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CYCLES OF MANY LENGTHS IN BALANCED BIPARTITE DIGRAPHS ON DOMINATING AND DOMINATED DEGREE CONDITIONS

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Abstract

In 2017, Adamus proved that a strong balanced bipartite digraph of order $2a$ with $a \geq 3$ is hamiltonian, if $d(u) + d(v) \geq 3a$ for every pair of dominating or dominated vertices $\{u, v\}$. In this paper, we characterize all non-hamiltonian bipartite digraphs when $d(u)+d(v) \geq 3a-1$ for every pair of dominating or dominated vertices $\{u, v\}$, consisting of one infinite family and four exceptional bipartite digraphs of order six. Using this result, we also prove that a strong balanced bipartite digraph of order $2a$ with $a \geq 4$ contains all cycles of lengths $2, 4, \ldots, 2a-2$ except for a single bipartite digraph, and also contains a hamiltonian path, if $d(u) + d(v) \geq 3a - 1$ for every pair of dominating or dominated vertices $\{u, v\}$. The bounds for $3a-1$ in two results are sharp. This partly settles the following problem when $l = a-1$ proposed by Adamus [*A Meyniel-type condition for bipancyclicity in balanced bipartitie digraphs*, Graphs Combin. 34 (2018) 703–709]. Whether for every $1 \leq l < a$ there is a $k(l)$, $k(l) \geq 1$, such that every strong balanced bipartite digraph of order $2a$ contains cycles of lengths $2, 4, \ldots, 2l$, whenever $d(u) + d(v) \geq 3a - k(l)$ for every pair of dominating or dominated vertices $\{u, v\}$.

Keywords: bipartite digraph, degree sum, bipancyclicity, hamiltonian cycle.

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1. Terminology and Introduction

In this paper, we consider finite digraphs without loops and multiple arcs. We shall assume that the reader is familiar with the standard terminology on digraphs and refer the reader to [8] for terminology not defined here. Let D be a digraph with vertex set $V(D)$ and arc set $A(D)$. Let x, y be distinct vertices in D. If $xy \in A(D)$, then we say that x dominates y and write $x \to y$. If $x \to y$ and $y \to x$, then we write $x \leftrightarrow y$. If x dominates y and y does not dominate x, then we write $x \mapsto y$. For some vertex z if $x \to z$ and $y \to z$, then we call the pair $\{x, y\}$ dominating. Likewise, if $z \to x$ and $z \to y$, then we call the pair $\{x, y\}$ dominated. We say that a pair of vertices $\{x, y\}$ is a good pair, if $\{x, y\}$ is dominating or dominated, otherwise, it is called a bad pair. For disjoint subsets X and Y of $V(D)$, $X \to Y$ means that every vertex of X dominates every vertex of Y and $X \Rightarrow Y$ means that there are no arcs from Y to X. For a vertex set $S \subseteq V(D)$, we denote by $N^+(S)$ the set of vertices in $V(D)$ dominated by the vertices of S; i.e., $N^+(S) = \{u \in V(D) : vu \in A(D)$ for some $v \in S\}$. Similarly, $N^-(S)$ denotes the set of vertices of $V(D)$ dominating vertices of S; i.e., $N^-(S) = \{u \in V(D) : uv \in A(D)$ for some $v \in S\}$. If $S = \{v\}$ is a single vertex, we denote by $d^+(v)$ (respectively, $d^-(v)$) the number of vertices in $V(D)$ dominated by v (respectively, dominating v). The degree of v is $d(v) = d^+(v) + d^-(v)$. For $S \subseteq V(D)$, we denote by $D[S]$ the subdigraph of D induced by the vertex set S. We denote by $d^+_S(v)$ (respectively, $d^-_S(v)$) the number of vertices in $D[S]$ dominated by v (respectively, dominating v). We set $d_S(v) = d^+_S(v) + d^-_S(v)$. For a pair of vertex sets X, Y of D, define $(X, Y) = \{xy \in A(D) : x \in X, y \in Y\}$. Let $\overleftrightarrow{a}(X, Y) = |(X, Y)| + |(Y, X)|$.

Let $P = y_0 y_1 \cdots y_k$ be a path or a cycle of D (note that $y_0 = y_k$ if P is a cycle). For $i \neq j$, $y_i, y_j \in V(P)$ we denote by $y_i P y_j$ the subpath of P from y_i to y_j, if it exists. If $0 < i \leq k$, then the predecessor of y_i on P is the vertex y_{i-1} and is also denoted by y_i^-. If $0 \leq i < k$, then the successor of y_i on P is the vertex y_{i+1} and is also denoted by y_i^+. Denote $(y_i^+)^+ = y_i^{++}$ and $(y_i^-)^- = y_i^{--}$. A cycle factor in D is a collection of vertex-disjoint cycles $C_1, C_2, \ldots, C_t$ such that $V(C_1) \cup V(C_2) \cup \cdots \cup V(C_t) = V(D)$.

A digraph D is said to be strongly connected or just strong, if for every pair of vertices x, y of D, there is a path from x to y and a path from y to x. A digraph D in which, for every pair of vertices $u, v \in V(D)$ precisely one of the arcs uv, vu belongs to $A(D)$ is called a tournament. A digraph D is complete, if for every pair of vertices x, y of D, both xy and yx are in $A(D)$. A digraph D is bipartite when $V(D)$ is a disjoint union of independent sets V_1 and V_2. It is called balanced if $|V_1| = |V_2|$. A matching from V_1 to V_2 is an independent set of arcs with origin in V_1 and terminus in V_2 (a set of arcs with no common end-vertices is called independent). If D is balanced, one says that such a matching is perfect if it consists of precisely $|V_1|$ arcs. A digraph is called semicomplete bipartite if for every pair of vertices x, y from distinct partite sets, xy or yx, or both is in $A(D)$. A digraph is called complete bipartite if for every pair of vertices x, y from distinct partite sets, both xy and yx are in $A(D)$. A complete bipartite digraph with partite sets of cardinalities a and b will be denoted by $K^*_{a,b}$. A digraph on $n \geq 2$ vertices containing cycles of all lengths $2, 3, \ldots, n$ is called pancyclic. A balanced bipartite digraph of order $2a$ is bipancyclic if it contains cycles of all even lengths $2, 4, \ldots, 2a$.

A digraph D is called hamiltonian if it contains a hamiltonian cycle, i.e., a cycle that includes every vertex of D. The problem of hamiltoncity of digraphs is one of central importance in graph theory and its applications. The following two results on the existence of hamiltonian cycles in digraphs are basic and famous.

Theorem 1 (Woodall, [21])**.** *Let D be a digraph of order n, where $n \geq 2$. If $d^+(u) + d^-(v) \geq n$ for every pair of vertices u, v such that there is no arc from u to v, then D is hamiltonian.*

Theorem 2 (Meyniel, [14])**.** *Let D be a strong digraph of order n, where $n \geq 2$. If $d(x) + d(y) \geq 2n - 1$ for all pairs of non-adjacent vertices x, y, then D is hamiltonian.*

In [1], Adamus *et al.* gave Woodall-type condition for hamiltonicity of balanced bipartite digraphs as follows.

Theorem 3 [1]**.** *Let D be a balanced bipartite digraph of order $2a$, where $a \geq 2$. If $d^+(u) + d^-(v) \geq a + 2$ for all u and v from the different partite set such that $uv \notin A(D)$, then D is hamiltonian.*

The first author of the present paper in [18] characterized all non-hamiltonian bipartite digraphs when reducing the bound by 1 in Theorem 3, consisting of only one exceptional bipartite digraph of order six. In [2], Adamus *et al.* gave Meyniel-type degree condition for hamiltonicity of balanced bipartite digraphs as follows.

Theorem 4 [2]**.** *Let D be a balanced bipartite digraph of order $2a$, where $a \geq 2$. Then D is hamiltonian provided one of the following holds.*

(a) *For every pair of non-adjacent vertices $u, v \in V(D)$, $d(u) + d(v) \geq 3a + 1$;*

(b) *D is strong and for every pair of non-adjacent vertices $u, v \in V(D)$, $d(u) + d(v) \geq 3a$.*

In [7], combining local structure of the digraph with conditions on the degrees of non-adjacent vertices, Bang-Jensen *et al.* raised the following conjectures.

