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BETA INVARIANT AND CHROMATIC UNIQUENESS OF WHEELS

Sooyeon Lee and Haidong Wu

Department of Mathematics
University of Mississippi
University, Mississippi

e-mail: slee27@olemiss.edu
hwu@olemiss.edu

Abstract

A graph G is chromatically unique if its chromatic polynomial completely determines the graph. An n-spoked wheel, W_n, is shown to be chromatically unique when $n \geq 4$ is even [S.-J. Xu and N.-Z. Li, *The chromaticity of wheels*, Discrete Math. 51 (1984) 207–212]. When n is odd, this problem is still open for $n \geq 15$ since 1984, although it was shown by different researchers that the answer is no for $n = 5, 7$, yes for $n = 3, 9, 11, 13$, and unknown for other odd n. We use the beta invariant of matroids to prove that if M is a 3-connected matroid such that $|E(M)| = |E(W_n)|$ and $\beta(M) = \beta(M(W_n))$, where $\beta(M)$ is the beta invariant of M, then $M \cong M(W_n)$. As a consequence, if G is a 3-connected graph such that the chromatic (or flow) polynomial of G equals to the chromatic (or flow) polynomial of a wheel, then G is isomorphic to the wheel. The examples for $n = 3, 5$ show that the 3-connectedness condition may not be dropped. We also give a splitting formula for computing the beta invariants of general parallel connection of two matroids as well as the 3-sum of two binary matroids. This generalizes the corresponding result of [T.H. Brylawski, *A combinatorial model for series-parallel networks*, Trans. Amer. Math. Soc. 154 (1971) 1–22].

Keywords: chromatic uniqueness of graphs, beta invariant, characteristic polynomial, 2-sum, 3-sum, matroids.

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1. Introduction

The notation and the terminology for the beta invariant, Tutte, and chromatic polynomial follows [19] and the matroid terminology not defined in the paper follow [20]. A graph is cosimple if it has no cut-edges. Let G be a graph. We use $P_G(\lambda)$ to denote its chromatic polynomial. A graph G is chromatically unique if whenever $P_G(\lambda) = P_H(\lambda)$ for a graph H, then $G \cong H$. If G is not isomorphic to H but $P_G(\lambda) = P_H(\lambda)$, then G and H are called chromatically equivalent (see, for example, [13]). There are many papers in the literature (see, for example, [10, 11, 13]) on chromatically unique graphs and chromatically equivalent graphs and many other result on graphs determined by other polynomials (see for example, [1, 5, 7, 18]). In general, it is difficult to determine if a graph is chromatically unique due to the lack of information that can be extracted from the chromatic polynomial of the given graph. For example, given a chromatic polynomial of a graph G, one cannot determine if the graph is 3-connected, or 2-connected but not 3-connected.

One of the open problems involving chromatic polynomial is on the chromatic uniqueness of the n-spoked wheel graph, W_n (for example, W_3 is isomorphic to K_4). Note some authors use W_n to denote a wheel with n vertices. Chao and Whitehead [11] proved that W_4 is chromatically unique but W_5 is not chromatically unique by providing a chromatically equivalent graph. Xu and Li [22] proved that for even $n \geq 4$, the wheels are chromatically unique. They also made a conjecture in the same paper that W_n is not chromatically unique for odd $n \geq 9$ [22]. Xu and Li [22] also provided a graph G which is chromatically equivalent to W_7 as well. Figure 1 shows two graphs G, from [11], and H, from [22], which are chromatically equivalent to W_5 and W_7, respectively. Xu and Li's conjecture was disproved as W_9 and W_{11} were shown to be chromatically unique. Read [21] proved that W_9 is chromatically unique by generating graphs which have the same properties as W_9, such as number of triangles, edges and vertices, and comparing the chromatic polynomials of these graphs using a computer. Li and Whitehead [17] provided a proof which does not depend on a computer for the chromatically uniqueness of W_9. Al-Rekaby and Khalaf [3] proved that W_{11} is chromatically unique without using a computer. Azarija [4] (Ph.D. thesis) proved that W_{11} and W_{13} are chromatically unique with the help of a computer. Azarija [4] also mentioned a conjecture, contrary to Xu and Li's conjecture [22], that for every odd $n \geq 9$, the wheel graph W_n is chromatically unique. The following are the results on the chromatic uniqueness of the wheels listed in chronological order.

