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2-NEARLY PLATONIC GRAPHS

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Abstract

A 2-nearly Platonic graph of type (k, d) is a k-regular plane graph with f faces, $f - 2$ of which are of size d and the remaining two are of sizes d_1, d_2, both different from d. Such a graph is called balanced if $d_1 = d_2$. We show that all connected 2-nearly Platonic graphs are balanced. This proves a recent conjecture by Keith, Froncek, and Kreher. We also show that any 2-nearly Platonic graph belongs to one of 15 well defined infinite classes. The latter states more precisely the statement of Deza, Dutour Sikirič, and Shtogrin from 2013, and of Froncek, Khorsandi, Musawi, and Qui from 2021 that there are only 14 such classes. Moreover, our short proof provides a complete characterization of all 2-nearly Platonic graphs.

Keywords: plane graph, Platonic solid, almost Platonic graph.

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1. Introduction and Terminology

Let us first present some of the basic definitions, notations and terminology used in this paper. Other terminology will be introduced as it naturally occurs in the text or is used according to Diestel's book [5]. Through this paper, we consider only finite simple graphs that are connected and planar.

A graph is said to be *planar* if it can be drawn in the plane such that each common point of two edges is a vertex. This drawing of a planar graph G is called a *plane graph* (or planar embedding of) G and can be regarded as a graph isomorphic to G. By this definition, we need some matters of the topology of the plane. Immediately, after deleting the points corresponding to the vertices and edges of a plane graph from the plane, we have some maximal open sets (or regions) of the points in the plane called *faces* of the plane graph. There exists exactly one unbounded region that we call the *outer face* of the plane graph and other faces are called *internal faces*. The *boundary* of a face is the set of points consisting of vertices and edges touching the face. A face is said to be *incident* with the vertices and edges in its boundary. Two faces are *adjacent* if their boundaries have an edge in common. Two faces of a plane graph are *touching* if they share at least one vertex.

The *degree* of a vertex v, denoted by $\deg_G(v)$ or simply by $\deg(v)$ when the underlying graph is understood, is the number of edges incident with the vertex, where any loop is counted twice. A graph is *r-regular* (or *r-valent*) if the degree of each vertex in G is r, and the graph is *regular* if it is r-regular for some r.

The *degree* or *size* of a face F of a connected plane graph is the number of edges incident with the face F, where any cut-edge is counted twice, and is denoted by $\deg_G(F)$ or $\deg(F)$ if the graph G is known from the content. Note that in the case when the planar graph is 2-connected, the boundary of any face is a cycle. We will use $f_r = f_r(G)$ to denote the number of faces of degree r in the graph G.

Let $G = (V(G), E(G), F(G))$ be a connected plane graph with vertex set $V(G)$, edge set $E(G)$, and face set $F(G)$. The well-known *Euler's formula* states that

$$|V(G)| - |E(G)| + |F(G)| = 2.$$

To distinguish our approach in this paper from that of [4, 8] we need to introduce two important notions which are studied in the Section 4.3 of the book [5]. First one is a *topological isomorphism* between two plane graphs H and H'. Its definition is rather complicated. We omit it here. Intuitively it is a homeomorphism from the plane R^2 to itself taking H onto H' (and keeping orientations of the corresponding faces). The second one is a *combinatorial isomorphism* of the graphs H and H'. It is a bijection $\sigma : V(H) \cup E(H) \cup F(H) \to V(H') \cup E(H') \cup F(H')$ that preserves incidence not only of vertices and edges but also of vertices and edges with faces. (Formally, we require that a vertex or edge $x \in V(H) \cup E(H)$ shall lie on the boundary of a face $F \in F(H)$ if and only if $\sigma(x)$ lies on the boundary of the face $\sigma(F)$.)

A graph G is *k-connected* if $|V(G)| > k$ and $G - X$ is connected for every set $X \subseteq V(G)$ with $|X| < k$.

