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ADJACENT VERTEX DISTINGUISHING TOTAL COLORING OF THE CORONA PRODUCT OF GRAPHS

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Abstract

An adjacent vertex distinguishing (AVD-)total coloring of a simple graph G is a proper total coloring of G such that for any pair of adjacent vertices u and v, we have $C(u) \neq C(v)$, where $C(u)$ is the set of colors given to vertex u and the edges incident to u for $u \in V(G)$. The AVD-total chromatic number, $\chi''_a(G)$, of a graph G is the minimum number of colors required for an AVD-total coloring of G. The AVD-total coloring conjecture states that for any graph G with maximum degree Δ, $\chi''_a(G) \leq \Delta+3$. The total coloring conjecture states that for any graph G with maximum degree Δ, $\chi''(G) \leq \Delta+2$, where $\chi''(G)$ is the total chromatic number of G, that is, the minimum number of colors needed for a proper total coloring of G. A graph G is said to be AVD-*total colorable* (total colorable), if G satisfies the AVD-total coloring conjecture (total coloring conjecture). In this paper, we prove that for any AVD-total colorable graph G and any total-colorable graph H with $\Delta(H) \leq \Delta(G)$, the corona product $G \circ H$ of G and H satisfies the AVD-total coloring conjecture. We also prove that the graph $G \circ K_n$ admits an AVD-total coloring using $(\Delta(G \circ K_n) + p)$ colors, if there is an AVD-total coloring of graph G using $(\Delta(G) + p)$ colors, where $p \in \{1, 2, 3\}$. Furthermore, given a total colorable graph G and positive integer r and p where $1 \leq p \leq 3$, we classify the corona graphs $G^{(r)} = G \circ G \circ \cdots \circ G$ ($r + 1$ times) such that $\chi''_a(G^{(r)}) = \Delta(G^{(r)}) + p$.

Keywords: adjacent vertex distinguishing total coloring, corona products.

2020 Mathematics Subject Classification: 05C15, 05C76.

1. Introduction

All graphs considered in this paper are finite, simple and undirected. We use the standard notation and terminology that can be found in the book of graph theory [18]. The *corona product* of two graphs G and H, denoted by $G \circ H$, is the graph obtained by taking one copy of G, called the *center graph* and $|V(G)|$ copies of H, called the *outer graph*, and making the ith vertex of G adjacent to every vertex of the ith copy of H, where $1 \leq i \leq |V(G)|$. The maximum degree of corona graph is denoted by $\Delta(G \circ H)$ and equals $\Delta(G \circ H) = |V(H)| + \Delta(G)$. An example is given in Figure 1 where the graph G is shown with black color, copies of graph H with blue color and the dotted edges are the newly added edges.

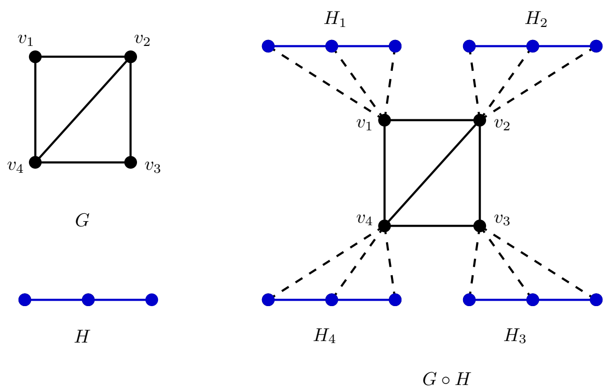

Figure 1. An example of corona product $G \circ H$.

A proper vertex (or edge) coloring of a graph $G = (V, E)$ is an assignment of colors to the vertices V (or edges E) of G such that no two adjacent vertices (or edges) get the same color. A proper *total coloring* of a graph G is an assignment of colors given to the vertices and the edges of the graph such that (i) two adjacent vertices receive different colors, (ii) two adjacent edges receive different colors and (iii) if a vertex x is incident to edge e, then x and e receive different colors. Through out the paper, whenever we say total coloring, we mean a proper total coloring, unless otherwise stated. The minimum integer k such that G has a total coloring using k colors is the *total chromatic number* of G and is denoted by $\chi''(G)$. Behzad [1] and Vizing [14] independently posed the *total coloring conjecture* which states that for any graph G, $\chi''(G) \leq \Delta + 2$ where Δ is the maximum degree of G. A graph G is said to be a *total colorable* graph if G satisfies the total coloring conjecture.

Consider a total coloring f of a graph $G = (V, E)$ and let $C(u)$ be the set of colors given to vertex u and the edges incident to u where $u \in V(G)$. A pair of adjacent vertices u and v satisfies the *adjacent vertex distinguishing property*

(AVD-*property*), if $C(u) \neq C(v)$. An *adjacent vertex distinguishing* (AVD-)*total coloring* of a graph G is a total coloring such that each pair of adjacent vertices u and v satisfies the AVD-property. We call a color c "available" for vertex v with respect to a total coloring, if $c \notin C(v)$. We denote the set of available colors on vertex v by $\bar{C}(v)$. Observe that if the AVD-property holds for an adjacent pair of vertices u and v, then $\bar{C}(u) \neq \bar{C}(v)$. The minimum integer k, such that G has an AVD-total coloring using k colors, is the *adjacent vertex distinguishing* (AVD-)*total chromatic number* of G and is denoted by $\chi''_a(G)$. This concept was introduced in 2005 by Zhang *et al.* [20]. They posed the AVD-*total coloring conjecture* which states that for any graph G, $\chi''_a(G) \leq \Delta(G)+3$. This conjecture has been proved for general graphs with $\Delta = 3$ by Wang [15] and Chen [2]. AVD-total colorings have been further studied for graphs with $\Delta = 3$ in [6, 9]. Papaioannou and Raftopoulou [12] proved the AVD-total coloring conjecture for 4-regular graphs. Recently, Lu *et al.* [8] validated the conjecture for all graphs with $\Delta = 4$. For planar graphs with $\Delta \geq 9$, the AVD-total coloring conjecture has been shown to be true [3–5, 17], while $\chi''_a(G) \leq \Delta + 2$ holds for planar graph G with $\Delta \geq 11$ [16, 19]. Wang and Huang [16] proved that if G is a planar graph with $\Delta \geq 14$, then $\chi''_a(G) = \Delta + 2$ if and only if G contains two adjacent vertices of maximum degree. Recently, Huo [7] extended this characterization to planar graphs with $\Delta = 13$. A graph G is said to be AVD-*total colorable* graph if G satisfies the AVD-total coloring conjecture.

For the total chromatic number of a graph, it is known that $\chi''(G) \geq \Delta + 1$. We say that a graph G is a *type-I-graph* if $\chi''(G) = \Delta + 1$. Mohan *et al.* [11] proved that the corona product of graphs of certain graph classes is a type-I graph. Later, Vignesh *et al.* [13] proved that for any two total colorable graphs G and H, the corona product $G \circ H$ is a type-I graph.

