integer and G be a graph with $\kappa'(G) \geq 3$ and $ess'(G) \geq s + 2$. If $v \in D_3(G)$, then $\kappa'(G - v) \geq 3$ and $ess'(G - v) \geq s + 1$.*

3. Proof of Theorem 8.

Let $s \geq 3$ be an integer, and let G be a connected, essentially s-edge-connected graph such that $L(G)$ is not a complete graph. Then for any edge $vx \in E(G)$ with $d_G(v) \in \{1, 2\}$, we have $d_G(x) \geq s+2-d_G(v)$. Following [19], the core of the graph G, written as G_0, is obtained by the following two operations repeatedly.

Operation 1. Delete each vertex of degree 1.

Operation 2. For each vertex y of degree 2 with $E_G(y) = \{xy, yz\}$, contract exactly one edge in $E_G(y)$. This amounts to deleting vertex y in G with $d_G(y) = 2$ and replacing xy and yz with a new edge xz.

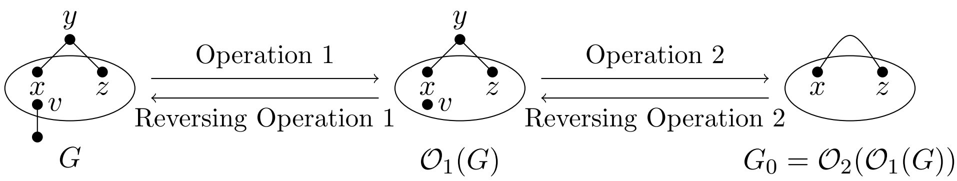

Figure 1. The core graph.

Let $\mathcal{O}_1(G)$ denote the graph obtained from G by applying Operation 1 to each vertex of degree 1, and $\mathcal{O}_2(G)$ the graph obtained from G by applying Operation 2 to each vertex of degree 2. Thus $G_0 = \mathcal{O}_2(\mathcal{O}_1(G))$. As shown in [19], we observe that G_0 is well-defined, and is 3-edge-connected and essentially s-edge-connected. By the definitions of these operations, any trail in G is contracted to a trail in G_0. Conversely, for any trail T' in G_0, there is a trail T in G such that T' is the contraction image of T. We call T a *lift* of T', or say that T' can be lifted to T.

In the rest of this section, we assume that G is a connected, essentially s-edge-connected graph, where $s \geq 6$ is an integer, and let $X \subseteq E(G)$ with $|X| \leq 3$. Let $H = G_0 - (E(G_0) \cap X)$. If H is not connected, then H contains an isolated vertex v with $E_G(v) = X = E_{G_0}(v)$ and $|X| = 3$, and $H - v$ is essentially s-edge-connected. If H is connected, then H is essentially $(s-3)$-edge-connected since G is essentially s-edge-connected. Let $G_X = \begin{cases} H, & \text{if } H \text{ is connected,} \\ H - v, & \text{if } H \text{ is not connected.} \end{cases}$
Then we have

(1) $\quad G_X$ is essentially $(s-3)$-connected.

Let $(G_X)_0$ be the core of G_X. Then

(2) $\quad (G_X)_0$ is 3-edge-connected and essentially $(s-3)$-edge-connected.

Theorem 17 (Theorem 4.1 of [13]). *Let G be an essentially 7-edge-connected graph. If $X \subseteq E(G)$ with $|X| \leq 3$, then $\tau((G_X)_0) \geq 2$.*

Lemma 18. *Let G be an essentially 7-edge-connected graph. Let $X \subseteq E(G)$ be a subset with $|X| \leq 3$ and $\{e_1, e_2\} \subseteq E(G) - X$. Then $G - X$ has an internally dominating (e_1, e_2)-trail.*

Proof. Let G_0 be the core of G. Notice that G is essentially 7-edge-connected. By (2),

(3) $\quad (G_X)_0$ is 3-edge-connected and essentially 4-edge-connected.

Claim 1. *Let $e = xy \in E(G)$. We assume that $d_{G_X}(y) \geq d_{G_X}(x)$ if $e \in E(G_0)$ but $e \notin E((G_X)_0)$; otherwise, we assume that $d_G(y) \geq d_G(x)$. Then $y \in V((G_X)_0)$. Therefore, $d_{(G_X)_0}(y) \geq 3$.*

Proof. Notice that there are three possibilities for the location of e: $e \in E((G_X)_0)$, $e \notin E(G_0)$, or $e \in E(G_0)$ and $e \notin E((G_X)_0)$. If $e \in E((G_X)_0)$, then both x and y are in $V((G_X)_0)$.

If $e \notin E(G_0)$, then since $d_G(y) \geq d_G(x)$, $d_G(x) \in \{1, 2\}$. As G is essentially 7-edge-connected, $d_G(x) + d_G(y) \geq 9$, and so $d_G(y) \geq 9 - d_G(x)$. Therefore, $d_{G_0}(y) \geq 7$ and $d_{G_X}(y) \geq 7 - 3 = 4$. This implies that $y \in V((G_X)_0)$.

If $e \in E(G_0)$ and $e \notin E((G_X)_0)$, then since $d_{G_X}(y) \geq d_{G_X}(x)$, we have $d_{G_X}(x) \in \{1, 2\}$. By (1), G_X is essentially 4-edge-connected. Then $d_{G_X}(x) + d_{G_X}(y) \geq 6$. Thus, $d_{G_X}(y) \geq 6 - d_{G_X}(x) \geq 4$. So $y \in V((G_X)_0)$. Claim 1 holds.

