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ON THE BROADCAST INDEPENDENCE NUMBER OF LOCALLY UNIFORM 2-LOBSTERS

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Abstract

Let G be a simple undirected graph. A broadcast on G is a function $f : V(G) \to \mathbb{N}$ such that $f(v) \leq e_G(v)$ holds for every vertex v of G, where $e_G(v)$ denotes the eccentricity of v in G, that is, the maximum distance from v to any other vertex of G. The cost of f is the value $\text{cost}(f) = \sum_{v \in V(G)} f(v)$. A broadcast f on G is independent if for every two distinct vertices u and v in G with $f(u) > 0$ and $f(v) > 0$, $d_G(u, v) > \max\{f(u), f(v)\}$, where $d_G(u, v)$ denotes the distance between u and v in G. The broadcast independence number of G is then defined as the maximum cost of an independent broadcast on G.

A caterpillar is a tree such that, after the removal of all leaf vertices, the remaining graph is a non-empty path. A lobster is a tree such that, after the removal of all leaf vertices, the remaining graph is a caterpillar. In [M. Ahmane, I. Bouchemakh and E. Sopena, *On the broadcast independence number of caterpillars*, Discrete Appl. Math. 244 (2018) 20–35], we studied independent broadcasts of caterpillars. In this paper, carrying on with this line of research, we consider independent broadcasts of lobsters and give an explicit formula for the broadcast independence number of a family of lobsters called locally uniform 2-lobsters.

Keywords: independence, broadcast independence, lobster.

2020 Mathematics Subject Classification: 05C12, 05C69.

1. Introduction

All the graphs we consider in this paper are simple and loopless undirected graphs. We denote by $V(G)$ and $E(G)$ the set of vertices and the set of edges of a graph G, respectively.

For any two vertices u and v of G, the *distance* $d_G(u,v)$ between u and v in G is the length (number of edges) of a shortest path joining u and v. The *eccentricity* $e_G(v)$ of a vertex v in G is the maximum distance from v to any other vertex of G. The minimum eccentricity in G is the *radius* $\mathrm{rad}(G)$ of G, while the maximum eccentricity in G is the *diameter* $\mathrm{diam}(G)$ of G.

A function $f : V(G) \to \{0,\ldots,\mathrm{diam}(G)\}$ is a *broadcast on* G if for every vertex v of G, $f(v) \leq e_G(v)$. The value $f(v)$ is called the *f-value of* v. Given a broadcast f on G, an *f-broadcast vertex* is a vertex v with $f(v) > 0$. The set of all f-broadcast vertices is denoted V_f^+. If $u \in V_f^+$ is a broadcast vertex, $v \in V(G)$ and $d_G(u,v) \leq f(u)$, we say that u *f-dominates* v. In particular, every f-broadcast vertex f-dominates itself. The *cost* $\mathrm{cost}(f)$ of a broadcast f on G is given by

$$\mathrm{cost}(f) = \sum_{v \in V(G)} f(v) = \sum_{v \in V_f^+} f(v).$$

A broadcast f on G is a *dominating broadcast* if every vertex of G is f-dominated by some vertex of V_f^+. The minimum cost of a dominating broadcast on G is the *broadcast domination number* of G, denoted $\gamma_b(G)$. A broadcast f on G is an *independent broadcast* if every f-broadcast vertex is f-dominated only by itself. The maximum cost of an independent broadcast on G is the *broadcast independence number* of G, denoted $\beta_b(G)$. An independent broadcast on G with cost β is an independent *β-broadcast*. An independent $\beta_b(G)$-broadcast on G is an *optimal* independent broadcast. Note here that any optimal independent broadcast is necessarily a dominating broadcast.

The notions of broadcast domination and broadcast independence were introduced by Erwin in his Ph.D. thesis [18] under the name of *cost domination* and *cost independence*, respectively. During the last decade, broadcast domination has been investigated by several authors (see e.g. [3,4,8–10,12–17,19–21,23–31]), while broadcast independence has attracted much less attention (see [2,11]), until the recent work of Bessy and Rautenbach. In [6], these authors prove that $\beta_b(G) \leq 4\alpha(G)$ for every graph G, where $\alpha(G)$ denotes the independence number of G, that is, the maximum cardinality of an independent set in G. In [6], they prove that $\beta_b(G) < 2\alpha(G)$ whenever G has girth at least 6 and minimum degree at least 3, or girth at least 4 and minimum degree at least 5. Answering questions posed in [22] and [17], they prove in [5] that deciding whether $\beta_b(G) \geq k$ for a given planar graph with maximum degree four and a given positive integer k is an NP-complete problem, and, using an approach based on dynamic programming, they prove that determining the value of $\beta_b(T)$ for a tree T of order n can be done in time $O(n^9)$.

Our goal, initiated in [2], is to give explicit formulas for $\beta_b(T)$, whenever T belongs to some particular subclass of trees, that can be computed in (hopefully) linear time. Recall that a *caterpillar* is a tree such that deleting all its pendent vertices leaves a simple path, called the *spine* of the caterpillar. A *lobster* is then a tree such that deleting all its pendent vertices leaves a caterpillar. The spine of such a lobster is the spine of the so-obtained caterpillar. A vertex belonging to the spine of a caterpillar, or of a lobster, is called a *spine-vertex* and an *internal* spine-vertex if it is not an end vertex of the spine. The *length* of a lobster L is the length (number of edges) of its spine.

Note that if L is a lobster of length 0, then the unique spine-vertex of L must be of degree at least 2, since otherwise, deleting all leaves of L would leave a single edge, which is not a caterpillar.

In [2], we gave an explicit formula for the broadcast independence number of caterpillars having no two consecutive internal spine vertices of degree 2. The aim of this paper is to pursue the study of independent broadcasts of trees by considering the case of locally uniform 2-lobsters.

Let G be a graph and $A \subset V(G)$, $|A| \geq 2$, be a set of pairwise antipodal vertices in G, that is, at distance $\operatorname{diam}(G)$ from each other. The function f defined by $f(u) = \operatorname{diam}(G) - 1$ for every vertex $u \in A$, and $f(v) = 0$ for every vertex $v \notin A$, is clearly an independent $|A|(\operatorname{diam}(G) - 1)$-broadcast on G.

Observation 1 (Dunbar *et al.* [17]). *For every graph G of order at least 2 and every set $A \subset V(G)$, $|A| \geq 2$, of pairwise antipodal vertices in G, $\beta_b(G) \geq |A|(\operatorname{diam}(G) - 1) \geq 2(\operatorname{diam}(G) - 1)$.*

In this paper, we determine the broadcast independence number of locally uniform 2-lobsters. The paper is organised as follows. We introduce in the next section the main definitions and a few preliminary results. We then consider in Section 3 the case of locally uniform 2-lobsters and prove our main result, which gives an explicit formula for the broadcast independence number of such lobsters. We then propose some concluding remarks in Section 4.

2. Preliminaries

Let G be a graph and H be a subgraph of G. Since $d_H(u, v) \geq d_G(u, v)$ for every two vertices $u, v \in V(H)$, every independent broadcast f on G satisfying $f(u) \leq e_H(u)$ for every vertex $u \in V(H)$ is an independent broadcast on H. Hence we have the following.

Observation 2. *If H is a subgraph of G and f is an independent broadcast on G satisfying $f(u) \leq e_H(u)$ for every vertex $u \in V(H)$, then the restriction f_H of f to $V(H)$ is an independent broadcast on H.*

For any independent broadcast f on a graph G, and any subgraph H of G, we denote by $f^*(H)$ the f-*value* of H defined as

$$f^*(H) = \sum_{v \in V(H)} f(v).$$

Observe that $f^*(G) = \mathrm{cost}(f)$.

The following lemma shows that, for any graph G of order at least 3, if v is a vertex of G having at least one pendent neighbour, then no independent broadcast f on G with $f(v) > 0$ can be optimal.

Lemma 3. *Let G be a graph of order at least 3 and v be a vertex of G having a pendent neighbour u. If f is an independent broadcast on G with $f(v) > 0$, then there exists an independent broadcast f' on G with $\mathrm{cost}(f') > \mathrm{cost}(f)$.*

Proof. The mapping f' defined by $f'(u) = f(v)+1$, $f'(v) = 0$ and $f'(w) = f(w)$ for every vertex $w \in V(G) \setminus \{u, v\}$ is clearly an independent broadcast on G with $\mathrm{cost}(f') > \mathrm{cost}(f)$. ■

The following lemma was given in [2]. However, we include its proof here for the sake of completeness.

Lemma 4. *Let T be a rooted tree of order at least 3, and v be any vertex of T such that the maximal subtree T_v of T, rooted at v and not containing any vertex of T whose distance from the root r of T is smaller than $d_T(v, r)$, has order at least 2. Let f be an optimal independent broadcast on T. If v is an f-broadcast vertex, then T_v contains at least one other f-broadcast vertex. In particular, this implies that if T_v is a subtree of height h, that is, $e_{T_v}(v) = h$, then $f(v) < h$.*

Proof. Suppose to the contrary that $f(v) > 0$ and $f(u) = 0$ for every vertex $u \in V(T_v) \setminus \{v\}$. Let $t = e_{T_v}(v)$ and $\bar{t} = e_{T-(T_v-v)}(v)$.

If $f(v) < t$, the independent broadcast f' given by $f'(w) = f(v)$ for some vertex w in T_v with $d_{T_v}(v, w) = t$ and $f'(u) = f(u)$ for every vertex $u \in V(T)\setminus\{w\}$ is such that $\mathrm{cost}(f') = \mathrm{cost}(f) + f(v)$, contradicting the optimality of f.

If $f(v) \geq \bar{t}$, then v is the unique f-broadcast vertex, which implies $\mathrm{cost}(f) < 2(\mathrm{diam}(T) - 1)$, again contradicting the optimality of f by Observation 1.

Hence $\bar{t} > f(v) \geq t$. Let now v' be any neighbour of v in T_v. Since $\bar{t} > f(v) \geq t$, we have $e_T(v') = e_T(v)+1 = \bar{t}+1 > f(v)+1$. The function f' defined by $f'(v) = 0$, $f'(v') = f(v)+1$ and $f'(u) = f(u)$ for every vertex $u \in V(T)\setminus\{v, v'\}$ is therefore an independent broadcast on T with $\mathrm{cost}(f') = \mathrm{cost}(f) + 1$, contradicting the optimality of f. This completes the proof. ■

In order to formally define locally uniform lobsters, and then locally uniform 2-lobsters, we introduce some notation.

Notation 5 ($\mathcal{S}_1$, $\mathcal{S}_2$)**.** A tree T rooted at a vertex r is of type $\mathcal{S}_1$ if every leaf of T is at distance 1 from r, which means that T is a star with center r. A tree T rooted at a vertex r is of type $\mathcal{S}_2$ if every leaf of T is at distance 2 from r.

Let L be a lobster with spine $v_0 \cdots v_k$, $k \geq 0$. The *subtree* of v_i, $0 \leq i \leq k$, denoted S_i, is the maximal subtree of L rooted at v_i that contains no spine-vertex except v_i. A *spine-subtree* of L is a subtree of some v_i, $0 \leq i \leq k$. A *branch* of a spine-subtree S_i is a maximal subtree of S_i containing v_i and exactly one neighbour of v_i. Therefore, if v_i has degree d in S_i, then S_i has d distinct branches.

A locally uniform lobster is then defined as follows.

Definition (Locally uniform lobster)**.** A lobster L is *locally uniform* if every spine-subtree of L is of type either $\mathcal{S}_1$ or $\mathcal{S}_2$. In other words, all branches of any spine-vertex have the same depth.

The following observation directly follows from this definition.

Observation 6. *If L is a locally uniform lobster with spine $v_0 \cdots v_k$, $k \geq 0$, then both spine-subtrees S_0 and S_k are of type $\mathcal{S}_2$, and thus* $\mathrm{diam}(L) = k + 4$.

Indeed, if S_0 or S_k is of type $\mathcal{S}_1$, then v_0 or v_k is a leaf of the caterpillar obtained by deleting all leaves of L, which implies that $v_0 \cdots v_k$ is not the spine of L, a contradiction.

Observe that Lemma 4 implies in particular the following result for locally uniform lobsters.

Corollary 7. *If L is a locally uniform lobster with spine $v_0 \cdots v_k$, $k \geq 0$, and f is an optimal independent broadcast on L, then the two following conditions hold.*

1. *If v is a vertex having a pendent neighbour, then $f(v) = 0$.*
2. *For every i, $0 \leq i \leq k$, $f(v_i) = 0$ if S_i is of type $\mathcal{S}_1$, and $f(v_i) \leq 1$ if S_i is of type $\mathcal{S}_2$.*

Moreover, the following lemma says that for every optimal independent broadcast on a locally uniform lobster with spine $v_0 \cdots v_k$, $k \geq 0$, both the spine-subtrees S_0 and S_k contain an f-broadcast vertex.

Lemma 8. *If L is a locally uniform lobster with spine $v_0 \cdots v_k$, $k \geq 0$, and f is an optimal independent broadcast on L, then $f^*(S_0) > 0$ and $f^*(S_k) > 0$.*

Proof. It is enough to prove the result for S_0. Assume to the contrary that $f^*(S_0) = 0$, and let v be a vertex of L that f-dominates the leaves of S_0. Since $f^*(S_0) = 0$, we necessarily have $f(v) \geq 4$ which implies that v is unique. By Corollary 7, v must be a leaf of L. Let ℓ be any leaf of S_0.