In this paper, we study the AVD-total coloring of the corona product of graphs. We prove the AVD-total coloring conjecture for corona product of an AVD-total colorable graph G and a total colorable graph H when $\Delta(G) \geq \Delta(H)$. We also prove the AVD-total coloring conjecture for graph $G \circ H$, when G is an AVD-total colorable graph and H is the complete graph K_n. Furthermore, given a total colorable graph G and a positive integer r, we classify the corona graphs $G^{(r)} = G \circ G \circ \cdots \circ G$ ($r + 1$ times) with respect to their AVD-total chromatic numbers. In particular, we characterize the corona graphs $G^{(r)}$ such that $\chi''_a(G^{(r)}) = \Delta(G^{(r)}) + p$, where p is a positive integer and $1 \leq p \leq 3$.

2. AVD-Total Coloring of the Corona Product of Graphs

In this section, we present the results on the corona product of two graphs $G \circ H$. Throughout the paper, we assume that $|V(G)| = m$, $|V(H)| = n$ and $V(G) =$

$\{v_1, v_2, \ldots, v_m\}$. Let $H_1, H_2, \ldots, H_m$ be the m copies of the graph H in the graph $G \circ H$ and $V(H_i) = \{x_{i1}, x_{i2}, \ldots, x_{in}\}$, for each i, $1 \leq i \leq m$. In [13], the total coloring of the corona product of two total colorable graphs G and H is examined. Therefore we focus on G and H being AVD-total colorable graphs. First, we consider the case, when $\Delta(G) \geq \Delta(H)$ and we prove that the AVD-total coloring conjecture holds for the corona product of two graphs G and H, where G and H are two AVD-total colorable graphs with $\Delta(G) \geq \Delta(H)$. Our result holds even if the graph H is a total colorable graph.

Note that the maximum degree of $G \circ H$, that is $\Delta(G \circ H) = |V(H)| + \Delta(G)$, depends on the maximum degree $\Delta(G)$ of G. Therefore, in the case where $\Delta(G) \geq \Delta(H)$, we start with an AVD-total coloring of the center graph G using $\Delta(G) + 3$ colors. Next, we totally color each subgraph H_i in the outer graph using the same set of colors, which is possible since $\Delta(G) \geq \Delta(H)$. Finally, we color the edges connecting the center graph to the outer graph using $|V(H)|$ new colors such that the obtained coloring is an AVD-total coloring using $\Delta(G \circ H) + 3$ colors.

On the other hand, for the case where $\Delta(H) > \Delta(G)$, we cannot use a similar approach. We cannot totally color the copies of H_i with the same set of $\Delta(G) + 3$ colors as used to color the center graph G, as $\Delta(H) > \Delta(G)$ holds.

Theorem 1. *Let G be an* AVD*-total colorable graph and H be an* AVD*-total or total colorable graph. If $\Delta(G) \geq \Delta(H)$, then $\chi''_a(G \circ H) \leq \Delta(G \circ H) + 3$.*

Proof. If G_c is a connected component of G, then $G_c \circ H$ is a maximal connected component of $G \circ H$, and the AVD-total colorings of different connected components do not depend on each other. Therefore, we assume without loss of generality that G is a connected graph. Since G is an AVD-total colorable graph, there exists an AVD-total coloring of G using $(\Delta(G) + 3)$ colors. Let $\mathcal{C} = \{1, 2, \ldots, \Delta(G) + 3\}$ be a set of $\Delta(G) + 3$ colors. Let f be an AVD-total coloring of G using $(\Delta(G) + 3)$ colors from the set $\mathcal{C}$. We assume that H is a total colorable graph; the case where H is AVD-total colorable is similar. Suppose that $|V(H)| = n \leq 2$. Note that in this case, H has at most one edge. We color the center graph G with respect to the coloring f and color the vertices and edge (if any) of each copy H_i with different colors from the set $\mathcal{C} \setminus \{f(v_i)\}$. Observe that it is always possible to color H_i this way as graph G is a connected graph and $\Delta(G) \geq \Delta(H)$. We color the n edges between vertex v_i and copy H_i with n new colors. Thus, we totally color the graph $G \circ H$ using $\Delta(G \circ H) + 3$, that is, $\Delta(G) + n + 3$ colors. It is easy to observe that the obtained coloring is an AVD-total coloring of $G \circ H$ as the coloring f is an AVD-total coloring of G.

Therefore, we assume that $n \geq 3$. Since H is a total colorable graph, there exists a total coloring of H using $(\Delta(H) + 2)$ colors. So, we take a total coloring g of each copy of H_i using $(\Delta(H) + 2)$ colors from the set $\mathcal{C}$, where $1 \leq i \leq m$. Note that since $\Delta(H) \leq \Delta(G)$, g requires at most $\Delta(H) + 2 \leq \Delta(G) + 3$ colors, while $\mathcal{C}$ has $\Delta(G) + 3$ colors in total. We use these colorings to construct an AVD-total coloring of $G \circ H$ using $(\Delta(G \circ H) + 3)$, that is, $(\Delta(G) + n + 3)$ colors[1]. First, we totally color the center graph G and color the edges of the subgraphs H_i according to the colorings f and g, respectively. Next, we color the vertices of each H_i and the uncolored edges connecting the subgraphs H_i to the center graph G, using n new colors. Let $\mathcal{C}' = \{c_1, c_2, \ldots, c_n\}$ be a set of n new colors. Formally, we define the AVD-total coloring $\phi\colon V(G \circ H) \cup E(G \circ H) \to \mathcal{C} \cup \mathcal{C}'$ as follows. For $1 \le i \le m$,

1. for each vertex $v_i \in V(G)$, $\phi(v_i) = f(v_i)$;
2. for each edge $v_iv_j \in E(G)$, $\phi(v_iv_j) = f(v_iv_j)$;
3. for each edge $x_{ij}x_{ik} \in E(H_i)$, $\phi(x_{ij}x_{ik}) = g(x_jx_k)$, where $x_jx_k \in E(H)$, $1 \le j, k \le n$;
4. for each vertex $x_{ij} \in V(H_i)$, $\phi(x_{ij}) = c_{j+1}$ for $1 \le j \le n-1$ and $\phi(x_{in}) = c_1$;
5. for each edge $\phi(v_ix_{ij}) = c_j$, for $1 \le j \le n$.

Note that the obtained coloring ϕ is a total coloring when restricted to any subgraph H_i or subgraph G. The coloring assigns n distinct colors to the n edges between a subgraph H_i and G, where $1 \le i \le m$. Therefore, the obtained coloring ϕ is a total coloring using $(\Delta(G) + n + 3)$ colors. Suppose that $u, v \in V(G \circ H)$ is a pair of adjacent vertices. Now, there are three cases to consider depending on whether u and v belong to G or H_i. (i) If $u, v \in V(G)$, then the AVD-property holds for the vertices u and v as the coloring ϕ restricted to G is an AVD-total coloring, and therefore $C(u) \ne C(v)$. (ii) If $u \in V(G)$ and $v \in V(H_i)$ for some i, $1 \le i \le m$, then $C(u)$ contains all colors of $\mathcal{C}'$, whereas $C(v)$ contains only two colors of $\mathcal{C}'$. This implies that the AVD-property holds for the vertices u and v as $n \ge 3$. (iii) If $u, v \in V(H_i)$, then $C(u)$ and $C(v)$ both contain exactly two colors from $\mathcal{C}'$. However, a different pair of colors from $\mathcal{C}'$ is used for each vertex of H_i. Thus, in every case the AVD-property holds for the vertices u and v. Hence, the obtained coloring is an AVD-total coloring of $G \circ H$ and $\chi''_a(G \circ H) \le \Delta(G) + n + 3 = \Delta(G \circ H) + 3$ as claimed. ■

In Figure 2, we illustrate the AVD-total coloring ϕ of $G \circ H$ produced by Theorem 1. Note that graphs G and H satisfy the condition $\Delta(G) \ge \Delta(H)$ of the theorem.