Conjecture 5 [7]**.** *Let D be a strong digraph of order n, where $n \geq 2$. Suppose that $d(x)+d(y) \geq 2n-1$ for every pair of dominating or dominated non-adjacent vertices $\{x, y\}$. Then D is hamiltonian.*

Conjecture 6 [7]**.** *Let D be a strong digraph of order n, where $n \geq 2$. Suppose that $d(x)+d(y) \geq 2n-1$ for every pair of dominated non-adjacent vertices $\{x, y\}$. Then D is hamiltonian.*

Conjecture 5 is still open. The authors of the present paper and Chang found an example in [16], which disproved Conjecture 6. In [3], Adamus gave dominating and dominated degree conditions for hamiltonicity of balanced bipartite digraphs.

Theorem 7 [3]**.** *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 3$. If $d(u) + d(v) \geq 3a$ for every pair of dominating or dominated vertices $\{u, v\}$, then D is hamiltonian.*

In Section 2, we replace $3a$ with $3a - 1$ in Theorem 7, then the following theorem holds.

Theorem 8. *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 3$. If $d(u) + d(v) \geq 3a - 1$ for every pair of dominating or dominated vertices $\{u, v\}$, then D is either hamiltonian, or isomorphic to a digraph in $\mathcal{H}_1$ (see Example 9 below), or isomorphic to one of the digraphs H_2, H_3, H_4, H_5 (see Example 10 below).*

Example 9. For an odd integer $a \geq 3$, let $\mathcal{H}_1$ be a set of bipartite digraphs with the following properties. For any digraph H_1 in $\mathcal{H}_1$, let V_1 and V_2 be partite sets of H_1 such that V_1 (respectively, V_2) is a disjoint union of S, R (respectively, U, W) with $|S| = |W| = \frac{a+1}{2}$, $|U| = |R| = \frac{a-1}{2}$ and $A(H_1)$ contains the following arcs.

(a) rw and wr, for all $r \in R$ and $w \in W$;

(b) us and su, for all $u \in U$ and $s \in S$;

(c) ws, for all $w \in W$ and $s \in S$;

(d) moreover, for any two vertices $w \in W$ and $s \in S$, $sw \notin A(D)$ and there exist $r \in R$ and $u \in U$ such that $ur \in A(H_1)$. For every $r \in R$, $d_U(r) \geq \frac{a-3}{2}$ and for every $u \in U$, $d_R(u) \geq \frac{a-3}{2}$.

Clearly, H_1 is strong. The degree sum of all pairs of vertices from the same partite set is greater than or equal to $3a-1$. Since $|N^+(S)| = |U| < |S|$, by König-Hall theorem, H_1 contains no perfect matching from V_1 to V_2. So H_1 contains no hamiltonian cycles. Note that $H_1[S \cup U]$ and $H_1[R \cup W]$ both are complete bipartite digraphs. Let s_1, s_2 (respectively, w_1, w_2) be two distinct vertices in S (respectively, W). There is a hamiltonian path P_1 (respectively, P_2) from s_1 (respectively, w_1) to s_2 (respectively, w_2) in $H_1[S \cup U]$ (respectively, $H_1[R \cup W]$). Then $w_1 P_2 w_2 s_1 P_1 s_2$ is a hamiltonian path of H_1. Moreover, H_1 contains cycles of all even lengths $2, 4, \ldots, 2a - 2$.

Example 10. Let H_2, H_3, H_4, H_5 be bipartite digraphs with partite sets $V_1 = \{x_1, x_2, x_3\}$ and $V_2 = \{y_1, y_2, y_3\}$. See Figures 1–4 on the next page. Undirected edges correspond to directed 2-cycles.

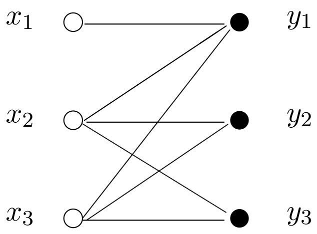

Figure 1. H_2.

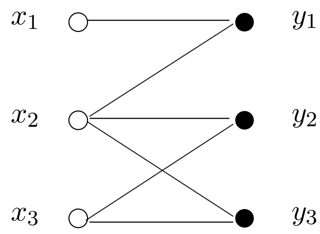

Figure 2. H_3.

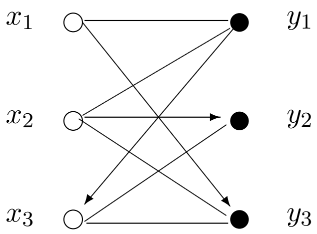

Figure 3. H_4.

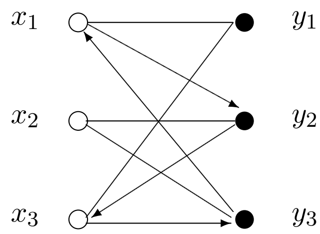

Figure 4. H_5.

Clearly, each of H_i is strong and the degree sum for every pair of dominating or dominated vertices is greater than or equal to 8. It is easy to check that each of H_i contains a cycle factor and possesses a hamiltonian path, but H_i is non-hamiltonian. Also note that each H_i, $2 \leq i \leq 5$, contains cycles of lengths $2, 4$.

From Example 9, we know that every digraph in $\mathcal{H}_1$ contains no cycle factors. Combining this with Theorem 8, we can obtain the following theorem.

Theorem 11. *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 4$. Suppose that D contains a cycle factor. If $d(u) + d(v) \geq 3a - 1$ for every pair of dominating or dominated vertices $\{u, v\}$, then D is hamiltonian.*

It is unknown whether the lower bound is sharp in Theorem 11. From Theorem 8, Examples 9 and 10, one easily derives the following immediate consequence.

Corollary 12. *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 3$. If $d(u) + d(v) \geq 3a - 1$ for every pair of dominating or dominated vertices $\{u, v\}$, then D possesses a hamiltonian path.*

The following example from [6] shows that the lower bound $3a-1$ in Corollary 12 is sharp.

Example 13. For $a \geq 3$ and $1 \leq l \leq a/2$, let $D(a, l)$ be a bipartite digraph with partite sets V_1 and V_2 such that V_1 (respectively, V_2) is a disjoint union of

S, R (respectively, U, W) with $|S| = |W| = a - l$, $|U| = |R| = l$, and $A(D(a, l))$ consists of the following arcs.

(a) ry and yr, for all $r \in R$ and $y \in V_2$;

(b) ux and xu, for all $u \in U$ and $x \in V_1$; and

(c) ws, for all $w \in W$ and $s \in S$.

Then $d(r) = d(u) = 2a$ for all $r \in R$ and $u \in U$, and $d(s) = d(w) = a + l$ for all $s \in S$ and $w \in W$. For even a, we have $d(x) + d(y) \geq 3a - 2$ for every pair of vertices x, y in $D(a, (a-2)/2)$. Notice that $D(a, (a-2)/2)$ is strong, but since the maximum matching from V_1 to V_2 has $a-2$ arcs, we have that $D(a, (a-2)/2)$ contains no hamiltonian paths and cycles of length $2a - 2$.

In [4], Adamus proved that the hypotheses of Theorem 7 imply bipancyclicity except for a single digraph.

Theorem 14 [4]**.** *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 3$. If $d(u) + d(v) \geq 3a$ for every pair of dominating or dominated vertices $\{u, v\}$, then either D is bipancyclic or it is a directed cycle of length $2a$.*

In the same paper, the author also proposed the following problem.

Problem 15 [4]**.** Whether for every $1 \leq l < a$ there exists a $k(l)$, $k(l) \geq 1$, such that every strong balanced bipartite digraph of order $2a$ contains cycles of all even lengths up to $2l$, provided $d(u) + d(v) \geq 3a - k(l)$ for every pair of dominating or dominated vertices $\{u, v\}$?

In Section 3, we shall prove that $k(a-1) = 1$ when $a \geq 4$ (see Theorem 16 below). In addition, for $l = 1$, we can take $k(1) = a - 1$ and the lower bound is sharp. In fact, if D is not a directed cycle, then there exists a vertex such that its degree is at least three. Therefore, there exists a good pair in D, say u and v. Since $d(u) + d(v) \geq 3a - (a-1) = 2a + 1$, we have $d(u) \geq a + 1$ or $d(v) \geq a + 1$. This implies that D contains a cycle of length 2. Note that the degree of every vertex in a bipartite tournament of order $2a$ is a and a bipartite tournament contains no cycles of length 2. Hence the lower bound is sharp for $l = 1$. From these, we conjecture $k(l) = a - l$ in Problem 15.