Theorem 1. (1) ([11], 1978) W_5 *is not chromatically unique.*

(2) ([22], 1984) *For every even* $n \geq 4$*, the wheel graph* W_n *is chromatically unique.*

(3) ([22], 1984) W_7 *is not chromatically unique.*

(4) ([21], 1988), ([17], 1992) W_9 *is chromatically unique.*

(5) ([3], 2014), ([4], 2016) W_{11} *and* W_{13} *are chromatically unique.*

In Section 2, as a corollary of the main result of this paper, we prove that if $P_G(\lambda) = P_{W_n}(\lambda)$, and G is 3-connected, then $G \cong W_n$. We actually prove a more general result for 3-connected matroids. In the third section, we give a splitting formula for the beta invariant of a generalized parallel connection across a 3-point line and 3-sum.

Let M be a matroid with rank function r, then the Tutte polynomial of M is defined as

$$T_G(x, y) = \sum_{F \subseteq E(G)} (x-1)^{r(E)-r(F)}(y-1)^{|F|-r(F)}.$$

Here $r(F)$ is the rank of the set F in the matroid.

The beta invariant, denoted $\beta(M)$, of M is a numerical invariant and was first introduced by Crapo [12]. The beta invariant can be obtained from the chromatic polynomial or from the Tutte polynomial of the matroid. The beta invariant of a matroid M is defined as follows:

$$\beta(M) = (-1)^{r(M)} \sum_{A \subseteq E(M)} (-1)^{|A|} r(A). \tag{1}$$

The chromatic (or characteristic) polynomial of the matroid M, denoted $P(M, \lambda)$, is defined by

$$P(M; \lambda) = \sum_{A \subseteq E(M)} (-1)^{|A|} \lambda^{r(M)-r(A)}. \tag{2}$$

For any graph G, there is a natural matroid $M(G)$ associated with G called the *cycle matroid of* G, where the set of circuits of the matroid $M(G)$ is the set of cycles of G. If M is a matroid and $M \cong M(G)$, then M is called a graphic matroid. When a matroid M is the cycle matroid of a graph G, then the chromatic polynomial of G can be obtained from the characteristic polynomial of $M(G)$ as follows:

$$P_G(\lambda) = \lambda^{\omega(G)} P(M(G); \lambda), \tag{3}$$

where $\omega(G)$ is the number of the components of G.

The beta invariant of the matroid M is related to the $P(M; \lambda)$ by the following identity

$$\beta(M) = (-1)^{r(M)+1} \left. \frac{dP(M; \lambda)}{d\lambda} \right|_{\lambda=1}. \tag{4}$$

Let $T_G(x, y)$ be the Tutte polynomial of the graph G with at least two edges. Then $\beta(M(G))$ is the coefficient of either x or y in $T_G(x, y)$.

Let G be any bridgeless graph, D be the set of directed edges of G after implementing arbitrary orientation, and A be an artribary Abelian group. An A-flow on G is a map f from D to A such that the total flow out of a vertex is equal to the total flow into the vertex. An A-flow f is a nowhere-zero flow (NZF) if $f(uv) \neq 0$ for any $uv \in D$.

For an Abelian group A, the number of NZF A-flows on G is independent of the structure of A, and depends only on the order of A, (see, for example [8]). This number is a polynomial of $|A|$, called the flow polynomial of G and denoted by $Q_G(x)$. This polynomial can be also obtained from the Tutte polynomial of G as follows. Suppose that G is a connected graph with n vertices and m edges. Then

$$Q_G(x) = (-1)^{m-n+1} T_G(0, 1-x). \tag{5}$$

Moreover, the beta invariant of $M(G)$ is given by the absolute value of the coefficient of $t = 1-x$ of the flow polynomial after we make the substitution $t = 1-x$.

The following is the deletion-contraction formula of Crapo [12] for the beta invariant.

Lemma 2. *For an element $e \in E(M)$ which is neither a loop nor a coloop of M, $\beta(M) = \beta(M\backslash e) + \beta(M/e)$.*

Matroid connectivity is defined as following. Let $(X, E(M) - X)$ be a partition of $E(M)$ of a matroid M. Then for a positive integer k, the pair $(X, E(M) - X)$ is a k-separation if $r(X) + r(E(M) - X) - r(M) \leq k - 1$ and $|X|$, $|E(M) - X| \geq k$. A matroid M is n-connected if it has no k-separation for all $k \in \{1, 2, \ldots, n-1\}$. Then the matroid M is 3-connected if M has no 1- or 2-separations. The following is our first main result.