Next, we aim to relax the condition $\Delta(G) \ge \Delta(H)$ and investigate the AVD-total chromatic number of the corona product $G \circ H$, where G is an AVD-total colorable graph and H is any total colorable graph. The following result is a first attempt towards this generalization. We consider G to be an AVD-total

[1]In fact, g requires at most $\Delta(H) + 2$ colors if H is total colorable, and at most $\Delta(H) + 3$ colors if it is AVD-total colorable. In both cases the number of needed colors is at most $\Delta(G) + 3$.

colorable graph and H to be the complete graph K_n on n vertices. Our proof uses the following known result.

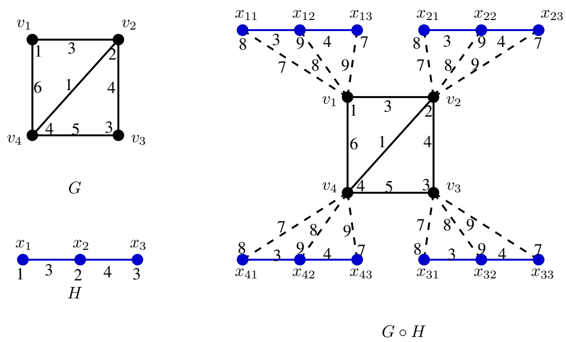

Figure 2. Illustration for the AVD-total coloring of $G \circ H$ produced by Theorem 1, where $\Delta(G) > \Delta(H)$.

Theorem 2 [20]**.** *For a complete graph K_n with n vertices,*

$$\chi''_a(K_n) = \begin{cases} n+1, & \text{if } n \text{ is even,} \\ n+2, & \text{if } n \text{ is odd.} \end{cases}$$

Theorem 3. *Let G be a connected graph with $m > 3$ vertices and K_n be a complete graph of order n. If there exists an* AVD*-total coloring of graph G using $\Delta(G)+p$ colors, then the graph $G \circ K_n$ has an* AVD*-total coloring using $\Delta(G \circ K_n) + p$ colors, where $p \in \{1, 2, 3\}$.*

Proof. As in the proof of Theorem 1, let G be a connected graph with $V(G) = \{v_1, v_2, \ldots, v_m\}$, where $m \geq 4$. Assume that there exists an AVD-total coloring of graph G using $\Delta(G) + p$ colors, where $p \in \{1, 2, 3\}$. The corona product $G \circ K_n$ has maximum degree $\Delta(G \circ K_n) = \Delta(G) + n$. We produce an AVD-total coloring of $G \circ K_n$ using $(\Delta(G) + n + p)$ colors as follows. The center graph is G and take m copies of K_n in outer graph, say K_n^i, where $1 \leq i \leq m$. Let $V(K_n^i) = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$, where $1 \leq i \leq m$. We start with an AVD-total coloring of the complete graph $(K_n^i + v_i)$ on $(n+1)$ vertices for each i, $1 \leq i \leq m$ and we remove the color of each vertex v_i, $1 \leq i \leq m$. Finally, we take an AVD-total coloring of the center graph G and make local modification to assure that the AVD-property holds for all pairs of adjacent vertices. There are two cases to consider depending on whether n is odd or even.

Case 1. n is odd. Let $\mathcal{C} = \{1, 2, \ldots, n+2\}$ be a set of $n+2$ colors. From Theorem 2, $\chi''_a(K_n^i + v_i) = n + 2$ as $n + 1$ is even, for $1 \leq i \leq m$. We take an AVD-total coloring ϕ of each complete subgraph $(K_n^i + v_i)$, $1 \leq i \leq m$ (which assigns the same color $\phi(v_i)$ and the same color set $C(v_i)$ to vertex v_i for every i, $1 \leq i \leq m$) using colors from the set $\mathcal{C}$ and we remove the color of every vertex v_i, $1 \leq i \leq m$. Now, observe that each vertex v_i of G has two available colors from the set $\mathcal{C}$, say $c_1, c_2 \in \mathcal{C} \setminus (C(v_i) \setminus \{\phi(v_i)\})$. Note that c_1 and c_2 are the same for all vertices v_i of G. These two colors can be used to color the uncolored edges and vertices of the center graph G. Let $\mathcal{C}' = \{n+3, n+4, \ldots, n+\Delta(G)+p\}$ be a set of $\Delta(G)+p-2$ new colors. We know that there exists an AVD-total coloring of graph G using $\Delta(G)+p$ colors. We take an AVD-total coloring f of the center graph with colors from $\{c_1, c_2\} \cup \mathcal{C}'$. Consider vertex v_i of G, for $1 \leq i \leq m$. Suppose that in the obtained coloring $f(v_i) \notin \{c_1, c_2\}$. In this case, the AVD-property holds for each pair of vertices consisting of v_i and any one of its neighbors. Suppose that in the obtained coloring, we have $f(v_i) \in \{c_1, c_2\}$. If for any vertex v_{ij} for $1 \leq j \leq n$ of the subgraph K_n^i we have $f(v_i) \neq \phi(v_{ij})$, then the AVD-property holds for v_i and v_{ij}. So it remains to consider the case when there exists a vertex $f(v_i) = \phi(v_{ij})$, that is vertices v_i and v_{ij} have the same color. In this case, we recolor vertex v_{ij} using any color from the set $\mathcal{C}'$. Since G is a connected graph with at least 4 vertices, $\mathcal{C}' \neq \emptyset$. Thus, we can always recolor vertex v_{ij} with some color from the set $\mathcal{C}'$. It follows that the obtained coloring of $G \circ K_n$ is an AVD-total coloring using $\Delta(G)+n+p$ colors.

Case 2. n is even. Let $\mathcal{C} = \{1, 2, \ldots, n+3\}$ be a set of $n+3$ colors. From Theorem 2, $\chi''_a(K_n^i + v_i) = n+3$ as $n+1$ is odd, for $1 \leq i \leq m$. We take the AVD-total coloring ϕ of each complete subgraph $(K_n^i + v_i)$, $1 \leq i \leq m$ (which assigns the same color $\phi(v_i)$ and the same color set $C(v_i)$ to vertex v_i for every i, $1 \leq i \leq m$) using colors from the set $\mathcal{C}$ and we remove the color of each vertex v_i, $1 \leq i \leq m$. Now, observe that each vertex v_i of G has three available colors from the set $\mathcal{C}$, say $c_1, c_2, c_3 \in \mathcal{C} \setminus (C(v_i) \setminus \{\phi(v_i)\})$. Note that c_1, c_2 and c_3 are the same for all vertices v_i of G. These three colors can be used to color the uncolored edges and vertices of the center graph G. Let $\mathcal{C}' = \{n+3, n+4, \ldots, n+\Delta(G)+p\}$ be a set of $\Delta(G)+p-3$ new colors. We know that there exists an AVD-total coloring of graph G using $\Delta(G)+p$ colors. We take an AVD-total coloring f of the center graph with colors from $\{c_1, c_2, c_3\} \cup \mathcal{C}'$. As in Case 1, the AVD-property holds for vertex v_i and any of its neighbors, except for the case where $f(v_i) \in \{c_1, c_2, c_3\}$ and vertex v_i of G has the same color with vertex v_{ij} of K_n^i, that is $f(v_i) = \phi(v_{ij})$, where $1 \leq i \leq m$, $1 \leq j \leq n$. In this case, we recolor vertex v_{ij} with some color from the set $\mathcal{C}'$. Since G is a connected graph with at least 4 vertices, $\mathcal{C}' \neq \emptyset$ and so we can always recolor vertex v_{ij} with some color from the set $\mathcal{C}'$. Thus the obtained coloring is a proper total coloring of graph $G \circ K_n$.

