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CUBIC GRAPHS HAVING ONLY k-CYCLES IN EACH 2-FACTOR

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The authors sincerely celebrate Professor Katsuhiro Ota in his 60th birth year.

Abstract

We consider the class of 2-connected cubic graphs having only k-cycles in each 2-factor, and obtain the following two results: (i) every 2-connected cubic graph having only 8-cycles in each 2-factor is isomorphic to a unique Hamiltonian graph of order 8; and (ii) a 2-connected cubic planar graph G has only k-cycles in each 2-factor if and only if $k = 4$ and G is the complete graph of order 4.

Keywords: cubic graph, 2-factor, Hamiltonian cycle, 2-factor Hamiltonian.

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1. Introduction

1.1. Definitions

All graphs in this paper are finite, simple, and undirected. Let G be a graph. We let $V(G)$ and $E(G)$ denote the *vertex set* and the *edge set* of G, respectively. For $x \in V(G)$, we let $N_G(x)$ denote the set of vertices of G adjacent to x.

A *cycle* is a connected 2-regular graph. The number of edges in a cycle is called its *length*, and a k-*cycle* is a cycle of length k. The vertex-disjoint union of cycles in a graph G is a 2-regular subgraph of G. A spanning 2-regular subgraph of G is called a 2-*factor* of G. Let K_n and $K_{m,n}$ denote the *complete graph* of order n and the *complete bipartite graph* with partite sets of order m and n, respectively. For terms and symbols not defined here, we refer the reader to [7].

1.2. Motivation of our investigations and main results

In this paper, we focus on graphs which have a 2-factor consisting of cycles of the same length. We begin with some known results.

Theorem 1 (Jackson and Yoshimoto [16]). *Every* 3*-connected cubic graph of order at least* 5 *has a* 2*-factor in which each component has length at least* 5.

Theorem 2 (Kündgen and Richter [17]). *Every* 2*-connected cubic graph without a Hamiltonian cycle either is the Petersen graph or has a* 2*-factor in which at least one component has length at least* 7.

In particular, the number of 5-cycles in 2-factors of graphs is deeply studied since it concerns the study of snarks [13, 14, 21], where a *snark* is a non-3-edge-colorable 2-connected cubic graph. Many important problems and conjectures can be reduced to snarks. Four Color Theorem, Tutte's 5-flow conjecture, Cycle Double Cover Conjecture and so on (cf. [5]). On the study of snarks, for example, the following result is already known (see Section 3 for the definition of cyclic edge-connectivity of a graph).

Theorem 3 (Lukot'ka *et al.* [20]). *For every positive integer m, there exists a nontrivial snark* (*i.e., one which is cyclically* 4*-edge-connected and has girth at least* 5) *with at least m* 5*-cycles in each* 2*-factor.*

We here consider the following problem.

Problem 4. For any integer $k \geq 3$, characterize a 2-connected cubic graph having only k-cycles in each 2-factor.

The case where $k = 5$ in Problem 4 is immediately obtained from Theorem 2, and it was independently proved by DeVos [8]. For the case where $k \in \{3, 4, 6\}$, we can easily obtain the following (a graph that all of its 2-factors are Hamiltonian is 2-*factor Hamiltonian*).

Theorem 5. (i) *There exists no* 2*-connected cubic graph having only* 3*-cycles in each* 2*-factor.*

(ii) *Every* 2*-connected cubic graph having only* 4*-cycles in each* 2*-factor is isomorphic to* K_4*.*

(iii) *Every* 2*-connected cubic graph having only* 6*-cycles in each* 2*-factor is isomorphic to* $K_{3,3}$*.*

Proof. By Theorem 2, for each $k \in \{3, 4, 6\}$, any 2-connected cubic graph having only k-cycles in each 2-factor is Hamiltonian, that is, each such graph consists of exactly k vertices. For $k = 3$, no such graph exists trivially. For $k = 4$ (respectively, 6), we see that K_4 (respectively, $K_{3,3}$) is the unique 2-factor Hamiltonian cubic graph (cf. [11]). ∎

Furthermore, we solve Problem 4 for $k = 8$ as follows (the graph H_8 is the cubic graph of order 8 shown in Figure 1; the proof is presented in Section 2).

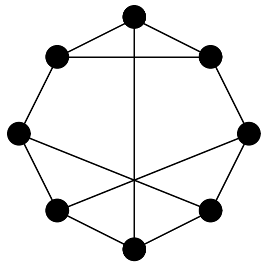

Figure 1. The graph H_8.

Theorem 6. *Every* 2*-connected cubic graph having only* 8*-cycles in each* 2*-factor is isomorphic to* H_8*.*

As in Theorems 5 and 6, for any even $k \geq 4$, a 2-connected cubic graph having only k-cycles in each 2-factor intuitively seems to be Hamiltonian. However, the *Coxeter graph* [7, Page 241] is an exception for this intuition. We confirm (by computer) that every 2-factor of the Coxeter graph consists of exactly two 14-cycles (cf. [4, Page 403]). It is also known that the Coxeter graph is hypohamiltonian (i.e., one which is not Hamiltonian but the graph obtained by removing any vertex is Hamiltonian) [6]. For $k \in \{10, 12\}$, we have not been able to find such an exception and any efficient technique to solve Problem 4.