Theorem 16. *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 4$. If $d(u) + d(v) \geq 3a - 1$ for every pair of dominating or dominated vertices $\{u, v\}$, then either D contains all cycles of lengths $2, 4, \ldots, 2a - 2$, or it is a directed cycle of length $2a$.*

Now we observe the following remark. Let D be a bipartite digraph with partite sets $V_1 = \{x_1, x_2, x_3\}$ and $V_2 = \{y_1, y_2, y_3\}$. The arc set $A(D)$ consists of the following 2-cycles $x_i \leftrightarrow y_i$ and $y_i \leftrightarrow x_{i+1}$, for $i = 1, 2, 3$, where the subscripts are taken modulo 3. Note that D satisfies the hypothesis of the degree sum condition of Theorem 16, but it contains no cycle of length $2a-2$. This shows that, in Theorem 16, the bound $a \geq 4$ is sharp.

Example 13 shows that the lower bound of the degree sum-condition in Theorem 16 is sharp. By Theorems 11 and 16, we can obtain the following corollary.

Corollary 17. *Let D be a strong balanced bipartite digraph of order $2a$, where $a \geq 4$. Suppose that D contains a cycle factor. If $d(u)+d(v) \geq 3a-1$ for every pair of dominating or dominated vertices $\{u,v\}$, then either D is bipancyclic or it is a directed cycle of length $2a$.*

The sharpness of the bound in Corollary 17 is unknown. For other recent results on the degree condition of balanced bipartite digraphs, see [5, 9, 10, 17, 20].

2. The Proof of Theorem 8

Before establishing the main theorem we present a series of structural lemmas and theorems which are useful in our proof.

Lemma 18 [2, 19]**.** *Let D be a balanced bipartite digraph with partite sets V_1 and V_2. Suppose that D is non-hamiltonian and contains a cycle factor. Let $C_1, C_2, \ldots, C_s$ be a cycle factor such that s is minimum possible. Then $\overleftrightarrow{a}(V(C_i), V(C_j)) \leq \frac{|V(C_i)|\cdot|V(C_j)|}{2}$, for all $i \neq j$. Furthermore, if $\overleftrightarrow{a}(V(C_i), V(C_j)) = \frac{|V(C_i)|\cdot|V(C_j)|}{2}$, then for any $u \in V(C_i) \cap V_q$ and $v \in V(C_j) \cap V_q$ with $q \in \{1,2\}$, $|A(D) \cap \{uv^+, vu^+\}| = 1$.*

Theorem 19 [11, 12]**.** *Let D be a strong semicomplete bipartite digraph. If D contains a cycle factor, then D is hamiltonian.*

Lemma 20 [19]**.** *Let D be a strong non-hamiltonian balanced bipartite digraph. Suppose that D contains a cycle factor $C_1 \cup C_2$. If $D[V(C_j)]$ is either a complete bipartite digraph, or a complete bipartite digraph minus one arc with $|V(C_j)| \geq 6$, for $j=1$ or $j=2$, then there exists $z \in V(C_{3-j})$ such that $d_{C_j}(z)=0$.*

Definition. Let D be a balanced bipartite digraph of order $2a$. For an integer k, we say that D satisfies the condition (M_k), when every pair of dominating or dominated vertices $\{u,v\}$ satisfies $d(u)+d(v) \geq 3a+k$.

In Lemmas 21, 22 and 23, we assume that D is a strong balanced bipartite digraph of order $2a$, where $a \geq 3$, which satisfies the condition (M_{-1}). Let V_1 and V_2 be the partite sets of D.

Lemma 21. *Either D contains a cycle factor or it is isomorphic to a digraph in $\mathcal{H}_1$ (see Example 9).*

Proof. Observe that D contains a cycle factor if and only if there exist both a perfect matching from V_1 to V_2 and a perfect matching from V_2 to V_1. By the König-Hall theorem, D contains a perfect matching from V_1 to V_2 and a perfect matching from V_2 to V_1 if and only if $|N^+(S)| \geq |S|$ for every $S \subseteq V_1$ and $|N^+(T)| \geq |T|$ for every $T \subseteq V_2$.

Suppose that there exists a non-empty set $S \subseteq V_1$ such that $|N^+(S)| < |S|$. Now we shall show that D is isomorphic to a digraph in $\mathcal{H}_1$. Note that $V_2 \setminus N^+(S) \neq \emptyset$. If $|S| = 1$, write $S = \{x\}$, then $|N^+(S)| < |S|$ implies that $d^+(x) = 0$. It is impossible in a strong digraph. Thus $|S| \geq 2$. If $|S| = a$, then every vertex from $V_2 \setminus N^+(S)$ has in-degree zero, which again contradicts strong connectedness of D. Therefore, $2 \leq |S| \leq a - 1$.

First we show that any two vertices in V_i, for $i = 1, 2$, form good pairs, $|S| = \frac{a+1}{2}$ and $|N^+(S)| = \frac{a-1}{2}$. Since D is strong and $|N^+(S)| < |S|$, there exist $s_1, s_2 \in S$ and $u \in N^+(S)$ such that $\{s_1, s_2\} \to u$. Thus $\{s_1, s_2\}$ forms a good pair. By the condition (M_{-1}) and the choice of S,

$$3a - 1 \leq d(s_1) + d(s_2) \leq 2(a + |N^+(S)|) \leq 2(a + |S| - 1). \tag{2.1}$$

This implies that $2|S| \geq a + 1$. Note that there are no arcs from S to $V_2 \setminus N^+(S)$ and $|V_2 \setminus N^+(S)| > |V_1 \setminus S|$. Since D is strong, there exist $w_1, w_2 \in V_2 \setminus N^+(S)$ and $r \in V_1 \setminus S$ such that $r \to \{w_1, w_2\}$. Thus $\{w_1, w_2\}$ forms a good pair. By the condition (M_{-1}) and the choice of S,

$$3a - 1 \leq d(w_1) + d(w_2) \leq 2a + 2(a - |S|) = 2(2a - |S|). \tag{2.2}$$

This implies that $2|S| \leq a + 1$. Thus $|S| = \frac{a+1}{2}$ and equalities hold everywhere in (2.1) and (2.2). In particular, $|N^+(S)| = |S| - 1 = \frac{a-1}{2}$, $d^-(s_1) = a$ and $d^+(w_1) = a$. Therefore, any two vertices in V_2 are dominating s_1 and any two vertices in V_1 are dominated by w_1, that is, any two vertices in V_i, for $i = 1, 2$, form good pairs.

For any $s_i, s_j \in S$ and $w_i, w_j \in V_2 \setminus N^+(S)$, we have that

$$3a - 1 \leq d(s_i) + d(s_j) \leq 2(|N^+(S)| + a), \tag{2.3}$$

and

$$3a - 1 \leq d(w_i) + d(w_j) \leq 2(2a - |S|). \tag{2.4}$$

Since $|S| = \frac{a+1}{2}$ and $|N^+(S)| = \frac{a-1}{2}$, equalities hold everywhere in (2.3) and (2.4). In particular, $d^-(s_i) = d^+(w_i) = a$, $d^-(w_i) = a - |S|$ and $d^+(s_i) = |N^+(S)|$. By the strong connectedness of D and the hypothesis of this lemma, D is isomorphic to a digraph in $\mathcal{H}_1$.

Analogously, if there exists a non-empty set $T \subseteq V_2$ such that $|N^+(T)| < |T|$, then D is isomorphic to a digraph in $\mathcal{H}_1$. This completes the proof. ■

Lemma 22. *Suppose that D is not a directed cycle of length $2a$. Then, for every vertex $x \in V(D)$, there exists a vertex $y \in V(D) \setminus \{x\}$ such that $\{x, y\}$ forms a good pair.*

Proof. Suppose, on the contrary, that there exists a vertex $x' \in V(D)$, say $x' \in V_1$, such that x' and any other vertex in V_1 form a bad pair. Since D is strong, there exists an (x', z)-path, for any $z \in V(D) \setminus \{x'\}$. Denote it by $P = p_1 p_2 \cdots p_l$, where $p_1 = x'$ and $p_l = z$.