Hence, the graph $G \circ K_n$ has an AVD-total coloring using $(\Delta(G \circ K_n) + p)$ colors. ■

In Figure 3 we illustrate the AVD-total coloring of $K_2 \circ K_3$ produced by Theorem 3 (Case 1).

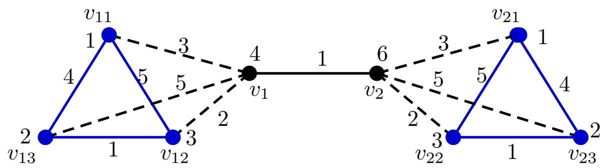

Figure 3. Illustration for the AVD-total coloring of $K_2 \circ K_3$ produced by Theorem 3.

Next, our interest is to compute the AVD-total chromatic number of graph $G \circ H$, when $H = G$ and G is a total colorable graph. First, we characterize the graphs $G \circ G$ such that $\chi''_a(G \circ G) = \Delta(G \circ G) + p$, where G is a total colorable graph and p is a positive integer $p \in \{1, 2\}$. Furthermore, we extend the characterization to the generalized corona graph $(G \circ G \circ \cdots \circ G(r+1 \text{ times}))$, where G is a total colorable graph and r is a positive integer. This generalized corona product of graphs, $(G \circ G \circ \cdots \circ G(r+1 \text{ times}))$ is called the *corona graph* of graph G and it is denoted by $G^{(r)}$. If G is a connected graph, then $G^{(0)} = G$, $G^{(1)} = G \circ G$, and $G^{(r)} = G^{(r-1)} \circ G$. Note that $\Delta(G^{(r)}) = \Delta(G) + nr$, where n is the order of graph G. For $G = K_3$ the corona graph $K_3^{(2)}$ is shown in Figure 4. First, we study the AVD-total chromatic number of corona graph $G^{(1)}$. Let the center graph of $G^{(1)}$ be G and let $G_1, \ldots, G_n$ be the n copies of G in the outer graph, where $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $V(G_i) = \{v_{i1}, v_{i2}, \ldots, v_{in}\}$, for $1 \leq i \leq n$. Observe that if $n = 1$, then $G^{(1)} \equiv K_2$ and hence, $\chi''_a(G^{(1)}) = \Delta(G^{(1)}) + 2$ by Theorem 2.

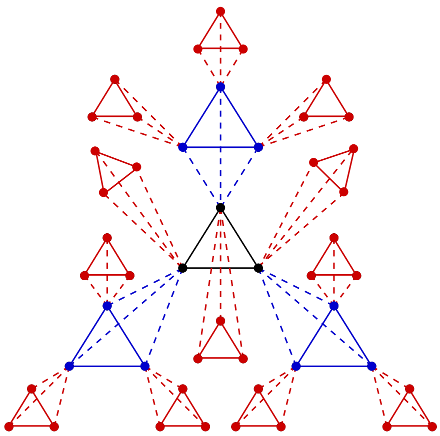

Figure 4. Corona graph $K_3^{(2)}$.

Theorem 4. *If G is a total-colorable graph with at least two vertices, then $\chi''_a(G^{(1)}) \leq \Delta(G^{(1)}) + 2$.*

Proof. Assume that $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $\mathcal{C} = \{1, 2, \ldots, \Delta(G)+2\}$ is a set of $\Delta(G)+2$ colors. Since G is a total colorable graph, there exists a total coloring f of G using the $\Delta(G) + 2$ colors of $\mathcal{C}$. We start by applying the total coloring f on the center graph G and on each copy G_i, $1 \leq i \leq n$. Let $\mathcal{C}' = \{c_1, c_2, \ldots, c_n\}$ be a set of n new colors. In order to produce an AVD-total coloring of $G^{(1)}$, we use $\mathcal{C}'$ to color the edges between G and each G_i and, finally, we modify the coloring in order to assure that the AVD-property holds for each pair of adjacent vertices. Now, each vertex v_i has at least one available color with respect to the coloring f from the set $\mathcal{C}$. Let $a_i \in \mathcal{C}$ be an available color of vertex v_i for each i, $1 \leq i \leq n$. Note that $a_i = a_j$ might hold for $1 \leq i \neq j \leq n$. Define the AVD-total coloring $\phi\colon V(G^{(1)}) \cup E(G^{(1)}) \to \mathcal{C} \cup \mathcal{C}'$ as follows.

1. For $1 \leq i, j \leq n$, $\phi(v_{ij}) = f(v_j)$.
2. For $1 \leq i, j, k \leq n$, $\phi(v_{ij}v_{ik}) = f(v_jv_k)$ where $v_jv_k \in E(G)$.
3. For $1 \leq i, j \leq n$, $\phi(v_iv_j) = f(v_iv_j)$ and $\phi(v_i) = c_i$.
4. For $1 \leq i, j \leq n$,
$$\phi(v_iv_{ij}) = \begin{cases} a_i, & \text{if } j = i, \\ c_j, & \text{otherwise.} \end{cases}$$

Note that the obtained coloring ϕ uses $(n-1)$ distinct colors from the set $\mathcal{C}'$ to color $(n-1)$ edges between any subgraph G_i and G, for $1 \leq i \leq n$. Hence, it is not difficult to verify that ϕ is a total coloring of $G^{(1)}$. Note that in $G^{(1)}$ the degree of vertex v_i is greater than the degree of vertex v_{ij} for any i, j. Hence, the AVD-property holds for the end vertices of each edge v_iv_{ij} going from G to G_i, $1 \leq i \leq n$. Next, observe that the AVD-property holds for any adjacent pair of vertices within G_i, as each vertex v_{ij} except for vertex v_{ii}, has exactly one color in their color set from the set $\mathcal{C}'$, which are mutually different. It remains to check the AVD-property for any pair of adjacent vertices v_i and v_k in the center graph G such that v_i and v_k have the same degree. Note that the color set of any vertex v_i is $C(v_i) = \mathcal{C}' \cup C_f(v_i) \setminus \{f(v_i)\} \cup \{a_i\}$, where $C_f(v_i)$ denotes the color set of vertex v_i with respect to the initial coloring f.

If for any adjacent pair the AVD-property is violated, then we will recolor some edge so that the AVD-property is satisfied on that pair of vertices. We will use Algorithm 1 to complete this step.