In the class of 2-connected cubic graphs having only cycles of the same length in each 2-factor, we have been able to find only non-planar graphs except for K_4, i.e., the Petersen graph, $K_{3,3}$, H_8, the Coxeter graph and graphs obtained from the five graphs by recursively performing the operation called a "star product" (for the definition of the star product, see Subsection 1.3). That is why we restrict ourselves to the class of planar graphs. Then, in fact, it turns out that only K_4 survives among the above graphs. (In Section 3, we prove a slightly stronger theorem than the following.)

Theorem 7. *Let $n \geq 4$ and $k \geq 3$ be integers with $n \geq k$, and G a 2-connected cubic planar graph of order n. Then G has only k-cycles in each 2-factor if and only if $k = 4$ and G is the complete graph of order 4.*

Moreover, in view of the fact that the orientable (respectively, non-orientable) genus of the Coxeter graph is equal to 3 (respectively, 6) [22], one might be able to consider the similar problems to Theorem 7 in surfaces with low genus (e.g., the projective plane and/or the torus).

1.3. Short survey of results related to ours

In this subsection, we survey some known results and problems on 2-factors of cubic graphs, which are related to our results.

Recall that a cubic graph G is 2-factor Hamiltonian if all 2-factors of G are Hamiltonian cycles. This concept is introduced by Funk *et al.* [11]. They proved that if a graph is 2-factor Hamiltonian, k-regular and bipartite, then $k \leq 3$. Aldred *et al.* [2] verified the same result for k-regular bipartite graphs with the more general property that all their 2-factors are isomorphic. Moreover, the following is also proved.

Proposition 8 (Funk *et al.* [11], Lemma 3.3)**.** *Let G be a 2-factor Hamiltonian cubic bipartite graph. Then G is 3-connected and $|V(G)| \equiv 2 \pmod 4$.*

Let G_1 and G_2 be cubic graphs, and let x and y be vertices of G_1 and G_2, respectively. A *star product* of G_1 and G_2, denoted by $G_1 * G_2$ (or $(G_1, x)*(G_2, y)$), is the graph obtained from G_1 and G_2 by removing x and y and adding three edges x_1y_1, x_2y_2, x_3y_3, where $N_{G_1}(x) = \{x_1, x_2, x_3\}$ and $N_{G_2}(y) = \{y_1, y_2, y_3\}$. Note that the star product is not always uniquely determined, in other words, the formation of the resulting graph depends on the choice of two vertices in the star product. It is known in [11] that if the resulting graph $G_1 * G_2$ is bipartite, then it is 2-factor Hamiltonian if and only if both G_1 and G_2 are 2-factor Hamiltonian. In the same paper, the following conjecture is proposed. The *Heawood graph* is the incidence graph (or the *Levi graph*) of the Fano plane $PG(2, 2)$, i.e., the graph is a cubic bipartite graph of order 14 [7, Page 244]. Observe that both $K_{3,3}$ and the Heawood graph are 2-factor Hamiltonian.

Conjecture 9 (Funk *et al.* [11], Conjecture 3.2)**.** *Let G be a 2-factor Hamiltonian cubic bipartite graph. Then G can be obtained from $K_{3,3}$ and the Heawood graph by a sequence of star products.*

By the results in [18, 19] (as already mentioned in [11]), in order to prove Conjecture 9, it would be sufficient to prove the following.

Conjecture 10 (Funk *et al.* [11])**.** *The Heawood graph is the only* 2*-factor Hamiltonian cyclically* 4*-edge-connected cubic bipartite graph of girth at least* 6*.*

Abreu *et al.* [1] extended some results on 2-factor Hamiltonian graphs to the more general class of pseudo 2-factor isomorphic graphs, where a graph G is *pseudo* 2*-factor isomorphic* if the parity of the number of cycles in a 2-factor is the same for all 2-factors of G. They propose conjectures similar to Conjectures 9 and 10, for pseudo 2-factor isomorphic graphs. However, Goedgebeur [12] found a counterexample for those conjectures by a computer search, and also verified that Conjecture 10 is true up to 40 vertices.

1.4. Construction of 2-factor Hamiltonian cubic graphs

In this subsection, we introduce a construction of 2-factor Hamiltonian cubic graphs with the help of a new lemma for the star product. As described in the previous subsection, Funk *et al.* [11] showed the following useful lemma.

Lemma 11 (Funk *et al.* [11])**.** *If a bipartite graph G can be represented as a star product $G = G_1 * G_2$, then G is* 2*-factor Hamiltonian if and only if both G_1 and G_2 are* 2*-factor Hamiltonian.*

We can easily obtain a 2-factor Hamiltonian bipartite cubic graph of order n for any $n \equiv 2 \pmod 4$, by a star product $K_{3,3} * K_{3,3} * \cdots * K_{3,3}$. For example, the left of Figure 2 is a star product of two $K_{3,3}$'s. To construct such graphs for $n \equiv 0 \pmod 4$, we prepare the following lemma. For a connected graph G, a *k-cut* is a subset $X \subseteq E(G)$ with $|X| = k$ such that $G - X$ is disconnected.