Let S denote the spine-subtree containing v. If S is of type $\mathcal{S}_1$ and $f(v) + d_L(\ell, v) > \text{diam}(L) + 1$, or S is of type $\mathcal{S}_2$ and $f(v) + d_L(\ell, v) > \text{diam}(L) + 3$, then v is the unique f-broadcast vertex of L, which contradicts the optimality of f by Observation 1. We now define the mapping f' on $V(L)$ given by $f'(u) = f(u)$ for every vertex $u \notin \{v, \ell\}$, $f'(v) = 0$, and $f'(\ell) = f(v) + d_L(\ell, v) - 2$ if v is adjacent to a spine-vertex, or $f'(\ell) = f(v) + d_L(\ell, v) - 4$ otherwise. From the above remark, we get $f'(\ell) \leq \text{diam}(L) - 1$ in both cases. The mapping f' is clearly an independent broadcast on L and, since $d_L(v, \ell) \geq 4$ if v is adjacent to a spine-vertex and $d_L(v, \ell) \geq 5$ otherwise, we get $\text{cost}(f') > \text{cost}(f)$, contradicting the optimality of f. ■

We now define 2-lobsters and locally uniform 2-lobsters.

Definition (2-lobster)**.** A lobster L is a 2*-lobster* if every spine-subtree of L has at least two branches.

Definition (Locally uniform 2-lobster)**.** A *locally uniform* 2*-lobster* is a 2-lobster which is locally uniform (see Figure 1).

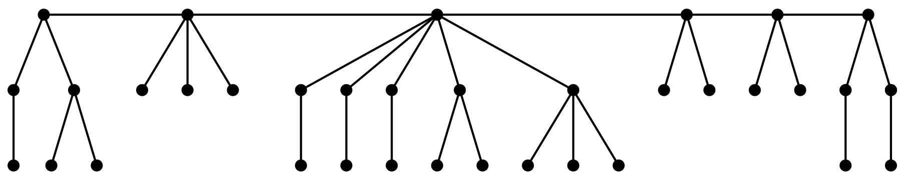


Figure 1. A sample locally uniform 2-lobster.

Due to their special structure, we can improve the lower bound on the broadcast independence number of locally uniform 2-lobsters of length $k \geq 1$.

Observation 9. *For every locally uniform* 2*-lobster L of length $k \geq 1$,*

$$\beta_b(L) \geq 2(k-1) + 12 = 2(\text{diam}(L) - 1) + 4.$$

To see that, consider the function f on $V(L)$ defined as follows. For each branch of S_0 and S_k, pick one leaf and set its f-broadcast value to 3, and, for each branch of every S_i, $1 \leq i \leq k-1$, if $k > 1$, pick one leaf and set its f-broadcast value to 1. The mapping f is clearly an independent broadcast on L and, since both S_0 and S_k are of type $\mathcal{S}_2$ and every spine-subtree of L has at least two branches, we get $\beta_b(L) \geq \text{cost}(f) \geq 2(k-1) + 12$.

3. Independent Broadcasts of Locally Uniform 2-Lobsters

In this section we determine the broadcast independence number of locally uniform 2-lobsters. Recall that by Observation 9, $\beta_b(L) > 2(\mathrm{diam}(L) - 1)$ for every locally uniform 2-lobster L of length $k \geq 1$ (the special case of locally uniform 2-lobsters of length 0 will be considered separately, in Lemma 16).

We first introduce some notation and define different types of spine-subtrees in Subsection 3.1. We then define the value $\beta^*(L)$ for every locally uniform 2-lobster L in Subsection 3.2, prove that every such lobster L admits an independent $\beta^*(L)$-broadcast in Subsection 3.3 and that it cannot admit any independent broadcast with cost strictly greater than $\beta^*(L)$ in Subsection 3.4. This allows us to finally state our main result in Subsection 3.5.

3.1. Different types of spine-subtrees

Let L be a locally uniform 2-lobster with spine $v_0 \cdots v_k$, $k \geq 0$. Two spine-subtrees S_i and S_{i+1}, $0 \leq i \leq k-1$, are called *neighbouring spine-subtrees*. Moreover, we say that S_i *precedes* S_{i+1}, and that S_{i+1} *follows* S_i. A *sequence* of p spine-subtrees, $p \geq 2$, is a sequence of consecutive spine-subtrees of the form $S_i \cdots S_{i+p-1}$ for some i, $0 \leq i \leq k-p+1$.

We will say that two independent broadcasts f_1 and f_2 on a locally uniform 2-lobster L are *similar* if their values on each spine-subtree of L are equal, that is, $f_1^*(S_i) = f_2^*(S_i)$ for every i, $0 \leq i \leq k$. Observe that any two similar independent broadcasts have the same cost.

A 1-*leaf* of L is a pendent vertex of L adjacent to a spine-vertex. A pendent vertex which is not a 1-leaf is a 2-*leaf* (recall that every pendent vertex is at distance at most 2 from a spine-vertex). An *only-leaf* is a leaf whose neighbour has only one leaf neighbour. Therefore, an only-leaf in a locally uniform 2-lobster is necessarily a 2-leaf, and is then called a 2-*only-leaf*. Two leaves having the same neighbour are said to be *sister-leaves*.

Notation 10 (λ_1, λ_2, λ_2^*)**.** For every i, $0 \leq i \leq k$, we denote by $\lambda_1(S_i)$, $\lambda_2(S_i)$ and $\lambda_2^*(S_i)$, the number of 1-leaves, of 2-leaves, and of 2-only-leaves of S_i, respectively. Moreover, we extend these three functions to the whole lobster L, by letting

$$\lambda_1(L) = \sum_{i=0}^{i=k} \lambda_1(S_i), \quad \lambda_2(L) = \sum_{i=0}^{i=k} \lambda_2(S_i), \quad \text{and} \quad \lambda_2^*(L) = \sum_{i=0}^{i=k} \lambda_2^*(S_i).$$

Let v_i be a spine-vertex of L with t non-spine neighbours, denoted $w_i^1, \ldots, w_i^t$. For every j, $1 \leq j \leq t$, the branch B_i^j of v_i is the maximal spine-subtree of S_i, rooted at v_i, containing the edge $v_i w_i^j$ but no edge $v_i w_i^{j'}$ with $j' \neq j$. We then define two types of branches.

Notation 11 ($\mathcal{H}_1$, $\mathcal{H}_2$, α_1, α_2, α_2^*)**.** A branch of a spine-subtree S is of type $\mathcal{H}_1$ if S is of type $\mathcal{S}_1$ (such a branch does not contain any 2-leaf), or of type $\mathcal{H}_2$ if S is of type $\mathcal{S}_2$ (such a branch does not contain any 1-leaf). For every spine-subtree S_i, $0 \le i \le k$, we denote by $\alpha_1(S_i)$ and $\alpha_2(S_i)$ the number of branches of S_i of type $\mathcal{H}_1$ and of type $\mathcal{H}_2$, respectively. Moreover, we denote by $\alpha_2^*(S_i)$ the number of branches of S_i of type $\mathcal{H}_2$ having at most two 2-leaves.

Since all branches of any spine-subtree of a locally uniform 2-lobster are of the same type, we get $\alpha_1(S_i) \ge 2$, $\alpha_2(S_i) = \alpha_2^*(S_i) = 0$, if S_i is of type $\mathcal{S}_1$, and $\alpha_1(S_i) = 0$, $\alpha_2(S_i) \ge 2$, $\alpha_2^*(S_i) \ge 0$, if S_i is of type $\mathcal{S}_2$.

Notation 12 (b_i)**.** For every i, $0 \le i \le k$, we denote by b_i the number of branches of the spine-subtree S_i.

Observe that $b_i = \deg_L(v_i) - 2$ if $1 \le i \le k-1$, and $b_i = \deg_L(v_i) - 1$ if $i \in \{0, k\}$.

In order to define various types of branches or spine-subtrees, we will use the following notation.

Notation 13 (Operators on types of branches or spine-subtrees)**.** Let $\mathcal{X}$, $\mathcal{Y}$ and $\mathcal{Z}$ be any types of branches or spine-subtrees. We then define the following types.

• $\overline{\mathcal{X}}$. A branch or spine-subtree S is of type $\overline{\mathcal{X}}$ if S is not of type $\mathcal{X}$.

• $\mathcal{X}|\mathcal{Y}$. A branch or spine-subtree S is of type $\mathcal{X}|\mathcal{Y}$ if S is of type $\mathcal{X}$ or $\mathcal{Y}$.

• $\mathcal{X}.\mathcal{Y}$, $\mathcal{X}\mathcal{Y}$. A sequence of two spine-subtrees SS' is of type $\mathcal{X}.\mathcal{Y}$, or simply $\mathcal{X}\mathcal{Y}$, if S is of type $\mathcal{X}$ and S' is of type $\mathcal{Y}$. More generally, a sequence of spine-subtrees $S_1 \cdots S_k$, $k \ge 3$, is of type $\mathcal{X}_1 \cdots \mathcal{X}_k$ if each S_i, $1 \le i \le k$, is of type $\mathcal{X}_i$.

• $\mathcal{X}[P_1, \ldots, P_p]$. For any properties $P_1, \ldots, P_p$, $p \ge 1$, a branch or spine-subtree S is of type $\mathcal{X}[P_1, \ldots, P_p]$ if S is a branch or spine-subtree of type $\mathcal{X}$ satisfying properties $P_1, \ldots, P_p$. For instance, a spine-subtree S of type $\mathcal{S}_2[\lambda_2 \ge 5, \alpha_2^* \le 3]$ is a spine-subtree of type $\mathcal{S}_2$ with at least five leaves and having at most three branches with at most two leaves, while a branch of type $\mathcal{H}_2[\lambda_2 = 3]$ is a branch of type $\mathcal{H}_2$ having three 2-leaves.

• $\langle\mathcal{X}\rangle\mathcal{Y}$, $\mathcal{Y}\langle\mathcal{Z}\rangle$, $\langle\mathcal{X}\rangle\mathcal{Y}\langle\mathcal{Z}\rangle$. A spine-subtree S is of type $\langle\mathcal{X}\rangle\mathcal{Y}$ (respectively, $\mathcal{Y}\langle\mathcal{Z}\rangle$) if S is a spine-subtree of type $\mathcal{Y}$ and the spine-subtree S' preceding S (respectively, following S) is of type $\mathcal{X}$ (respectively, $\mathcal{Z}$). A spine-subtree S is then of type $\langle\mathcal{X}\rangle\mathcal{Y}\langle\mathcal{Z}\rangle$ if S is of type $\langle\mathcal{X}\rangle\mathcal{Y}$ and of type $\mathcal{Y}\langle\mathcal{Z}\rangle$.

• $\emptyset$. Slightly abusing the notation, we use the symbol $\emptyset$ to denote an "empty spine-subtree", so that, for instance, a spine-subtree S is of type $\langle\emptyset\rangle\mathcal{Y} =$ (respectively, $\mathcal{Y}\langle\emptyset\rangle$), if $S = S_0$ (respectively, $S = S_k$) and S is of type $\mathcal{Y}$.

• $\{\mathcal{X}_1 \cdots \mathcal{X}_p\}^+$, $\{\mathcal{X}_1 \cdots \mathcal{X}_p\}^*$. For any types of spine-subtrees $\mathcal{X}_1, \ldots, \mathcal{X}_p$, $p \ge 1$, a sequence of spine-subtrees $S_i, \ldots, S_{i+pj}$, $0 \le i \le k - pj$, $0 \le j \le \lfloor \frac{k-i}{p} \rfloor$, is of type

$\{\mathcal{X}_1 \cdots \mathcal{X}_p\}^+$, if every spine-subtree S_ℓ, $i \le \ell \le i+pj$ is of type $\mathcal{X}_{\ell-i \pmod p)+1}$, and none of the sequences $S_{i-p}, \ldots, S_i, \ldots, S_{i+pj}$ and $S_i, \ldots, S_{i+pj}, \ldots, S_{i+pj+p}$ is of type $\{\mathcal{X}_1 \cdots \mathcal{X}_p\}^+$ (the sequence is thus maximal). Moreover, we will denote by $\{\mathcal{X}_1 \cdots \mathcal{X}_p\}^*$ the type $\emptyset | \{\mathcal{X}_1 \cdots \mathcal{X}_p\}^+$, corresponding to either an empty sequence or a sequence of type $\{\mathcal{X}_1 \cdots \mathcal{X}_p\}^+$.

Our aim now is twofold. We will first construct, for any locally uniform 2-lobster L, an independent broadcast f^* on L with $\mathrm{cost}(f^*) = \beta^*(L)$, for some value $\beta^*(L)$, and then prove that the value $\beta^*(L)$ is the optimal cost of an independent broadcast on L.

The independent broadcast f^* will be constructed in four steps, that is, we will construct a sequence of independent broadcasts $f_1, \ldots, f_4$, with $\mathrm{cost}(f_i) \le \mathrm{cost}(f_{i+1})$ for every i, $1 \le i \le 3$, and then set $f^* = f_4$. Each step will consist in modifying the broadcast values of some vertices, according to the type of the spine-subtree, or of the sequence of spine-subtrees, they belong to.