□

For $i = 1, 2$, denote $e_i = x_i y_i$ in such a way that if $e_i \in E(G_0)$ but $e_i \notin E((G_X)_0)$, then the labeling of x_i and y_i satisfies $d_{G_X}(y_i) \geq d_{G_X}(x_i)$; otherwise we label x_i and y_i so that $d_G(y_i) \geq d_G(x_i)$. Let

$$Q = \begin{cases} (G_X)_0(e_1, e_2), & \text{if } e_1, e_2 \in E((G_X)_0), \\ (G_X)_0(e_i), & \text{if } \{e_1, e_2\} \cap E((G_X)_0) = \{e_i\}, \\ (G_X)_0, & \text{otherwise} \end{cases}$$

and

$$v_i = \begin{cases} v(e_i), & \text{if } e_i \in E((G_X)_0), \\ y_i, & \text{otherwise.} \end{cases}$$

By Theorem 17, $\tau((G_X)_0) \geq 2$ and so $F(Q) \leq 2$. By Theorem 14(iii) and (3), Q is collapsible. By Theorem 13(iii),

$$\text{(4)} \qquad Q \text{ has a spanning } (v_1, v_2)\text{-trail } T_1.$$

Let T_2 be the lift of T_1 in G_X and let T_3 be the lift of T_2 in $(G - X)(e_1, e_2)$. Let T be a trail obtained from T_3 by replacing v_i by e_i. Then T is an (e_1, e_2)-trail of $G - X$. Let $T = w_1 f_1 w_2 f_2 \cdots f_k w_k$, where $f_1 = e_1$ and $f_k = e_2$, and let $\mathcal{I} = \{w_2, w_3, \ldots, w_{k-1}\}$. Then $V(T_1) - \{v(e_1), v(e_2)\} \subseteq \mathcal{I}$. To show that T is an internally dominating (e_1, e_2)-trail in $G - X$, it suffices to show that every edge $e = xy$ of $G - X$ is incident with an internal vertex of T, i.e., either $x \in \mathcal{I}$ or $y \in \mathcal{I}$.

We assume that $d_{G_X}(y) \geq d_{G_X}(x)$ if $e \in E(G_0)$ but $e \notin (G_X)_0$; otherwise, we assume that $d_G(y) \geq d_G(x)$. By Claim 1, $y \in V((G_X)_0)$. By (4), $y \in V(T_1)$. As $d_{(G_X)_0}(y) \geq 3$, $y \notin \{v(e_1), v(e_2)\}$ and so $y \in V(T_1) - \{v(e_1), v(e_2)\} \subseteq \mathcal{I}$. ■

Lemma 19. *Every* 7*-connected line graph is* 3*-hamiltonian-connected.*

Proof. Lemma 19 follows from Lemma 18 and Theorem 11. ■

***Proof of Theorem* 8.** By Lemma 19, Theorem 8 holds when $s = 3$. We assume that $s \geq 4$ and that Theorem 8 holds for smaller values of s. Let G be a graph with $\kappa(L(G)) \geq s+4$. For any $S \subseteq V(L(G))$ with $|S| \leq s$, pick $v_{e_0} \in S$. Assume that the edge in G corresponding to v_{e_0} in $L(G)$ is e_0. Let $G^* = G - e_0$. Since $\kappa(L(G)) \geq s+4$, $\kappa(L(G^*)) = \kappa(L(G) - v_{e_0}) \geq s+3$. It follows by induction that as $L(G^*)$ is $(s+3)$-connected, $L(G^*)$ is $(s-1)$-hamiltonian-connected, and so $L(G) - S = L(G^*) - (S - \{v_{e_0}\})$ must be hamiltonian-connected. It follows by definition that $L(G)$ is s-hamiltonian-connected. ■

4. Graphs with Property $\mathcal{K}(s)$ and Proof of Theorem 9

Throughout this section, we assume that $s \geq 0$ is an integer. Following [15], we shall introduce a property of graphs which will play an important role in our arguments.

([15]) Let $\mathcal{K}$ denote the graph family such that a (connected) graph G is in $\mathcal{K}$ if and only if G satisfies each of the following.

(KS1) For any $w \in D_3(G)$, the subgraph induced by $N_G(w)$ contains at least one edge.

(KS2) Let $w \in N_G(x_1) \cap N_G(x_2)$, where $x_1, x_2 \in D_3(G)$ and $x_1x_2 \notin E(G)$. If $N_G(w) = \{x_1, x_2, v\}$, then either $vx_1 \notin E(G)$ or $vx_2 \notin E(G)$.

(KS3) Let $w_1, w_2 \in N_G(x_1) \cap N_G(x_2)$, where $x_1, x_2 \in D_3(G)$ and $x_1x_2 \notin E(G)$. If $w_1w_2 \in E(G)$, then $N_G(w_1) \cup N_G(w_2) \subseteq N_G(x_1) \cup N_G(x_2) \cup \{x_1, x_2\}$.

By definition, $K_4 \in \mathcal{K}$ and every claw-free graph satisfies (KS1) and (KS3). Since (KS2) is violated for the graphs W_4 and W_5, we have $W_4, W_5 \notin \mathcal{K}$. For an integer $s \geq 0$, a graph G is said to have *Property* $\mathcal{K}(s)$ if G is in $\mathcal{K} - \{K_4\}$ and satisfies both $\kappa'(G) \geq 3$ and $ess'(G) \geq s+4$.

Lemma 20 [15]**.** *If the graph G has Property $\mathcal{K}(s)$, then there is a set $\triangle(G)$ of edge-disjoint triangles in G such that $D_3(G) \subseteq V(L)$, where L is the subgraph induced by $\bigcup_{K\in\triangle(G)} E(K)$, and $D_3(G) \cap V(K) \neq \emptyset$ for each $K \in \triangle(G)$.*

Let G have Property $\mathcal{K}(s)$ and $v \in D_3(G)$. By Lemma 20, there is a triangle in $\triangle(G)$ that contains v. We denote this triangle by $\triangle_v$. Thus, for $v, u \in D_3(G)$, we have either $E(\triangle_v) = E(\triangle_u)$ or $E(\triangle_v) \cap E(\triangle_u) = \emptyset$. Fix a given subset $E' \subseteq E(G)$. Define $\triangle'(G) = \{\triangle_v \in \triangle(G) : v \in D_3(G)$ and $E(\triangle_v) \cap E' = \emptyset\}$ and $\triangle^*(G) = \triangle(G) - \triangle'(G)$. Then $\triangle(G) = \triangle'(G)$ if $E' \cap E(\triangle(G)) = \emptyset$. Let $G_1 = G/\triangle(G)$ and $G_1^* = G/\triangle'(G)$ be the graphs obtained from G by contracting the edges in $\triangle(G)$ and $\triangle'(G)$, respectively. Thus if $E' \cap E(\triangle(G)) = \emptyset$, then $G_1 = G_1^*$. We call G_1 a $\triangle$-contraction of G and G_1^* a $\triangle$-contraction of G with respect to E'. Since G is 3-edge-connected and essentially $(s+4)$-edge-connected, we have

(5) $\kappa'(G_1) \geq 4$ and $ess'(G_1) \geq s+4$, and $\kappa'(G_1^*) \geq 3$ and $ess'(G_1^*) \geq s+4$.