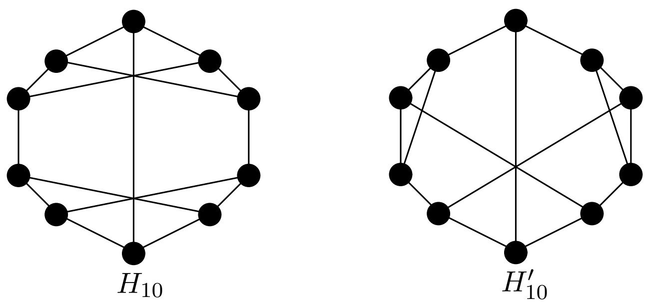

Figure 2. Two 2-factor Hamiltonian cubic graphs of order 10.

Lemma 12. *Let G_1 and G_2 be* 2*-factor Hamiltonian cubic graphs, and let G_2 be bipartite. Then the star product $G = G_1 * G_2$ is* 2*-factor Hamiltonian.*

Proof. Let $S = \{u_1v_1, u_2v_2, u_3v_3\}$ be the 3-cut generated by the star product, where $u_i \in V(G_1)$ and $v_i \in V(G_2)$ for each $i \in \{1, 2, 3\}$. Since G_1, G_2 and G are all 3-edge-colorable, G has a 2-factor. Let F be a 2-factor of G. Since G_2 is bipartite cubic, the two partite sets of G_2 have the same number of vertices. So F must contain an at least one edge in S, and hence F passes exactly two edges in S. Moreover, since both G_1 and G_2 are 2-factor Hamiltonian, F is a Hamiltonian cycle (otherwise, G_1 or G_2 has a 2-factor consisting of at least two cycles). ■

By using Lemma 12, we can construct a 2-factor Hamiltonian non-bipartite cubic graph of order n for any $n \equiv 0 \pmod 4$, by a star product $K_4 * K_{3,3} * K_{3,3} * \cdots * K_{3,3}$. Observe that H_8 is obtained from K_4 and $K_{3,3}$ by a star product.

We also see that the right of Figure 2 is obtained from two K_4's and $K_{3,3}$ by applying a star product two times. The 2-factor hamiltonicity of the resulting graph H'_{10} obtained from $K_4 * K_{3,3}$ and K_4 by a star product is not guaranteed only by Lemmas 11 and 12. However, we easily check that H'_{10} is 2-factor Hamiltonian. Thus, for the non-bipartite case, we need more observations for a star product to explain all 2-factor Hamiltonian cubic graphs obtained by star products.

On the other hand, we know that there exists a 2-factor Hamiltonian cubic graph which cannot be represented as a star product. Recall that the Heawood graph is one such graph. Moreover, so is the *triplex graph* [23]. Fouquet *et al.* [10] provided a general construction of such graphs, namely $FS(j, k)$, as follows. Prepare k disjoint $K_{1,3}$'s, where $H_i = \{x_i, y_i, z_i, t_i\}$ $(0 \leq i \leq k-1)$ is a $K_{1,3}$ with center vertex t_i, and let $T = \bigcup_{i=0}^{k-1}\{t_i\}$. Then let G be the graph obtained from the above H_i's by joining three pairs of pendant vertices of H_i and H_{i+1} for each i (where each subscript is taken modulo k). Up to isomorphism, the matching joining vertices of degree 1 of H_{k-1} and those of H_0 determines the graph G. This construction produces essentially three distinct graphs, namely $FS(1, k)$, $FS(2, k)$ and $FS(3, k)$, where $FS(j, k)$ is the graph with j cycles induced by $V(FS(j, k)) \setminus T$. Observe that $FS(j, k)$ is a simple cubic graph for $k \geq 3$ and that if k is odd, then $FS(2, k)$ is known as the *flower snark* [15]. It is proved in [10] that $FS(j, k)$ is 2-factor Hamiltonian if and only if k is odd and $j \in \{1, 3\}$. Furthermore, if $k \geq 3$ is odd and $j = 1$ (or if $k \geq 5$ is odd and $j = 3$), then $FS(j, k)$ cannot be represented as a star product since it is cyclically 4-edge-connected. In particular, $FS(1, 3)$ is the triplex graph. Therefore, there are infinitely many 2-factor Hamiltonian cubic (non-bipartite) graphs which cannot be represented as a star product.

2. Proof of Theorem 6

Let G be a 2-connected cubic graph having only 8-cycles in each 2-factor. If $|V(G)| = 8$, then since there are only five connected cubic graphs of order 8, we can easily check that exactly one of them is 2-factor Hamiltonian, i.e., G is isomorphic to H_8. So we suppose that $|V(G)| > 8$, for contradiction.