Claim. *The obtained coloring ϕ from the* Algorithm 1 *is an* AVD*-total coloring.*

Proof. First we show that the coloring obtained by Algorithm 1 is a total coloring. Let v_k is a neighbor of vertex v_i with least index such that $\bar{C}(v_k) = \bar{C}(v_i)$ and Algorithm 1 picks a color $c' \in \bar{C}(v_k)$. Since $\bar{C}(v_k) = \bar{C}(v_i)$, color c' is also

Algorithm 1 Recoloring Algorithm

function *Recoloring*(G, ϕ)
for $i \leftarrow 1$ **to** $n-1$ **do**
  for $k \leftarrow i+1$ **to** n **do**
    if $v_k \in N(v_i)$ *and* $\bar{C}(v_k) = \bar{C}(v_i)$ **then**
      Pick a color $c' \in \bar{C}(v_k)$;
      Recolor $\phi(v_{ik}) = c_k$;
      Recolor $\phi(v_i v_{ik}) = c'$;
      $C(v_i) = \mathcal{C}' \setminus \{c_k\} \cup C_f(v_i) \setminus \{f(v_i)\} \cup \{a_i, c'\}$;
      break;

available on vertex v_i. It implies that $c' \in \mathcal{C}$ as $\bar{C}(v_i) \subset \mathcal{C}$. Since we recolor vertex v_{ik} with color c_k, after recoloring $c' \in \bar{C}(v_{ik})$. It implies that the coloring obtained after the recoloring of edge $f(v_i v_{ik})$ with color c' remains proper total coloring of the graph $G^{(1)}$. Recall that in $G^{(1)}$ the degree of any vertex v_i is greater than the degree of any vertex v_{ij}, for any i, j, $1 \leq i, j \leq n$. Therefore, the AVD-property always holds for such pairs of adjacent vertices. Additionally, at any iteration i, recoloring takes place on one edge whose endpoints belong to the graph G_i and the center graph G. Therefore, we only need to prove the following claim.

Subclaim. *At the end of iteration i, the* AVD*-property holds on any pair of adjacent vertices within the subgraph G_i and any pair of adjacent vertices from the set $\{v_1, v_2, \ldots, v_i\}$.*

Proof. Assume that at the start of iteration i, the AVD-property holds for each adjacent pair of vertices from the set $\{v_1, v_2, \ldots, v_{i-1}\}$. If at the ith iteration, recoloring does not take place, then our claim is trivially true. Suppose that at the ith iteration, recoloring takes place. It implies that at the beginning of the ith iteration, there exists some vertex $v_k \in N(v_i)$ such that $\bar{C}(v_i) = \bar{C}(v_k)$, where k is the least such index and $k > i$. Since we have used color c' on edge $v_i v_{ik}$ and color c' is still available on vertex v_k, $C(v_i) \neq C(v_k)$. Note that even after Algorithm 1, $C(v_{ii}) \cap \mathcal{C}' = \emptyset$ and $C(v_{ik}) \cap \mathcal{C}' = c_k$. Therefore, $C(v_{ii}) \neq C(v_{ik})$. Observe that even after the recoloring, exactly one color belongs to the color set $C(v_{ij})$ from the set $\mathcal{C}'$ which is distinct for each value of j, for $i \neq j \neq k$. Therefore, $C(v_{ik}) \neq C(v_{ij})$ for any $1 \leq j \leq n$, $j \neq k$. Hence, the AVD-property holds on any pair of adjacent vertices within the subgraph G_i.

Observe that the recoloring step can cause violation of the AVD-property only on the pair of vertices v_i and its some neighbor from the set $\{v_1, v_2, \ldots, v_{i-1}\}$. Assume for a contradiction that at the end of iteration i, the AVD-property fails for the adjacent vertices v_i and v_j, where $j < i$, i.e., at the end of iteration i,

$C(v_i) = C(v_j)$. Since after recoloring v_i has a new available color c_k, color c_k is also available on vertex v_j. It implies that $v_jv_k \in E(G^{(1)})$ and at iteration j, recoloring took place on edge v_jv_{jk}. Since at the end of iteration i the color sets of the vertices v_i and v_j are the same and color c_k is available on both the vertices, $C(v_j) = C(v_i) = C(v_k)$ at the start of iteration j as well. This implies that Algorithm 1 must have chosen vertex $v_i \in N(v_j)$ for the recoloring step as $j < i < k$. It implies that color c_i is available on vertex v_j but not color c_k, which is a contradiction. Hence, at the end of iteration i the AVD-property holds on any adjacent pair of vertices from the set $\{v_i, v_2, \ldots, v_i\}$. It completes the proof of the subclaim.

Therefore, the coloring obtained from Algorithm 1 is an AVD-total coloring of the graph $G^{(1)}$ using $(\Delta + n + 2)$ colors. □

Thus, we obtained an AVD-total coloring ϕ of the graph $G^{(1)}$ using $\Delta(G^{(1)})+$ 2, that is, $\Delta(G) + n + 2$ colors. Hence, $\chi''_a(G^{(1)}) \leq \Delta(G^{(1)}) + 2$. ■

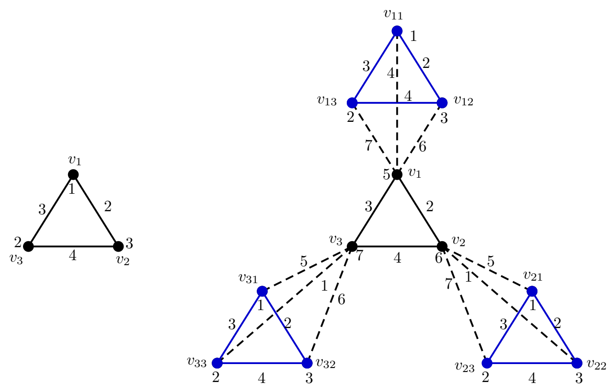

Figure 5. Illustration for the AVD-total coloring of $K_3^{(1)}$ produced by Theorem 4.

In Figure 5, we illustrate the AVD-total coloring of graph $K_3 \circ K_3$ produced by Theorem 4.

Theorem 5. *Let G be a total-colorable graph with at least two vertices. If G has no two adjacent vertices with degree $\Delta(G)$, then $\chi''_a(G^{(1)}) = \Delta(G^{(1)}) + 1$.*

Proof. Let $\mathcal{C} = \{1, 2, \ldots, \Delta(G) + 2\}$ be a set of $\Delta(G) + 2$ colors. Since G is a total colorable graph, there exists a total coloring f of G using the $\Delta(G) + 2$ colors of $\mathcal{C}$. Note that each vertex v_i in G must have at least one available color from the set $\mathcal{C}$ corresponding to the coloring f. Let $a_i \in \mathcal{C}$ be an available color of vertex v_i. Note that $a_i = a_j$ might hold for $1 \leq i \neq j \leq n$. Without loss of generality, assume that v_n is a vertex of maximum degree Δ, up to a relabeling of the vertices. We start by applying the total coloring f on the center graph G

and on each copy G_i, $1 \le i \le n$. Let $\mathcal{C}' = \{c_1, c_2, \ldots, c_{n-1}\}$ be a set of $n-1$ new colors. In order to produce an AVD-total coloring of $G^{(1)}$, we use $\mathcal{C}'$ to color the edges between G and each G_i and, finally, we modify the coloring in order to assure that the AVD-property holds for each pair of adjacent vertices. Define a partial AVD-total coloring assignment, $\phi : V(G^{(1)}) \cup E(G^{(1)}) \to \mathcal{C} \cup \mathcal{C}'$ as follows.