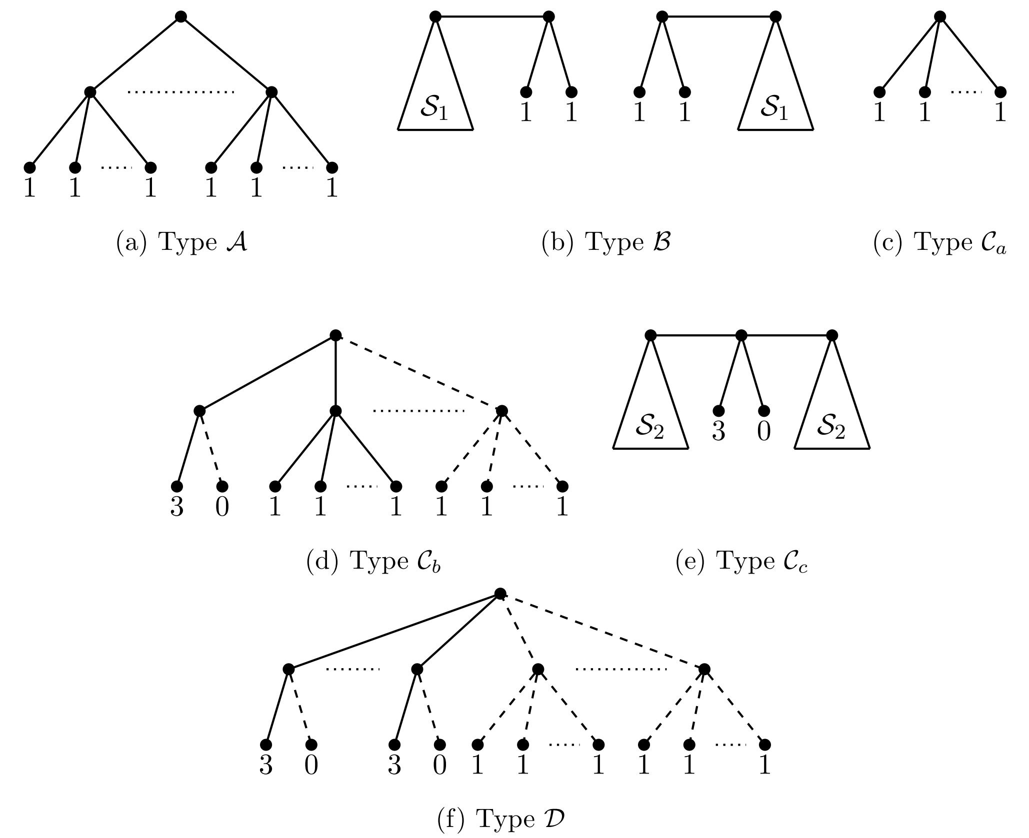


Figure 2. Spine-subtrees of given special types.

We now introduce the specific types of spine-subtrees, or types of sequences of spine-subtrees, that will be used. All these types are illustrated in Figure 2 (do not consider the depicted broadcast values yet, they will be discussed later, in Claim 18).

Definition ($\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}_a$, $\mathcal{C}_b$, $\mathcal{C}_c$, $\mathcal{C}$, $\mathcal{D}$)**.** We define the following types of spine-subtrees.

• $\mathcal{A} = \mathcal{S}_2[\alpha_2^* = 0, \alpha_2 \geq 2]$. A spine-subtree of type $\mathcal{A}$ is a spine-subtree of type $\mathcal{S}_2$ with at least two branches of type $\mathcal{H}_2$, all of them having at least three leaves.

• $\mathcal{B} = \langle \mathcal{S}_1 \rangle\ \mathcal{S}_1[\lambda_1 = 2] \mid \mathcal{S}_1[\lambda_1 = 2]\ \langle \mathcal{S}_1 \rangle$. A spine-subtree of type $\mathcal{B}$ is a spine-subtree of type $\mathcal{S}_1$ with two leaves, having at least one neighbouring spine-subtree of type $\mathcal{S}_1$.

• $\mathcal{C}_a = \mathcal{S}_1[\lambda_1 \geq 3]$. A spine-subtree of type $\mathcal{C}_a$ is a spine-subtree of type $\mathcal{S}_1$ with at least three leaves.

• $\mathcal{C}_b = \mathcal{S}_2[\alpha_2^* = 1, \alpha_2 \geq 2]$. A spine-subtree of type $\mathcal{C}_b$ is a spine-subtree of type $\mathcal{S}_2$ having at least two branches of type $\mathcal{H}_2$ with exactly one of them having at most two leaves.

• $\mathcal{C}_c = \langle \mathcal{S}_2 \rangle\ \mathcal{S}_1[\lambda_1 = 2]\ \langle \mathcal{S}_2 \rangle$. A spine-subtree of type $\mathcal{C}_c$ is a spine-subtree of type $\mathcal{S}_1$ with two leaves having two neighbouring spine-subtrees of type $\mathcal{S}_2$.

• $\mathcal{C} = \mathcal{C}_a \mid \mathcal{C}_b \mid \mathcal{C}_c$.

• $\mathcal{D} = \mathcal{S}_2[\alpha_2^* \geq 2]$. A spine-subtree of type $\mathcal{D}$ is a spine-subtree of type $\mathcal{S}_2$ with at least two branches having at most two leaves.

The following observation directly follows from the previous definition, considering the neighbouring requirements, and will be useful later.

Observation 14. *A spine-subtree of type $\mathcal{B}$ or $\mathcal{C}_a$ cannot have a spine-subtree of type $\mathcal{C}_c$ as a neighbouring spine-subtree.*

It is not difficult to check that the set of types $\{\mathcal{A}, \mathcal{B}, \mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c, \mathcal{D}\}$ induces a partition of the spine-subtrees of any locally uniform 2-lobster (with possibly empty parts), as stated in the following proposition.

Proposition 15. *Let $\mathcal{T} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c, \mathcal{D}\}$, and L be any locally uniform 2-lobster. Every spine-subtree of L belongs to exactly one type in $\mathcal{T}$.*

3.2. Definition of $\beta^*(L)$

We are now able to define the value $\beta^*(L)$ for any locally uniform 2-lobster L, which will be proven to be the optimal cost of an independent broadcast on L. The value $\beta^*(L)$ will be expressed as a formula involving the number of 1-leaves, 2-leaves and 2-only-leaves, and the number of spine-subtrees, or sequences of spine-subtrees, of types defined in the previous subsection, appearing in L.

Finally, recall that $\lambda_1(L)$, $\lambda_2(L)$ and $\lambda_2^*(L)$ denote the number of 1-leaves, of 2-leaves and of 2-only-leaves in L, respectively. We are now able to define $\beta^*(L)$.

Definition ($\beta^*(L)$)**.** Let L be a locally uniform 2-lobster. We then let

$$\beta^*(L) = \nu_1(L) + \nu_2(L) + \nu_3(L) + \nu_4(L),$$

where

- $\nu_1(L) = \lambda_1(L) + \lambda_2(L) + \lambda_2^*(L)$ is the total number of leaves in L, where each 2-only-leaf is counted twice;
- $\nu_2(L)$ is the number of branches of type $\mathcal{H}_2$ in L with at most two 2-leaves;
- $\nu_3(L)$ is the number of spine-subtrees of type $\mathcal{C}_c$ in L;
- $\nu_4(L)$ is the sum, taken over all sequences of spine-subtrees $\mathbf{S}$ in L of type

$$\left\langle\overline{\mathcal{C}_c} \cdot (\emptyset|\mathcal{A}|\mathcal{B}|\mathcal{C}_a)\right\rangle \, \mathcal{A} \cdot \{(\mathcal{C}|\mathcal{A}) \cdot \mathcal{A}\}^* \, \left\langle(\emptyset|\mathcal{A}|\mathcal{B}|\mathcal{C}_a) \cdot \overline{\mathcal{C}_c}\right\rangle,$$

of the value

$$\frac{\ell(\mathbf{S}) + 1}{2} - \#_{\mathcal{C}_b,\mathcal{C}_c}(\mathbf{S}),$$

where $\ell(\mathbf{S})$ denotes the number of spine-subtrees in $\mathbf{S}$, and $\#_{\mathcal{C}_b,\mathcal{C}_c}(\mathbf{S})$ the number of spine-subtrees of type $\mathcal{C}_b$ or $\mathcal{C}_c$ in $\mathbf{S}$.

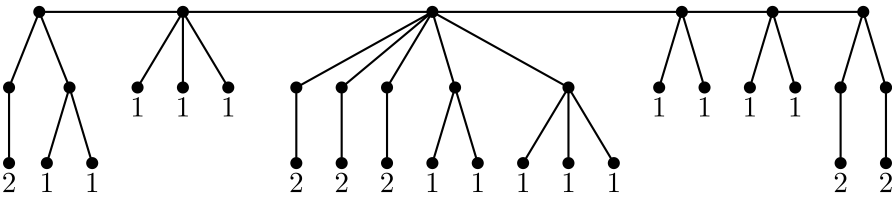
3.3. Lower bound

We will now prove that every locally uniform 2-lobster admits an independent broadcast f with $\text{cost}(f) = \beta^*(L)$. We consider the case of locally uniform 2-lobsters of length 0 separately.

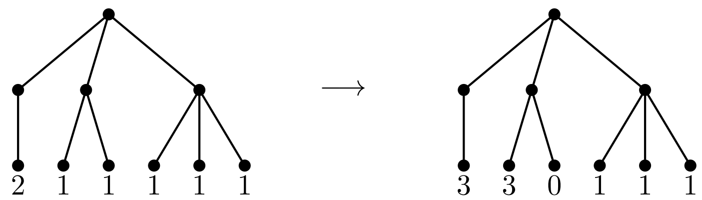
Lemma 16. *If L is a locally uniform 2-lobster of length $k = 0$, then there exists an independent broadcast f on L with $\text{cost}(f) = \beta^*(L)$, thus implying $\beta_b(L) \geq \beta^*(L)$.*

Proof. Recall that since $k = 0$, $L = S_0$ is necessarily of type $\mathcal{S}_2$ (Observation 6) and has at least two branches, so that $\text{diam}(L) = 4$. We construct an independent broadcast f on L as follows, by considering each branch separately. Let B be any branch of L. If B has at most two leaves, then we set $f(\ell) = 3$ for one leaf ℓ of B, and $f(\ell') = 0$ for the sister-leaf ℓ' of ℓ, if any (such a branch contributes 2 to $\nu_1(L)$ and 1 to $\nu_2(L)$). If B has at least three leaves, then we set $f(\ell) = 1$ for every leaf ℓ of B (such a branch contributes its number of leaves, that is $\lambda_2(B)$, to $\nu_1(L)$). Finally, if S_0 is of type $\mathcal{A}$, then we set $f(v_0) = 1$ (in that case, we have $\nu_4(L) = 1$).

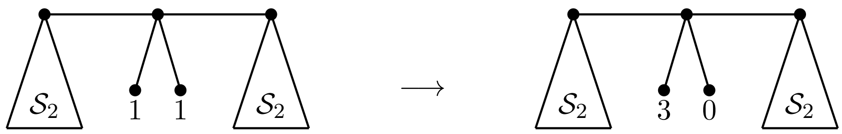
We then have $\text{cost}(f) = \nu_1(L) + \nu_2(L) + \nu_4(L)$, and thus, since $\nu_3(L) = 0$, $\text{cost}(f) = \beta^*(L)$. ∎



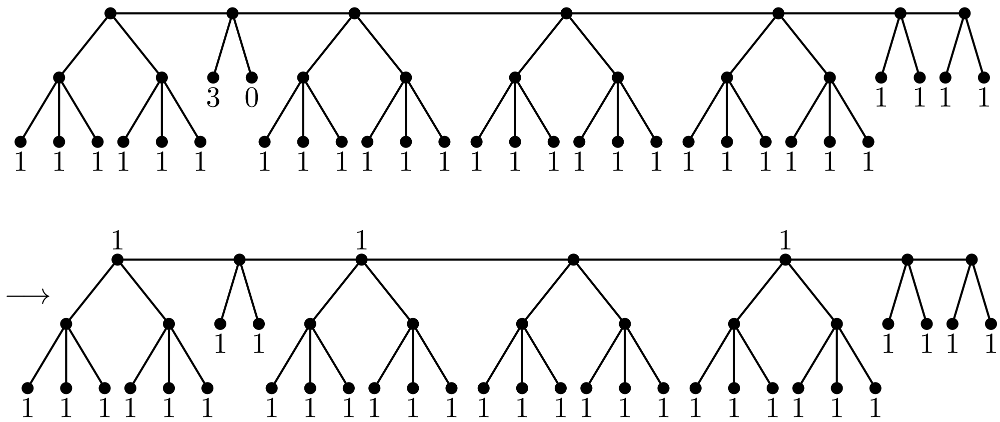
(a) The independent broadcast f_1 on a sample locally uniform 2-lobster



(b) From f_1 to f_2



(c) From f_2 to f_3



(d) From f_3 to f_4, for a sequence of type $\langle\emptyset\rangle\mathcal{A}\cdot\mathcal{C}_c\cdot\mathcal{A}\cdot\mathcal{A}\cdot\mathcal{A}\langle\mathcal{B}\cdot\mathcal{B}\rangle$ (cost increases by 2)

Figure 3. Proof of Lemma 17: from f_1 to f_4.