Let $w \in V_2$ be arbitrary. If p_1 and w are adjacent, then since p_1 and any other vertex in V_1 form a bad pair, we have $d(w) \leq a$. In particular, $d(p_2) \leq a$. If p_1 and w are not adjacent, then $d(w) \leq 2a - 2$. Thus, for any $y' \in V_2 \setminus \{p_2\}$, either $d(p_2) + d(y') \leq 2a < 3a - 1$ or $d(p_2) + d(y') \leq 3a - 2 < 3a - 1$, which implies that p_2 and any vertex in V_2 form a bad pair. Continuing this process, we can obtain that $p_l = z$ and any other vertex from the same partite set form a bad pair. By the arbitrariness of z, it follows that D contains no good pair. Therefore, for every $v \in V(D)$, $d^-(v) = d^+(v) = 1$. This together with strong connectedness of D implies that D is a directed cycle of length $2a$, contrary to the hypothesis of this lemma. ■

Lemma 23. *Suppose that D is not a directed cycle of length $2a$. For any $u \in V(D)$, $d(u) \geq a - 1$.*

Proof. By Lemma 22, there exists a vertex $v \in V(D) \setminus \{u\}$ such that $\{u, v\}$ forms a good pair. Then, by the condition (M_{-1}), $d(u) + d(v) \geq 3a - 1$. This together with $d(v) \leq 2a$ implies $d(u) \geq a - 1$. ■

Proof of Theorem 8. Suppose that D is not isomorphic to a digraph in $\mathcal{H}_1$. By Lemma 21, D contains a cycle factor $C_1, C_2, \ldots, C_s$. Assume that s is minimum possible and D is not hamiltonian. So $s \geq 2$. Without loss of generality, assume that $|V(C_1)| \leq |V(C_2)| \leq \cdots \leq |V(C_s)|$. Clearly, $|V(C_1)| \leq a$. Denote $\overline{C}_1 = D - V(C_1)$. By Lemma 18, the following holds

$$
\begin{aligned}
& \overleftrightarrow{a}(V(C_1) \cap V_1, V(\overline{C}_1)) + \overleftrightarrow{a}(V(C_1) \cap V_2, V(\overline{C}_1)) \\
(2.5) \qquad & = \overleftrightarrow{a}(V(C_1), V(\overline{C}_1)) = \sum_{i=2}^{s} \overleftrightarrow{a}(V(C_1), V(C_i)) \\
& \leq \frac{|V(C_1)|(2a - |V(C_1)|)}{2}.
\end{aligned}
$$

Without loss of generality, we may assume that

$$
(2.6) \qquad \overleftrightarrow{a}(V(C_1) \cap V_1, V(\overline{C}_1)) \leq \frac{|V(C_1)|(2a - |V(C_1)|)}{4},
$$

as otherwise

$$\overleftrightarrow{a}(V(C_1)\cap V_2, V(\overline{C}_1)) \le \frac{|V(C_1)|(2a-|V(C_1)|)}{4}. \tag{2.7}$$

To complete the proof, we first give the following several claims.

Claim 1. *Let $q\in\{1,2\}$, $i\neq j\in\{1,2,\dots,s\}$, $u\in V(C_i)\cap V_q$ and $v\in V(C_j)\cap V_q$. Then each of the following holds.*

(a) $|A(D)\cap\{uv^+, vu^+\}|\le 1$;

(b) *If $d(u)\ge 2a-1$, then $|A(D)\cap\{u^-v, vu^+\}|\le 1$;*

(c) *If $d(u)=2a$, then $|A(D)\cap\{u^-v, vu^+\}|=0$.*

(d) *If there exist t vertices in $V(C_i)\cap V_q$ such that their degree are greater than or equal to $2a-1$, then $d(v)\le 2a-t$.*

(e) *If $d(u)\ge 2a-1$ and $d(v)\ge 2a-1$, then $d(u)=d(v)=2a-1$, $d_{C_i}(u)=|V(C_i)|$ and $d_{C_j}(v)=|V(C_j)|$.*

Proof. (a) If $|A(D)\cap\{uv^+, vu^+\}|=2$, then C_i can be merged with C_j, i.e., $uv^+C_jvu^+C_iu$ is a cycle, a contradiction to the minimality of s. Hence, $|A(D)\cap\{uv^+, vu^+\}|\le 1$.

(b) If $|A(D)\cap\{u^-v, vu^+\}|=2$, then, by (a), $v^-\nrightarrow u$ and $u\nrightarrow v^+$, which means $d(u)\le 2a-2$, a contradiction. Therefore, the statement (b) holds.

(c) Since $d(u)=2a$, we have $v^-\to u$ and $u\to v^+$. By (a), $u^-\nrightarrow v$ and $v\nrightarrow u^+$. Hence, $|A(D)\cap\{u^-v, vu^+\}|=0$.

(d) By (b), it is obvious.

(e) By (b), $|A(D)\cap\{u^-v, vu^+\}|\le 1$ and $|A(D)\cap\{v^-u, uv^+\}|\le 1$. So the result holds. □

Claim 2. *Let u,v belong to the same partite set such that $\{u,v\}$ forms a bad pair. Then $d(u)+d(v)\le 2a$.*

Proof. Since $\{u,v\}$ forms a bad pair, $d^+(u)+d^+(v)\le a$ and $d^-(u)+d^-(v)\le a$. Therefore, $d(u)+d(v)\le 2a$. □

Claim 3. *If there exists $u\in V_q$ for $q\in\{1,2\}$ such that $d(u)\le a$ or $d(u)\ge 2a-1$, then every pair of vertices in V_{3-q} forms a good pair.*

Proof. Clearly, if $d(u)\ge 2a-1$, then every pair of vertices in V_{3-q} forms a good pair. If $d(u)\le a$, we can find another vertex in V_q such that its degree is greater than or equal to $2a-1$. In fact, by Lemma 22, there exists $v\in V_q$ such that $\{u,v\}$ forms a good pair. Thus, $d(v)\ge 3a-1-d(u)\ge 2a-1$. □

Claim 4. *If there exists $u\in V(C_i)\cap V_q$ such that $d(u)=a-1$, for $i\in\{1,\dots,s\}$ and $q\in\{1,2\}$, then each of the following holds.*

(a) *u and every vertex in $V(\overline{C}_i) \cap V_{3-q}$ are not adjacent.*

(b) *If $a \geq 4$ and $|V(C_i)| \leq a+1$, then, for any $w \in (V(C_i) \cap V_q) \setminus \{u\}$, $\{u, w\}$ forms a good pair and $d(w) = 2a$.*

Proof. (a) Suppose, on the contrary, that u and some vertex in $V(\overline{C}_i) \cap V_{3-q}$, say v, are adjacent. Then $\{u, v^-\}$ is dominating or $\{u, v^+\}$ is dominated. Thus, $d(v^+) \geq 3a-1-d(u) = 2a$ or $d(v^-) \geq 3a-1-d(u) = 2a$, which is a contradiction to Claim 1(c).

(b) By Claim 4(a), $d(u) = d_{C_i}(u) = a-1$. Let $w \in (V(C_i) \cap V_q) \setminus \{u\}$ be arbitrary. If $d_{C_i}(u) + d_{C_i}(w) \geq |V(C_i)| + 1$, then $\{u, w\}$ forms a good pair by Claim 2. Now assume that $d_{C_i}(u) + d_{C_i}(w) \leq |V(C_i)|$. Then $2 \leq d_{C_i}(w) \leq |V(C_i)| - d_{C_i}(u) \leq a+1-(a-1) = 2$, which means $|V(C_i)| = a+1$. As $a \geq 4$, we have $a+1$ is even and $|V(C_i)| = a+1 \geq 6$. If $u \to w^+$ or $w^- \to u$, then $\{u, w\}$ forms a good pair. Now assume that $u \nrightarrow w^+$ and $w^- \nrightarrow u$. Thus, $a-1 = d(u) = d_{C_i}(u) \leq |V(C_i)| - 2 = a-1$, which implies that $u \leftrightarrow z^+$ or $u \leftrightarrow z^-$, for any $z \in V(C_i) \cap V_q \setminus \{u, w\}$. Therefore, $\{u, z\}$ forms a good pair and so $d(z) \geq 3a-1-(a-1) = 2a$. By Claim 1(c), for any $x \in V_q \setminus V(C_i)$, $z^- \nrightarrow x$ and $x \nrightarrow z^+$. Clearly, $\{z^+, z^-\}$ forms a good pair. Thus, $3a-1 \leq d(z^+) + d(z^-) \leq 2(a - \frac{a+1}{2}) + 4(\frac{a+1}{2} - 2) + \overleftrightarrow{a}(\{z^+, z^-\}, \{u, w\})$. From this, we obtain $\overleftrightarrow{a}(\{z^+, z^-\}, \{u, w\}) \geq 6$, which implies that $\{u, w\}$ forms a good pair and $d(w) \geq 3a-1-d(u) \geq 2a$. So $d(w) = 2a$. □