By Theorem 14(iv), for any $X \subseteq E(G_1)$ with $|X| \leq 2$, $\tau(G_1 - X) \geq 2$, and so $F(G_1 - X) = 0$. Let t be the number of different triangles in $\triangle^*(G)$ and let $\triangle^*(G) = \{\triangle_{v_1}, \ldots, \triangle_{v_t}\}$ with $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$. Then $\{v_1, \ldots, v_t\} \subseteq D_3(G_1^*)$ and $E(\triangle_{v_i}) \cap E' \neq \emptyset$ for $i = 1, \ldots, t$ (Figure 2). Since G_1^* is 3-edge-connected and essentially 4-edge-connected, we have either $d_{G_1^*}(u_i) \geq 4$ or $d_{G_1^*}(w_i) \geq 4$. Without loss of generality, we assume that

$$d_{G_1^*}(w_i) \geq 4. \tag{6}$$

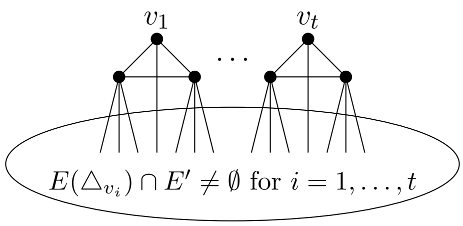

Figure 2. $G_1^* = G_1/\triangle'(G)$.

Lemma 21. (i) *If $t = 0$, then $G_1^* = G_1$.*

(ii) *If $t = 1$, then for any edge $e \in E(G_1^*)$, $\tau(G_1^* - e) \geq 2$.*

(iii) *If $s = 0$ and $t = 2$, then $\tau(G_1^*) \geq 2$.*

(iv) *If $s \geq 1$ and $t = 2$, then for any $e \in E(G_1^*)$, $\tau(G_1^* - e) \geq 2$.*

Proof. If $t = 0$, then $\triangle^*(G) = \emptyset$. Thus $G_1^* = G_1$. If $t = 1$, then $\triangle^*(G) = \{\triangle_{v_1}\}$ with $V(\triangle_{v_1}) = \{v_1, u_1, w_1\}$. By (6), $d_{G_1^*}(w_1) \geq 4$. Let Q_1 be the graph obtained from G_1^* by adding the new edge v_1u_1. Actually this new edge and the edge v_1u_1 in the triangle $\triangle_{v_1}$ are parallel. We denote this new edge by $(v_1u_1)'$. Then Q_1 is 4-edge-connected. Thus, for any edge $e \in E(G_1^*)$, $\tau(G_1^* - e) = \tau(Q_1 - \{e, (v_1u_1)'\}) \geq 2$.

If $t = 2$, then $\triangle^*(G) = \{\triangle_{v_1}, \triangle_{v_2}\}$ with $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}(i = 1, 2)$. By (6), $d_{G_1^*}(w_i) \geq 4$. If $s = 0$, then we set Q_2 to be the graph obtained from G_1^* by adding the new edges v_1v_2 and u_1u_2. Thus, Q_2 is 4-edge-connected. So $\tau(G_1^*) = \tau(Q_2 - \{v_1v_2, u_1u_2\}) \geq 2$. If $s \geq 1$, then G_1^* is essentially 5-edge-connected. Thus, for $x \in \{u_1, w_1, u_2, w_2\}$, $d_{G_1^*}(x) \geq 4$. Let Q_3 be the graph obtained from G_1^* by adding the new edge v_1v_2. Then Q_3 is 4-edge-connected. So for any edge $e \in E(G_1^*)$, $\tau(G_1^* - e) = \tau(Q_3 - \{v_1v_2, e\}) \geq 2$. ■

The next lemma will be used in the proof of Theorem 9. For any edge subset X of G with $|X| = s$, to prove that $L(G)$ is s-hamiltonian connected, it suffices to prove that for any two edges $e_1, e_2 \in G - X$, $G - X$ has an internally dominating (e_1, e_2)-trail. By Theorem 8, we only need to consider $s \in \{0, 1, 2\}$.

Lemma 22. *Let X be an edge subset of G, and $s = |X| \in \{0, 1, 2\}$. If G satisfies Property $\mathcal{K}(s)$, then for any two edges $e_1, e_2 \in G - X$, $G - X$ has an internally dominating (e_1, e_2)-trail T such that $V(G) - V(T) \subseteq \bigcup_{i=3}^{s+3} D_i(G)$.*

Proof. Let $E' = X \cup \{e_1, e_2\}$ and let G_1 be a $\triangle$-contraction of G and G_1^* be a $\triangle$-contraction of G with respect to E'. If $\triangle^*(G) \neq \emptyset$, then we assume that $\triangle^*(G) = \{\triangle_{v_1}, \ldots, \triangle_{v_t}\}$ with $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$ $(i = 1, \ldots, t)$. Thus $E(\triangle_{v_i}) \cap E' \neq \emptyset$ and $t \in \{0, 1, \ldots, 2 + s\}$. By (5), we have

$$D_i(G_1^*) \subseteq D_i(G) \text{ for } i = 3, \ldots, s + 3. \tag{7}$$

Since a triangle is collapsible, to prove Lemma 22, by Proposition 12 and Theorem 13(iv), it suffices to prove that $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that

$$V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \bigcup_{i=3}^{s+3} D_i(G_1^*). \tag{8}$$

Claim 1. *If $s = 0$, then for any two edges e', e'' in G_1^*, $G_1^*(e', e'')$ is collapsible. Therefore, $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T.*

Proof. Since $s = 0$, we have $X = \emptyset$ and $t \in \{0, 1, 2\}$. Thus $G_1^* - X = G_1^*$. By (5), G_1^* is 3-edge-connected and essentially 4-edge-connected. Therefore, $G_1^*(e', e'')$ is 2-edge-connected. Let G' be the reduction of $G_1^*(e', e'')$. By Lemma 21(i)–(iii), $\tau(G_1^*) \geq 2$. Thus $F(G_1^* - \{e', e''\}) \leq 2$. As $F(G_1^* - \{e', e''\}) = F(G_1^*(e', e''))$, we have $F(G_1^*(e', e'')) \leq 2$. Since $G_1^*(e', e'')$ has only two vertices of degree two and since G_1^* is essentially 4-edge-connected, G' has at most two vertices of degree two. By Theorem 14(iii), $G' = K_1$. So $G_1^*(e', e'')$ is collapsible. By Theorem 13(iii), $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T. Claim 1 holds. □