Lemma 17. *Every locally uniform* 2*-lobster* L *of length* $k \geq 1$ *admits an independent broadcast* f *with* $\text{cost}(f) = \beta^*(L)$*, thus implying* $\beta_b(L) \geq \beta^*(L)$*.*

Proof. We will construct a sequence of four independent broadcasts $f_1, \ldots, f_4$ on L, step by step, such that $\text{cost}(f_4) = \beta^*(L)$. Each independent broadcast f_i, $2 \leq i \leq 4$, is obtained by possibly modifying the independent broadcast f_{i-1}, and is such that $\text{cost}(f_i) \geq \text{cost}(f_{i-1})$. Moreover, for each independent broadcast f_i, $1 \leq i \leq 3$, we will have $f_i(v_j) = 0$ for every spine-vertex v_j, $0 \leq j \leq k$, while we may have $f_4(v_j) = 1$.

These modifications are illustrated in Figure 3, where dashed edges represent optional edges. These figures should help the reader to see that each mapping f_i is a valid independent broadcast on L.

Step 1. Let f_1 be the mapping defined by $f_1(u) = 2$ if u is an 2-only-leaf, $f_1(u) = 1$ if u is a leaf which is not an only-leaf and $f_1(u) = 0$ otherwise (see Figure 3(a)).

Clearly, f_1 is an independent broadcast on L and, since leaves are the only f_1-broadcast vertices, with f_1-values either 2 or 1 depending on whether there are 2-only-leaves or not, respectively, we have

$$\text{cost}(f_1) = \lambda_1(L) + \lambda_2(L) + \lambda_2^*(L) = \nu_1(L).$$

Step 2. We modify f_1 as follows, to obtain f_2. For every branch B_i^j of type $\mathcal{H}_2[\lambda_2 \leq 2]$, $0 \leq i \leq k$, $1 \leq j \leq b_i$, such that S_i is a spine-subtree of type $\mathcal{S}_2$, we let $f_2(\ell) = 3$ for one leaf ℓ of B_i^j, and $f_2(\ell') = 0$ for the sister-leaf ℓ' of ℓ, if any (see Figure 3(b)).

Again, f_2 is an independent broadcast on L and, since the value of each branch with at most two 2-leaves of a spine-subtree of type $\mathcal{S}_2$ has been increased by one, we have

$$\text{cost}(f_2) = \text{cost}(f_1) + \nu_2(L).$$

Step 3. We modify f_2 as follows, to obtain f_3. For every spine-subtree S of type $\mathcal{C}_c$, we let $f_3(\ell) = 3$ for one leaf ℓ of S, and $f_3(\ell') = 0$ for the sister-leaf ℓ' of ℓ (see Figure 3(c)). Note here that setting $f_3(\ell) = 3$ is allowed, since L does not contain any vertex x with $f_3(x) > 0$ at distance at most 3 from the leaves of S.

Again, f_3 is an independent broadcast on L, and, since the broadcast value of every considered spine-subtree S has been increased by 1, we have

$$\text{cost}(f_3) = \text{cost}(f_2) + \nu_3(L).$$

After these three steps, it is easy to check that the f_3-values are as follows.

Claim 18. *After Step* 3*, only leaves of* L *are broadcast vertices, and their values, depending on the type of the spine-subtree they belong to, are those values depicted in Figure* 2*.*

Before describing the last step, we introduce some terminology.

A spine-subtree S *exceeds by* e, for some integer $e \geq 1$, if S contains a 1-leaf with broadcast value $e+1$, or a 2-leaf with broadcast value $e+2$. Therefore, if a spine-subtree S_i, $0 \leq i \leq k$, of a locally uniform 2-lobster L exceeds by $e \geq 1$, then none of the spine vertices $v_{i-e}, \dots, v_i, \dots, v_{i+e}$ can be a broadcast vertex.

We can then partition the set $\mathcal{T} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}_a, \mathcal{C}_b, \mathcal{C}_c, \mathcal{D}\}$ in three parts $\mathcal{E}_0$, $\mathcal{E}_1$ and $\mathcal{E}_2$, corresponding to the types of spine-subtrees that do not exceed, or that exceed by 1 or 2, respectively, after Step 3 (as given in Claim 18). In order to be complete, we will also say that the "empty subtree", of type $\emptyset$, does not exceed. Therefore, we have

$$\mathcal{E}_0 = \{\emptyset, \mathcal{A}, \mathcal{B}, \mathcal{C}_a\}, \quad \mathcal{E}_1 = \{\mathcal{C}_b, \mathcal{D}\} \text{ and } \mathcal{E}_2 = \{\mathcal{C}_c\}.$$

Moreover, we denote by $\overline{\mathcal{E}_i}$ the complement of $\mathcal{E}_i$ for every i, $0 \leq i \leq 2$, that is, $\overline{\mathcal{E}_i} = (\mathcal{T} \cup \{\emptyset\}) \setminus \mathcal{E}_i$ and, for any two sets of types of spine-subtrees $\mathcal{X} = \{\mathcal{X}_1, \dots, \mathcal{X}_p\}$ and $\mathcal{Y} = \{\mathcal{Y}_1, \dots, \mathcal{Y}_q\}$, we denote by $\mathcal{X}.\mathcal{Y}$ the set of types $\{\mathcal{X}_i.\mathcal{Y}_j : 1 \leq i \leq p,\ 1 \leq j \leq q\}$.

Let S be a spine-subtree of type $\mathcal{A}$. By *increasing* S *by one*, we mean giving the broadcast value 1 to the root of S (observe that only leaves of S are f_3-broadcast vertices, and that $f_3(\ell) = 1$ for every such leaf ℓ).

Let now S be a spine-subtree of type $\mathcal{C}_b$ or $\mathcal{C}_c$. By *decreasing* S *by one*, we mean the following (recall that the broadcast values assigned by f_3 are depicted in Figure 2).

- If S is of type $\mathcal{C}_b$, then we give the broadcast value 2 to one leaf of the (unique) branch of type $\mathcal{H}_2[\lambda_2 \leq 2]$, and the broadcast value 0 to its sister-leaf, if any.
- If S is of type $\mathcal{C}_c$, then we give the broadcast value 1 to each of the two leaves of S.

Observe that after having been decreased by one, a spine-subtree of type $\mathcal{C}_b$ or $\mathcal{C}_c$ does no longer exceed.

We are now able to describe the fourth step of the proof. The key idea of this last step is to increase by one some spine-subtrees of type $\mathcal{A}$, and decrease by one some spine-subtrees of type $\mathcal{C}_b$ or $\mathcal{C}_c$, provided that this results in a strict increasing of the cost of the current independent broadcast on L.

Step 4. We modify f_3 as follows, to obtain f_4. For every sequence $\mathbf{S}$ of spine-subtrees $A_0X_1A_1 \cdots X_pA_p$, $p \geq 0$, of type

$$\mathcal{T}_4 = \langle \overline{\mathcal{E}_2} \cdot \mathcal{E}_0 \rangle\ \mathcal{A} \cdot \{(\mathcal{C}|\mathcal{A}) \cdot \mathcal{A}\}^*\ \langle \mathcal{E}_0 \cdot \overline{\mathcal{E}_2} \rangle,$$

we decrease by one each spine-subtree X_i of type $\mathcal{C}_b$ or $\mathcal{C}_c$, $1 \leq i \leq p$, and increase by one each spine-subtree A_j, $0 \leq j \leq p$ (see Figure 3(d)). Note that this can be done since none of the spine-subtrees X_i, $1 \leq i \leq p$, exceeds and no spine-subtree outside the sequence can prevent us from doing so on the extremal spine-subtrees A_0 and A_p.

The broadcast value of the whole sequence $\mathbf{S}$ is thus increased by $p+1$, that is $\frac{\ell(\mathbf{S})+1}{2}$, minus the number of spine-subtrees of type $\mathcal{C}_b$ or $\mathcal{C}_c$, that is $\#_{\mathcal{C}_b,\mathcal{C}_c}(\mathbf{S})$. (Note that since the number of spine-subtrees of type $\mathcal{C}_b$ or $\mathcal{C}_c$ is at most p, this broadcast value always increases.) Therefore, doing the above modification for every sequence of spine-subtrees of type $\mathcal{T}_4$, the so-obtained independent broadcast f_4 satisfies

$$\text{cost}(f_4) = \text{cost}(f_3) + \nu_4(L).$$

We finally get $\text{cost}(f_4) = \beta^*(L)$, as required. This completes the proof. ■

3.4. Upper bound

We first prove that, for every locally uniform 2-lobster L, we can choose an optimal independent broadcast on L that satisfies some given properties.

The next lemma shows that every locally uniform 2-lobster L of length $k \geq 1$ admits an optimal independent broadcast $\tilde{f}$ such that the $\tilde{f}$-values of the leaves in the spine-subtrees S_0 and S_k are at most 3.

Lemma 19. *If L is a locally uniform 2-lobster of length $k \geq 1$, then there exists an optimal independent broadcast $\tilde{f}$ on L such that $\tilde{f}(\ell) \leq 3$ for every leaf ℓ of S_0 and S_k.*

Proof. Let f be any optimal independent broadcast on L. Recall that, by Observation 6, both spine-subtrees S_0 and S_k must be of type $\mathcal{S}_2$. Also note that, by symmetry, it is enough to prove the result for S_0. If S_0 has at least two broadcast leaves, then the broadcast value of each of them is at most 3, since every two such leaves are at distance at most 4 from each other. We thus only need to consider the case when S_0 has a unique broadcast leaf. Moreover, we can assume that the broadcast value of this leaf is at least 7, since otherwise, by setting the broadcast value of any two leaves at distance 4 from each other to 3, we would get either a broadcast satisfying the requirement of the lemma, or a contradiction with the optimality of the broadcast. Therefore, we get that the result holds if $k \leq 3$ since, in that case, $\text{diam}(L) \leq 7$, which implies $f(v) \leq 6$ for every vertex v of L since f is maximal.

The proof now is by contradiction. Let L be a counter-example to the lemma, of length $k \geq 4$, with $\text{diam}(L) > 7$, and f be an optimal independent broadcast on L which minimizes the value of $f(\ell) = \alpha$, where ℓ is the (unique) f-broadcast leaf of S_0. We thus have $\alpha \geq 7$.

Observe that at least one vertex at distance $\alpha + 1$ from ℓ must be an f-broadcast vertex, since otherwise we could increase the value of $f(\ell)$ by 1, contradicting the optimality of f. Let x denote any such vertex. The spine-subtrees

$S_1, \ldots, S_{\alpha-4}$ do not contain any f-broadcast vertex (since every such vertex is f-dominated by ℓ), and x is either a 2-leaf of $S_{\alpha-3}$, a 1-leaf of $S_{\alpha-2}$, or the spine-vertex $v_{\alpha-1}$.

We consider four cases, depending on whether these vertices are f-broadcast vertices or not. For each of these cases, we assume that none of the previous cases occurs.

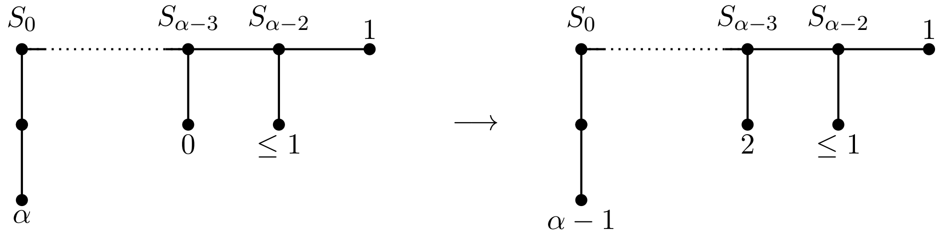
1. *$v_{\alpha-1}$ is an f-broadcast vertex.* In this case, by Corollary 7, we know that $f(x) = 1$. Consider the spine-subtree $S_{\alpha-3}$. If $S_{\alpha-3}$ is of type $\mathcal{S}_1$, then all its vertices are f-dominated by ℓ, and the mapping g defined by $g(\ell) = \alpha - 1$, $g(\ell_{\alpha-3}) = 2$ for one leaf $\ell_{\alpha-3}$ of $S_{\alpha-3}$ and $g(v) = f(v)$ for every other vertex v of L is clearly an independent broadcast on L with $\text{cost}(g) = \text{cost}(f) - 1 + 2 = \text{cost}(f) + 1$, which contradicts the optimality of f (see Figure 4(a)). Now, if $S_{\alpha-3}$ is of type $\mathcal{S}_2$, then $f(\ell_{\alpha-3}) \leq 3$ for every leaf $\ell_{\alpha-3}$ of $S_{\alpha-3}$. Therefore, the mapping g defined by $g(\ell) = \alpha - 2$, $g(\ell_{\alpha-4}) = 2$ for one leaf $\ell_{\alpha-4}$ of $S_{\alpha-4}$ and $g(v) = f(v)$ for every other vertex v of L is clearly an independent broadcast on L, with $\text{cost}(g) = \text{cost}(f) - 2 + 2 = \text{cost}(f)$, which contradicts the minimality of α (see Figure 4(b), where $S_{\alpha-4}$ is of type $\mathcal{S}_1$, the case $S_{\alpha-4}$ of type $\mathcal{S}_2$ being similar). Therefore, $v_{\alpha-1}$ cannot be an f-broadcast vertex.