Claim 5. *If $|V(C_1)| \geq 4$, then it is impossible that there exist two vertices from the same partite set in $V(C_i)$ for $i \in \{1, 2, \ldots, s\}$ such that their degrees are both equal to a.*

Proof. Suppose not. Without loss of generality, assume that there exist two vertices $u, w \in V(C_i) \cap V_1$ such that $d(u) = d(w) = a$. Write $C_i = x_1 y_1 \cdots x_m y_m x_1$ and $C_j = u_1 v_1 \cdots u_n v_n u_1$, for some $j \neq i$, where $x_k, u_l \in V_1$ and $y_k, v_l \in V_2$ with $k \in \{1, \ldots, m\}$ and $l \in \{1, \ldots, n\}$. Since $|V(C_1)| \geq 4$, we have $|V(C_i)| \geq 4$ and $|V(C_j)| \geq 4$. For convenience, assume $u = x_1$ and $w = x_r$. By $d(x_1) + d(x_r) = 2a < 3a-1$, we know that $\{x_1, x_r\}$ forms a bad pair, which means that, for any $y \in V_2$, $\overleftrightarrow{a}(\{x_1, x_r\}, y) \leq 2$. Combining this with $d(x_1) + d(x_r) = 2a$, we have $\overleftrightarrow{a}(\{x_1, x_r\}, y) = 2$.

Let $f \in \{1, r\}$. According to Claim 1(a) and the condition (M_{-1}), we can obtain the following observation. If $x_f \to v_i$ for some $v_i \in V(C_j) \cap V_2$, then $\{x_f, u_i\}$ forms a good pair, $u_i \nrightarrow y_f$ and $d(u_i) = 2a-1$; if $v_i \to x_f$, then $\{x_f, u_{i+1}\}$ forms a good pair, $y_{f-1} \nrightarrow u_{i+1}$ and $d(u_{i+1}) = 2a-1$.

By $\overleftrightarrow{a}(\{x_1, x_r\}, v_1) = 2$ and $\{x_1, x_r\}$ forming a bad pair, we can, without loss of generality, assume that $x_1 \to v_1$, and $v_1 \to x_1$ or $v_1 \to x_r$. By the above observation, $u_1 \nrightarrow y_1$, $d(u_1) = d(u_2) = 2a-1$, and $y_m \nrightarrow u_2$ or $y_{r-1} \nrightarrow u_2$, which implies that $d_{C_j}(u_1) = d_{C_j}(u_2) = |V(C_j)|$ and $u_2 \to y_1$. However,

$u_2y_1C_ix_1v_1u_1v_2C_jv_nu_2$ is a cycle with vertex set $V(C_i) \cup V(C_j)$, a contradiction to the minimality of s. $\square$

We now consider the following two cases.

Case 1. $|V(C_1)| = 2$. Let $V(C_1) \cap V_1 = \{x_1\}$ and $V(C_1) \cap V_2 = \{y_1\}$. By (2.5), we have that

$$d(x_1) + d(y_1) \leq 2a + 2. \tag{2.8}$$

By (2.6), $d(x_1) \leq a+1$. From this and Lemma 23, we have $a-1 \leq d(x_1) \leq a+1$.

Claim 6. *If $d(x_1) + d(y_1) = 2a + 2$, then x_1 (respectively, y_1) and every vertex in $V_1 \setminus \{x_1\}$ (respectively, $V_2 \setminus \{y_1\}$) are dominating (respectively, dominated).*

Proof. By $d(x_1)+d(y_1) = 2a+2$, we have that $d_{C_i}(x_1)+d_{C_i}(y_1) = \frac{|V(C_1)|\cdot|V(C_i)|}{2}$, for $i \neq 1$. Let y be a vertex in $V(C_i) \cap V_2$ and y^- denote the predecessor of y on C_i. By Lemma 18, we have $|A(D) \cap \{x_1y, y^-y_1\}| = 1$. So $\{x_1, y^-\}$ is dominating and $\{y_1, y\}$ is dominated. $\square$

Now, according to the degree of x_1, we consider the following three subcases.

Subcase 1.1. $d(x_1) = a-1$. By Claim 4, x_1 and every vertex in $V_2 \setminus \{y_1\}$ are not adjacent, that is, $a - 1 = d(x_1) = d_{C_1}(x_1) = 2$. So $a = 3$. If x_1 and every vertex in $V_1 \setminus \{x_1\}$ form good pairs, then by the condition (M_{-1}), for any $x \in V_1 \setminus \{x_1\}$, $d(x) = 2a$. Clearly, D is isomorphic to H_2. Now assume that x_1 and some vertex in $V_1 \setminus \{x_1\}$ form a bad pair. By Claim 3, $4 = a + 1 \leq d(y) \leq 2a - 2 = 4$, for any $y \in V_2$, that is, $d(y) = 4$. Clearly, D is isomorphic to H_3.

Subcase 1.2. $d(x_1) = a$. By Claim 3, every pair of vertices in V_2 forms a good pair.

First assume that there exist two vertices in V_1 such that they form a bad pair. By Claim 3 and (2.8), we have that

$$a + 1 \leq d(y_1) \leq a + 2. \tag{2.9}$$

Since $d(x_1) = a$ and $x_1 \leftrightarrow y_1$, there exists a vertex $y' \in V_2 \setminus \{y_1\}$ such that x_1 and y' are not adjacent. Then by (2.9) and the condition (M_{-1}), $d(y') \geq 3a - 1 - d(y_1) \geq 2a - 3$. Denote $D' = D - \{x_1, y_1\}$. So $d_{D'}(y') = d_D(y') \geq 2a - 3 = 2(a - 1) - 1$, which means that every pair of vertices in $V_1 \setminus \{x_1\}$ forms a good pair. So it must be that x_1 and some vertex in $V_1 \setminus \{x_1\}$ forms a bad pair. Let the vertex be x_2 and assume that $x_2 \in V(C_i)$. Since $\{x_1, x_2\}$ forms a bad pair, we have $|A(D) \cap \{x_2y_1, x_1x_2^+\}| = 0$ and $|A(D) \cap \{x_2^-x_1, y_1x_2\}| = 0$. Then, by Claim 1(a), $\overleftrightarrow{a}(V(C_1), V(C_i)) \leq |V(C_i)| - 2$. By Lemma 18, we obtain $d(x_1) + d(y_1) = \overleftrightarrow{a}(V(C_1), V(\overline{C}_1)) + 4 \leq |V(C_i)| - 2 + 2a - |V(C_i)| - 2 + 4 = 2a$. So $d(y_1) \leq a$, a contradiction to (2.9).

Next assume that any two vertices in V_1 form good pairs. By the condition (M_{-1}),

$$d(x) \geq 2a - 1, \text{ for any } x \in V_1 \setminus \{x_1\}. \tag{2.10}$$

So y_1 and every vertex in V_1 are adjacent. From this with (2.8), $a+1 \leq d(y_1) \leq a+2$. If $d(y_1) = a+1$, then $D[V(\overline{C}_1)]$ is a complete bipartite digraph. If $d(y_1) = a+2$, then $D[V(\overline{C}_1)]$ is a complete bipartite digraph minus one arc. Clearly, $D[V(\overline{C}_1)]$ contains a hamiltonian cycle. Hence $s = 2$. Note that $d_{C_2}(x_1) = a - 2 > 0$ and $d_{C_2}(y_1) \geq a - 1 > 0$. By Lemma 20, $D[V(C_2)]$ is a complete bipartite digraph minus one arc with $|V(C_2)| = 4$ and $d(y_1) = a+2$. So the equality holds in (2.5). Write $C_2 = x_2y_2x_3y_3x_2$, where $x_i \in V_1$ and $y_i \in V_2$, for $i = 2, 3$. Without loss of generality, assume $x_2 \mapsto y_2$. By (2.10), we have $x_2 \leftrightarrow y_1$. By Claim 1(c), $y_3 \nrightarrow x_1$ and $x_1 \nrightarrow y_2$. Clearly $y_2 \nrightarrow x_1$, for otherwise D contains a hamiltonian cycle. So, x_1 and y_2 are not adjacent. By $d(x_1) = a$, we have $x_1 \mapsto y_3$. By Claim 1(a), $x_3 \nrightarrow y_1$. By $d(x_3) \geq 2a - 1$, $y_1 \mapsto x_3$. Note that D is isomorphic to H_4.