Let $c : E(G) \to \{1,2,3\}$ be a 3-edge coloring of G and let u_0v_0 be an edge with color 3. Observe that for each $i, j \in \{1,2,3\}$ with $i \neq j$, the graph induced by edges with colors i and j is a 2-factor consisting only of 8-cycles. So let $C_1 = u_0u_1u_2 \cdots u_7u_0$ and $C_2 = v_0v_1v_2 \cdots v_7v_0$ be two $(1,2)$-cycles of length 8, where an *(i,j)-cycle* is a cycle induced by edges with colors i and j. Without loss of generality, we may suppose that u_iu_{i+1} and v_iv_{i+1} have color 1 (respectively, 2) if i is even (respectively, odd). We also consider two 8-cycles $C_3 = u_1u_0v_0v_1w_1w_2w_3w_4u_1$ and $C_4 = u_7u_0v_0v_7z_1z_2z_3z_4u_7$; C_3 is a $(1,3)$-cycle in which four edges u_0v_0, v_1w_1, w_2w_3 and w_4u_1 have color 3 and the others have color 1, and C_4 is a $(2,3)$-cycle in which four edges u_0v_0, v_7z_1, z_2z_3 and z_4u_7 have color 3 and the others have color 2. We here divide the proof into two cases.

Case 1. $E(C_1) \cap E(C_3) = \{u_0u_1\}$ or $E(C_2) \cap E(C_3) = \{v_0v_1\}$. By symmetry, we may suppose that $E(C_1) \cap E(C_3) = \{u_0u_1\}$. Here we switch two colors 1 and 3 of C_3. By the hypothesis of Theorem 6, the spanning subgraph of G induced by edges with colors 1 and 2 in the resulting 3-edge coloring is a 2-factor. However, since u_0u_1 has color 3 and u_0v_0, u_1w_4 have color 1, the $(1,2)$-cycle containing vertices in $V(C_1)$ has length at least 10, a contradiction.

By this observation, one of w_1w_2 and w_3w_4 is in $E(C_1)$ and the other is in $E(C_2)$. In other words, we can deduce the following claim.

Claim 13. (i) *$V(C_3) \subset V(C_1) \cup V(C_2)$.*
(ii) *$V(C_4) \subset V(C_1) \cup V(C_2)$.*

Proof. (i) It has been already shown above that every vertex of C_3 is in $V(C_1) \cup V(C_2)$.

(ii) By the symmetry of colors 1 and 2, one can see that $V(C_4) \subset V(C_1) \cup V(C_2)$ like in (i). $\square$

Case 2. One of w_1w_2 and w_3w_4 is in $E(C_1)$ and the other is in $E(C_2)$. We switch two colors 1 and 3 of C_3 and let c' be the resulting 3-edge coloring of G. In this case, we consider two 8-cycles D and D' induced by edges with colors 1 and 2 in c', each of which consists of eight vertices among C_1 and C_2. (Since only the colors of C_3 are switched, we see from Claim 13(i) that such cycles exist.) Note that, in D and D', in order to traverse between $V(C_1)$ and $V(C_2)$, u_0v_0, v_1w_1, w_2w_3 or w_4u_1 must be taken, which are the edges of C_3 with color 1 in c'. By the assumption of this case and symmetry, we may suppose that D contains u_0v_0, and hence $\{u_0, u_7, v_0, v_7\} \subset V(D)$.

Claim 14. (i) *Both D and D' have exactly two edges in $\{u_0v_0, v_1w_1, w_2w_3, w_4u_1\}$.*

(ii) *Either* $D = u_3u_4u_5u_6u_7u_0v_0v_7u_3$ (*the left of Figure* 3) *or* $D = u_5u_6u_7u_0v_0v_7v_6v_5u_5$ (*the right of Figure* 3) *holds.*

Proof. Let $A = \{u_0v_0, v_1w_1, w_2w_3, w_4u_1\}$.

(i) Obviously D contains at least two edges in A. If D' contains at most one edge in A, then D' consists of at most 6 vertices of either $V(C_1) \setminus \{u_0, u_7\}$ or $V(C_2) \setminus \{v_0, v_7\}$, and hence, its length is less than 8, a contradiction.

(ii) By (i), the cycle D consists of

- four edges in $E(C_1) \cup E(C_2)$ with color 2 in both c and c';
- two edges in A (with color 1 in c'), one of which is u_0v_0; and
- two edges in $E(C_1) \cup E(C_2)$ with color 1 in both c and c'.

Hence by symmetry, the statement holds. □

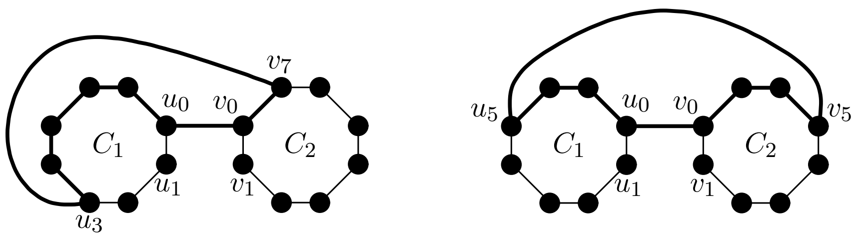

Figure 3. The 8-cycle D in Subcases 2.1 and 2.2.