2. *Both a 2-leaf $\ell_{\alpha-3}$ of $S_{\alpha-3}$ and a 1-leaf $\ell_{\alpha-2}$ of $S_{\alpha-2}$ are f-broadcast vertices.* In this case, we necessarily have $f(\ell_{\alpha-3}) \leq 3$ and $f(\ell_{\alpha-2}) \leq 3$. Therefore, the mapping g defined by $g(\ell) = \alpha - 2$, $g(\ell_{\alpha-4}) = 3$ for one leaf $\ell_{\alpha-4}$ of $S_{\alpha-4}$ and $g(v) = f(v)$ for every other vertex v of L is clearly an independent broadcast on L with $\text{cost}(g) = \text{cost}(f) - 2 + 3 = \text{cost}(f) + 1$, which contradicts the optimality of f (see Figure 4(c), where again $S_{\alpha-4}$ is of type $\mathcal{S}_1$, the case $S_{\alpha-4}$ of type $\mathcal{S}_2$ being similar).

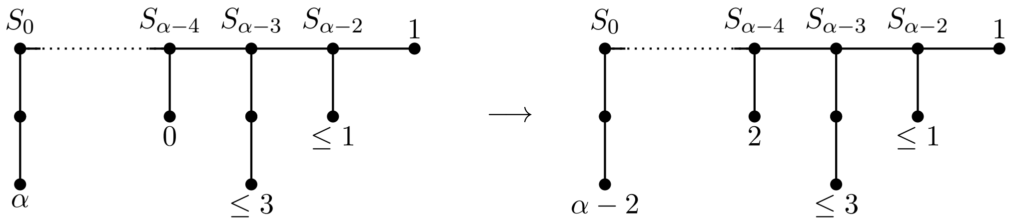
3. *A 1-leaf $\ell_{\alpha-2}$ of $S_{\alpha-2}$ is an f-broadcast vertex.* We let $\beta = f(\ell_{\alpha-2})$. We necessarily have $\beta \leq \alpha$. Consider the mapping g defined by $g(\ell) = g(\ell') = 3$, where ℓ' is a 2-leaf of S_0 not belonging to the same branch as ℓ (recall that S_0 has at least two branches), $g(\ell_j) = 2$ for one leaf ℓ_j of each spine-subtree S_j, $1 \leq j \leq \alpha - 3$, $g(\ell_{\alpha-2}) = 2$, and $g(v) = f(v)$ for every other vertex v of L. The mapping g is clearly an independent broadcast on L (recall that $v_{\alpha-1}$ is not an f-broadcast vertex) with $\text{cost}(g) = \text{cost}(f) - \alpha - \beta + 6 + 2(\alpha - 2) = \text{cost}(f) + (\alpha - \beta) + 2$, which contradicts the optimality of f (see Figure 4(d), where $S_1, \ldots, S_{\alpha-3}$ are of type $\mathcal{S}_1$, all other cases being similar).

4. *A 2-leaf $\ell_{\alpha-3}$ of $S_{\alpha-3}$ is an f-broadcast vertex.* Note first that if $\alpha - 3 = k$, then the optimality of f implies $f(\ell_{\alpha-3}) = \alpha$, so that $\text{cost}(f) = 2\alpha = 2(\text{diam}(L) - 1)$, in contradiction with Observation 9. We thus have $\alpha - 3 < k$.

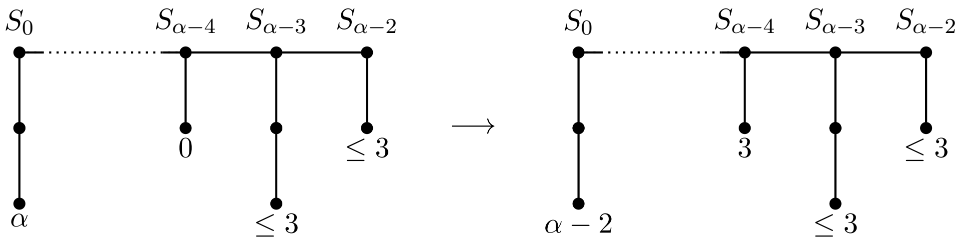
We let $\beta = f(\ell_{\alpha-3})$. We necessarily have $\beta \leq \alpha$. Similarly as in the previous case, consider the mapping g defined by $g(\ell) = g(\ell') = 3$, where ℓ' is a 2-leaf of S_0 not belonging to the same branch as ℓ, $g(\ell_j) = 2$ for one leaf ℓ_j of each spine-subtree S_j, $1 \leq j \leq \alpha - 4$, $g(\ell_{\alpha-3}) = 3$, and $g(v) = f(v)$ for every other



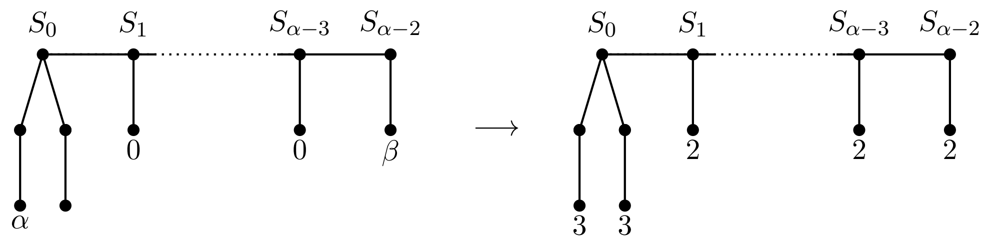
(a) Case 1, $S_{\alpha-3}$ is of type $\mathcal{S}_1$



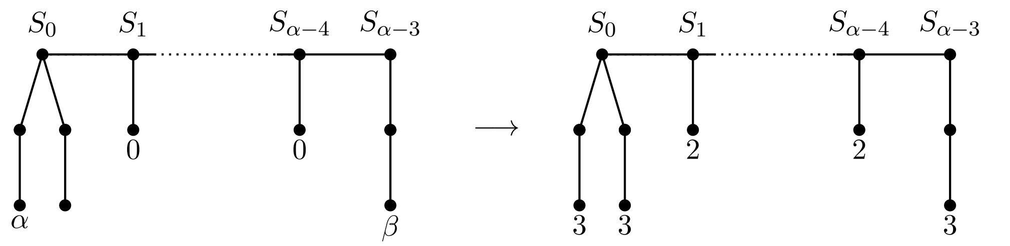
(b) Case 1, $S_{\alpha-3}$ is of type $\mathcal{S}_2$ (with $S_{\alpha-4}$ being of type $\mathcal{S}_1$)



(c) Case 2 (with $S_{\alpha-4}$ being of type $\mathcal{S}_1$)



(d) Case 3 (with $S_1, \ldots, S_{\alpha-3}$ being of type $\mathcal{S}_1$)



(e) Case 4 (with $S_1, \ldots, S_{\alpha-4}$ being of type $\mathcal{S}_1$)

Figure 4. Independent broadcasts for the proof of Lemma 19 (except for S_0 when needed, only one branch per spine-subtree is depicted).

vertex v of L. The mapping g is clearly an independent broadcast on L with $\text{cost}(g) = \text{cost}(f) - \alpha - \beta + 6 + 2(\alpha - 4) + 3 = \text{cost}(f) + (\alpha - \beta) + 1$, which again contradicts the optimality of f (see Figure 4(e), where $S_1, \ldots, S_{\alpha-4}$ are of type $\mathcal{S}_1$, all other cases being similar).

We thus obtain a contradiction in each case, which implies that no counter-example to the lemma exists. This completes the proof. ■

Let f be any independent broadcast on a locally uniform 2-lobster L and S_i be any spine-subtree of L. Recall that $f^*(S_i)$ denotes the broadcast value of S_i, that is, the sum of the broadcast values of the vertices of S_i. The next lemma shows that every locally uniform 2-lobster L of length $k \geq 1$ admits an optimal independent broadcast $\tilde{f}$ on L such that every spine-subtree of L contains an $\tilde{f}$-broadcast vertex.

Lemma 20. *If L is a locally uniform 2-lobster of length $k \geq 1$, then there exists an optimal independent broadcast $\tilde{f}$ on L such that $\tilde{f}^*(S_i) > 0$ for every i, $0 \leq i \leq k$.*

Proof. Since the result directly follows from Lemma 8 if $k = 1$, we only need to consider the case $k > 1$. Assume to the contrary that there does not exist any such independent broadcast $\tilde{f}$, and let f be any optimal independent broadcast on L that maximises the number of f-broadcast leaves. Let i be the smallest index such that $f^*(S_i) = 0$, and ℓ_i be any leaf of S_i. By Lemma 8, we can assume that f has be chosen in such a way that $1 \leq i < k$ holds.

Suppose first that S_i is of type $\mathcal{S}_2$, so that ℓ_i is a 2-leaf. From the choice of S_i, we know that $f^*(S_{i-1}) > 0$. Moreover, we claim that S_{i-1} is of type $\mathcal{S}_2$ and that, for every leaf ℓ_{i-1} of S_{i-1}, $f(\ell_{i-1}) \leq 4$. Indeed, this follows from Lemma 19 if $i = 1$. If $i \geq 2$, S_{i-1} cannot be of type $\mathcal{S}_1$ since otherwise any vertex that f-dominates ℓ_i would also f-dominate all the leaves of S_{i-1} (such a dominating vertex must belong to some S_j with $j > i$). Thus S_{i-1} is of type $\mathcal{S}_2$ and the fact that $f^*(S_{i-2}) > 0$ implies $f(\ell_{i-1}) \leq 4$ for every leaf ℓ_{i-1} of S_{i-1}. Therefore, ℓ_i is necessarily f-dominated by a vertex $y \in S_j$, for some $j > i$, with $f(y) = d_L(y, \ell_i)$ ($f(y) > d_L(y, \ell_i)$ would imply that y dominates the leaves of S_{i-1}, contradicting the independence of f). Consider the mapping g defined by $g(y) = f(y) - 1$, $g(\ell_i) = 1$, and $g(w) = f(w)$ for every other vertex w of L. The mapping g is an independent broadcast on L with $\text{cost}(g) = \text{cost}(f)$, which contradicts the maximality of the number of f-broadcast leaves.

Suppose now that S_i is of type $\mathcal{S}_1$, so that ℓ_i is a 1-leaf, and that ℓ_i is f-dominated by a unique vertex y. If $y \in S_{i-1}$, by Lemma 19, we cannot have $y \in S_0$, which gives $i \geq 2$. Moreover, we necessarily have $f(y) = d_L(y, \ell_i)$, since otherwise y would dominate a leaf of S_{i-2}, contradicting the independence of f. Now, the mapping g defined by $g(y) = f(y) - 1$, $g(\ell_i) = 1$, and $g(w) = f(w)$ for every other vertex w of L, is an independent broadcast on L, with $\text{cost}(g) = \text{cost}(f)$, which contradicts the maximality of the number of f-broadcast leaves. If $y \in S_j$ for some $j > i$, then we necessarily have either $f(y) = d_L(y, \ell_i)$, if S_{i-1} is of type $\mathcal{S}_1$, or $d_L(y, \ell_i) \leq f(y) \leq d_L(y, \ell_i) + 1$, if S_{i-1} is of type $\mathcal{S}_2$. Therefore, the mapping g defined by $g(y) = d_L(y, \ell_i) - 1$, $g(\ell_i) = 1 + f(y) - d_L(y, \ell_i)$, and $g(w) = f(w)$ for every other vertex w of L, is an independent broadcast on L, with $\text{cost}(g) \geq \text{cost}(f)$, which again contradicts the maximality of the number of f-broadcast leaves.

Suppose finally that S_i is of type $\mathcal{S}_1$ and that ℓ_i is f-dominated by two distinct vertices y_1 and y_2, with $y_1 \in S_{i_1}$ and $y_2 \in S_{i_2}$. Note that we necessarily have, without loss of generality, $i_1 = i - 1$ and $i < i_2$. We claim that $f(y_1) \geq 3$ and $f(y_2) \geq 3$. Indeed, if, say, $f(y_1) = 2$, then $y_1 = v_{i-1}$, which contradicts Corollary 7. The case $f(y_2) = 2$ is similar. Moreover, we clearly have either $f(y_1) = 3$ and $f(y_2) = d_L(y_2, \ell_i)$, if S_{i-1} is of type $\mathcal{S}_1$, or $f(y_1) = 4$ and $d_L(y_2, \ell_i) \leq f(y_2) \leq d_L(y_2, \ell_i) + 1$, if S_{i-1} is of type $\mathcal{S}_2$. We consider two cases, depending on the value of $f(y_2)$.

1. $f(y_2) = d_L(y_2, \ell_i)$. In that case, the mapping g defined by $g(y_1) = f(y_1) - 1$, $g(y_2) = f(y_2) - 1$, $g(\ell_i) = 2$, and $g(w) = f(w)$ for every other vertex w of L, is an independent broadcast on L, with $\text{cost}(g) = \text{cost}(f)$, which contradicts the maximality of the number of f-broadcast leaves.

2. $f(y_2) = d_L(y_2, \ell_i) + 1$. In that case, we necessarily have that S_{i-1} is of type $\mathcal{S}_2$ and $f(y_1) = 4$ on one hand, and $f(y_2) \geq 4$ on the other hand, which implies that $f^*(S_{i_2+1}) = 0$, if $i_2 < k$.

If y_2 is not a 1-leaf of S_{i+1}, then the mapping g defined by $g(y_1) = 3$, $g(y_2) = f(y_2) - 2$, $g(\ell_i) = 3$, and $g(w) = f(w)$ for every other vertex w of L, is an independent broadcast on L, with $\text{cost}(g) = \text{cost}(f)$, which contradicts the maximality of the number of f-broadcast leaves.