Subcase 1.3. $d(x_1) = a + 1$. By (2.8) and Lemma 23, $a - 1 \leq d(y_1) \leq a + 1$. The case $d(y_1) = a - 1$ or $d(y_1) = a$ is similar to Subcases 1.1 and 1.2. Thus we assume that $d(y_1) = a + 1$. Hence

$$d(x_1) + d(y_1) = 2a + 2 \tag{2.11}$$

and the equality in (2.5) holds. Therefore, $\overleftrightarrow{a}(V(C_1), V(C_j)) = |V(C_j)|$. By Lemma 18, for any $x_i, y_i \in V(C_2)$,

$$|A(D) \cap \{x_iy_1, x_1y_i\}| = 1 \text{ and } |A(D) \cap \{y_{i-1}x_1, y_1x_i\}| = 1. \tag{2.12}$$

By Claim 6, x_1 (respectively, y_1) and every vertex in $V_1 \setminus \{x_1\}$ (respectively, $V_2 \setminus \{y_1\}$) form good pairs (respectively, good pairs). By the condition (M_{-1}),

$$d(u) \geq 2a - 2, \text{for every } u \in V(D) \setminus \{x_1, y_1\}. \tag{2.13}$$

Since $2(2a - 2) = 4a - 4 > 2a$, by Claim 2, every pair of vertices in $V_1 \setminus \{x_1\}$ or $V_2 \setminus \{y_1\}$ forms a good pair.

Assume that $|V(C_2)| = 2$. Write $C_2 = x_2y_2x_2$, where $x_2 \in V_1$ and $y_2 \in V_2$. Analogously, we can also assume that $d(x_2) = d(y_2) = a + 1$. Thus, $2a - 2 \leq d(x_2) = a + 1$, which implies that $a = 3$ and so $s = 3$. Write $C_3 = x_3y_3x_3$, where $x_3 \in V_1$ and $y_3 \in V_2$. Analogously, we can also assume that $d(x_3) = d(y_3) = a+1$. By Theorem 19, there exist two vertices from different partite sets such that they are not adjacent, say x_1 and y_2. By $d(x_1) = d(y_2) = a + 1 = 4$, we have that $x_1 \leftrightarrow y_3$ and $x_3 \leftrightarrow y_2$. It is easy to see that x_2 and y_3 are not adjacent since $s = 3$. Then $d(x_2) = a + 1 = 4$ implies that $x_2 \leftrightarrow y_1$. Clearly, $x_1y_1x_2y_2x_3y_3x_1$ is a hamiltonian cycle in D, a contradiction.

Next assume that $|V(C_2)| \geq 4$. From this, $|V(C_i)| \geq 4$, for $i = 3, \ldots, s$. Let $D' = D - \{x_1, y_1\}$ and $a' = a - 1$.

First we claim that $s = 2$. It suffices to show that D' is hamiltonian. For $a' = 2$ and $a' = 3$, it is obvious. According to (2.13),

$$d_{D'}(u) \geq 2a - 4 = 2a' - 2. \tag{2.14}$$

Thus, for any two non-adjacent vertices u and v in D', $d_{D'}(u)+d_{D'}(v) \geq 2(2a'-2)$. If $a' \geq 5$, then $2(2a'-2) \geq 3a'+1$. By Theorem 4(a), D' is hamiltonian. If $a' = 4$, then $2(2a'-2) \geq 3a'$. If D' is strong, then by Theorem 4(b), D' is hamiltonian. Next assume that D' is not strong. In this case, $s = 3$ and C_2, C_3 are both 4-cycles. Write $C_2 = x_2y_2x_3y_3x_2$, where $x_i \in V_1$ and $y_i \in V_2$, for $i = 2, 3$. Since D' is not strong, without loss of generality, assume that $C_2 \Rightarrow C_3$. So $d_{D'}(x_2) \leq 2 + 4 = 2a' - 2$ and $d_{D'}(y_2) \leq 2a' - 2$. Combining this with (2.14), we have that $d_{D'}(x_2) = d_{D'}(y_2) = 2a' - 2 = 2a - 4$. By (2.13), $x_2 \leftrightarrow y_1$ and $x_1 \leftrightarrow y_2$. This means that C_1 can be merged with C_2, a contradiction. Hence $s = 2$.

Write $C_2 = x_2y_2 \cdots x_ay_ax_2$, where $x_i \in V_1$ and $y_i \in V_2$, for $i = 2, \ldots, a$. By Lemma 20, there exists a vertex $z \in V(C_2)$ such that $d_{C_1}(z) = 0$, say x_2, namely, x_2 and y_1 are not adjacent. From this and $d(x_2) \geq 2a - 2$, we have that $d(x_2) = 2a - 2$, implying $x_2 \leftrightarrow y_i$, for $i = 2, \ldots, a$. Since y_1 and x_2 are not adjacent, by (2.12), we have $y_a \to x_1$ and $x_1 \to y_2$.

First consider the case when $a = 3$. By $d_{C_2}(x_1) = d_{C_2}(y_1) = a - 1$, we have $y_3 \mapsto x_1$ and $x_1 \mapsto y_2$ and $y_1 \leftrightarrow x_3$. Thus $x_3y_1x_3$, $x_1y_2x_2y_3x_1$ is a cycle factor of D. By $x_1 \leftrightarrow y_1$ and Claim 1(a), we have $y_3 \nrightarrow x_3$ and $x_3 \nrightarrow y_2$, and so $x_3 \mapsto y_3$ and $y_2 \mapsto x_3$. Note that D is isomorphic to the digraph H_5.

Next consider the case when $a \geq 4$. Assume that $x_a \to y_1$. By (2.12), $x_1 \nrightarrow y_a$. Furthermore, for any $i \in \{2, \ldots, a-1\}$, $y_a \nrightarrow x_{i+1}$, otherwise $x_ay_1x_1y_2C_2y_ix_2y_ax_{i+1}C_2x_a$ is a hamiltonian cycle, a contradiction. Thus, $d(y_a) \leq 2a-3$, a contradiction to (2.13). Now we assume $x_a \nrightarrow y_1$. Since $d_{C_2}(y_1) = a-1$ and y_1 and x_2 are not adjacent, there exists a vertex $x_i \in \{x_3, \ldots, x_{a-1}\}$ such that $x_i \to y_1$. Take $r = \max\{i : 3 \leq i \leq a-1 \text{ and } x_i \to y_1\}$. By the choice of r, for every $j \in \{r+1, \ldots, a\}$, $x_j \nrightarrow y_1$. Then by (2.12), $x_1 \to y_j$. If $x_j \to y_2$, then $x_ry_1x_1y_jC_2x_2y_rC_2x_jy_2C_2x_r$ is a hamiltonian cycle, a contradiction. Hence $x_j \nrightarrow y_2$. Combining this with $x_j \nrightarrow y_1$ and $d(x_j) \geq 2a-2$, we have $d(x_j) = 2a-2$. Hence $x_j \to \{y_j, y_{j-1}\} \to x_j$, which implies that the converse of $y_rC_2y_a$ is a directed path, that is, $y_ax_ay_{a-1} \cdots x_{r+1}y_r$ is a directed path. So $x_ry_1x_1y_ax_ay_{a-1} \cdots x_{r+1}y_rx_2C_2x_r$ is a hamiltonian cycle, a contradiction.

Case 2. $|V(C_1)| \geq 4$. In this case, $a \geq 4$. Let $x', x'' \in V(C_1) \cap V_1$ and $y', y'' \in V(C_1) \cap V_2$ be distinct and chosen so that $\overleftrightarrow{a}(\{x', x''\}, V(\overline{C}_1))$ and $\overleftrightarrow{a}(\{y', y''\},$

$V(\overline{C}_1))$ are minimal respectively. According to (2.6),

$$d_{\overline{C}_1}(x') + d_{\overline{C}_1}(x'') = \overleftrightarrow{a}(\{x', x''\}, V(\overline{C}_1)) \leq 2a - |V(C_1)|. \tag{2.15}$$

Claim 7. *$|V(C_1)| = a - 1$ or $|V(C_1)| = a$.*

Proof. Suppose, on the contrary, that $|V(C_1)| \leq a - 2$. By (2.15), $d(x') + d(x'') \leq 2|V(C_1)| + 2a - |V(C_1)| \leq 3a - 2$, which means $\{x', x''\}$ forms a bad pair. According to Claim 4(b), we can obtain that $d(x') \neq a - 1$ and $d(x'') \neq a - 1$. These together with Lemma 23 and Claim 2 imply that $d(x') = a = d(x'')$, a contradiction to Claim 5. Therefore $|V(C_1)| = a - 1$ or $|V(C_1)| = a$. □

By Claim 7, $s = 2$ and $|V(C_2)| = a + 1$ or $|V(C_2)| = a$.