Theorem 3. *Let M be a 3-connected matroid such that $|E(M)| = |E(W_n)|$ and $\beta(M) = \beta(M(W_n))$. Then $M \cong M(W_n)$.*

In [14], de Mier and Noy proved that any wheel W_n is completely determined by its Tutte polynomial $T_{W_n}(x, y)$. Our above result shows that if we know the graph is 3-connected with the right number of edges of the graph, then we need only one coefficient (that is, $\beta(M)$), which is the coefficient of x (or y), rather than the whole Tutte polynomial to determine the wheel graph.

Corollary 4. *Let G be a 3-connected graph such that G and W_n have the same chromatic or flow polynomial. Then $G \cong W_n$.*

In Figure 1, the graphs G and H are chromatically equivalent to W_5 and W_7, respectively. Both G and H are 2-connected but not 3-connected. This shows 3-connectedness condition in our above results may not be dropped. We can also see from these two examples that both G and H have the same number of vertices and edges as W_5 and W_7, respectively. In fact, when two simple graphs have the same chromatic polynomial, then these two graphs have the same number of edges and vertices. It is straightforward to show that the same property applies to two simple matroids with the same chromatic polynomial. And a similar result holds for the flow polynomial of a cosimple matroid as well. It is easily shown that $P(M;\lambda)$ is a polynomial with degree $r(M)$ and with integer coefficients alternating in sign beginning with 1, $-|E(M)|, \ldots$. Thus the next proposition is clearly true. The following propositions will be used in the proof in the next section.

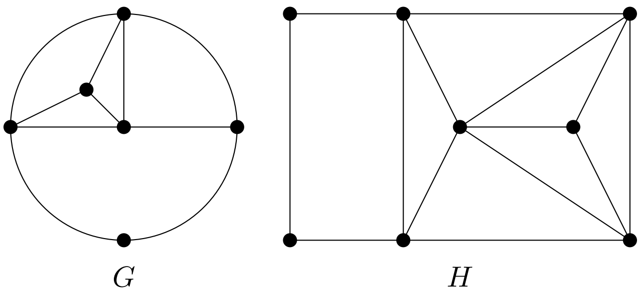

Figure 1. $P_G(\lambda) = P_{W_5}(\lambda)$ and $P_H(\lambda) = P_{W_7}(\lambda)$.

Proposition 5. *Let M and N be simple matroids such that $P(M;\lambda) = P(N;\lambda)$. Then $r(M) = r(N)$ and $|E(M)| = |E(N)|$.*

Proposition 6 [15]**.** *Let G and H be connected cosimple graphs with at least one edge such that $Q_G(x) = Q_H(x)$. Then $|V(G)| = |V(H)|$ and $|E(G)| = |E(H)|$.*

In the last section, we provide a splitting formula for the beta invariant of generalized parallel connection of two matroids, and the 3-sum of two binary matroids. These are well-known operations for matroids and have numerous applications (see [19]). A matroid is binary if it can be represented by a matroid in the field of two elements. Let M_1 and M_2 be matroids such that $M_1|T$, the submatroid restricted on T, and $M_2|T$ are equal, where $T = E(M_1) \cap E(M_2)$. Let $N = M_1|T$ and suppose that $si(N)$, the simplified matroid associated with N, is a modular flat of the matroid $si(M_1)$ (see [19] for the definition of the modular flat). The *generalized parallel connection* $P_N(M_1, M_2)$ of M_1 and M_2 across N is the matroid on $E(M_1) \cup E(M_2)$ whose flats are those subsets X of $E(M_1) \cup E(M_2)$ such that $X \cap E(M_1)$ is a flat of M_1, and $X \cap E(M_2)$ is a flat of M_2. When T is a triangle of both M_1 and M_2 and both M_1 and M_2 have at least 7 elements, the 3-sum of two binary matroids, $M_1 \oplus_3 M_2$ is defined as $M = P_T(M_1, M_2) \backslash T$.

The following is our second main result.