Claim 2. *Assume that $s = 1$. Let $X = \{f\}$.*

(i) *If $t = 0$, then $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T.*

(ii) *If $t \in \{1, 2, 3\}$, then $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $|V((G_1^* - X)(e_1, e_2)) - V(T)| \leq 1$ and $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq D_3(G_1^*)$. Furthermore, if $t \in \{1, 2\}$ and $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v\}$, then $E_G(v) = \{f, e_1, e_2\}$.*

Proof. As $s = 1$, G_1^* is 3-edge-connected and essentially 5-edge-connected. Thus, $(G_1^* - f)(e_1, e_2)$ is 2-edge-connected and $|D_2((G_1^* - f)(e_1, e_2))| \leq 3$. If $t = 0$, then $G_1^* = G_1$ and so $F((G_1^* - f)(e_1, e_2)) \leq 1$. By Theorem 14(ii), $(G_1^* - f)(e_1, e_2)$ is collapsible. So $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T.

If $t \in \{1, 2\}$, then by Lemma 21(ii) and Lemma 21(iv), $F((G_1^* - f)(e_1, e_2)) \leq 2$. Let G' be the reduction of $(G_1^* - f)(e_1, e_2)$. Since $|D_2((G_1^* - f)(e_1, e_2))| \leq 3$

and since G_1^* is essentially 5-edge-connected, we have $|D_2(G')| \le 4$. By Theorem 14(iii), $G' \in \{K_1, K_{2,2}, K_{2,3}, K_{2,4}\}$. If $G' = K_1$, by Theorem 13(iii), $(G_1^* - f)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' \in \{K_{2,3}, K_{2,4}\}$, then $v(e_1), v(e_2) \in D_2(G')$. By Theorem 13(iv), $(G_1^* - f)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' = K_{2,2}$, then $v(e_1), v(e_2) \in D_2(G')$ such that $v(e_1), v(e_2)$ are not adjacent. Let $V(G') - \{v(e_1), v(e_2)\} = \{v, v'\}$. Then either $PI(v)$ or $PI(v')$ is trivial. Without loss of generality, we assume that $PI(v)$ is trivial. So f is incident to v, and $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $V((G_1^* - f)(e_1, e_2)) - V(T) = \{v\}$, where $E_G(v) = \{e_1, e_2, f\}$.

If $t = 3$, then $f \in X'$ and $\triangle^*(G) = \{\triangle_{v_1}, \triangle_{v_2}, \triangle_{v_3}\}$, where for $1 \le i \le 3$, $v_i \in D_3(G_1^*)$. Without loss of generality, we assume that $f \in E(\triangle_{v_1})$, $e_1 \in E(\triangle_{v_2})$ and $e_2 \in E(\triangle_{v_3})$. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 4-edge-connected. By Claim 1, $(G_1^* - v_1)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail T'. Let $T = \begin{cases} (T' - f) + \{v_1u_1, v_1w_1\}, & \text{if } f \in E(T') \\ T', & \text{otherwise} \end{cases}$. Then T is a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ such that $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \{v_1\} \subseteq D_3(G_1^*)$. Claim 2 holds. □

Claim 3. *If $s = 2$, then $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \bigcup_{i=3}^{5} D_i(G_1^*)$.*

Proof. Since $s = 2$, G_1^* is 3-edge-connected and essentially 6-edge-connected. Let $X = \{f_1, f_2\}$. Then $G_1^* - X$ is connected and essentially 4-edge-connected. So $(G_1^* - X)(e_1, e_2)$ is connected. As $s = 2$, we have $t \in \{0, 1, 2, 3, 4\}$.

Claim 3.1. *If $G_1^* - X$ is not 2-edge-connected, then Claim 3 holds.*

Proof. Assume that $G_1^* - X$ is not 2-edge-connected. Let e be a cut edge of $G_1^* - X$, and let H_1 and H_2 be components of $(G_1^* - X) - e$. Then $\{f_1, f_2, e\}$ is a 3-edge cut of G_1^*. As G_1^* is essentially 6-edge-connected, we may assume that $V(H_1) = \{v_1\}$. Then $E_{G_1^*}(v_1) = \{f_1, f_2, e\}$. Thus $t \le 3$. Consider $G_1^* - v_1$. Then $d_{G_1^* - v_1}(x) \ge 4$ for any $x \in N_{G_1^*}(v_1)$. Since $t \le 3$, $G_1^* - v_1$ contains at most two vertices of degree 3. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected. Thus $\tau((G_1^* - v) - e_1) \ge 2$. This implies that $F((G_1^* - v) - \{e_1, e_2\}) \le 1$ and so $F((G_1^* - v_1)(e_1, e_2)) \le 1$. By Theorem 14(ii), $(G_1^* - v_1)(e_1, e_2)$ is collapsible. Let T be a spanning $(v(e_1), v(e_2))$-trail of $(G_1^* - v_1)(e_1, e_2)$. Then T is a dominating $(v(e_1), v(e_2))$-trail of $(G_1^* - X)(e_1, e_2)$ with $V(G_1^*) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. Claim 3.1 holds. □

By Claim 3.1, we may assume that $G_1^* - X$ is 2-edge-connected. Then $(G_1^* - X)(e_1, e_2)$ is also 2-edge-connected. If $t = 0$, then $f_1, f_2, e_1, e_2 \in E(G_1)$ and $G_1^* = G_1$. So $(G_1 - X)(e_1, e_2)$ has at most three vertices of degree 2. Let G' be the reduction of $(G_1 - X)(e_1, e_2)$. Then $|D_2(G')| \le 4$. By (5), G_1 is 4-edge-connected.