A *Platonic solid* of type (k, d) is a convex three-dimensional polyhedron all vertices of which are of degree k and all faces of which are of degree d. Using Euler's formula we get that

$$(k, d) \in \{(3,3), (3,4), (4,3), (3,5), (5,3)\}.$$

The class of Platonic solids consists of the well-know five polyhedra: the tetrahedron (of type $(3,3)$), the cube (of type $(3,4)$), the octahedron (of type $(4,3)$), the dodecahedron (of type $(3,5)$), and the icosahedron (of type $(5,3)$).

The investigation of properties of Platonic solids has a long history. There is no reliable information about their first mention. However, they are still attractive for mathematicians, chemists, and others. In the last two centuries, many of authors have paid attention to these polyhedra and they have extended the study to convex and concave polytopes, see, e.g., [9].

From a combinatorial point of view a convex polyhedron can be studied as a plane graph as there is a correspondence between the graph of a polyhedron determined by its vertices and edges and a planar graph. Namely, Steinitz's theorem (see, e.g. [9]) states that a graph G with at least four vertices is the graph of vertices and edges of a convex polyhedron if and only if G is planar and 3-connected.

A connected k-regular plane graph with f faces is a *t-nearly Platonic graph* of type (k, d) if $f > 2t$, $f - t$ of its faces are of size d, and the remaining t faces are of sizes other than d. The faces of size d are often called *common faces*, and the remaining ones *exceptional* faces. When $t \geq 2$ and all exceptional faces are of the same size, then the graph is called a *balanced t-nearly Platonic graph.*

In 1967, Grünbaum [9] considered 3-regular connected planar graphs. For a 3-regular connected plane graph and $k \in \{2, 3, 4, 5\}$, he proved that if the degree of all faces but t of them is divisible by k then $t \geq 2$ and if $t = 2$ two exceptional faces do not have a common edge [9]. In 1968, Malkevitch proved the same results for 4-regular and 5-regular 3-connected plane graphs [19]. Several papers are devoted to the study of this topic, but all of them have considered plane graphs such that the sizes of all faces but some exceptional ones are multiple of k and $k \in \{2, 3, 4, 5\}$ (see [2, 10–12]). For a more general approach to 3-connected k-regular plane graphs see [13, 16], and [14].

Keith, Froncek, and Kreher [17, 18] and Froncek, Khorsandi, Musawi, and Qiu [7] proved recently that there are no 1-nearly Platonic graphs.

Deza, Dutour Sikirič, and Shtogrin [4] classified for each admissible pair (k, d) all possible degrees of the exceptional faces of balanced 3-nearly Platonic graphs and sketched a proof of the completeness of the list. Froncek and Qiu [6] provided a detailed proof of existence of infinite families of such graphs for each listed exceptional degrees based on the combinatorial isomorphism.

There are two well-known infinite classes of 3-connected 2-nearly Platonic graphs known also as n-prisms and n-antiprisms, $n \geq 3$. The n-prism, denoted by D_n, is obtained from two cycles $C_n = x_1 \cdots x_n x_1$ and $C'_n = y_1 \cdots y_n y_1$ by adding edges $x_i y_i, 1 \leq i \leq n$. The n-antiprism, denoted by A_n, is obtained from two cycles $C_n = x_1 \cdots x_n x_1$ and $C'_n = y_1 \cdots y_n y_1$ by adding edges $x_i y_i, x_i y_{i+1}, x_n y_1$, $1 \leq i \leq n$.

There are 14 known classes of balanced connected 2-nearly Platonic (plane) graphs (see [4, 8]). The authors of [4] provide a list and a sketch of a proof that their list is complete. Froncek *et al.* in [8], give a very extensive proof of the existence of this list constituted according to the combinatorial isomorphism of plane graphs.

So the above mentioned result, that 14 such classes exist, is correct from this point of view. In [17], Keith, Froncek, and Kreher conjectures that every connected 2-nearly Platonic graph must be balanced.