Theorem 7. *Let M_1 and M_2 be two binary matroids such that $E(M_1) \cap E(M_2) = \{s, p, q\}$ and $M = M_1 \oplus_3 M_2$. Then $\beta(M) = \beta(M_1)\beta(M_2) - \beta(M_1/s)\beta(M_2/s) - \beta(M_1/p)\beta(M_2/p) - \beta(M_1/q)\beta(M_2/q)$.*

2. Chromatic Uniqueness of Wheels

There are results on characterizing matroids with specific beta invariants. For example, the beta invariant of a matroid M is always non-negative, and if M has at least two elements, $\beta(M) > 0$ if and only if M is connected [12]. Also, if M has at least two elements, $\beta(M) = 1$ if and only if M is a series-parallel network [9]. Now we characterize 3-connected matroids with fixed beta invariant $\beta(M) \geq 2$ and with maximum number of elements. A matroid whirl W^n, first defined by Tutte [23], is a non-graphic matroid closely related to the wheel graph W_n (see [19]). It is known that $\beta(W_{n+1}) = n$ and $\beta(W^{n+1}) = n + 1$ [12].

We will use the following well-known Tutte's Wheels and Whirls Theorem (see Oxley [20, Theorem 8.8.4]).

Theorem 8. *Let M be a 3-connected matroid other than a wheel or a whirl. Then there exists a 3-connected matroid M_0, which is a wheel of rank at least three or a whirl of rank at least two, and a sequence $M_0, M_1, \ldots, M_n$ of 3-connected matroids with $M_n \cong M$ such that M_i is a single-element deletion or single-element contraction of M_{i+1} for all i in $\{0, 1, \ldots, n-1\}$.*

Theorem 9. *Let $\mathcal{M}_n$ be the set of 3-connected matroids N with $\beta(N) = n$, $n > 1$. Then $|E(M)| = \max\{|E(N)| : N \in \mathcal{M}_n\}$ if and only if $M \cong W_{n+1}$.*

Proof. Note that the only nontrivial 3-connected matroids with less than 4 elements are $U_{0,1}, U_{1,1}, U_{1,2}, U_{1,3}, U_{2,3}$, but each of these matroid has beta invariant at most one. Thus for each $M \in \mathcal{M}_n$, as $\beta(M) = n > 1$, we have that M has at least four elements. Assume to the contrary that $M \in \mathcal{M}_n$, $M \ncong W_{n+1}$, and $|E(M)| = \max\{|E(N)| : N \in \mathcal{M}_n\}$. We will show that $|E(W_{n+1})| > |E(M)|$. Note that M is neither a wheel nor a whirl. Indeed, if M is a wheel W_t, then as $\beta(W_t) = t - 1$ [12, Proposition 10], we conclude that $t = n + 1$; a contradiction. If M is a whirl W^t for some t, then $n = t$ as $\beta(W^t) = t$ [12, Proposition 10]. This is a contradiction as $|E(W_{n+1})| > |E(W^n)|$ but $\beta(W_{n+1}) = \beta(W^n) = n$. By Theorem 8, there exists a sequence of 3-connected matroids $M_0, \ldots, M_k$, such that $M_0 \cong W$ and $M_k \cong M$, where W is a wheel of rank r at least three or a whirl of rank r at least two. Since M is not a wheel or a whirl, $k > 0$ and $|E(M)| = |E(W)| + k = 2r + k$.

Since $W \cong M_0$, if W is a wheel, then $\beta(M_0) = r(M_0) - 1$. If W is a whirl, then $\beta(M_0) = r(M_0)$. Let $e \in E(M_{i+1}) \backslash E(M_i)$ for some $0 \leq i < k$. Then by the deletion-contraction formula for the beta invariant, $\beta(M_{i+1}) = \beta(M_{i+1} \backslash e) +$