By Theorem 14(iv), we have $\tau(G_1 - X) \geq 2$ and so $F((G_1 - X)(e_1, e_2)) \leq 2$. Since $(G_1-X)(e_1, e_2)$ is 2-edge-connected and G_1 is essentially 6-edge-connected, $G' \in \{K_1, K_{2,2}, K_{2,3}\}$. If $G' = K_1$, by Theorem 13(iii), $(G_1 - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' = K_{2,3}$, then $v(e_1), v(e_2) \in D_2(G')$. By Theorem 13(iv), $(G_1 - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' = K_{2,2}$, then $v(e_1), v(e_2) \in D_2(G')$ such that $v(e_1), v(e_2)$ are not adjacent. Let $V(G') - \{v(e_1), v(e_2)\} = \{v, v'\}$. Then either $PI(v)$ or $PI(v')$ is trivial. Without loss of generality, we assume that $PI(v)$ is trivial. So f_1, f_2 are incident to v and $(G_1 - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $V((G_1 - X)(e_1, e_2)) - V(T) = \{v\} \subseteq D_4(G_1)$, where $E_G(v) = \{e_1, e_2, f_1, f_2\}$. Next we just need to consider $t \in \{1, 2, 3, 4\}$.

Claim 3.2. *If $f_1, f_2 \in E(G_1)$, then Claim 3 holds.*

Proof. In this case, $|\{e_1, e_2\} \cap E(G_1)| \leq 1$ and $t \in \{1, 2\}$. Without loss of generality, we assume that $e_2 \notin E(G_1)$. We consider two cases.

Case 1. $t = 1$. Then $\triangle^*(G) = \{\triangle_{v_1}\}$ with $v_1 \in D_3(G_1^*)$ and $V(\triangle_{v_1}) = \{v_1, u_1, w_1\}$, and $e_2 \in E(\triangle_{v_1})$. Let $E_{G_1^*}(v_1) = \{v_1u_1, v_1w_1, v_1z\}$. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected. Since G_1^* is essentially 6-edge-connected, we have $d_{G_1^*-v_1}(y) \geq 4$ for $y \in \{u_1, w_1, z\}$. Thus $G_1^* - v_1$ is 4-edge-connected, and so $\tau((G_1^* - v_1) - \{f_1, f_2\}) \geq 2$.

If $e_2 \in \{v_1u_1, v_1w_1\}$, assume that $e_2 = v_1w_1$. Let $a = \begin{cases} z, & \text{if } e_1 = v_1z, \\ u_1, & \text{if } e_1 = v_1u_1, \\ v(e_1), & \text{otherwise} \end{cases}$ and let $H = \begin{cases} (G_1^* - v_1) - \{f_1, f_2\}, & \text{if } e_1 \in \{v_1z, v_1u_1\} \\ ((G_1^* - v_1) - \{f_1, f_2\})(e_1), & \text{otherwise} \end{cases}$. Then $F(H) \leq 1$ and H is 2-edge-connected. By Theorem 14(ii), H has a spanning (w_1, a)-trail. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^*-X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$.

If $e_2 \notin \{v_1u_1, v_1w_1\}$, then $e_2 = u_1w_1$. If $e_1 \in E_{G_1^*}(v_1)$, then $e_1 = v_1b$, where $b \in \{w_1, u_1, z\}$. As $F(((G_1^* - v_1) - \{f_1, f_2\})(e_2)) \leq 1$, $((G_1^* - v_1) - \{f_1, f_2\})(e_2)$ has a spanning $(b, v(e_2))$-trail. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. If $e_1 \notin E_{G_1^*}(v_1)$, then as $F(((G_1^* - v_1) - \{f_1, f_2\})(e_1)) \leq 1$, $((G_1^* - v_1) - \{f_1, f_2\})(e_1)$ has a spanning $(w_1, v(e_1))$-trail T'. If $e_2 \notin E(T')$, then this trail T' can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. If $e_2 \in E(T')$, then $T = T' - \{w_1u_1\} + \{v_1u_1, v_1w_1, w_1v(e_2)\}$ is a spanning $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$.

Case 2. $t = 2$. Then $e_1, e_2 \notin E(G_1)$ and $\triangle^*(G) = \{\triangle_{v_1}, \triangle_{v_2}\}$ with $v_i \in D_3(G_1^*)$ and $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$ $(i = 1, 2)$. For $i = 1, 2$, we assume that $e_i \in E(\triangle_{v_i})$, $E_{G_1^*}(v_i) = \{v_iu_i, v_iw_i, v_iz_i\}$, and let x_i be the vertex on which $\triangle_{v_i}$ is contracted in G_1. If $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$ is not connected, then $z_1 = z_2$ and f_1, f_2, x_1z_1, x_2z_2 are incident to z_1 of degree 4. Thus $\{f_1, f_2, v_1z_1, v_2u_2, v_2w_2\}$ is an essential 5-edge cut in G_1^*, a contradiction. So $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$ is connected. Similarly, $G_1 - \{f_1, f_2, x_1z_1\}$ and $G_1 - \{f_1, f_2, x_2z_2\}$ are 2-edge-connected.

Consider $G_1 - \{f_1, f_2, x_1z_1\}$. By (5), G_1 is 4-edge-connected. By Theorem 14(iv), $F(G_1 - \{f_1, f_2, x_1z_1\}) \le 1$ and $F(G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}) \le 2$. By Theorem 14(ii), $G_1 - \{f_1, f_2, x_1z_1\}$ is collapsible. Thus $G_1 - \{f_1, f_2, x_1z_1\}$ has a spanning (x_1, x_2)-trail T. If $e_2 \ne u_2w_2$, then by Lemma 15, $(G_1^* - X)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T with $V(G_1^*) - V(T) \subseteq \{v_1, v_2\} \subseteq D_3(G_1^*)$. So we may assume that $e_2 = u_2w_2$. Similarly, we assume that $e_1 = u_1w_1$. By Lemma 15, $\{f_1, f_2\} \cap \{x_1z_1, x_2z_2\} = \emptyset$.