By Claim 14, we further divide the case into two cases.

Subcase 2.1. $D = u_3u_4u_5u_6u_7u_0v_0v_7u_3$. The edges u_3v_7 and w_2w_3 coincide. If $w_1w_2 = v_6v_7$ and $w_3w_4 = u_3u_2$ (thus $v_1v_6 \in E(G)$), then G has multiple edges u_1u_2 and u_1w_4, a contradiction. So $w_1w_2 = u_2u_3$ and $w_3w_4 = v_7v_6$. We consider the $(2, 3)$-cycle $C_4 = u_7u_0v_0v_7z_1z_2z_3z_4u_7$ in the original 3-edge coloring c (see the left of Figure 4). Now $z_1 = u_3$ and $z_2 = u_4$, and the edge z_3z_4 is an edge in $E(C_1) \cup E(C_2)$ by Claim 13(ii).

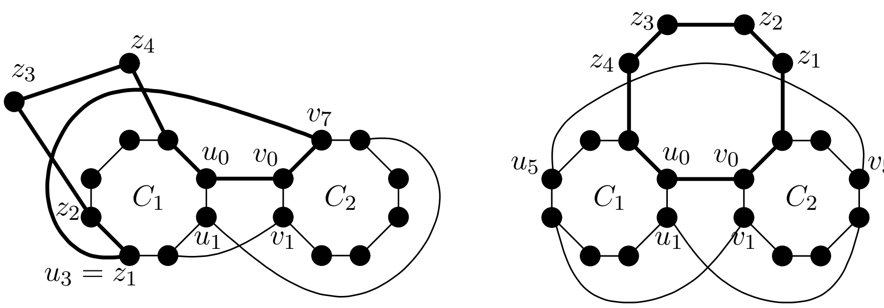

Figure 4. The $(2, 3)$-cycle C_4 in Subcases 2.1 and 2.2.

Claim 15. (i) *The edges z_3z_4 and v_3v_4 cannot coincide.*
(ii) *The edges z_3z_4 and u_5u_6 cannot coincide.*

Proof. (i) First, suppose that $z_3z_4 = v_3v_4$. By switching colors 2 and 3 of C_4 in c, the spanning subgraph induced by edges with colors 1 and 2 (in the resulting edge coloring) has the cycle of length 16 (i.e., $u_0v_0v_1v_2v_3u_4u_5u_6u_7v_4v_5v_6v_7u_3u_2u_1$), a contradiction. Second, suppose that $z_3z_4 = v_4v_3$. Similarly, we obtain the cycle of length 16 (i.e., $u_0v_0v_1v_2v_3u_7u_6u_5u_4v_4v_5v_6v_7u_3u_2u_1$), a contradiction.

(ii) First, suppose that $z_3z_4 = u_5u_6$. Then multiple edges appear, a contradiction. Second, suppose that $z_3z_4 = u_6u_5$. Then the spanning subgraph induced by edges with colors 1 and 3 in the original 3-edge coloring c has the cycle of length 4 (i.e., $u_4u_5u_7u_6$), a contradiction. □

By Claim 15, z_3z_4 must be one of $\{u_1u_2, v_1v_2, v_5v_6\}$, which contradicts that G is cubic.

Subcase 2.2. $D = u_5u_6u_7u_0v_0v_7v_6v_5u_5$. The edges u_5v_5 and w_2w_3 coincide. If $w_1w_2 = v_4v_5$ and $w_3w_4 = u_5u_4$ and if switching two colors 1 and 3 of C_3, then the spanning subgraph induced by edges with colors 1 and 2 in the resulting 3-edge coloring contains two $(1,2)$-cycles $u_1u_2u_3u_4u_1$ and $v_1v_2v_3v_4v_1$ of length 4, a contradiction. So $w_1w_2 = u_4u_5$ and $w_3w_4 = v_5v_4$.

We here consider the $(2,3)$-cycle $C_4 = u_7u_0v_0v_7z_1z_2z_3z_4u_7$ in c (see the right of Figure 4). By Claim 13(ii), the edge z_1z_2 is an edge in $E(C_1) \cup E(C_2)$. For each edge e with color 2 in c in $\{u_1u_2, u_3u_4, u_5u_6, v_1v_2, v_3v_4, v_5v_6\}$, one end vertex of e is incident to an edge of C_3. Thus, some vertex $x \in V(C_1) \cup V(C_2)$ must be of degree at least 4 (i.e., x is incident to at least one each edge of C_3 and C_4), which contradicts that G is cubic.

3. PROOF OF THEOREM 7

In this section, we shall prove the following stronger theorem, and hence Theorem 7 follows as a corollary.