Otherwise, that is, if y_2 is a 1-leaf of S_{i+1} and $f(y_2) = 4$, we cannot give to ℓ_i the broadcast value 3, since ℓ_i would then dominate y_2. Observe that since S_{i+1} is of type $\mathcal{S}_1$, we have $i + 1 < k$, so that S_{i+2} exists.

If S_{i+2} is of type $\mathcal{S}_2$, then the leaves of S_{i+2} are necessarily f-dominated only by y_2. Let ℓ_{i+2} be any leaf of S_{i+2}. In that case, the mapping g defined by $g(y_1) = 3$, $g(\ell_i) = 2$, $g(y_2) = 2$, $g(\ell_{i+2}) = 1$, and $g(w) = f(w)$ for every other vertex w of L, is an independent broadcast on L, with $\text{cost}(g) = \text{cost}(f)$, which contradicts the maximality of the number of f-broadcast leaves.

Suppose finally that S_{i+2} is of type $\mathcal{S}_1$, and let ℓ_{i+2} denote any 1-leaf of S_{i+2}. Note that y_2 f-dominates ℓ_{i+2}. If ℓ_{i+2} is f-dominated only by y_2, then the mapping g defined by $g(y_1) = 3$, $g(\ell_i) = 2$, $g(y_2) = 2$, $g(\ell_{i+2}) = 2$, and $g(w) = f(w)$ for every other vertex w of L, is an independent broadcast on L, with $\text{cost}(g) > \text{cost}(f)$, which contradicts the optimality of f. Otherwise, let z be the other vertex of L which f-dominates ℓ_{i+2}. Then, the mapping g defined by $g(y_1) = 3$, $g(\ell_i) = 2$, $g(y_2) = 2$, $g(\ell_{i+2}) = 2$, $g(z) = f(z) - 1$, and $g(w) = f(w)$ for every other vertex w of L, is an independent broadcast on L, with $\text{cost}(g) = \text{cost}(f)$, which contradicts the maximality of the number of f-broadcast leaves. This completes the proof. ■

From Lemma 20 and Corollary 7, we get the following corollary.

Corollary 21. *If L is a locally uniform 2-lobster of length $k \geq 1$, then there exists an optimal independent broadcast $\tilde{f}$ on L such that, for every spine-subtree S_i of L, $0 \leq i \leq k$, and every vertex x of S_i, $\tilde{f}(x) \leq 1$ if $x = v_i$, $\tilde{f}(x) \leq 3$ if x is a 1-leaf of S_i, and $\tilde{f}(x) \leq 4$ if x is a 2-leaf of S_i.*

Proof. Let $\tilde{f}$ be an optimal independent broadcast on L such that $\tilde{f}^*(S_i) > 0$ for every i, $0 \leq i \leq k$. The existence of $\tilde{f}$ is guaranteed by Lemma 20. If $x = v_i$, then $\tilde{f}(x) \leq 1$ follows from Corollary 7. Otherwise, assuming that the claimed bound on $\tilde{f}(x)$ is not satisfied would imply $\tilde{f}^*(S) = 0$ for a neighbouring spine-subtree S of S_i, in contradiction with our assumption on $\tilde{f}$. ■

The next two lemmas show that every locally uniform 2-lobster L of length $k \geq 1$ admits an optimal independent broadcast $\tilde{f}$ on L such that the $\tilde{f}$-value of every spine-subtree S of L is bounded from above by a value depending on the type of S.

Recall that $\mathcal{T}_4$ denotes the type of sequence used in Step 4 in the proof of Lemma 17, that is

$$\mathcal{T}_4 = \langle \overline{\mathcal{E}_2} \cdot \mathcal{E}_0 \rangle \ \mathcal{A} \cdot \{(\mathcal{C}|\mathcal{A}) \cdot \mathcal{A}\}^* \ \langle \mathcal{E}_0 \cdot \overline{\mathcal{E}_2} \rangle.$$

In the following, when we say that a spine-subtree S_i *appears as an $\mathcal{A}$-spine-subtree* (respectively, *as a $\mathcal{C}$-spine-subtree*) *in a sequence of type $\mathcal{T}_4$*, we mean that $S_i = A_j$ (respectively, $S_i = X_j$) for some j, $0 \leq j \leq p$ (respectively, $1 \leq j \leq p$), in the corresponding sequence $A_0 X_1 A_1 \cdots X_p A_p$.

Lemma 22. *If L is a locally uniform 2-lobster of length $k \geq 1$, then there exists an optimal independent broadcast $\tilde{f}$ on L such that, for every spine-subtree S_i of L, $0 \leq i \leq k$, $\tilde{f}$ satisfies the following properties.*

1. $\tilde{f}^*(S_i) > 0$.
2. *If* $\tilde{f}^*(S_i) = \lambda_1(S_i)$, *or* $\tilde{f}^*(S_i) = \lambda_2(S_i)$, *then* $\tilde{f}(\ell) = 1$ *for every leaf of* S_i.
3. *If S_i is of type $\mathcal{S}_1$, then*
   (a) $\tilde{f}^*(S_i) \leq \lambda_1(S_i)$ *if S_i is of type $\mathcal{B}$ or $\mathcal{C}_a$,*
   (b) $\tilde{f}^*(S_i) \leq 3$ *if S_i is of type $\mathcal{C}_c$,*
   (c) $\tilde{f}^*(S_i) \leq 2$ *if S_i is of type $\mathcal{C}_c$ and S_i belongs to a sequence of type $\mathcal{T}_4$.*
4. *If S_i is of type $\mathcal{S}_2$, then*
   (a) $\tilde{f}^*(S_i) \leq \lambda_2(S_i) + 1$ *if S_i is of type $\mathcal{A}$,*
   (b) $\tilde{f}^*(S_i) \leq \lambda_2(S_i) + \lambda_2^*(S_i) + \alpha_2^*(S_i)$ *if S_i is of type $\mathcal{C}_b$ or $\mathcal{D}$,*
   (c) $\tilde{f}^*(S_i) \leq \lambda_2(S_i) + \lambda_2^*(S_i) + \alpha_2^*(S_i) - 1$ *if S_i is of type $\mathcal{C}_b$ and S_i belongs to a sequence of type $\mathcal{T}_4$.*
5. *If S_i is not of type $\mathcal{A}$, then* $\tilde{f}(v_i) = 0$.

Proof. Thanks to Lemma 20, we know that we can choose an independent broadcast $\tilde{f}$ on L which satisfies item 1. By Corollary 21, we get that the $\tilde{f}$-value of every 1-leaf is at most 3, and that the $\tilde{f}$-value of every 2-leaf is at most 4. This observation will be implicitly used all along the proof.

Note also that if $\tilde{f}^*(S_i) = \lambda_1(S_i)$, or $\tilde{f}^*(S_i) = \lambda_2(S_i)$, then we can obviously modify $\tilde{f}$, in order to satisfy item 2, without modifying its cost.

We now prove that $\tilde{f}$ can be chosen in such a way that it satisfies all the other items of the lemma. Let S_i be any spine-subtree of L.

Note first that the above observation already proves item 3(b). Moreover, observe that we necessarily have $\tilde{f}^*(S_i) \leq \lambda_1(S_i)$ if S_i is of type $\mathcal{B}$ or $\mathcal{C}_a$, since in each of these cases, the only way to attain the value $\lambda_1(S_i)$ is to have one leaf ℓ of S_i with $\tilde{f}(\ell) = \tilde{f}^*(S_i)$, which would imply that a neighbouring spine-subtree of S_i has no $\tilde{f}$-broadcast vertex, in contradiction with item 1. This proves item 3(a).

Suppose now that S_i is of type $\mathcal{A}$. If the broadcast value of a 2-leaf of S_i is 2, then its at least two sister-leaves cannot be $\tilde{f}$-broadcast vertices since this would contradict the independence of $\tilde{f}$. Therefore, the greatest possible value of $\tilde{f}^*(S_i)$ is obtained when the spine-vertex v_i and all the 2-leaves of S_i have $\tilde{f}$-value 1. This gives $\tilde{f}^*(S_i) \leq \lambda_2(S_i) + 1$, which proves item 4(a).

Suppose now that S_i is of type $\mathcal{C}_b$ or $\mathcal{D}$. Observe first that, if $\tilde{f}(v_i) = 1$, then the broadcast value of every leaf of S_i is 1. The optimality of $\tilde{f}$ then implies that S_i has a unique branch B with two leaves and no branch with a unique leaf, since otherwise we could set to 0 the broadcast value of v_i and to 3 the broadcast value of one leaf of every such branch, and thus increase the cost of $\tilde{f}$. (Note also that, for the same cost, we can set $\tilde{f}(v_i) = 0$ and set to 0 and 3 the broadcast value of the two leaves of this branch. This remark will be useful in the next paragraph.) In that case, we thus have $\tilde{f}^*(S_i) = \lambda_2(S_i) + 1 \leq \lambda_2(S_i) + \lambda_2^*(S_i) + \alpha_2^*(S_i)$, which proves item 4(b). Suppose now $\tilde{f}(v_i) = 0$ and let B be any branch of S_i. The optimality of $\tilde{f}$ then implies the following. If B has one or two 2-leaves, the $\tilde{f}$-value of one of theses leaves is 3 (otherwise, we would have $\tilde{f}^*(B) \leq 2$). If B has at least three leaves, the largest possible value of $\tilde{f}^*(B)$ is $\lambda_2(B)$, since as soon as a 2-leaf has a broadcast value at least 2, none of its sister-leaves can be a broadcast vertex. (Note that if B has three 2-leaves, then either one of them has $\tilde{f}$-value 3, or, for the same cost, each of them has $\tilde{f}$-value 1.) Therefore, $\tilde{f}^*(S_i) \leq \lambda_2(S_i) + \lambda_2^*(S_i) + \alpha_2^*(S_i)$, which proves item 4(b).

Suppose now that S_i is of type $\mathcal{C}_b$ or $\mathcal{C}_c$, and belongs to some sequence of type $\mathcal{T}_4$. In such a sequence, each spine-subtree of type $\mathcal{C}$ is associated with one of its neighbouring spine-subtrees of type $\mathcal{A}$, in such a way that no spine-subtree of type $\mathcal{A}$ is associated with two distinct spine-subtrees of type $\mathcal{C}$. Let S_i' denote the spine-subtree associated with S_i (we have $S_i' \in \{S_{i-1}, S_{i+1}\}$). On the one hand, from the above discussion about spine-subtrees of type $\mathcal{A}$, we know that their largest possible broadcast value can be attained only if their spine-vertex has broadcast value 1. On the other hand, from the above discussion about spine-subtrees of type $\mathcal{C}_b$ or $\mathcal{C}_c$ (when proving items 4(b) and 3(b), respectively), we know that their largest possible broadcast value can be attained only if one leaf ℓ of the unique branch B of S_i having at most two leaves has a broadcast value of 3. Therefore, S_i and S_i' cannot get their largest possible broadcast value both at the same time. We thus need either to remove the broadcast value of the spine-vertex of S_i', or to give the broadcast value 2 to ℓ if ℓ is a 2-only-leaf, or 1 to each 2-leaf of B otherwise. This second choice proves that the optimal independent broadcast $\tilde{f}$ can be chosen in order to satisfy items 4(c) and 3(c).

It remains to prove item 5. If S_i is of type $\mathcal{S}_1$, the result follows from Lemma 3. We thus only need to consider the case when S_i is of type $\mathcal{C}_b$ or $\mathcal{D}$. Suppose that $\tilde{f}(v_i) = 1$ (we cannot have $\tilde{f}(v_i) > 1$ by Corollary 7). If S_i is of type $\mathcal{C}_b$, then we can set $\tilde{f}(v_i) = 0$, $\tilde{f}(\ell) = 3$ for a 2-leaf ℓ of S_i belonging to the unique branch of S_i having at most two leaves, and $\tilde{f}(\ell') = 0$ for the sister-leaf ℓ' of ℓ, if any. Such a modification does not decrease the cost of $\tilde{f}$ and we are done. If S_i is of type $\mathcal{D}$, then we cannot have $\tilde{f}(v_i) = 1$, since this would contradict the optimality of $\tilde{f}$, as the previous modification can be done on at least two branches of S_i having at most two 2-leaves. This completes the proof. ∎

Lemma 23. *If L is a locally uniform 2-lobster of length $k \geq 1$, then there exists an optimal independent broadcast $\tilde{f}$ on L such that, for every spine-subtree S_i of L, $0 \leq i \leq k$, $\tilde{f}$ satisfies the following properties.*

1. *$\tilde{f}$ satisfies all the items of Lemma* 22,
2. *$\tilde{f}^*(S_i) \leq \lambda_2(S_i)$ if S_i is of type $\mathcal{A}$ and S_i does not appear as an $\mathcal{A}$-spine-subtree in a sequence of type $\mathcal{T}_4$.*

Proof. Thanks to Lemma 22, we know that we can choose an independent broadcast on L which satisfies item 1. So consider such a broadcast $\tilde{f}$ on L. Recall that by item 5 of Lemma 22, $\tilde{f}(v_i) = 0$ for every spine-subtree S_i which is not of type $\mathcal{A}$. Moreover, by item 4(a) of Lemma 22, if S_i is a spine-subtree of L of type $\mathcal{A}$, then $\tilde{f}^*(S_i) \leq \lambda_2(S_i) + 1$, and, as observed in the proof of that lemma, the only way to attain this value is to give a broadcast value of 1 to the spine-vertex and to all the 2-leaves of S_i.