1. For $1 \le i, j, k \le n$, $\phi(v_{ij}) = f(v_j)$ and $\phi(v_{ij}v_{ik}) = f(v_jv_k)$, for $v_jv_k \in E(G)$.
2. For $1 \le i, j \le n$, $\phi(v_iv_j) = f(v_iv_j)$.
3. For $1 \le i \le n-1$, $\phi(v_i) = c_i$.
4. For $1 \le i \le n$,

$$\phi(v_iv_{ij}) = \begin{cases} a_i, & \text{if } j = i, \\ c_j, & \text{if } 1 \le j \le n-1 \quad \text{and} \quad j \ne i. \end{cases}$$

Now, v_n is the only uncolored vertex and the remaining uncolored edges are $\{v_iv_{in} | 1 \le i \le n-1\ \}$. We show how to color edge v_iv_{in} for $i < n$. Note that since vertex v_{in} is a Δ-degree vertex of G_i, there exists exactly one color from the set $\mathcal{C}$ in $\bar{C}(v_{in})$, in fact this color is a_n. There are three cases.

Case I. $f(v_i) \in \bar{C}(v_{in})$ i.e., $a_n = f(v_i)$. In this case, we assign $\phi(v_iv_{in}) = f(v_i)$.

Case II. $f(v_i)$ is used for some edge incident to v_{in}. Suppose that there exists an edge in subgraph G_i, say $v_{in}v_{i\ell}$ $(i \ne \ell)$, such that $\phi(v_{in}v_{i\ell}) = f(v_i)$. In this scenario, we recolor edge $v_{in}v_{i\ell}$ with color c_i and $\phi(v_iv_{in}) = f(v_i)$, where $1 \le i \le n-1$.

Case III. $f(v_i) = f(v_n)$. It implies that $\phi(v_{in}) = f(v_i)$. In this case, we recolor vertex v_{in} with color c_{i-1} (index $i-1$ is taken over modulo $(n-1)$) and assign $\phi(v_iv_{in}) = f(v_i)$.

In Figure 6, examples are given to depict the colorings for each case. Note that the obtained coloring does not violate the total coloring properties.

Finally, we color v_n, which is the only uncolored vertex. Let $f(v_n) = c$, where $c \in \mathcal{C}$. To color vertex v_n with color c, we have to make sure that in subgraph G_n there is no vertex colored with color c. We know that $\phi(v_{nn}) = c$. Since v_{nn} is a vertex of degree Δ and $\phi(v_{nn}) = c$, there exist at most $n - \Delta \le n-1$, vertices in subgraph G_n with initial color $\phi(v_{nj}) = c$. Hence, there exists at least one index k, such that v_{nk} has color $\phi(v_{nk}) = c_k$, such that $c_k \ne c$. We recolor all vertices v_{nj} of color c with color c_k and assign $\phi(v_n) = c$. Note that any two vertices of G_n have different subsets of colors from $\mathcal{C}'$ in their color sets.

It is easy to verify that the obtained coloring is a total coloring of $G^{(1)}$ using $(\Delta + n + 1)$ colors. First of all, note that in $G^{(1)}$ the degree of vertex v_i is greater than the degree of vertex v_{ij} for any i, j. Hence the AVD-property is satisfied

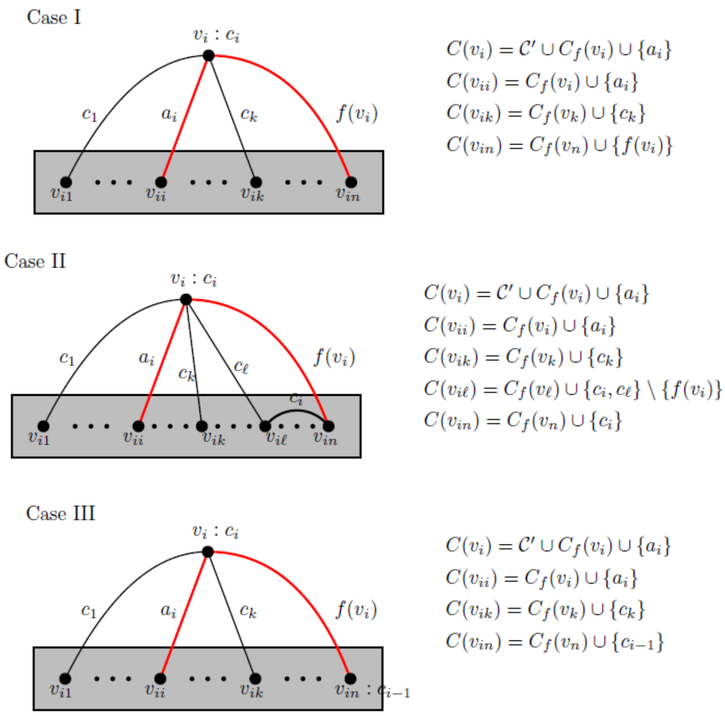

Figure 6. Case analysis for the initial coloring assignment in the proof of Theorem 5.

for such pairs of adjacent vertices v_i and v_{ij}, where $1 \leq i, j \leq n$. Next, since v_{in} is a vertex with degree Δ in G_i and it has no neighbor with degree Δ, the AVD-property holds on vertex v_{in} and any neighbor of v_{in} in G_i, for all $1 \leq i \leq n$. Note that the color set of vertex v_{ij} has at most two colors from the set $\mathcal{C}'$ which is different for each vertex v_{ij}, for $1 \leq j \leq n$, $j \neq i$ and the color set of vertex v_{ii} does not contain any color from the set $\mathcal{C}'$, for $1 \leq i \leq n-1$ (refer to Figure 6). It implies that the AVD-property holds for every pair of adjacent vertices in G_i, for $1 \leq i \leq n$. Observe that for any vertex v_i with maximum degree $\Delta(G^{(1)})$, the color set of v_i contains all the colors from the sets $\mathcal{C}$ and $\mathcal{C}'$. According to the hypothesis, G has no two adjacent vertices with the maximum degree, and so $G^{(1)}$ does not have any adjacent pair of vertices having the maximum degree. Therefore, any pair of adjacent vertices containing a vertex of maximum degree satisfies the AVD-property.

It remains to check the AVD-property for any pair of adjacent vertices v_i and v_k in the center graph G such that v_i and v_k have the same degree. If for any adjacent pair the AVD-property violates, then we will recolor some vertices so that the AVD-property is satisfied on that pair of vertices. We will use Algorithm 2 to complete this step. Since vertex v_n is a vertex of degree Δ in G, v_n is a maximum degree vertex in graph $G^{(1)}$. Since v_n has no neighbor of maximum degree, the AVD-property holds for vertex v_n and any of its neighbors. Therefore, in Algorithm 2, i runs from 1 to $n-2$. Note that the color set of any vertex v_i for i $(1 \leq i < n)$, is $C(v_i) = \mathcal{C}' \cup C_f(v_i) \cup \{a_i\}$, where $C_f(v_i)$ denotes the color of vertex v_i with respect to the initial coloring f.