Our approach to the decomposition of the class of all connected 2-nearly Platonic graphs according to the topological isomorphism and *fundamental bricks* (defined latter) shows that every 2-nearly Platonic graph belongs to one of 15 well defined classes and that all connected 2-nearly Platonic graphs are balanced. The latter proves the conjecture of Keith, Froncek, and Kreher [17]. The former states more precisely the statement of Deza, Dutour Sikirič, and Shtogrin [4] and Froncek *et al.* [8]. Our result provides the same 13 infinite classes as that of [8] that all connected 2-nearly Platonic graphs are balanced and splits the fourteenth class of [8] into two disjoint ones. Moreover, our approach gives a complete characterization of all 2-nearly Platonic graphs and provides a method how to construct all of them.

2. Results

The following theorem is proved in [17].

Theorem 1. *There are no* 1*-nearly Platonic graphs.*

The proof of the next theorem can be found in [3] and [7].

Theorem 2. *Every connected* 2*-nearly Platonic graph is* 2*-connected.*

The following useful lemma is proved in [8]. For convenience, we repeat its proof.

Lemma 3. *Let G be a* 2*-nearly Platonic graph of type (k, d). Then $3 \leq d \leq 5$ for $k = 3$ and $d = 3$ for $k \in \{4, 5\}$.*

Proof. Let $|V(G)| = n$, $|E(G)| = m$, $|F(G)| = f$, and d_1 and d_2 be sizes of two exceptional faces. The graph G is k-regular so we have $kn = 2m$. On the other hand, $2m = (f-2)d + d_1 + d_2$ and by Euler's formula $f - 2 = m - n = \frac{1}{2}(k-2)n$. Therefore, $kn = \frac{1}{2}(k-2)nd + d_1 + d_2$ or $d = \frac{2k}{k-2} - \frac{2(d_1+d_2)}{(k-2)n}$ which implies that $d < \frac{2k}{k-2}$. Now, $3 \leq d \leq 5$ for $k = 3$ and $d = 3$ for $k \in \{4, 5\}$. ∎

It is easy to see from Lemma 3 that the only possible types of 2-nearly Platonic graphs are

$$(k, d) \in \{(3, 3), (3, 4), (4, 3), (3, 5), (5, 3)\}.$$

The main result of this paper is the following.

Theorem 4. *Any* 2*-nearly Platonic graph is balanced and belongs to exactly one of* 15 *well defined infinite classes of plane graphs.*

In Sections 3 and 4 we state and give proofs of Theorem 5 and Theorem 6 from which we obtain Theorem 4.

3. On 2-Connected But Not 3-Connected 2-Nearly Platonic Graphs

In this section we give the proof of our main result, Theorem 4, for the classes of graphs with properties named in the section title.

Let G be a plane graph and F_1 and F_2 be two its distinct faces touching in a vertex z. The *splitting the vertex z with respect to the faces F_1 and F_2* is a local operation changing G to a new plane graph G' in which the vertex z of G is replaced by two non-adjacent vertices z_1 and z_2 of G' and the faces F_1 and F_2 are changed to one new face F' of G' so that $\deg_{G'}(z_1) + \deg_{G'}(z_2) = \deg_G(z)$ and $\deg_{G'}(F') = \deg_G(F_1) + \deg_G(F_2)$. The remaining vertices and faces of G keep their degrees in the corresponding vertices and faces of G'. Observe, that $|V(G')| = |V(G)| + 1$, $|E(G')| = |E(G)|$, and $|F(G')| = |F(G)| - 1$.