Suppose that there exists in L a spine-subtree S_i of type $\mathcal{A}$, that does not appear as an $\mathcal{A}$-spine-subtree in any sequence of type $\mathcal{T}_4$, and such that $\tilde{f}(v_i) = 1$, which implies $\tilde{f}(\ell) = 1$ for every leaf ℓ of S_i since $\tilde{f}$ is optimal. Such a spine-subtree will be called a *bad spine-subtree*. Moreover, suppose that S_i is the leftmost such bad spine-subtree of L. We claim that the broadcast $\tilde{f}$ can be modified, without decreasing its cost, in such a way that either the number of bad spine-subtrees in L strictly decreases, or this number is still the same but the index of the leftmost bad spine-subtree in L strictly increases, which will prove item 2.

All along this proof, we will modify the independent broadcast $\tilde{f}$ on some spine-subtrees, according to their type. By applying the *standard modification of* $\tilde{f}$ on a spine-subtree S_j of L, we mean the following.

- If S_j is a bad spine-subtree, then we set $\tilde{f}(v_j) = 0$.
- If S_j is of type $\mathcal{A}$, S_j is not a bad spine-subtree, and $\tilde{f}(v_j) = 0$, then we set $\tilde{f}(v_j) = 1$.
- If S_j is of type $\mathcal{C}_b$, then we set $\tilde{f}(\ell) = 3$ for a 2-leaf ℓ of the unique branch of S_j having at most two leaves, and $\tilde{f}(\ell') = 0$ for the sister-leaf of ℓ, if any.
- If S_j is of type $\mathcal{C}_c$, then we set $\tilde{f}(\ell) = 3$ for a 1-leaf ℓ of S_j, and $\tilde{f}(\ell') = 0$ for the sister-leaf of ℓ.
- If S_j is of type $\mathcal{D}$, then we set $\tilde{f}(\ell) = 3$ for one 2-leaf ℓ of each branch of S_j having at most two 2-leaves, and $\tilde{f}(\ell') = 0$ for its sister-leaf ℓ', if any.

Note here that if we apply the standard modification to a spine-subtree S_j of type $\mathcal{A}$, then $\tilde{f}^*(L)$ decreases by one if S_j is a bad spine-subtree, and increases by one if S_j is not bad and v_j is not a broadcast vertex. Note also that, in particular, $\tilde{f}^*(S_j)$ increases by one if S_j is of type $\mathcal{A}$ and has a neighbouring bad spine-subtree so that, in such a case, $\tilde{f}^*(L)$ does not decrease. Moreover, $\tilde{f}^*(L)$ increases by one if S_j is of type $\mathcal{C}_b$ or $\mathcal{C}_c$, and by at least 2 if S_j is of type $\mathcal{D}$. Moreover, in order to get an independent broadcast after having applied the standard modification, we need to ensure that the spine-vertices of the neighbouring spine-subtrees of

S_j (if any) are not broadcast vertices whenever S_j is of type $\mathcal{C}_b$ or $\mathcal{D}$, or of type $\mathcal{A}$ and not a bad spine-subtree, and that the same property holds for the two neighbouring spine-subtrees on each side (if any), when S_j is of type $\mathcal{C}_c$. In the rest of the proof, this property of the broadcast function will be referred to as the *independence property.*

We first consider the case when S_i belongs to some sequence of type $\mathcal{T}_4$, but not as an $\mathcal{A}$-spine-subtree, which implies that the length of this sequence is at least 3. Let $A_0X_1A_1 \cdots X_pA_p$, $p \geq 1$, denote the corresponding sequence. Every spine-subtree corresponding to some X_j, $1 \leq j \leq p$, is surrounded by two spine-subtrees of type $\mathcal{A}$. If any such spine-subtree S_α (corresponding to some X_j) is of type $\mathcal{A}$, then having $\tilde{f}(v_\alpha) = 1$ (in other words, if S_α is a bad spine-subtree) would imply that none of $v_{\alpha-1}$ and $v_{\alpha+1}$ is an $\tilde{f}$-broadcast vertex. Hence, the number of bad spine-subtrees in this sequence is at most the number of $\mathcal{A}$-spine-subtrees minus one. We can thus apply the standard modification of $\tilde{f}$ on all the spine-subtrees of type $\mathcal{A}$ without decreasing the cost of $\tilde{f}$. Moreover, the independence property holds since the left neighbour of A_0 and the right neighbour of A_p, if any, are of type $\mathcal{E}_0$, which implies that their spine-vertex is not a broadcast vertex, either by item 3(a) of Lemma 22 (for types $\mathcal{B}$ and $\mathcal{C}_a$), or since the leftmost bad spine-subtree belongs to the sequence (for type $\mathcal{A}$).

We suppose now that S_i does not belong to any sequence of type $\mathcal{T}_4$. The following claims will be useful in the sequel.

Claim 24. *If a bad spine-subtree S_j, not belonging to any sequence of type $\mathcal{T}_4$, has a neighbouring spine-subtree of type $\mathcal{D}$, then we can modify $\tilde{f}$, without decreasing its cost, in such a way that the number of bad spine-subtrees strictly decreases.*

Proof. Suppose that S_{j+1} exists and is of type $\mathcal{D}$ (the case when S_{j-1} exists and is of type $\mathcal{D}$ is similar). Note that since $\tilde{f}(v_j) = 1$, we necessarily have $\tilde{f}(\ell) \leq 2$ for every leaf ℓ of S_{j+1}. If S_{j+2} does not exist, or if S_{j+2} exists and $\tilde{f}(v_{j+2}) = 0$, then we can apply the standard modification of $\tilde{f}$ on S_j and S_{j+1} (the independence property clearly holds), thus contradicting the optimality of $\tilde{f}$ since $\tilde{f}^*(S_j)$ decreases by 1 and $\tilde{f}^*(S_{j+1})$ increases by at least 2 since S_{j+1} has at least two branches with at most two leaves. Finally, if S_{j+2} exists and $\tilde{f}(v_{j+2}) = 1$, we get that S_{j+2} is also a bad spine-subtree of L. In that case, we can apply the standard modification of $\tilde{f}$ on S_j, S_{j+1} and S_{j+2} (again, the independence property clearly holds), without decreasing the cost of $\tilde{f}$ since S_{j+1} has at least two branches with at most two leaves. □

Claim 25. *Let S_i be the leftmost bad spine-subtree in L that does not belong to any sequence of type $\mathcal{T}_4$. If S_{i+1} is of type $\mathcal{A}$ and S_{i+2} is of type $\mathcal{B}$, $\mathcal{C}_a$ or $\mathcal{C}_c$, then we can modify $\tilde{f}$, without decreasing its cost, in such a way that the index of the leftmost bad spine-subtree in L strictly increases.*

Proof. Since $\tilde{f}(v_i) = 1$, S_{i+1} cannot be bad. Moreover, $\tilde{f}(\ell) \leq 2$ for every leaf ℓ of S_{i+2}, so that we can assume $\tilde{f}(\ell) = 1$ for every such leaf since S_{i+2} is of type $\mathcal{B}$, $\mathcal{C}_a$ or $\mathcal{C}_c$ (S_{i+2} has only 1-leaves), which will ensure the independence property. We can thus apply the standard modification of $\tilde{f}$ on S_i and S_{i+1}, without decreasing the cost of $\tilde{f}$, since $\tilde{f}^*(S_i)$ decreases by one and $\tilde{f}^*(S_{i+1})$ increases by one, but strictly increasing the index of the leftmost bad spine-subtree in L. □

Claim 26. *If S_i is the leftmost bad spine-subtree in L that does not belong to any sequence of type $\mathcal{T}_4$, then we can assume that S_{i+1} is of type $\mathcal{C}_b$ or $\mathcal{C}_c$.*

Proof. If S_{i-1} is of type $\mathcal{C}_b$, then, since $\tilde{f}(v_i) = 1$, we get that $\tilde{f}(\ell) \leq 2$ for every 2-leaf of S_{i-1} (which will ensure the independence property) and we can thus apply the standard modification of $\tilde{f}$ on S_i and S_{i-1}, without decreasing the cost of $\tilde{f}$, since $\tilde{f}^*(S_i)$ decreases by one and $\tilde{f}^*(S_{i-1})$ increases by one.

If S_{i-1} is of type $\mathcal{C}_c$, then, since $\tilde{f}(v_i) = 1$, we get that $\tilde{f}(\ell) = 1$ for every 1-leaf of S_{i-1}. Moreover, we claim that $\tilde{f}(v_{i-2}) = 0$, and $\tilde{f}(v_{i-3}) = 0$ if S_{i-3} exists. On the one hand, $\tilde{f}(v_{i-2}) = 1$ would imply that S_{i-2} is a $\mathcal{A}$-spine-subtree of a sequence of type $\mathcal{T}_4$ (since S_i is the leftmost bad spine-subtree), and thus $S_{i-1}S_i$ should be of type $\mathcal{E}_0\overline{\mathcal{E}_2}$, a contradiction since S_{i-1} is of type $\mathcal{C}_c$. On the other hand, $\tilde{f}(v_{i-3}) = 1$ would imply that S_{i-3} is a $\mathcal{A}$-spine-subtree of a sequence of type $\mathcal{T}_4$, and thus $S_{i-2}S_{i-1}$ should be of type $\mathcal{E}_0\overline{\mathcal{E}_2}$, again contradicting the fact that S_{i-1} is of type $\mathcal{C}_c$. This, together with the fact that S_{i-2} must be of type $\mathcal{S}_2$, will ensure the independence property and, similarly as before, we can thus apply the standard modification of $\tilde{f}$ on S_i and S_{i-1}, without decreasing the cost of $\tilde{f}$.

Thanks to Claim 24, we can thus assume that S_{i-1} either does not exist or is of type $\mathcal{A}$, $\mathcal{C}_a$ or $\mathcal{B}$ (these two latter cases imply that S_{i-2} cannot be of type $\mathcal{C}_c$, and is thus necessarily of type $\mathcal{B}$ or $\mathcal{C}_a$). If S_{i-1} is of type $\mathcal{A}$ and S_{i-2} is of type $\mathcal{C}_c$, then, similarly as above, we can apply the standard modification of $\tilde{f}$ on S_i and S_{i-2}, without decreasing the cost of $\tilde{f}$.

In all the remaining cases, S_i is of type $\langle\overline{\mathcal{E}_2}.\mathcal{E}_0\rangle\mathcal{A}$, so that we necessarily have that either S_{i+1} is of type $\mathcal{C}_b$ or $\mathcal{C}_c$, or S_{i+1} is of type $\mathcal{A}$ and S_{i+2} is of type $\mathcal{C}_c$, since otherwise S_i would be a sequence of type $\mathcal{T}_4$. But the case when S_{i+1} is of type $\mathcal{A}$ and S_{i+2} is of type $\mathcal{C}_c$ is covered by Claim 25. This concludes the proof.

□

Thanks to Claims 24 and 26, we can assume that S_{i-1} is not of type $\mathcal{D}$, and that S_{i+1} is of type either $\mathcal{C}_b$ or $\mathcal{C}_c$. We consider these two latter cases separately when S_{i+2} is not a bad spine-subtree, or together otherwise. Note that since $\tilde{f}(v_i) = 1$, we can assume that $\tilde{f}(\ell) \leq 2$, or $\tilde{f}(\ell) = 1$, for every leaf ℓ of S_{i+1} depending on whether S_{i+1} if of type $\mathcal{C}_b$ or $\mathcal{C}_c$, respectively, which will partly ensure the independence property.

1. *S_{i+1} is of type $\mathcal{C}_b$ and S_{i+2} is not a bad spine-subtree.* In that case, the independence property clearly holds and we can apply the standard modification of $\tilde{f}$ on S_i and S_{i+1}. We are then done since the cost of $\tilde{f}$ is not modified and the index of the leftmost bad spine-subtree in L strictly increases.

2. *S_{i+1} is of type $\mathcal{C}_c$ and S_{i+2} is not a bad spine-subtree.* In this case, S_{i+2} necessarily exists and is of type $\mathcal{S}_2$. If S_{i+3} is not a bad spine-subtree, then we can apply the standard modification of $\tilde{f}$ on S_i and S_{i+1} and we are done again (the fact that S_{i+2} could have some leaf with broadcast value 3 does not compromise the independence property). Otherwise, S_{i+2} must be of type $\mathcal{C}_b$ or $\mathcal{A}$, thanks to our assumption based on Claim 24. We thus have two cases to consider.

(a) S_{i+3} is a bad spine-subtree and S_{i+2} is of type $\mathcal{C}_b$. In that case, we get that $\tilde{f}(\ell) \leq 2$ for every 2-leaf ℓ of S_{i+2}, which ensures the independence property, so that we can apply the standard modification of $\tilde{f}$ on S_i, S_{i+1}, S_{i+2} and S_{i+3}, without decreasing the cost of $\tilde{f}$. Indeed, both $\tilde{f}^*(S_i)$ and $\tilde{f}^*(S_{i+3})$ decrease by one, while both $\tilde{f}^*(S_{i+1})$ and $\tilde{f}^*(S_{i+2})$ increase by one.