Algorithm 2 Recoloring Algorithm

function *Recoloring*(G, ϕ)
for $i \leftarrow 1$ **to** $n-2$ **do**
  for $k \leftarrow i+1$ **to** $n-1$ **do**
    if $v_k \in N(v_i)$ *and* $\bar{C}(v_k) = \bar{C}(v_i)$ **then**
      Pick a color $c' \in \bar{C}(v_k)$;
      Recolor $\phi(v_i v_{ik}) = c'$;
      $C(v_i) = \mathcal{C}' \setminus \{c_k\} \cup C_f(v_i) \cup \{a_i, c'\}$;
      break;

Claim. *The obtained coloring ϕ from the* Algorithm 2 *is an* AVD-*total coloring.*

Proof. First we show that the coloring obtained by Algorithm 2 is a total coloring. Let v_k is a neighbor of vertex v_i with least index such that $\bar{C}(v_k) = \bar{C}(v_i)$ and Algorithm 2 picks a color $c' \in \bar{C}(v_k)$. It implies that $c' \in \bar{C}(v_{ik})$. Since $\bar{C}(v_k) = \bar{C}(v_i)$, color c' is also available on vertex v_i. It implies that $c' \in \mathcal{C}$ as $\bar{C}(v_i) \subset \mathcal{C}$. Therefore, the coloring obtained by the recoloring of edge $f(v_i v_{ik})$ with color c', remains a proper total coloring of the graph $G^{(1)}$. Recall that in $G^{(1)}$ the degree of any vertex v_i is greater than the degree of any vertex v_{ij}, for any i, j, $1 \leq i, j \leq n$. Therefore, the AVD-property always holds for such pairs of adjacent vertices. Additionally, at any iteration i, recoloring takes place on one edge whose endpoints belong to the graph G_i and the center graph G. Therefore, we only need to prove the following claim.

Subclaim. *At the end of iteration i, the* AVD-*property holds on any pair of adjacent vertices within the subgraph G_i and any pair of adjacent vertices from the set $\{v_1, v_2, \ldots, v_i\}$.*

Proof. Assume that at the start of iteration i, the AVD-property holds for each adjacent pair of vertices from the set $\{v_1, v_2, \ldots, v_{i-1}\}$. If at the ith iteration, recoloring does not take place, then our claim is trivially true. Suppose that at the ith iteration, recoloring takes place. It implies that at the beginning of the ith iteration, there exists some vertex $v_k \in N(v_i)$ such that $\bar{C}(v_i) = \bar{C}(v_k)$, where k is the least such index and $k > i$. Since we have used color c' on edge $v_i v_{ik}$ and color c' is still available on vertices v_{ii} and v_k, $C(v_{ii}) \neq C(v_{ik})$ and $C(v_i) \neq C(v_k)$. After the application of Algorithm 2, the color set $C(v_{ik})$ has one less color from the set $\mathcal{C}'$ as $i < n$. However, the color set $C(v_{ij})$ contain at least one color from the set $\mathcal{C}'$, where $j \neq k$, $j \neq i < n$. If prior to recoloring by Algorithm 2, the color set $C(v_{ik})$ contained only one color from the set $\mathcal{C}'$, then after recoloring the color set $C(v_{ik})$ has no color from the set $\mathcal{C}'$. Therefore, in this case $C(v_{ik}) \neq C(v_{ij})$ for any $1 \leq j \leq n-1$. Next assume that prior to recoloring by Algorithm 2, the color set $C(v_{ik})$ contained two colors from the set $\mathcal{C}'$, then vertex $C(v_{ik}) \cap \mathcal{C}' = \{c_i, c_k\}$ by the recoloring in Case II (see Figure 6). Therefore, after recoloring by Algorithm 2 the set $C(v_{ik})$ contains one color, that is, c_i from the set $\mathcal{C}'$ which is not present on any other vertex v_{ij} $(j \neq k)$ except for vertex v_{in}. Therefore, $C(v_{ik}) \neq C(v_{ij})$ for any $1 \leq j \leq n-1$, $j \neq k$ and $C(v_{ik}) \neq C(v_{in})$ because $d(v_{in}) = \Delta(G)+1$ whereas $d(v_{ik}) \neq \Delta(G)+1$. Hence, the AVD-property holds on any pair of adjacent vertices within the subgraph G_i.

For the second part of the subclaim, assume that the AVD-property is violated for the pair of vertices v_i and one of its neighbors from the set $\{v_1, v_2, \ldots, v_{i-1}\}$. Let at the end of iteration i, the AVD-property fails for the adjacent vertices v_i and v_j, where $j < i$, i.e., at the end of iteration i, $C(v_i) = C(v_j)$. Since after recoloring, vertex v_i has a new available color c_k, then color c_k is also available for vertex v_j. It implies that $v_j v_k \in E(G^{(1)})$ and at iteration j, the recoloring process took place on edge $v_j v_{jk}$. Since at the end of iteration i the color sets of the vertices v_i and v_j are the same and color c_k is available on both the vertices, $C(v_j) = C(v_i) = C(v_k)$ at the start of iteration j as well. This implies that Algorithm 2 would have choosen vertex $v_i \in N(v_j)$ for the recoloring step as $j < i < k$. In that case, color c_i would be available for vertex v_j but not color c_k, which is a contradiction. Hence, at the end of iteration i the AVD-property holds on any adjacent pair of vertices from the set $\{v_i, v_2, \ldots, v_i\}$. It completes the proof of the subclaim.

Thus, the coloring obtained from Algorithm 2 is an AVD-total coloring of the graph $G^{(1)}$ using $(\Delta + n + 1)$ colors. □

Hence, we obtained an AVD-total coloring ϕ of the graph $G^{(1)}$ using $\Delta(G) + n + 1$ colors. Therefore, $\chi''_a(G^{(1)}) = \Delta(G) + n + 1 = \Delta(G^{(1)}) + 1$. ■

We know that if any two maximum degree vertices are adjacent, then $\chi''_a(G) \geq \Delta(G) + 2$. Therefore, the next corollary follows from Theorem 4 and Theorem 5.

Corollary 6. *Let G be a total colorable graph with at least two vertices. Then $\chi''_a(G^{(1)}) = \Delta(G^{(1)}) + 2$, if there exists a pair of adjacent vertices with degree $\Delta(G)$ in G; otherwise $\chi''_a(G^{(1)}) = \Delta(G^{(1)}) + 1$.*

Next, we extend the above results obtained for $G^{(1)}$ to $G^{(r)}$, where r is any positive integer. Let G be a graph of order n and $V(G) = \{v_1, v_2, \ldots, v_n\}$. To get $G^{(i)}$, we take one copy of G with vertex set $V(G)$ denoted as G^i_j for each vertex

x_j of $G^{(i-1)}$ with $V(G^i_j) = \left\{v^i_{j1}, v^i_{j2}, \dots, v^i_{jn}\right\}$ and add the edges $v^i_{jh}x_j$, where, $1 \leq h \leq n$ and $1 \leq j \leq |V(G^{(i-1)})|$. An example $K_3^{(2)}$ is given in Figure 4.