Let G be a 2-nearly Platonic graph of type (k, d) with two exceptional faces F_1 and F_2. Choose a minimum 2-vertex cut $\{u, v\}$, i.e., a 2-vertex cut so that a component K of the graph $G - \{u, v\}$ has the minimum number of vertices from among all possible 2-vertex-cuts of G. Then the block $H = H(u, v)$ induced on the vertex set $V(H(u, v)) = V(K) \cup \{u, v\}$ has the following properties. It is 2-connected, all vertices of H lie on or in the interior (respectively, in the exterior) of a separating cycle C determined by two internally vertex disjoint u, v-paths P_1 and P_2 whose all internal vertices are from the vertex set $V(K)$ (Compare with [15].). All vertices of H, except for the vertices u and v, are of degree k and all internal faces of H are of degree d. For the vertices u and v we have $\deg_H(u) = a \geq 2$, $\deg_H(v) = b \geq 2$. Observe that H is a subgraph of the Platonic solid of type (k, d) and that it is isomorphic to the the Platonic solid of type (k, d) with one edge deleted or it is isomorphic to the Platonic solid of type (k, d) with one vertex, say z, split with respect to two distinct faces incident to z, to two vertices u' and v'. By case analysis one can easily check that otherwise on C or in the interior of C a vertex w of $\deg_H(w) \neq k$, $u \neq w \neq v$, and/or a face of size different from d appears which would be a contradiction.

Case 1. In the former case we have $\deg_H(u) = a = \deg_H(v) = b = k - 1$. Both paths P_1 and P_2 have length $d - 1$ and the graph $H + uv$ is the Platonic solid of type (k, d). This means that any nontrivial block of the graph G is isomorphic to the block H and, therefore G has a decomposition into graphs $H^+ = H(u, v) + v' + vv'$ and can be obtained, e.g., by replacing any edge of the cycle C_t of length $t \geq 2$ with the graph H^+ in a suitable way. In this case we have $d_1 = d_2 = td$.

Case 2. In the latter case we have $\deg_H(u') = a \geq 2$, $\deg_H(v') = b \geq 2$, and $a + b = k$. Let $(a, b) = (2, 2)$ for $k = 4$ and $(a, b) = (3, 2)$ or $(a, b) = (2, 3)$ for $k = 5$. In all these cases both paths P_1 and P_2 have length 3. This means that any nontrivial block of the graph G is isomorphic to the block H and G has a decomposition into the graphs $H = H(u, v)$ and can be obtained, e.g., by replacing any edge of the cycle C_t of length $t \geq 2$ with the graph $H = H(u, v)$ in a suitable way. The result is a 2-nearly Platonic graph with $d_1 = d_2 = 3t$. Note that for $t = 1$ we have the Platonic solid of type (k, d).

From the above considerations we have the following.

Theorem 5. Any 2-connected but not 3-connected 2-nearly Platonic graph is balanced. There are seven infinite classes of such 2-nearly Platonic graphs. Namely, one of type $(3, 3)$, one of type $(3, 4)$, one of type $(3, 5)$, two of type $(4, 3)$, and two of type $(5, 3)$.

4. 3-Connected 2-Nearly Platonic Graphs

Let G be a 3-connected 2-nearly Platonic graph of type (k, d) with two exceptional faces F_1 and F_2. Let F_1 (respectively, F_2) be the outer (respectively, the inner) face. We denote their respective boundaries by $x_1, x_2, \ldots, x_n$, and $y_1, y_2, \ldots, y_m$ in clockwise order. We define the distance between F_1 and F_2 as

$$\mathrm{dist}(F_1, F_2) = \min\{\mathrm{dist}(x_i, y_j) | x_i \in F_1, y_j \in F_2\}.$$

Let the vertices x_1 and y_1 be chosen so that

$$\mathrm{dist}(F_1, F_2) = \mathrm{dist}(x_1, y_1) = l.$$

Claim 1. *If G is a 3-connected 2-nearly Platonic graph with exceptional faces F_1 and F_2, then* $\mathrm{dist}(F_1, F_2) \geq 1$.