$\beta(M_{i+1}/e)$. Since M_{i+1} is 3-connected, both $M_{i+1}\backslash e$ and M_{i+1}/e are connected and both $\beta(M_{i+1}\backslash e)$, $\beta(M_{i+1}/e) \geq 1$. Also either $M_{i+1}\backslash e \cong M_i$ or $M_{i+1}/e \cong M_i$. Combined with the deletion contraction formula, for each $i \in \{0, 1, \ldots, k-1\}$,

$$\beta(M_{i+1}) \geq \beta(M_i) + 1. \tag{6}$$

Thus, by applying induction on (6),we get

$$n = \beta(M) = \beta(M_k) \geq \beta(M_0) + k. \tag{7}$$

If W is a rank r whirl, then $\beta(M_0) = r$ and by (7), $\beta(M_k) = n \geq r + k$. Similarly, if W is a rank r wheel, $\beta(M_0) = r - 1$ and $\beta(M_k) \geq r - 1 + k$. Therefore $n = \beta(M_k) \geq r-1+k$. Let $s \in \mathbb{N}$ such that $r+s = n+1$. Then $\beta(W_{n+1}) = r+s-1$ and since $n \geq r - 1 + k$, we have that $s \geq k > 0$. Therefore, $|E(W_{n+1})| = 2r + 2s > 2r + k = |E(M)|$. Thus $\beta(W_{n+1}) = \beta(M)$ but $|E(W_{n+1})| > |E(M)|$, a contradiction. ■

Theorem 3 is an immediate consequence of Theorem 9 as only a wheel has the maximum number of elements among all 3-connected matroids with a fixed beta invariant. Now we prove Corollary 4.

***Proof of Corollary* 4.** If G is a 3-connected graph such that $P_G(\lambda) = P_{W_n}(\lambda)$, then $|E(G)| = |E(W_n)|$ and $\beta(M(G)) = \beta(W_n)$, where $M(G)$ is a cycle matroid of the graph G. Thus $M(G) \cong M(W_n)$. As both G and W_n are 3-connected, $G \cong W_n$ by Whitney's 2-isomorphism theorem [24]. Suppose that a 3-connected graph G and W_n have the same flow polynomial. By Proposition 5, the beta invariant is the coefficient of t in $Q_G(x)$ after we make the substitution $t = 1-x$. Hence if $Q_G(x) = Q_{W_n}(x)$, we have that $\beta(M) = \beta(W_n)$. Then again, by applying Proposition 6 and Theorem 9, we conclude that $G \cong W_n$. ■

The last corollary can be also proved using a result of the authors [16, Theorem 1.6]. For the open problem that whether W_{2n+1} is chromatically unique, Corollary 4 reduces the problem to the 2-connected but not 3-connected case. In this case, one can decompose such a graph into a 2-sum of two minors of G first. For definitions of 2-sum and 3-sum, see the next section. In the next section, we will derive a formula to find the beta invariants of the 3-sum of two binary matroids. Since graphic matroids are also binary, the results apply to graphic matroids as well. In the end of the next section, we will discuss a possible strategy of studying whether W_{2n+1} is chromatically unique for small n using beta invariant of the 2-sum of two graphic matroids by considering the 2-connected but not 3-connected case.

3. Beta Invariant of Generalized Parallel Connection

In this section, we prove a result on the computation of the beta invariant of a generalized parallel connection and as a corollary, 3-sum of two binary matroids. Since graphs are special binary matroids, the result will apply to graphs as well. Suppose $E(M_1) \cap E(M_2) = T$, $cl_{M_1}(T)$ is a modular flat of M_1, and every non-loop element of $cl_{M_1}(T) - T$ is parallel to some element of T, and $M_1|T = M_2|T$. Let $P_T(M_1, M_2)$ be the generalized parallel connection of M_1 and M_2 across the common set T. From now on, we will use $cl_1(T)$ instead of $cl_{M_1}(T)$ for simplicity. When both M_1 and M_2 are binary and $|E_1|$, $|E_2| \geq 7$, then recall that $P_T(M_1, M_2)\backslash T$ is called 3-sum of M_1 and M_2 and is denoted $M_1 \oplus_3 M_2$. When $T = \{p\}$, then $P_T(M_1, M_2)$ is called parallel connection of M_1 and M_2 with respect to p and is denoted by $P(M_1, M_2)$, and $P(M_1, M_2)\backslash T$ is called 2-sum of M_1 and M_2 and is denoted by $M_1 \oplus_2 M_2$. Brylawski [9, Theorem 6.16(vi)] proved the following result.

Theorem 10 [9]**.** *Suppose that p is neither a loop nor a coloop of M_1 and M_2, then $\beta(P(M_1, M_2)) = \beta(M_1)\beta(M_2)$.*

Using Theorem 10, Oxley proved the following theorem [20, Proposition 2.5].