Theorem 16. *Let $n \geq 4$ and $k \geq 3$ be integers with $n \geq k$, G a 2-connected cubic planar graph of order n, and $v \in V(G)$. If every 2-factor of G consists of a cycle through v of length at most k and a disjoint union of k-cycles, then $k = 4$ and G is isomorphic to K_4.*

Note that our proof strongly depends on Theorem 17 shown by Diwan [9] (see Theorem in Page 251, the first paragraph after the theorem in Page 251 and the Cases 4 and 5 in the proof in Pages 252–258). A *cyclic k-cut* S of a connected graph G is one such that at least 2 components of $G - S$ have cycles. A connected graph G containing two disjoint cycles is *cyclically k-edge-connected* if there is no cyclic l-cut of G with $l \leq k-1$. Let $k(H)$ be the *cyclic edge-connectivity* of H, which is the maximum integer m such that H is cyclically m-edge-connected. A k*-face* is a face bounded by a closed walk of length k, and two faces are *adjacent* if their boundary walks share an edge.

Theorem 17 (Diwan [9])**.** *Let G be a 2-connected cubic planar graph except K_4.*

(i) *If $k(G) = 4$ and G does not contain a 4-cycle, then G has a perfect matching which contains four edges corresponding to a cyclic 4-cut of G.*

(ii) *If $k(G) = 5$, then G has a 2-factor F satisfying one of the following:*

  (a) *a 5-cycle f_1 in F is a 5-face of G; or*

  (b) *an 8-cycle f_2 in F is the symmetric difference of boundaries of two adjacent 5-faces of G.*

***Proof of Theorem* 16.** If $n = 4$, then G is isomorphic to K_4 and hence the theorem holds. So, we suppose that G is a minimal counterexample, i.e., it satisfies all the assumptions in the theorem but $G \neq K_4$, and let v be a vertex as in the statement.

It is well known by the Four Color Theorem [3] (in particular, Tait's theorem [24]) that every 2-connected cubic planar graph has a 3-edge coloring. Let $c : E(G) \to \{1, 2, 3\}$ be a 3-edge coloring of G and let $G_{i,j}$ be the subgraph of G induced by the edges with colors i and j for any two distinct $i, j \in \{1, 2, 3\}$. Since each $G_{i,j}$ is a 2-factor of G, every (i, j)-cycle in $G_{i,j}$, other than one through v, is of length k, which is an alternating cycle with colors i and j. Thus, k (and the length of the cycle through v) must be even.

Claim 18. *G is 3-edge connected.*

Proof. Suppose to the contrary that G has a 2-cut $S = \{e_1, e_2\}$. Since each $G_{i,j}$ consists of cycle(s), e_1 and e_2 are colored by the same color, say 1. Let R be a connected component of $G - S$ that does not contain v. Observe that $V(R)$ must be a multiple of k by considering $(2, 3)$-cycles in R. Let C be the $(1, 2)$-cycle containing e_1 and e_2. Since all $(1, 2)$-cycles in R are of length k, the length of C must be more than k, a contradiction. □

Claim 19. *In every 3-edge coloring of G, all edges in any 4- or 6-cut of G cannot be colored by the same color.*

Proof. Let $S = \{e_1, e_2, e_3, e_4\}$ be a 4-cut of G and R be a component of $G - S$ that does not contain v. Suppose that all edges in S are colored by the same color, say 1. Observe that $V(R)$ must be a multiple of k by considering $(2, 3)$-cycles in R. Let C be the $(1, 2)$-cycle containing e_1. One can see that C contains 4 or exactly 2 edges in S. If C contains all edges in S, then the length of C must be more than k by considering $(1, 2)$-cycles in R, a contradiction. If C

contains exactly 2 edges in S, then switch the colors 1 and 2 of C in c. Let c' be the resulting 3-edge coloring of G and let C_2 be the $(2,3)$-cycle (colored in c') containing e_1 in G. Then the length of C_2 must be more than k by considering $(2,3)$-cycles (colored in c') in R, a contradiction.

Let S be a 6-cut of G and suppose that all edges in S are colored by the same color, say 1. In this case there are only two possibilities: all edges in S are contained in exactly one $(1,2)$-cycle, or there exists a $(1,2)$-cycle that contains exactly two edges in S. In both cases we can use the similar argument above, and we get a contradiction. □

Claim 20. *G has none of the following configurations:*

(i) *two adjacent* 3*-faces;*

(ii) *a* 4*-face;*

(iii) *a* 5*-face adjacent to a* 3*-face;*

(iv) *a* 6*-face adjacent to at least two* 3*-faces;*

(v) *a* 7*-face adjacent to at least three* 3*-faces;*

(vi) *an* 8*-face adjacent to at least four* 3*-faces.*

Proof. (i) If G contains two adjacent 3-faces $v_1v_2v_3$ and $v_2v_3v_4$, then G has a 2-cut $S = \{v_1u_1, v_4u_4\}$ since G is not isomorphic to K_4, where u_i is the neighbor of v_i other than v_2 and v_3 for $i \in \{1,4\}$. This is contrary to Claim 18.