Proof. Assume that there is 3-connected 2-Platonic graph of type (k, d) with $\mathrm{dist}(F_1, F_2) = 0$. Then the faces F_1 and F_2 share exactly one edge uv or exactly one common vertex z (because of 3-connectivity). If, in the former case, we delete the edge uv we get the graph H described in the previous Section, Case 1, which together with the edge uv leads to G being the Platonic solid of type (k, d); a contradiction. If, in the latter case, we split the vertex z with respect to the faces F_1 and F_2 to two vertices u and v, we get the graph H investigated in the case 2 of the previous section. The converse operation to splitting, the identification of the vertices u and v, leads again to the Platonic solid of type (k, d) which does not contain any exceptional face; a contradiction. ∎

Let $P = z_1 z_2 \cdots z_r$ be a path from a vertex z_1 of F_1 to a vertex z_r of F_2. The *pattern of P with respect to the faces F_1 and F_2* is the sequence $t_1, t_2, \ldots, t_r$ of ordered triplets $t_i = (\alpha_G(z_i), \beta_G(z_i), \gamma_G(z_i))$, $i \in \{1, \ldots, r\}$, where $\alpha_G(z_i)$ (respectively, $\gamma_G(z_i)$) denotes the number of edges going to right (respectively, to left) from the vertex z_i in G when going from the vertex z_1 to the vertex z_r along the path P. The value $\beta_G(z_i)$ denotes the number of edges of the path P incident to the vertex z_i. Notice that $\alpha_G(z_i) + \beta_G(z_i) + \gamma_G(z_i) = \deg_G(z_i)$.

Let $P_1 = x_1 v_1 \cdots v_{l-1} y_1$ be an x_1, y_1-path of length l between F_1 and F_2. Let $P_2 = x_a w_1 \cdots w_{l-1} y_b$ be another path of length l between F_1 and F_2 having the same pattern as P_1. Let $Q_1 = x_1 x_2 \cdots x_a$ (respectively, $Q_2 = y_1 y_2 \cdots y_b$) be a subpath of the boundary cycle of F_1 (respectively, of F_2). Then the subgraph D of G bounded by the cycle $C = P_1 \cup Q_1 \cup P_2 \cup Q_2$ has the following properties.

1. All faces inside of C are of degree d.
2. All vertices of D, except for the ones of the paths P_1 and P_2, are of degree k.
3. For the vertices of the paths P_1 and P_2 we have
$$\deg_D(x_1) + \deg_D(x_a) = \deg_D(y_1) + \deg_D(y_b) = k + 1$$
$$\deg_D(v_i) + \deg_D(w_i) = k + 2$$
for any $i \in \{1, \ldots, l-1\}$.
4. The subgraph D is 2-connected.

The subgraph $B_l(k, d)$ is called a *fundamental brick* of the type (k, d) if it is of minimum order with respect to the properties 1, 2, 3, and 4.

Note. If there is no other path in G of the same pattern as P_1, we consider P_1 again, in the role of P_2. In this case Q_1 is the cycle around F_1 and Q_2 the cycle around F_2.

Below we show that the fundamental bricks are enforced (unavoidable) configurations with respect to the face F_1, its clockwise orientation (given by subscripts of vertices of F_1), its distance l from the face F_2, the pair (k, d), and that $2 \leq a = b \leq d + 1$. The fundamental brick $B = B_l(k, d)$ can be found uniquely when starting with the path P_1, checking (drawing) only vertices of degree k and faces of size d, continuing along the path Q_1 in the clockwise order, taking into consideration the pattern of P_1, namely the degrees $\deg_B(x_1)$, $\deg_B(y_1)$, and $\deg_G(v_i), i \in \{1, \ldots, l-1\}$. The procedure finishes when the first path P_2 of the same pattern as the path P_1 has, is found out.

A fundamental brick B is present in any 3-connected 2-nearly Platonic graph G. It is possible to delete it from the graph G and obtain a smaller 3-connected 2-nearly Platonic graph G' of the same type or a Platonic graph of the same type. The *deletion* of B means removing from G all vertices and edges of B except those from the paths P_1 and P_2 and next identifying the corresponding vertices and edges of the paths P_1 and P_2. The resulting new graph G' has a new path P_1^* of the same pattern as P_1.