Theorem 11. *Let M be a matroid and suppose that $\beta(M) = k > 1$. Then either*

(i) *M is a series-parallel extension of a 3-connected matroid N such that $\beta(N) = k$, or*

(ii) *$M = M_1 \oplus_2 M_2$ for some matroids M_1 and M_2, and $\beta(M) = \beta(M_1)\beta(M_2)$ each having $\beta(M_i) < k$ for $i = 1, 2$.*

We prove the generalizations of the last two results.

Theorem 12. *Let M_1 and M_2 be matroids and $M = P_T(M_1, M_2)$, the generalized parallel connection of M_1 and M_2 across a triangle T. Then $\beta(M) = \beta(M_1)\beta(M_2)$.*

The proof easily follows from the next result from the fact that $\beta(M_1 \mid T) = 1$ as T is a triangle.

Theorem 13. *Let M_1 and M_2 be matroids and $M = P_T(M_1, M_2)$, the generalized parallel connection of M_1 and M_2 across T. If $\beta(M_1|T) \neq 0$, then $\beta(M) = \frac{\beta(M_1)\beta(M_2)}{\beta(M_1|T)}$.*

Proof. Let M_1 and M_2 be matroids and $M = P_T(M_1, M_2)$, the generalized parallel connection of M_1 and M_2 across T. Here we assume that $cl_1(T)$ is a modular flat of M_1 and every non-loop element of $cl_{M_1}(T) - T$ is parallel to some elements of T, and $M_1|T = M_2|T$.

We prove the theorem by induction on $|E(M_1)-cl_1(T)|$. If $|E(M_1)-cl_1(T)| = 0$, then M is obtained from M_2 by possibly adding loops and parallel elements to some elements in T. It is straightforward to verify that the theorem holds. Assume $|E(M_1) - cl(T)| > 0$ and take $e \in E(M_1) - cl_1(T)$. If e is a loop, then $e \in cl_1(T)$; a contradiction. Suppose that e is a coloop of M. This is true if and only if e is in no circuits of M. Moreover, a circuit of M_1 is also a circuit of M. Thus e is in no circuit of M_1, and thus e is also a coloop of M_1. Therefore neither M nor M_1 is connected, and both sides of $\beta(M) = \frac{\beta(M_1)\beta(M_2)}{\beta(M_1|T)}$ are zero and the theorem holds. Thus we assume that e is neither a loop nor a coloop of M. Again it is straightforward to show that e is neither a loop nor a coloop of M_1 also. Thus by the deletion-contraction formula and induction,

$$\begin{aligned}\beta(M) &= \beta(M\backslash e) + \beta(M/e) = \beta(P_T(M_1\backslash e, M_2)) + \beta(P_T(M_1/e, M_2))\\ &= \frac{\beta(M_1\backslash e)\beta(M_2)}{\beta(M_1|T)} + \frac{\beta(M_1/e)\beta(M_2)}{\beta(M_1|T)}\\ &= \frac{(\beta(M_1\backslash e) + \beta(M_1/e))\beta(M_2)}{\beta(M_1|T)} = \frac{\beta(M_1)\beta(M_2)}{\beta(M_1|T)}.\end{aligned}$$

This completes the proof. ∎

Next, we compute the beta invariant of 3-sum of two binary matroids M_1 and M_2.

Theorem 14. *Let M_1 and M_2 be two binary matroids such that $E(M_1)\cap E(M_2) = \{s, p, q\}$ and $M = M_1 \oplus_3 M_2$. Then $\beta(M) = \beta(M_1)\beta(M_2) - \beta(M_1/s)\beta(M_2/s) - \beta(M_1/p)\beta(M_2/p) - \beta(M_1/q)\beta(M_2/q)$.*

Proof. Let $M = P_T(M_1, M_2)$ where $T = \{s, p, q\}$ is a triangle of both M_1 and M_2. Note that $M_1 \oplus_3 M_2 = P_T(M_1, M_2)\backslash T$. Then by Corollary 12 and the deletion-contraction formula,

$$\beta(M_1)\beta(M_2) = \beta(M) = \beta(P_T(M_1, M_2)\backslash s) + \beta(P_T(M_1, M_2)/s). \tag{1}$$