Notice that $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$ is connected and $F(G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}) \le 2$. Let G' be the reduction of $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$. By Theorem 14(iii), $G' \in \{K_1, K_2, K_{2,\ell}\}$. Since G_1 is 4-edge-connected and essentially 6-edge-connected, and since x_1 and x_2 are the vertices on which $\triangle_{v_1}$ and $\triangle_{v_2}$ are contracted in G_1, we have $G' \ne K_{2,\ell}$. If $G' = K_1$, then $G_1 - \{f_1, f_2, x_1z_1, x_2z_2\}$ has a spanning (x_1, x_2)-trail. By Lemma 15(ii), $(G_1^* - X)(e_1, e_2)$ has a spanning $(v(e_1), v(e_2))$-trail. If $G' = K_2$, then we assume that $G' = ab$. Thus either $PI(a)$ or $PI(b)$ is trivial. Without loss of generality, we assume that $PI(a)$ is trivial. Since G_1^* is essentially 6-edge-connected, we have $a \in V(G_1^*)$. As $G_1 - \{f_1, f_2, x_1z_1\}$ and $G_1 - \{f_1, f_2, x_2z_2\}$ are 2-edge-connected, we have $E_{G_1}(a) \cap \{x_1z_1, x_2z_2\} \ne \emptyset$. Without loss of generality, we assume that $a = z_1$. Thus $x_1 \in V(PI(b))$. Since $E_{G_1^*}(a) \cup \{v_1u_1, v_1w_1\} - \{z_1v_1\}$ is essentially edge-cut of G_1^* and since G_1^* is essentially 6-edge-connected, $d_{G_1}(a) \ge 5$. Thus $E_{G_1}(a) = \{ab, z_1x_1, z_2x_2, f_1, f_2\}$, $a = z_1 = z_2$ and $x_1, x_2 \in PI(b)$. Let T be a spanning (x_1, x_2)-trail in $PI(b)$. As $e_1 = u_1w_1$ and $e_2 = u_2w_2$, by Lemma 15(ii), $(G_1^* - X)(e_1, e_2) - a$ has a spanning $(v(e_1), v(e_2))$-trail T'. This trail T' is a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{a\} \subseteq D_5(G_1^*)$. We finish the proof of Claim 3.2. □

By Claim 3.2, we may assume that $f_1 \notin E(G_1)$. In addition, we assume that $\triangle_{v_1} \in \triangle^*(G)$ such that $f_1 \in E(\triangle_{v_1})$. Let $E_{G_1^*}(v_1) = \{v_1u_1, v_1w_1, v_1z_1\}$, where $V(\triangle_{v_1}) = \{v_1, u_1, w_1\}$.

Claim 3.3. *If $E_{G_1^*}(v_1) \cap \{e_1, e_2\} = \emptyset$, then Claim 3 holds.*

Proof. In this case, $e_1, e_2 \in E(G_1^* - v_1)$. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected. If $f_2 \notin E_{G_1^*}(v_1)$, then $f_2 \in E(G_1^* - v_1)$. By Claim 2, $((G_1^* - v_1) - f_2)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T' such that $V(((G_1^* - v_1) - f_2)(e_1, e_2)) - V(T') \subseteq \{y\} \subseteq D_3(G_1^* - v_1)$. Let $T = \begin{cases} (T' - f_1) + \{v_1u_1, v_1w_1\}, & \text{if } f_1 \in E(T') \\ T', & \text{othererwise} \end{cases}$. Then T is a dominating $(v(e_1),$

$v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ such that $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \{v_1, y\} \subseteq D_3(G_1^*)$. If $f_2 \in E_{G_1^*}(v_1)$, then by Claim 2, $((G_1^* - v_1) - u_1w_1)(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T such that $V(((G_1^* - v_1) - f_2)(e_1, e_2)) - V(T) \subseteq \{y\} \subseteq D_3(G_1^* - v_1)$. This trail T is also a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ such that $V((G_1^* - X)(e_1, e_2)) - V(T) \subseteq \{v_1, y\} \subseteq D_3(G_1^*)$. Claim 3.3 holds. □

By Claim 3.3, we assume that $e_1 \in E_{G_1^*}(v_1)$. Let $e_1 = v_1b_1$, where $b_1 \in \{w_1, u_1, z_1\}$. Consider f_2. If $f_2 \in E(G_1)$, then $t \in \{1, 2\}$; if $f_2 \notin E(G_1)$, then we may assume that $f_2 \in E(\triangle_{v_i})$ for some $\triangle_{v_i} \in \triangle^*(G)$. Thus by Claim 3.3, $e_2 \in E_{G_1^*}(v_i)$. So we still have $t \in \{1, 2\}$.

Claim 3.4. *If* $t = 1$*, then Claim* 3 *holds.*

Proof. In this case, $\triangle^*(G) = \{\triangle_{v_1}\}$. Since G_1^* is 3-edge-connected and essentially 6-edge-connected and since v_1 is only the vertex of degree three, $G_1^* - v_1$ is 4-edge-connected. So

$$\tau((G_1^* - v_1) - \{f_2, u_1w_1\}) \geq 2. \tag{9}$$

If $e_2 \in E_{G_1^*}(v_1)$, then we assume that $e_2 = v_1b_2$, where $b_2 \in \{u_1, w_1, z_1\} - \{b_1\}$. By (9), $(G_1^* - v_1) - \{f_2, u_1w_1\}$ is collapsible. Thus $(G_1^* - v_1) - \{f_2, u_1w_1\}$ has a spanning (b_1, b_2)-trail. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. If $e_2 \notin E_{G_1^*}(v_1)$, by (9), $F(((G_1^* - v_1) - \{f_2, u_1w_1\})(e_2)) \leq 1$. By Theorem 14(ii), $((G_1^* - v_1) - \{f_2, u_1w_1\})(e_2)$ is collapsible. Thus $((G_1^* - v_1) - \{f_2, u_1w_1\})(e_2)$ has a spanning $(b_1, v(e_2))$-trail. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. Claim 3.4 holds. □

By Claim 3.4, we assume that $t = 2$. Then $\triangle^*(G) = \{\triangle_{v_1}, \triangle_{v_2}\}$, where $v_i \in D_3(G_1^*)$, $V(\triangle_{v_i}) = \{v_i, u_i, w_i\}$ and $\{e_2, f_2\} \cap E(\triangle_{v_2}) \neq \emptyset$. Let $E_{G_1^*}(v_2) = \{v_2u_2, v_2w_2, v_2z_2\}$. Since G_1^* is essentially 6-edge-connected, $d_{G_1^*}(x) \geq 5$ for $x \in \{u_1, w_1, u_2, w_2\}$. By Lemma 16, $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected. Since v_2 is the only vertex of degree 3 in $G_1^* - v_1$, we have $\tau((G_1^* - v_1) - \{f_2\}) \geq 2$, and so $F((G_1^* - v_1) - \{u_1w_1, f_2\}) \leq 1$.