The repeated use of the deletion of the fundamental brick B from the graph G leads to a Platonic graph of the same type as G has (in this case $n = m$) or to a contradiction with Theorem 1.

The fundamental brick can be used also conversely, for constructions of all 2-nearly Platonic graphs of type (k, d).

Next we show how the fundamental bricks look depending on the type of the graph G. We distinguish several cases.

Case 1. Type $(3, 3)$. As no 3-connected 3-regular planar graph (except the tetrahedron) contains two adjacent triangular faces (there would be a 2-vertex cut), there is no 3-connected 2-nearly Platonic graph of type $(3, 3)$.

Case 2. Type $(3, 4)$. If a 3-connected 3-regular planar graph contains a vertex incident to three faces of degree four, then it is either the graph of the cube or it contains either at least one face of degree at least five or a 2-vertex cut. A contradiction. This implies that $l = 1$ and the fundamental brick $B_1(3, 4)$ consists of exactly one face of degree four. This shows that G is exactly an n-prism with $|V(F_1)| = |V(F_2)| = n$ for some $n \geq 4$.

Case 3. Type $(4, 3)$. As $\text{dist}_G(F_1, F_2) = \text{dist}(x_1, y_1) \geq 1$ (by Claim 1), each edge of any exceptional face is adjacent to a triangular face and (by Euler's formula, see e.g. [9] or [16]) $f_3 = m + n$. From this we immediately have $n = m, l = 1$ and that G is the n-antiprism for some $n \geq 4$. Hence $B_1(4, 3)$ consists of two adjacent triangular faces.

Case 4. Type $(3, 5)$.

Subcase 4.1. If $\text{dist}_G(F_1, F_2) = \text{dist}_G(x_1, y_1) = 1$, then the fundamental brick $B_1(3, 5)$ is unique and is enforced starting from the edge x_1y_1. It is bounded by the cycle $C = x_1x_2 \cdots x_6y_6y_5 \cdots y_1x_1$ and contains of 10 pentagonal faces. All its vertices, up to the vertices x_1, x_6, y_1, y_6 which are of degree 2, are of degree 3. Using this brick one can easily construct a 3-connected 2-nearly Platonic graph with $n = m = 5t$, $t \geq 2$, from the t-prism by replacing each its quadrangular face with the brick $B_1(3, 5)$.

Subcase 4.2. Let $\text{dist}_G(x_1, y_1) \geq 2$. Observe that there is no 3-connected 2-nearly Platonic graph with $\text{dist}_G(F_1, F_2) = 2$. Otherwise we have $\text{dist}_G(F_1, F_2) = 1$. Let $\text{dist}_G(F_1, F_2) \geq 3$. As $f_5 = m + n$ (by Euler's formula, see e.g. [9] or [13]) and the fact that the exceptional face F_i is incident to $\deg_G(F_i)$ faces of degree 5, we immediately have $n = m$ and $l = 3$. Clearly, $B_3(3, 5)$ consists of two adjacent pentagonal faces and $a = b = 2$. The graph G is a generalization of the dodecahedron for some $n \geq 3$, $n \neq 5$. It can be obtained from the n-prism D_n by replacing each edge x_iy_i of it by the path $x_iu_iz_iy_i$ and then by inserting the edge u_iz_{i+1} for any $i \in \{1, \ldots, n\}$, subscripts modulo n.

Case 5. Type $(5, 3)$.