(ii) Suppose that G contains a 4-face $v_1v_2v_3v_4$. Let u_i be the neighbor of v_i other than v_{i-1} and v_{i+1} for $i \in \{1,2,3,4\}$ with indices taken modulo 4 (the u_i's are not necessarily distinct). Let $S = \{v_1u_1, v_2u_2, v_3u_3, v_4u_4\}$. By Claim 19 and symmetry, we may suppose that $c(v_1u_1) = c(v_2u_2) = 1$ and $c(v_3u_3) = c(v_4u_4) = 2$. Let C be a $(1,2)$-cycle containing v_1u_1. If C does not contain v_3u_3, then switch the colors 1 and 2 of C in c. In the resulting 3-edge coloring, all edges in the 4-cut S are colored by 2, which contradicts Claim 19. If C contains v_3u_3, then C is represented as $u_1v_1v_2u_2 \cdots u_3v_3v_4u_4 \cdots u_1$ by the planarity of G. So C can be divided into two cycles $u_1v_1v_4u_4 \cdots u_1$ and $u_2v_2v_3u_3 \cdots u_2$, which creates a new 2-factor of G having two cycles of length less than k, a contradiction.

(iii) Suppose that G contains a 5-face adjacent to a 3-face, namely $v_1v_2v_3v_4v_6$ and $v_4v_5v_6$. Let u_i be the neighbor of v_i other than v_{i-1} and v_{i+1} for $i \in \{1,2,3,5\}$ with indices taken modulo 6 (the u_i's are not necessarily distinct). Let $S = \{v_1u_1, v_2u_2, v_3u_3, v_5u_5\}$. By Claim 19 and symmetry, we may suppose that $c(v_1u_1) = c(v_2u_2) = 1$ and $c(v_3u_3) = c(v_5u_5) = 2$. Let C be a $(1,2)$-cycle containing v_1u_1; thus C contains v_2u_2. If C does not contain v_3u_3, then switch the colors 1 and 2 of C in c. In the resulting 3-edge coloring, all edges in the 4-cut S are colored by 2, which contradicts Claim 19. If C contains v_3u_3, then C is represented as $u_1v_1v_2u_2 \cdots u_3v_3v_4v_6v_5u_5 \cdots u_1$ by the planarity of G. So C can be divided into two cycles $u_1v_1v_6v_4v_5u_5\cdots u_1$ and $u_2v_2v_3u_3\cdots u_2$, which creates a new 2-factor of G having two cycles of length less than k, a contradiction.

(iv) Suppose that G contains a 6-face adjacent to at least two 3-faces. By Claim 20(i) and symmetry, it can be divided into two cases as follows: (iv.1) A 6-face $v_1v_2v_3v_5v_6v_8$ and two 3-faces $v_3v_4v_5$ and $v_6v_7v_8$. Let u_i be the neighbor of v_i other than v_{i-1} and v_{i+1} for $i \in \{1,2,4,7\}$ with indices taken modulo 8 (the u_i's are not necessarily distinct). Let $S = \{v_1u_1, v_2u_2, v_4u_4, v_7u_7\}$. By Claim 19 and symmetry, we may suppose that one of the following holds: $c(v_1u_1) = c(v_2u_2) = 1$ and $c(v_4u_4) = c(v_7u_7) = 2$, or $c(v_1u_1) = c(v_7u_7) = 1$ and $c(v_2u_2) = c(v_4u_4) = 2$. In both cases, let C be a $(1,2)$-cycle containing v_1u_1; thus C contains v_2u_2 or v_7u_7. If C does not contain v_4u_4, then switch the colors 1 and 2 of C in c. In the resulting 3-edge coloring, all edges in the 4-cut S are colored by 2, contrary to Claim 19. If C contains v_4u_4, then C can be divided into two cycles like in the case (iii), which creates a new 2-factor of G having two cycles of length less than k, a contradiction. (iv.2) A 6-face $v_1v_2v_4v_5v_6v_8$ and two 3-faces $v_2v_3v_4$ and $v_6v_7v_8$. Let u_i be the neighbor of v_i other than v_{i-1} and v_{i+1} for $i \in \{1,3,5,7\}$ with indices taken modulo 8 (the u_i's are not necessarily distinct). Let $S = \{v_1u_1, v_3u_3, v_5u_5, v_7u_7\}$. By Claim 19 and symmetry, we may suppose that $c(v_1u_1) = c(v_3u_3) = 1$ and $c(v_5u_5) = c(v_7u_7) = 2$. In this case, considering a $(1,2)$-cycle containing v_1u_1, we can get a contradiction like in the case (iv.1).

(v) and (vi) One can also get a contradiction like in the previous case. We leave the proofs to the reader. □

Let H be the cubic planar graph obtained from G by contracting each 3-face of G to a single vertex (by Claims 18 and 20(i), there is no multiple edge in H). Moreover, since H is clearly 2-connected cubic planar, H has a 2-factor F. We denote by F_G the 2-factor of G obtained from F by a suitable extension (i.e., for a contracted vertex v_f of degree 3 corresponding to a 3-face $f = xyz$ and two edges av_f, v_fb in F where $ay, zb \in E(G)$, F_G passes four edges ay, yx, xz and zb instead of av_f and v_fb; see Figure 5).