Consider $G_1^* - v_1$. Then $(G_1^* - v_1) - \{u_1w_1, f_2\}$ is essentially 3-edge-connected. If $(G_1^* - v_1) - \{u_1w_1, f_2\}$ has a cut edge f', then we assume that H_1 and H_2 are components of $(G_1^* - v_1) - \{u_1w_1, f_2, f'\}$. Since G_1^* is essentially 6-edge-connected, we have either H_1 or H_2 is trivial. Without loss of generality, we assume that $V(H_1) = \{u_1\}$. Then $E_{G_1^*}(u_1) = \{u_1v_1, u_1w_1, f_2, f'\}$ and $(E_{G_1^*}(u_1) \cup E_{G_1^*}(v_1)) - \{u_1v_1\}$ is an essential 5-edge cut in G_1^*, a contradiction. So $(G_1^* - v_1) - \{u_1w_1, f_2\}$ is 2-edge-connected.

If $e_2 \in E_{G_1^*}(v_1) \cup \{u_1w_1\}$, then $f_2 \in E(\triangle_{v_2})$. We assume that e_2 is incident to b_3, where $b_3 \in \{u_1, w_1, z_1\}$. By Theorem 14(ii), $(G_1^* - v_1) - \{u_1w_1, f_2\}$ is collapsible. Thus $(G_1^* - v_1) - \{u_1w_1, f_2\}$ has a spanning (b_1, b_3)-trail T'. Thus $T = v(e_1)T'v(e_2)$ is a dominating $(v(e_1), v(e_2))$-trail in $(G_1^* - X)(e_1, e_2)$ such that $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. So we may assume that $e_2 \notin E_{G_1^*}(v_1) \cup \{u_1w_1\}$.

As $\tau((G_1^* - v_1) - \{f_2\}) \geq 2$, $F(((G_1^* - v_1) - \{u_1w_1, f_2\})(e_2)) \leq 2$. Let G' be the reduction of $((G_1^* - v_1) - \{u_1w_1, f_2\})(e_2)$. By Theorem 14(iii), $G' \in \{K_1, K_{2,\ell}\}$ $(\ell \geq 3)$. Since $d_{G_1^* - v_1}(u_1) \geq 4$ and $d_{G_1^* - v_1}(w_1) \geq 4$ and since $G_1^* - v_1$ is 3-edge-connected and essentially 5-edge-connected, $G' = K_1$. Notice that $e_1 = v_1b_1$. Then $((G_1^* - v_1) - \{u_1w_1, f_2\})(e_2)$ has a spanning $(b_1, v(e_2))$-trail T. This trail can be extended to a dominating $(v(e_1), v(e_2))$-trail T in $(G_1^* - X)(e_1, e_2)$ with $V((G_1^* - X)(e_1, e_2)) - V(T) = \{v_1\} \subseteq D_3(G_1^*)$. We finish the proof of Claim 3.

□

We finish the proof of Lemma 22. ■

We need one more notation. Let $e = xy \in E(W_5)$ with $x, y \in D_3(W_5)$ and let H be a graph and $e' = x'y' \in E(H)$. Define a new graph $H \oplus W_5$ to be a graph obtained from the disjoint union of $H - e'$ and W_5 by identifying x and x' to form a new vertex, also called x, and by identifying y and y' to form a new vertex, also called y.

Lemma 23 [15]**.** *Suppose that $s \geq 0$ and that G is a claw-free graph such that $\kappa(L(G)) \geq s + 4$. Let G_0 be the core of G and let $w_1, w_2, w_3 \in D_3(G_0)$ be vertices with $N_{G_0}(w_2) = \{w_1, w_3, v\}$. If $vw_1, vw_3 \in E(G_0)$, then each of the following holds.*

(i) *$s = 0$.*

(ii) *Either $G = G_0 \in \{K_4, W_4, W_5\}$, or there exists a subgraph H of G with $\kappa'(H) \geq 3$ and $ess'(H) \geq 4$ such that $G_0 = H \oplus W_5$ (see Figure 3).*

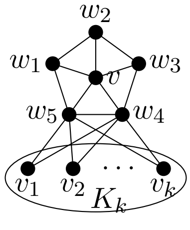

Figure 3. $K_{k+2} \oplus W_5$ in Lemma 23.

***Proof of Theorem* 9.** Let X be any edge subset of G with $|X| = s$. To prove that $L(G)$ is s-hamiltonian connected, it suffices to prove that for any two edges

$e_1, e_2 \in G - X$, $G - X$ has an internally dominating (e_1, e_2)-trail. By Theorem 8, we assume that $s \in \{0, 1, 2\}$. Let G_0 be the core of G. Then it suffices to assume that $X \cup \{e_1, e_2\} \subseteq E(G_0)$, and to show $G_0 - X$ has an internally dominating (e_1, e_2)-trail T with $V(G_0) - V(T) \subseteq \bigcup_{i=3}^{s+3} D_i(G_0)$. By contradiction, we assume that G is a counterexample to Theorem 9 with $|V(G_0)|$ minimized. Then there exist edges $X \cup \{e_1, e_2\} \subseteq E(G_0)$ such that $(G_0 - X)(e_1, e_2)$ does not have a dominating $(v(e_1), v(e_2))$-trail T with

$$V((G_0 - X)(e_1, e_2)) - V(T) \subseteq \bigcup_{i=3}^{s+3} D_i(G_0). \tag{10}$$

By (10) and Theorem 13(iii), we assume that $G_0 \notin \{K_4, W_4, W_5\}$ and G_0^* is not collapsible. By Lemma 22, G_0 does not have Property $\mathcal{K}(s)$. As G_0 is claw-free, (KS2) is violated. Thus there exist $w_1, w_2, w_3 \in D_3(G_0)$ with $N_{G_0}(w_2) = \{w_1, w_3, v\}$ and $vw_1, vw_3 \in E(G_0)$. By Lemma 23, we have $s = 0$ and $G_0 = H \oplus W_5$ for a subgraph H of G_0 with $\kappa'(H) \geq 3$ and $ess'(H) \geq 4$. Assume that $V(W_5) = \{v, w_1, \ldots, w_5\}$ with $w_4w_5 \in E(H) \cap E(W_5)$, as depicted in Figure 3. As H is claw-free, every 3-edge-cut of H has at least one edge in a 3-cycle. By Theorem 13(v), for any two edges $e', e'' \in E(H)$, $H(e', e'')$ is collapsible. Thus H and $H(e')$ are collapsible.