(b) S_{i+3} is a bad spine-subtree and S_{i+2} is of type $\mathcal{A}$. Suppose first that either $i = 0$ or the sequence $S_{i-2}S_{i-1}$ is of type $\overline{\mathcal{E}_2}.\mathcal{E}_0$. In that case, S_{i+4} must exist and must be of type $\mathcal{C}_c$ (each of its two 1-leaves having broadcast value 1), since otherwise the sequence $S_iS_{i+1}S_{i+2}$ would be of type $\mathcal{T}_4$. Observe then that S_{i+3} is somehow "in the same situation as S_i".

Let $L' = S'_1S'_2S'_3S'_4 \cdots S'_s$, $s \geq 5$, be the maximal subsequence of L, starting at S_i (that is, $S'_1 = S_i$), whose type is a prefix of $(\mathcal{A}.\mathcal{C}_c.\mathcal{A})^*$ (considered as a word), and such that S'_j is a bad spine-subtree if $j \equiv 1 \pmod 3$ (which gives that we can assume $\tilde{f}(\ell) = 1$ for each 1-leaf of each spine-subtree of type $\mathcal{C}_c$, thus ensuring the independence property). We then have three cases, according to the value of (s mod 3).

(i) If $s \equiv 1 \pmod 3$, which means that S_{i+s} is not of type $\mathcal{C}_c$, we get that $S_i \cdots S_{i+s-1}$ is a sequence of type $\mathcal{T}_4$, a contradiction.

(ii) If $s \equiv 0 \pmod 3$, which means that S_{i+s} is not a bad spine-subtree, we get that $\tilde{f}(v_{i+s}) = 0$ (if S_{i+s} exists), which will ensure the independence property when modifying S_{i+s-1}. We can thus apply the standard modification of $\tilde{f}$ on all the spine-subtrees S'_j of L' with $j \not\equiv 0 \pmod 3$, without decreasing the cost of $\tilde{f}$ (we have as many bad spine-subtrees, whose $\tilde{f}^*$-value decreases by one, as spine-subtrees of type $\mathcal{C}_c$, whose $\tilde{f}^*$-value increases by one).

(iii) If $s \equiv 2 \pmod 3$, which means that S_{i+s} is not of type $\mathcal{A}$, we get that S_{i+s} is of type $\mathcal{C}_b$ or $\mathcal{D}$. If S_{i+s+1} is not a bad spine-subtree, then we can apply the standard modification of $\tilde{f}$ on all the spine-subtrees S'_j of L' with $j \not\equiv 0 \pmod 3$ (the fact that S_{i+s} can have a 2-leaf with broadcast value 3 does not compromise the independence property), without decreasing the cost of $\tilde{f}$, as in the previous subcase. If S_{i+s+1} is a bad spine-subtree, which implies that S_{i+s} is of type $\mathcal{C}_b$

by Claim 24, and that $\tilde{f}(\ell) \leq 2$ for every 2-leaf ℓ of S_{i+s}, thus ensuring the independence property, we can apply the standard modification of $\tilde{f}$ on S_{i+s}, S_{i+s+1}, and on all the spine-subtrees S'_j of L' with $j \not\equiv 0 \pmod 3$, without decreasing the cost of $\tilde{f}$ since $\tilde{f}^*(S_{i+s})$ increases by one, $\tilde{f}^*(S_{i+s+1})$ decreases by one, while the modifications on the S'_j's cancel each other out, as previously.

Suppose now that $i > 0$ and the sequence $S_{i-2}S_{i-1}$ is not of type $\overline{\mathcal{E}_2}.\mathcal{E}_0$. We first consider the case when S_{i-1} is not of type $\mathcal{E}_0$, which means that S_{i-1} is of type $\mathcal{C}_b$ or $\mathcal{C}_c$ (recall that S_{i-1} cannot be of type $\mathcal{D}$ by Claim 24). Since $\tilde{f}(v_i) = 1$, we have $\tilde{f}(\ell) \leq 2$ for every leaf ℓ of S_{i-1}. Moreover, since S_i is the leftmost bad spine-subtree, we necessarily have $\tilde{f}(v_{i-2}) = 0$ if S_{i-2} exists, as well as $\tilde{f}(v_{i-3}) = 0$ if S_{i-3} exists (only needed if S_{i-1} is of type $\mathcal{C}_c$), which will ensure the independence property. Therefore, we can apply the standard modification on S_{i-1} and S_i without decreasing the cost of $\tilde{f}$ since $\tilde{f}^*(S_{i-1})$ increases by one and $\tilde{f}^*(S_i)$ decreases by one.

Suppose finally that S_{i-1} is of type $\mathcal{E}_0$, which implies that S_{i-2} exists and is of type $\mathcal{C}_c$, so that S_{i-1} must be of type $\mathcal{S}_2$ and thus of type $\mathcal{A}$. Since $\tilde{f}(v_i) = 1$, we have $\tilde{f}(\ell) \leq 2$ for every leaf ℓ of S_{i-2}, and, since S_i is the leftmost bad spine-subtree, $\tilde{f}(v_{i-3}) = 0$ and $\tilde{f}(v_{i-4}) = 0$ (if S_{i-4} exists), which will ensure the independence property. Therefore, we can apply the standard modification on S_{i-2} and S_i without decreasing the cost of $\tilde{f}$ since $\tilde{f}^*(S_{i-2})$ increases by one and $\tilde{f}^*(S_i)$ decreases by one.

3. S_{i+1} *is of type* $\mathcal{C}_b$ *or* $\mathcal{C}_c$*, and* S_{i+2} *is a bad spine-subtree.* Suppose first that either $i = 0$ or the sequence $S_{i-2}S_{i-1}$ is of type $\overline{\mathcal{E}_2}.\mathcal{E}_0$. In this case, we get that either S_{i+3} is of type $\mathcal{C}_b$ or $\mathcal{C}_c$, or S_{i+3} is of type $\mathcal{A}$ and S_{i+4} is of type $\mathcal{C}_c$, since otherwise $S_iS_{i+1}S_{i+2}$ would be a sequence of type $\mathcal{T}_4$. Therefore, S_{i+2} is "in the same situation as S_i".

Let $L' = S'_1S'_2S'_3 \cdots S'_s$, $s \geq 4$, be the maximal subsequence of L, starting at S_i (that is, $S'_1 = S_i$), whose type is a prefix of $(\mathcal{A}.(\mathcal{C}_b|\mathcal{C}_c))^*$ (considered as a word), and such that S'_j is a bad spine-subtree if $j \equiv 1 \pmod 2$ (which gives, for every leaf ℓ of a spine-subtree S'_j in this subsequence with $j \equiv 0 \pmod 2$, $\tilde{f}(\ell) \leq 1$ if S'_j is of type $\mathcal{C}_c$, and $\tilde{f}(\ell) \leq 2$ if S'_j is of type $\mathcal{C}_b$). We then have two cases, according to the value of (s mod 2).

(a) If $s \equiv 1 \pmod 2$, which means that S_{i+s} is not of type $\mathcal{C}_b$ nor $\mathcal{C}_c$, we get that S_{i+s} is of type $\mathcal{A}$ and S_{i+s+1} is of type $\mathcal{C}_c$, since otherwise L' would be a sequence of type $\mathcal{T}_4$. In that case, since S_{i+s-1} is a bad spine-subtree, we get that no leaf of S_{i+s+1} has $\tilde{f}$-value 3, thus ensuring the independence property, so that we can apply the standard modification of $\tilde{f}$ on the spine-subtrees $S_i, \ldots, S_{i+s}$, without decreasing the cost of $\tilde{f}$, since we have as many bad spine-subtrees as spine-subtrees of type $\mathcal{C}_b$ or $\mathcal{C}_c$ minus one, and one spine-subtree of type $\mathcal{A}$ which is not bad.

(b) If $s \equiv 0 \pmod 2$, which means that S_{i+s} is not a bad spine-subtree (thus ensuring that the independence property will hold when modifying S_{i+s-1}), we consider two cases.

If S_{i+s-1} is of type $\mathcal{C}_b$, then we can apply the standard modification of $\tilde{f}$ on the spine-subtrees $S_i, \ldots, S_{i+s-1}$, without decreasing the cost of $\tilde{f}$ since we have as many bad spine-subtrees as spine-subtrees of type $\mathcal{C}_b$ or $\mathcal{C}_c$.

If S_{i+s-1} is of type $\mathcal{C}_c$, we get that S_{i+s} is of type either $\mathcal{A}$ (but not a bad spine-subtree), $\mathcal{C}_b$ or $\mathcal{D}$. If S_{i+s+1} is not a bad spine-subtree, then, again, we can apply the standard modification of $\tilde{f}$ on the spine-subtrees $S_i, \ldots, S_{i+s-1}$, without decreasing the cost of $\tilde{f}$ as before. If S_{i+s+1} is a bad spine-subtree and S_{i+s} is not of type $\mathcal{A}$, then we necessarily have $\tilde{f}(\ell) \leq 2$ for every 2-leaf ℓ of S_{i+s}, which will ensure the independence property, so that we can apply the standard modification of $\tilde{f}$ on the spine-subtrees $S_i, \ldots, S_{i+s+1}$, without decreasing the cost of $\tilde{f}$ (again, we have as many bad spine-subtrees as spine-subtrees of type $\mathcal{C}_b$ or $\mathcal{C}_c$). If S_{i+s+1} is a bad spine-subtree and S_{i+s} is of type $\mathcal{A}$, then, by applying the standard modification of $\tilde{f}$ on S_{i+s} and S_{i+s+1} (the independence property clearly holds there), so that the cost of $\tilde{f}$ remains the same, we get a new subsequence L', whose length has been increased by 1, so that we now have $s \equiv 1 \pmod 2$ and the previous case applies.

Suppose now that $i > 0$ and the sequence $S_{i-2}S_{i-1}$ is not of type $\overline{\mathcal{E}_2}.\mathcal{E}_0$. In that case, the proof is the same as in case 2(b), since the standard modification we use only applies to S_i, S_{i-1}, and possibly S_{i-2}, and the independence property is not compromised when S_{i-1} is of type $\mathcal{C}_c$ as $\tilde{f}(v_{i+1}) = 0$. This completes the proof. ∎

3.5. Main result

We are now able to prove the main result of our paper.

Theorem 27. *For every locally uniform 2-lobster L of length $k \geq 0$, $\beta_b(L) = \beta^*(L)$.*

Proof. If $k = 0$, the result follows from Lemma 16, observing that the independent broadcast built in its proof reaches the upper bounds on the broadcast values stated in Lemma 22. We can thus assume $k \geq 1$. By Lemma 17, we know that there exists an independent broadcast f on L with $\mathrm{cost}(f) = \beta^*(L)$. Let f be the independent broadcast on L constructed in the proof of Lemma 17. We claim that for every spine-subtree S_i of L, $f^*(S_i)$ equals the upper bound given in Lemmas 22 or 23, which will prove the theorem.

1. If S_i is of type $\mathcal{B}$ or $\mathcal{C}_a$, then $f^*(S_i)$ has been set to $\lambda_1(S_i)$ in Step 1, and never modified in the following steps.

2. If S_i is of type $\mathcal{C}_b$, then $f^*(S_i)$ has been set to $\lambda_2(S_i) + \lambda_2^*(S_i) + \alpha_2^*(S_i)$ in Steps 1 and 2. Moreover, if S_i belongs to some sequence of type $\mathcal{T}_4$, then $f^*(S_i)$ has been decreased by 1 in Step 4.

3. If S_i is of type $\mathcal{C}_c$, then $f^*(S_i)$ has been set to 3 in Steps 1 and 3. Moreover, if S_i belongs to some sequence of type $\mathcal{T}_4$, then $f^*(S_i)$ has been decreased by 1 in Step 4.

4. If S_i is of type $\mathcal{D}$, then $f^*(S_i)$ has been set to $\lambda_2(S_i) + \lambda_2^*(S_i) + \alpha_2^*(S_i)$ in Steps 1 and 2, and never modified in the following steps.

5. Finally, if S_i is of type $\mathcal{A}$, then $f^*(S_i)$ has been set to $\lambda_1(S_i)$ in Step 1. Moreover, if S_i belongs to some sequence of type $\mathcal{T}_4$, then $f^*(S_i)$ has been increased by 1 in Step 4. This concludes the proof. ■

4. Concluding Remarks

In this paper, we have given an explicit formula for the broadcast independence number of a subclass of lobsters, called locally uniform 2-lobsters. Moreover, it is easily seen that computing the value $\beta^*(L)$ for a locally uniform 2-lobster L of length k can be done in linear time (simply processing the spine-subtrees $S_0, \ldots, S_k$ in that order), which improves the result of Bessy and Rautenbach [5] for this particular subclass of trees.

A natural question, as a first step, would be to extend our result to the whole subclass of locally uniform lobsters. In fact, we were able to give an explicit formula for every such lobster not containing any spine-subtree of type $\mathcal{Z}$, that is, having exactly one branch and three 2-leaves (see [1]). However, the proof is then quite involved and we thus decided to only consider in this paper the restricted class of locally uniform 2-lobsters. Determining when the optimal broadcast value of a spine-subtree of type $\mathcal{Z}$ is 3 or 4 appears to be not so easy.

The more general question of giving an explicit formula for the broadcast independence number of the whole class of lobsters is certainly more challenging.

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