By [19, Proposition 11.4.14(viii)], $P_T(M_1, M_2)/s = P_{T/s}(M_1/s, M_2/s)$, thus using Theorem 10, we deduce that

$$\begin{aligned}\beta(P_T(M_1, M_2)/s) &= \beta(P_{T/s}(M_1/s, M_2/s)) = \beta(si(P_{T/s}(M_1/s, M_2/s)))\\ &= \beta(P(M_1/s, M_2/s)) = \beta(M_1/s)\beta(M_2/s).\end{aligned} \tag{2}$$

Applying the deletion-contraction formula again for the matroid $P_T(M_1, M_2)\backslash s$, we obtain

$$\begin{aligned}\beta(P_T(M_1, M_2)\backslash s) &= \beta(P_T(M_1, M_2)\backslash s, p) + \beta(P_T(M_1, M_2)\backslash s/p)\\ &= \beta(P_T(M_1, M_2)\backslash T) + \beta(P_T(M_1, M_2)\backslash s, p/q)\\ &\quad +\beta(P_T(M_1, M_2)\backslash s/p).\end{aligned} \tag{3}$$

However,

$$\begin{aligned}\beta(P_T(M_1,M_2)\backslash s,p/q) &= \beta(P_T(M_1,M_2)/q\backslash s,p) \\ &= \beta(M_1/q\oplus_2 M_2/q)=\beta(M_1/q)\beta(M_2/q),\end{aligned} \tag{4}$$

and similarly, we can show that

$$\beta(P_T(M_1,M_2)\backslash s/p)=\beta(M_1/p)\beta(M_2/p). \tag{5}$$

Now combining equations (3)–(5) and then with (1) and (2) we obtain the required result. ■

As an application, we compute the beta invariant of $AG(3,2)=F_7\oplus_3 F_7$. Using the above theorem we obtain $\beta(AG(3,2))=\beta(F_7)\beta(F_7)-3(\beta(F_7/e))^2=9-3=6$.

The generalized parallel connection is the generalization of k-clique-sum of graphs. As an application for graphs, we can compute the beta invariant of the graphic matroids obtained by taking the k-clique sum of two graphs using Theorem 13. If G is the k-clique sum of graphs H and J, where the edges in the common clique of H and J are not deleted, then $\beta(M(G))=\frac{\beta(M(H))\beta(M(J))}{\beta(M(K_k))}$ where K_k is the complete graph with k vertices.

For the open problem that whether W_{2n+1} is chromatically unique, Corollary 4 reduces the problem to the 2-connected but not 3-connected case. The referee asked if $P(G)=P(W_{15})$, and G is not chromatic unique, can we conclude that G must have a vertex of degree 2?

Here is a possible way one can use the beta invariant to study whether W_{2k+1} is chromatically unique where k is small, and in particular, to answer referee's above question. We use W_{15} as an example. Assume that $P(G)=P(W_{15})$, and G is not chromatically unique. Then G is simple, 2-connected with 16 vertices and 30 edges, and $\beta(M(G)=\beta(W_{15})=14$. Assume that the minimum degree of G is at least 3. Then by Theorem 11, either $M(G)$ is a serial-parallel extension of $M(H)$, where H is a 3-connected graph with the same beta invariant, or $M(G)$ is the 2-sum of two graphic matroids $M(S)$ and $M(T)$ each having at least three edges, with beta invariant of 2 and 7, respectively. In the former case, G must be isomorphic to H (and thus 3-connected) as $\delta(G)\geq 3$ and G is simple. By Corollary 4, $G\cong W_{15}$. So we assume that the latter case happens. By [20, Theorem 2.2], the graph S is a series-parallel extension of K_4, and T is a series-parallel extension of one of five 3-connected graphs, each having at most eight vertices [6, Figure 4]. Now considering these various series-parallel extensions (the fact that $\delta(G)\geq 3$ will reduce the number of cases), we then compute the chromatic polynomials of possible graphs G using Sage, and compare these to the chromatic polynomial of W_{15}. If we none of these graphs have the same chromatic polynomial as $P(W_{15})$, then we get a contradiction and we conclude that $\delta(G) = 2$. If one of these graphs does have the same chromatic polynomial as $P(W_{15})$, then we find a graph $G \ncong W_{15}$, as G is not 3-connected, with the same chromatic polynomial of W_{15}. This will involve some case checking and computations. In particular, we need to determine some 3-connected graphs with beta invariant larger than 9.

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