Theorem 7. *Let G be a total colorable graph with at least two vertices and r be a positive integer. Then $\chi''_a(G^{(r)}) = \Delta(G^{(r)}) + 2$, if there exists a pair of adjacent vertices with degree $\Delta(G)$ in G; otherwise $\chi''_a(G^{(r)}) = \Delta(G^{(r)}) + 1$.*

Proof. We prove the result by induction on r. We already proved that the result is valid for $r = 1$. Observe that if G has a pair of adjacent vertices with degree $\Delta(G)$, then $G^{(r)}$ also has a pair of adjacent vertices with degree $\Delta(G^{(r)})$ which belongs to the center subgraph G. Assume that the result is true for $r = i-1 \geq 1$. Now, we have to show that the result is also true for $r = i$. To get $G^{(i)}$, we add a copy of G for each vertex x_j of $G^{(i-1)}$, denoted as G^i_j, where $1 \leq j \leq |V(G^{(i-1)})|$.

Let $\chi''_a(G^{(i-1)}) = \Delta(G^{(i-1)}) + p = \Delta(G) + n(i-1) + p$, where $p \in \{1, 2\}$. Assume that f is an AVD-total coloring of $G^{(i-1)}$ using $(\Delta(G)+n(i-1)+p)$ colors and $\mathcal{C}$ is the set of colors used in f. Let $C_{i-1}(v)$ be the color set of $v \in V(G^{(i-1)})$ with respect to f. Let $\mathcal{C}'_i$ be a set of n new colors. We shall extend the AVD-total coloring f to $G^{(i)}$ using colors from the set $\mathcal{C}'_i$. Therefore, we have to color all vertices and edges of each newly added copy G^i_j, as well as color the edges between G^i_j and each vertex of $V(G^{(i-1)})$, where $1 \leq j \leq |V(G^{(i-1)})|$.

We know that G is a total colorable graph. So, we take a total coloring of each copy G^i_j using $(\Delta(G) + 2)$ colors from the set $\mathcal{C}$, where $1 \leq j \leq |V(G^{(i-1)})|$. Note that for a fixed j, there are n edges between G^i_j and $G^{(i-1)}$, and all these edges are incident to exactly one vertex in $V(G^{(i-1)})$. We color the n edges which connect the vertices of G^i_j to a vertex of $G^{(i-1)}$ with n distinct colors from the set $\mathcal{C}'_i$. Let ϕ be the obtained coloring of graph $G^{(i)}$. Now, if there exists some vertex $u \in G^i_j$ such that $\phi(u) = \phi(v_j)$, where $v_j \in G^{(i-1)}$ and v_j is adjacent to every vertex of the subgraph G^i_j, then we recolor vertex u with any color from the set $\mathcal{C}$, which has not been used on the subgraph G^i_j. Next, we claim that at most $n/2$ vertices of G^i_j have to be recolored and so we will always have enough colors for recoloring. Suppose that $\phi(u) = \phi(v_j)$ holds for more than half of the vertices of G^i_j, then a permutation of the colors used for the total coloring of G^i_j would guarantee that $\phi(u) \neq \phi(v_j)$ for at least half of the vertices of G^i_j. Eventually, fewer than half of the vertices of G^i_j would need to be recolored.

Observe that the obtained coloring of $G^{(i)}$ after such recoloring step, is a total coloring. Let the updated set of colors of vertex v be $C_i(v)$, for every vertex $v \in V(G^{(i)})$. Since from each vertex $v^i_{jh} \in G^i_j$ there is exactly one edge that connects v^i_{jh} to $G^{(i-1)}$ and we have colored that edge with a new color from $\mathcal{C}'_i$, $C_i(v^i_{jh})$ has exactly one color from the set $\mathcal{C}'_i$ which is different for each h, $1 \leq h \leq n$ and $1 \leq j \leq |V(G^{(i-1)})|$. Thus, the AVD-property holds for every pair of adjacent vertices in G_j^i. Note that we have colored all the edges between any G_j^i and $G^{(i-1)}$, from the set $\mathcal{C}_i'$. Therefore, for any vertex $x \in V(G^{(i-1)})$, $C_i(x) = C_{i-1}(x) \cup \mathcal{C}_i'$. Since $C_{i-1}(x) \neq C_{i-1}(y)$ in coloring f for any pair of adjacent vertices $x, y \in V(G^{(i-1)})$, $C_i(x) \neq C_i(y)$. Therefore, AVD-property holds for each pair of adjacent vertices in $G^{(i)}$. Finally, the AVD-property holds for the vertices v_{jh}^i and x, where $1 \leq h \leq n$, $1 \leq j \leq |V(G^{(i-1)})|$ and $x \in V(G^{(i-1)})$ because the degree of vertex x is always greater than the degree of vertex v_{jh}^i. Hence, the obtained coloring is an AVD-total coloring of $G^{(i)}$ using $(\Delta(G) + i \cdot n + p)$ colors. Thus, we proved that if $\chi_a''(G^{(i-1)}) = \Delta(G^{(i-1)}) + p$ then $\chi_a''(G^{(i)}) = \Delta(G^{(i)}) + p$, where $p \in \{1, 2\}$.

Hence, for any positive integer r, $\chi_a''(G^{(r)}) = \Delta(G^{(r)}) + 2$ if there exists a pair of adjacent vertices with degree $\Delta(G)$ in G, otherwise $\chi_a''(G^{(r)}) = \Delta(G^{(r)}) + 1$. ■

3. Conclusion

Our work raises several interesting questions that could be further investigated. In particular, we proved that the AVD-total coloring conjecture holds for the corona product $G \circ H$ of any two AVD-total colorable graphs G and H, if $\Delta(G) \geq \Delta(H)$. We also proved that the AVD-total coloring conjecture holds for the corona product $G \circ K_n$, where G is an AVD-total colorable graph. Furthermore, given a total colorable graph G and a positive integer r, we classified the corona graphs $G^{(r)} = G \circ G \circ \cdots \circ G$ ($r+1$ times) with respect to their AVD-total chromatic numbers. However, the verification of the AVD-total coloring conjecture is an open question for the corona product $G \circ H$, of general graphs G and H. It would be interesting to characterize the graphs $G \circ H$ according to their AVD-total chromatic number, when G or H is restricted to some special classes of graphs such as complete graphs, cycles or trees.

Recall that for a bipartite graph, the total chromatic number and the AVD-total chromatic number can take values $\Delta + 1$ or $\Delta + 2$ at most. It is known that the problem of classifying bipartite graphs with total chromatic number $\Delta + 1$ and $\Delta + 2$ is NP-complete even when the given graph is 3-regular and bipartite [10]. However, the classification of bipartite graphs with AVD-total chromatic number 4 and 5 is polynomial time solvable when the given graph has maximum degree 3 [9]. Therefore, it is an interesting open question whether the classification problem for the AVD-total chromatic number of general bipartite graphs is NP-complete.

As a final remark, AVD-total colorings have been extensively studied for planar graphs [3–5, 17]. However the verification of the AVD-total coloring conjecture is an open question for planar graphs with $\Delta < 9$. Additionally, it is known that $\chi_a''(G) \leq \Delta(G) + 2$ for planar graphs with $\Delta \geq 11$ [16, 19]. It would be an interesting open question whether $\chi_a''(G) \leq \Delta(G) + 2$, for all planar graphs.

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