If $\{e_1, e_2\} \cap E(W_5) = \emptyset$, then by the minimality of G_0, $H(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T_1 with $V(H(e_1, e_2)) - V(T_1) \subseteq D_3(H)$. Thus the subgraph induced by $E(T_1) \cup \{vw_5, vw_4, w_5w_1, w_1w_2, w_2w_3, w_3w_4\}$ is a dominating $(v(e_1), v(e_2))$-trail in G_0^*, contrary to (10). If $e_1, e_2 \in E(W_5)$, then by inspection, $W_5(e_1, e_2)$ has a dominating $(v(e_1), v(e_2))$-trail T_2 that contains either w_4 or w_5. As H is collapsible, H has a spanning eulerian subgraph T_3. Thus $T_4 = G_0^*[(E(T_2) - E(T_3)) \cup (E(T_3) - E(T_2))]$ is a dominating $(v(e_1), v(e_2))$-trail in G_0^* with $V(G_0^*) - V(T_4) \subseteq D_3(G_0)$, contrary to (10). Thus we assume that $e_1 \in E(H) - E(W_5)$ and $e_2 \in E(W_5) - E(H)$. By Theorem 13(ii), $W_5(e_2)$ is collapsible. By Theorem 13(v), $H(e_1)$ is collapsible. Thus G_0^* is collapsible, a contradiction. ■

Acknowledgments

The authors are indebted to an anonymous reviewer for providing insightful comments which help to improve the manuscript. The research of Xiaoling Ma is supported by the Natural Science Foundation of China (No. 11701490), the Youth-Talent Project of Xinjiang Province (No. 2019Q016), and the Natural Science Foundation of Xinjiang Province (No. 2021D01C069).

References

[1] J.A. Bondy and U.S.R. Murty, Graph Theory with Applications (Elsevier Science Publishing Co. Inc. New York NY, 1976).
https://doi.org/10.1007/978-1-349-03521-2

[2] P.A. Catlin, *A reduction method to find spanning Eulerian subgraphs*, J. Graph Theory **12** (1988) 29–44.
https://doi.org/10.1002/jgt.3190120105

[3] P.A. Catlin, Z. Han and H.-J. Lai, *Graphs without spanning closed trails*, Discrete Math. **160** (1996) 81–91.
https://doi.org/10.1016/S0012-365X(95)00149-Q

[4] P.A. Catlin, H.-J. Lai and Y. Shao, *Edge-connectivity and edge-disjoint spanning trees*, Discrete Math. **309** (2009) 1033–1040.
https://doi.org/10.1016/j.disc.2007.11.056

[5] Z.-H. Chen, H.-J. Lai, W. Shiu and D.Y. Li, *An s-hamiltonian line graph problem*, Graphs Combin. **23** (2007) 241–248.
https://doi.org/10.1007/s00373-007-0727-y

[6] F. Harary and C.St.J.A. Nash-Williams, *On eulerian and hamiltonian graphs and line graphs*, Canad. Math. Bull. **8** (1965) 701–710.
https://doi.org/10.4153/CMB-1965-051-3

[7] F. Jaeger, *A note on sub-Eulerian graphs*, J. Graph Theory **3** (1979) 91–93.
https://doi.org/10.1002/jgt.3190030110

[8] T. Kaiser, Z. Ryjáček and P. Vrána, *On* 1*-Hamilton-connected claw-free graphs*, Discrete Math. **321** (2014) 1–11.
https://doi.org/10.1016/j.disc.2013.12.009

[9] T. Kaiser and P. Vrána, *Hamilton cycles in* 5*-connected line graphs*, European J. Combin. **33** (2012) 924–947.
https://doi.org/10.1016/j.ejc.2011.09.015

[10] M. Kriesell, *Every* 4*-connected line graph of claw-free graphs are hamiltonian-connected*, J. Combin. Theory Ser. B **82** (2001) 306–315.
https://doi.org/10.1006/jctb.2001.2040

[11] R. Kužel and L. Xiong, *Every* 4*-connected line graph is hamiltonian if and only if it is hamiltonian-connected*, in: R. Kučzel: Hamiltonian Properties of Graphs, Ph.D. Thesis, (U.W.B. Pilsen, 2004).

[12] H.-J. Lai, Y. Liang and Y. Shao, *On s-hamiltonian-connected line graphs*, Discrete Math. **308** (2008) 4293–4297.
https://doi.org/10.1016/j.disc.2007.07.120

[13] H.-J. Lai and Y. Shao, *On s-hamiltonian line graphs*, J. Graph Theory **74** (2013) 344–358.
https://doi.org/10.1002/jgt.21713

[14] H.-J. Lai, Y. Shao, G. Yu and M. Zhan, *Hamiltonian connectedness in* 3*-connected line graphs*, Discrete Appl. Math. **157** (2009) 982–990.
https://doi.org/10.1016/j.dam.2008.02.005

[15] H.-J. Lai, M. Zhan, T. Zhang and J. Zhou, *On s-hamiltonian line graphs of claw-free graphs*, Discrete Math. **342** (2019) 3006–3016.
https://doi.org/10.1016/j.disc.2019.06.006

[16] M.M. Matthews and D.P. Sumner, *Hamiltonian results in $K_{1,3}$-free graphs*, J. Graph Theory **8** (1984) 139–146.
https://doi.org/10.1002/jgt.3190080116

[17] Z. Ryjáček and P. Vrána, *Line graphs of multigraphs and Hamilton-connectedness of claw-free graphs*, J. Graph Theory **66** (2011) 152–173.
https://doi.org/10.1002/jgt.20498

[18] Z. Ryjáček and P. Vrána, *A closure for* 1*-Hamilton-connectedness in claw-free graphs*, J. Graph Theory **75** (2014) 358–376.
https://doi.org/10.1002/jgt.21743

[19] Y. Shao, Claw-Free Graphs and Line Graphs, Ph.D. Dissertation (West Virginia University, 2005).

[20] C. Thomassen, *Reflections on graph theory*, J. Graph Theory **10** (1986) 309–324.
https://doi.org/10.1002/jgt.3190100308

[21] S. Zhan, *Hamiltonian connectedness of line graphs*, Ars Combin. **22** (1986) 89–95.

Received 27 September 2021
Revised 27 December 2021
Accepted 27 December 2021
Available online 8 February 2022