Subcase 5.1. Let $\mathrm{dist}_G(F_1, F_2) = \mathrm{dist}_G(x_1, y_1) \geq 2$. Then $f_3 = 3m + 3n$ (by Euler's formula, see e.g. [9] or [14]). Because each exceptional face F_i is touching $3\deg_G(F_i)$ triangular faces we immediately have $l = 2, n = m$ and $B_2(5, 3)$ being enforced by a cycle $C = x_1x_2w_1y_2y_1v_1x_1$ having inside a vertex z adjacent to all vertices of C except for the vertex y_1. Also the edge v_1y_2 is present in $B_2(5, 3)$. Using the fundamental brick $B_2(5, 3)$ one can easily construct a 3-connected, balanced 2-nearly Platonic graph with exceptional n-gonal faces for every $n \geq 4$. It can be obtained from the n-prism D_n by replacing each edge x_iy_i with the paths $x_iu_iy_i$, followed by inserting a vertex z_i in the interior of the face bounded be the cycle $x_ix_{i+1}u_{i+1}y_{i+1}y_iu_ix_i$ and then adding the following edges: u_iy_{i+1}, z_iu_i, z_ix_i, z_ix_{i+1}, z_iu_{i+1}, z_iy_{i+1} for any $i \in \{1, \ldots, n\}$.

Subcase 5.2. Let $\mathrm{dist}_G(F_1, F_2) = \mathrm{dist}_G(x_1, y_1) = 1$. To look for a fundamental brick B we start with the edge x_1y_1 and with $\deg_B(x_1) = r$, $r \in \{2, 3, 4\}$ and $\deg_B(y_1) = s, s \in \{2, 3, 4\}$. Observe, that the cases when $r = s = 2$ and $r = s = 4$ cannot appear because otherwise a quadrangular face would appear in G. As $\deg_B(x_1) + \deg_B(x_a) = 6$, it is enough, without loss of generality, to consider only the cases $r \in \{3, 4\}$. In all possible subcases the fundamental brick B is bounded by a cycle $C = x_1x_2x_3x_4y_4y_3y_2y_1x_1$ having six vertices $z_1, z_2, z_3, z_4, z_5, z_6$ in the interior.

Subcase 5.2.1. Let $\deg_B(x_1) = 4$ and $\deg_B(y_1) = 3$. Then the fundamental brick (in this case B^1) contains the following edges in the interior of C: x_1z_1, x_1z_4, x_2z_1, x_2z_2, x_2z_3, x_3z_3, x_3y_3, x_3y_4, y_1z_4, y_2z_4, y_2z_5, y_2z_6, y_3z_6, y_3z_3, z_1z_2, z_1z_4, z_1z_5, z_2z_5, z_2z_3, z_2z_6, z_3z_6, z_4z_5, z_5z_6.

Subcase 5.2.2. Let $\deg_B(x_1) = 4$ and $\deg_B(y_1) = 2$. Then the fundamental brick B^2 contains the following edges in the interior of C: x_1z_1, x_1y_2, x_2z_1, x_2z_2, x_2z_3, x_3z_3, x_3z_6, x_3y_4, y_2z_1, y_2z_4, y_3z_4, y_3z_5, y_3z_6, y_4z_6, z_1z_2, z_1z_4, z_2z_3, z_2z_5, z_2z_4, z_4z_5, z_5z_6, z_3z_5, z_3z_6.

Subcase 5.2.3. Let $\deg_B(x_1) = 3$ and $\deg_B(y_1) = 3$. Then the fundamental brick B^3 contains the following edges in the interior of C: x_1z_1, x_2z_1, x_2z_2, x_2z_3, x_3z_3, x_3z_5, x_3z_6, x_4z_6, y_1z_1, y_2z_1, y_2z_2, y_2z_4, y_3z_4, y_3z_5, y_3z_6, y_4z_6, z_1z_2, z_2z_4, z_2z_3, z_3z_4, z_3z_5, z_4z_5, z_5z_6.

If we take a t-prism, $t \geq 2$, and replace each quadrangular face with the fundamental brick B^j, $j \in \{1, 2, 3\}$ we get three infinite classes $\mathcal{B}_j$ of 3-connected balanced 2-nearly Platonic graphs of type $(5, 3)$.