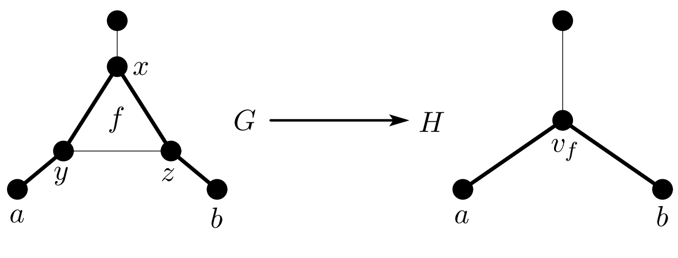

Figure 5. The 2-factor F_G of G obtained from a 2-factor F of H.

Note that the parity of the length of a cycle C in F_G is the same as that of the cycle in F corresponding to C, and recall that every cycle in each 2-factor has even length. On the one hand, H is 3-edge-connected since G is 3-edge-connected by Claim 18. On the other hand, by the construction of H and Claim 20(ii)–(vi), one can check that H has neither a 3-face nor a 4-face, and hence H has a 5-face by Euler's formula. Thus, we consider the following three cases divided by the cyclic edge-connectivity $k(H)$: $3 \le k(H) \le 5$.

Case 1. $k(H) = 5$. By Theorem 17(ii), H has a 2-factor F satisfying one of the following: (ii.a) a 5-cycle f_1 in F is a 5-face of H; or (ii.b) an 8-cycle f_2 in F is the symmetric difference of boundaries of two adjacent 5-faces of H. If the condition (ii.a) holds, then F_G has an odd cycle (extended from f_1), a contradiction. Otherwise, i.e., if (ii.b) holds, then the cycle in F_G extended from f_2 is separated by six edges from other component of F_G, since f_2 is an 8-cycle containing exactly one chord (see Figure 6). Then the six edges correspond to a (cyclic) 6-cut S of G. The six edges of S are colored by the same color in the 3-edge coloring induced by the 2-factor F_G, contrary to Claim 19.

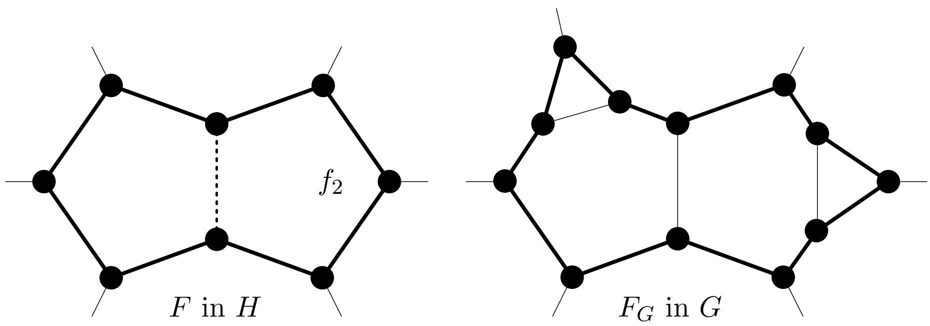

Figure 6. An 8-cycle f_2 in F, and the cycle in F_G extended from f_2.

Case 2. $k(H) = 4$. Recall that H has no 4-face (and hence no 4-cycle). By Theorem 17(i), H has a perfect matching M which contains the four edges corresponding to a cyclic 4-cut S of H. By the construction of H, S is also a cyclic 4-cut of G. Let $F = G - M$ be the 2-factor of H. If the 2-factor F_G of G consists of all but one k-cycles, which are all even cycles, then F_G induces a 3-edge coloring of G such that the four edges of S are colored by the same color, contrary to Claim 19.

Case 3. $k(H) = 3$. Let S be a cyclic 3-cut of H. By the construction of H, S is also a cyclic 3-cut of G. Let R_1 be the component of $G - S$ containing a vertex v and let R_2 be the other one. We denote by G/R_1 (respectively, G/R_2) the graph obtained from G by contracting R_1 (respectively, R_2) to a single vertex, say r_1 (respectively, r_2). By construction, both G/R_1 and G/R_2 are 2-connected cubic planar graphs.

Recall that H has no 3-face (and hence no 3-cycle), and notice that since S is a cyclic 3-cut in H, G/R_1 is not isomorphic to K_4. Since $|V(G/R_1)| < |V(G)|$, by inductive hypothesis, G/R_1 has a 2-factor F_1 which does not satisfy the following. A cycle passing r_1 is of length at most k and the other cycles are of length k. Let e_1 and e_2 be two edges incident to r_1 used in F_1. Observe that G/R_2 has a 2-factor F_2 containing both e_1 and e_2 since G/R_2 has a 3-edge coloring. Moreover, $F_1 \cup F_2$ forms a 2-factor F of G. However, by the assumption of F_1, F does not satisfy the assumption in the statement of the theorem (because one of the cycles of F in R_2, that does not contain v, is not of length k; or the cycle of F containing e_1 and e_2 is of length more than k), a contradiction. ■

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