Claim 8. *Let $q, i \in \{1, 2\}$. Any two vertices in $V(C_i) \cap V_q$ form a good pair.*

Proof. Let $u, v \in V(C_i) \cap V_q$ be arbitrary. By Lemma 23, $d(u) \geq a - 1$ and $d(v) \geq a - 1$. If one of the degree of u, v equals to $a - 1$, then, by Claim 4, $\{u, v\}$ forms a good pair. So $d(u) \geq a$ and $d(v) \geq a$. According to Claim 5, $d(u) \geq a + 1$ or $d(v) \geq a + 1$. Hence $d(u) + d(v) \geq 2a + 1$. By Claim 2, $\{u, v\}$ forms a good pair. □

By Claim 8, $\{x', x''\}$ forms a good pair. Therefore, by (2.15), we have

$$\begin{aligned} 3a - 1 &\leq d(x') + d(x'') \leq 2|V(C_1)| + d_{C_2}(x') + d_{C_2}(x'') \\ &\leq 2|V(C_1)| + 2a - |V(C_1)| = 2a + |V(C_1)| \leq 3a. \end{aligned} \tag{2.16}$$

By (2.16) and Claim 7, we have that $d_{C_2}(x') + d_{C_2}(x'') = 2a - |V(C_1)|$ or $d_{C_2}(x') + d_{C_2}(x'') = 2a - |V(C_1)| - 1$.

Claim 9. *If $d_{C_2}(x') + d_{C_2}(x'') = 2a - |V(C_1)|$, then $d_{C_2}(y') + d_{C_2}(y'') \leq 2a - |V(C_1)|$.*

Proof. If $d_{C_2}(x') + d_{C_2}(x'') = 2a - |V(C_1)|$, then the equality holds in (2.6). Further, by (2.5), the inequality holds in (2.7). Then $d_{C_2}(y') + d_{C_2}(y'') \leq 2a - |V(C_1)|$. □

Claim 10. *For any $x \in V(D)$, $d(x) \geq a$.*

Proof. Suppose not. According to Lemma 23, there exists a vertex z in $V(D)$ such that $d(z) = a - 1$. Without loss of generality, assume that $z \in V(C_i) \cap V_1$ for $i = 1$ or $i = 2$. By Claim 4, for any $w \in (V(C_i) \cap V_1) \setminus \{z\}$, $d(w) = 2a$. In particular, $d(z^{++}) = d(z^{--}) = 2a$. For any $u \in V(C_{3-i}) \cap V_1$, by Claim 1(c), $z^+ \nrightarrow u$ and $u \nrightarrow z^-$. Hence $d(u) \leq 2a - 2$ and so $d(z) + d(u) \leq 3a - 3$, which means $\{z, u\}$ forms a bad pair. So $u \nrightarrow z^+$ and $z^- \nrightarrow u$. From these, we have that z^+ and z^- are not adjacent to u. By the arbitrariness of u, we get that z^+ and z^- are not adjacent to every vertex of $V(C_{3-i}) \cap V_1$. By Claim 3, $\{z^+, z^-\}$ forms a good pair.

So $3a-1 \le d(z^+)+d(z^-) = d_{C_i}(z^+)+d_{C_i}(z^-) \le 2|V(C_i)| \le 2(a+1) < 3a-1$, a contradiction. Hence, for any $x \in V(D)$, $d(x) \ge a$. $\square$

Claim 11. *Any two vertices from the same partite set form good pairs.*

Proof. Suppose that there exists a pair of vertices $\{u,v\}$ from the same partite set such that $\{u,v\}$ forms a bad pair. By Claim 10, $d(u) \ge a$ and $d(v) \ge a$. Combining this with Claim 2, $d(u) = d(v) = a$. By Claim 8, $u \in V(C_1)$, $v \in V(C_2)$, or $u \in V(C_2)$, $v \in V(C_1)$. Write $C_1 = x_1y_1\cdots x_my_mx_1$ and $C_2 = x_{m+1}y_{m+1}\cdots x_ay_ax_{m+1}$, where $x_i \in V_1$ and $y_i \in V_2$, for $i=1,2,\dots,a$. Without loss of generality, assume that $u = x_1$, $v = x_{m+1}$. By Claims 2 and 5, it is not difficult to see that $\{x_1,x_{m+1}\}$ is the unique bad pair. So $d(x_j) \ge 2a-1$ for any $x_j \in V_1 \setminus \{x_1,x_{m+1}\}$. By Claim 1(d), we can deduce that $a=4$ and $|V(C_1)| = |V(C_2)| = 4$. By Claim 1(e), $d(x_2) = d(x_4) = 2a-1$, $x_2 \to \{y_1,y_2\} \to x_2$ and $x_4 \to \{y_3,y_4\} \to x_4$. If $y_1 \to x_4$ and $y_3 \to x_2$, then $y_3x_2y_2x_1y_1x_4y_4x_3y_3$ is a hamiltonian cycle, a contradiction. Without loss of generality, assume that $y_1x_4 \notin A(D)$. By $d(x_4) = 2a-1$, we have $x_4 \mapsto y_1$, $x_4 \leftrightarrow y_2$. From this and Claim 1(a), we have $x_2 \not\to y_4$, $x_1 \not\to y_4$ and $y_3 \not\to x_1$. By $d(x_2) = 2a-1$, we have $y_3 \to x_2$. Since x_1 and x_3 are neither dominating nor dominated, we have $x_1 \not\to y_3$ and $y_4 \not\to x_1$. Hence, y_3 and y_4 are not adjacent to x_1. By $d(x_1) = a = 4$, we have $x_1 \to \{y_1,y_2\} \to x_1$. Then $y_3x_2y_1x_1y_2x_4C_2y_3$ is a hamiltonian cycle, a contradiction. $\square$

By Claim 11, any two vertices from the same partite set form good pairs.

Claim 12. *For any $u \in V(C_2)$, $d_{C_1}(u) > 0$.*

Proof. Suppose, on the contrary, that there exists $u_0 \in V(C_2)$ such that $d_{C_1}(u_0) = 0$. If $u_0 \in V(C_2) \cap V_1$, then by the condition (M_{-1}) and (2.15), $2(3a-1) \le 2d(u_0)+d(x')+d(x'') \le 2|V(C_2)|+2|V(C_1)|+2a-|V(C_1)| \le 6a-4$, a contradiction. So $u_0 \in V(C_2) \cap V_2$. If $d_{C_2}(y')+d_{C_2}(y'') \le 2a-|V(C_1)|$, then similar to the above argument, we can obtain a contradiction. Now assume that $d_{C_2}(y')+d_{C_2}(y'') > 2a-|V(C_1)|$. By Claim 9 and (2.15), $d_{C_2}(x')+d_{C_2}(x'') = 2a-|V(C_1)|-1$. Then, by (2.16), we can deduce that $|V(C_1)| = a$. So $|V(C_2)| = a$. Then $d(u_0) = d_{C_2}(u_0) \le |V(C_2)| = a$, which implies that $d(y) \ge 3a-1-d(u_0) \ge 2a-1$, for any $y \in V_2 \setminus \{u_0\}$, which can obtain a contradiction according to Claim 1(d). $\square$

By Lemma 20 and Claim 12, $D[V(C_1)]$ is not a complete bipartite digraph. Recall that $d_{C_2}(x')+d_{C_2}(x'') = 2a-|V(C_1)|$ or $d_{C_2}(x')+d_{C_2}(x'') = 2a-|V(C_1)|-1$.