It is easy to see that $\mathcal{B}_j \cap \mathcal{B}_3 = \emptyset$ for $j \in \{1, 2\}$.

We need to show that $\mathcal{B}_1 \cap \mathcal{B}_2 = \emptyset$. To do this consider a plane (embedding of) graph $G \in \mathcal{B}_1$ (respectively, $H \in \mathcal{B}_2$).

From the above considerations we know that any fundamental brick B of every 2-nearly Platonic graph from case 5.2 is uniquely determined by the exceptional face F_1, its (clockwise) orientation, and the ordered pair $(\deg_B(x_1), \deg_B(y_1))$.

On the other hand, different fundamental bricks can give the same 2-nearly Platonic graph. One can easily check that the 2-nearly Platonic graph G, determined by the fundamental brick B^1, defined by the ordered pair $(4, 3)$ with respect to the face F_1 and its orientation, is also determined by the fundamental bricks B_1^1 and B_2^1 defined by the ordered pairs $(3, 2)$ and $(2, 4)$, respectively, but not any other fundamental brick.

Analogously, the 2-nearly Platonic graph H, determined by the fundamental brick B^2, defined by the ordered pair (4, 2) with respect to the face F_1 and its orientation, is also determined by the fundamental bricks B_1^2 and B_2^2 defined by the ordered pairs $(2, 3)$ and $(3, 4)$, respectively, but no any other fundamental brick.

Of course, all the above mentioned fundamental bricks are considered with the respect to the face F_1 and its (clockwise) orientation.

Consider an axis of symmetry not intersecting G (respectively, H) in the plane. Let $\bar{G}$ (respectively, $\bar{H}$) be the image of G (respectively, of H) according to this axial symmetry. Evidently, the graphs G and $\bar{G}$ (respectively, the graphs H and $\bar{H}$) are combinatorially isomorphic but, by the axial symmetry, the clockwise orientation of F_1 is changed to the counterclockwise orientation of the corresponding face $\bar{F}_1$ in $\bar{G}$. This means that the plane graphs G and $\bar{G}$ are distinct (respectively, plane graphs H and $\bar{H}$ are distinct) and, hence, not topologically isomorphic as plane graphs.

If we consider the clockwise orientation of the face $\bar{F}_1$ we find out that the axial symmetric graph $\bar{G}$ (respectively, $\bar{H}$) is determined by the fundamental bricks defined by any of the ordered pairs $(3, 4), (2, 3)$, and $(4, 2)$ and by no other one (respectively, by any of the ordered pairs $(4, 3), (3, 2)$, and $(2, 4)$ and no other one). This means that $\bar{H} \in \mathcal{B}_1$ and $\bar{G} \in \mathcal{B}_2$.

As the sets of fundamental bricks that determine the 2-nearly Platonic graphs $G \in \mathcal{B}_1$ and $H \in \mathcal{B}_2$ are disjoint, the classes $\mathcal{B}_1$ and $\mathcal{B}_2$ are disjoint as well.

From the above considerations we have the following.

Theorem 6. *Any* 3*-connected* 2*-nearly Platonic graph is balanced. There are eight infinite classes of* 3*-connected* 2*-nearly Platonic graphs. Namely, one of type* $(3, 4)$*, one of type* $(4, 3)$*, two of type* $(3, 5)$*, and four of type* $(5, 3)$*.*

Remark 1. A 3-dimensional polyhedron, which has no plane symmetry, is called *chiral*, see e.g., [1], p. 301, or [20]. Regular polyhedra corresponding to 2-nearly Platonic graphs from the classes $\mathcal{B}_1$ and $\mathcal{B}_2$ are chiral. Moreover, to any convex polyhedron $G \in \mathcal{B}_1$ there is a convex polyhedron $G' \in \mathcal{B}_2$ mirror symmetric to G and vice versa.

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