First suppose $d_{C_2}(x')+d_{C_2}(x'') = 2a-|V(C_1)|$. By (2.6), $d_{C_2}(x_i)+d_{C_2}(x_j) = 2a-|V(C_1)|$, for any $x_i,x_j \in V(C_1) \cap V_1$. Since $D[V(C_1)]$ is not a complete bipartite digraph, there exists a vertex $x^* \in V(C_1) \cap V_1$ such that $d_{C_1}(x^*) \leq |V(C_1)| - 1$. For any $x_k \in (V(C_1) \cap V_1) \setminus \{x^*\}$,

$$3a - 1 \leq d(x^*) + d(x_k) \leq (2|V(C_1)| - 1) + 2a - |V(C_1)| \leq 3a - 1.$$

So $|V(C_1)| = a$, $d_{C_1}(x^*) = |V(C_1)| - 1$ and $d_{C_1}(x_k) = |V(C_1)|$, which implies that $D[V(C_1)]$ is a complete bipartite digraph minus one arc. According to Lemma 20 and Claim 12, $|V(C_1)| = a = 4$. Write $C_1 = x_1y_1x_2y_2x_1$ and $C_2 = x_3y_3x_4y_4x_3$, where $x_i \in V_1$ and $y_i \in V_2$, for $i = 1, 2, 3, 4$. According to Claim 9, $d_{C_2}(y_1) + d_{C_2}(y_2) \leq 2a - |V(C_1)| = a$. Then $3a - 1 \leq d(y_1) + d(y_2) \leq 2a - 1 + a = 3a - 1$ implies that $d_{C_2}(y_1) + d_{C_2}(y_2) = 2a - |V(C_1)| = a$, which means $d_{C_1}(x_3) + d_{C_1}(x_4) = 2a - |V(C_2)| = a$. By symmetry, we can deduce that $D[V(C_2)]$ is a complete bipartite digraph minus one arc. Note that $d(x_1)+d(x_2) = 3a - 1 = 11$ and $d(x_3) + d(x_4) = 3a - 1 = 11$. Without loss of generality, assume $d(x_1) \leq 5$ and $d(x_3) \leq 5$. Then $d(x_1) + d(x_3) \leq 10 = 3a - 2$, a contradiction.

Now suppose $d_{C_2}(x') + d_{C_2}(x'') = 2a - |V(C_1)| - 1$. From (2.16), we have $|V(C_1)| = a$ and $d_{C_1}(x') = d_{C_1}(x'') = a$. If $a = 4$, then $D[V(C_1)]$ is a complete bipartite digraph, a contradiction. Next assume that $a \geq 6$. By Lemma 20 and Claim 12, $D[V(C_1)]$ is neither a complete bipartite digraph nor a complete bipartite digraph minus one arc. Denote $V(C_1) \cap V_1 = \{x_1, x_2, \ldots, x_{\frac{a}{2}}\}$ and without loss of generality, assume that $d_{C_2}(x_1) \leq d_{C_2}(x_2) \leq \cdots \leq d_{C_2}(x_{\frac{a}{2}})$. Observe that $\{x_1, x_2\} = \{x', x''\}$. By the choice of x_1 and x_2 and $d_{C_2}(x_1) + d_{C_2}(x_2) = a - 1$, we know that $d_{C_2}(x_1) \leq \frac{a}{2} - 1$. Denote $d_{C_2}(x_1) = \frac{a}{2} - k$, with $k \geq 1$. So $d_{C_2}(x_2) = \frac{a}{2} + k - 1$ and $d_{C_2}(x_i) \geq \frac{a}{2} + k - 1$, for $i = 2, \ldots, \frac{a}{2}$. By (2.6),

$$\sum_{i=1}^{a/2} d_{C_2}(x_i) \leq \frac{a^2}{4}. \tag{2.17}$$

Since $D[V(C_1)]$ is neither a complete bipartite digraph nor a complete bipartite digraph minus one arc, either there exists a vertex $x_i \in V(C_1) \cap V_1$ such that $d_{C_1}(x_i) \leq a - 2$ or there exist at least two vertices x_i and x_j such that $d_{C_1}(x_i) = a - 1$ and $d_{C_1}(x_j) = a - 1$. For any $x_t \in (V(C_1) \cap V_1) \setminus \{x_1\}$, if $d_{C_1}(x_t) \leq a - l$, then

$$\begin{aligned} d_{C_2}(x_t) &\geq 3a - 1 - d(x_1) - d_{C_1}(x_t) \\ &\geq 3a - 1 - \left(\frac{3a}{2} - k\right) - (a - l) = \frac{a}{2} + k + l - 1. \end{aligned} \tag{2.18}$$

If there exists a vertex $x_i \in V(C_1) \cap V_1$ such that $d_{C_1}(x_i) \leq a - 2$, then, by (2.18), $d_{C_2}(x_i) \geq \frac{a}{2} + k + 1$. Therefore,

$$\sum_{i=1}^{a/2} d_{C_2}(x_i) \geq \Big(\frac{a}{2}-k\Big)+\Big(\frac{a}{2}-2\Big)\Big(\frac{a}{2}+k-1\Big)+\frac{a}{2}+k+1$$
$$=\frac{a^2}{4}+\Big(\frac{a}{2}-2\Big)(k-1)+1>\frac{a^2}{4},$$

a contradiction to (2.17).

Now assume there exist at least two vertices x_i and x_j such that $d_{C_1}(x_i) = a-1$ and $d_{C_1}(x_j) = a-1$. Since $d_{C_1}(x_1) = d_{C_1}(x_2) = a$, we have $a \geq 8$. By $\sum_{i=1}^{\frac{a}{2}} d_{C_2}(x_i) \geq \frac{a}{2}-k+(\frac{a}{2}-1)(\frac{a}{2}+k-1) = \frac{a^2}{4}+\frac{(a-4)k-(a-2)}{2}$ and (2.17), we have $k \leq 1$ and so $k=1$. By (2.18), $d_{C_2}(x_i) \geq \frac{a}{2}+k$ and $d_{C_2}(x_j) \geq \frac{a}{2}+k$. Therefore,

$$\sum_{i=1}^{a/2} d_{C_2}(x_i) \geq \Big(\frac{a}{2}-k\Big)+\Big(\frac{a}{2}-3\Big)\Big(\frac{a}{2}+k-1\Big)+2(\frac{a}{2}+k)$$
$$=\Big(\frac{a}{2}-1\Big)+\Big(\frac{a}{2}-3\Big)\frac{a}{2}+2\Big(\frac{a}{2}+1\Big)=\frac{a^2}{4}+1,$$

a contradiction to (2.17). We have considered all cases and completed the proof of the theorem. ■

3. The Proof of Theorem 16

The proof of Theorem 16 will be based on Theorems 24, 28 and Lemmas 25, 26 and 27.

Theorem 24 [15]**.** *Let D be a strong digraph of order n, where $n \geq 3$. If $d(u)+d(v) \geq 2n$ for all pairs of non-adjacent vertices u, v in D, then D is either pancyclic, or a tournament, or n is even and D is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$.*

Lemma 25 [6]**.** *Let D be a bipartite digraph on $n \geq 3$ vertices, which contains a cycle C of length $2r$ with $2 \leq 2r < n$. Let x be a vertex not contained in C. If $d_C(x) \geq r+1$, then D contains cycles of every even length m, $2 \leq m \leq 2r$, through x.*

From now on, we assume that D is a strong balanced bipartite digraph of order $2a$ where $a \geq 4$, which satisfies the condition (M_{-1}). To prove Theorem 16, by Theorems 8 and Example 9, it suffices to consider the case when D contains a hamiltonian cycle. Now assume D contains a hamiltonian cycle C and D is not a cycle of length $2a$. If there exists a vertex $x \in V(D)$ such that $d^+(x) = a$ or $d^-(x) = a$, then D contains cycles of all even lengths. Now we assume that

$$d^+(u) \leq a-1 \text{ and } d^-(u) \leq a-1, \text{ for every } u \in V(D). \tag{3.1}$$

By Lemma 22, for every $u \in V(D)$, there exists a vertex $v \in V(D) \setminus \{u\}$ such that $\{u, v\}$ forms a good pair. By (3.1), $d(v) \leq 2a - 2$. Hence

$$d(u) \geq 3a - 1 - (2a - 2) = a + 1. \tag{3.2}$$

Now for the digraph D, we will prove the following two lemmas.

Lemma 26. *Every pair of vertices $\{u, v\}$ from the same partite set in D forms a good pair. Moreover $d^+(u) + d^+(v) \geq a + 1$ and $d^-(u) + d^-(v) \geq a + 1$.*

Proof. Let u and v be two vertices from the same partite sets. By (3.2), $d(u) \geq a + 1$ and $d(v) \geq a + 1$. Thus, $2a + 2 \leq d(u) + d(v) = (d^+(u) + d^+(v)) + (d^-(u) + d^-(v))$, which implies $d^+(u) + d^+(v) \geq a + 1$ or $d^-(u) + d^-(v) \geq a + 1$. Therefore, $\{u, v\}$ forms a good pair. From this and (3.1), we have $3a - 1 \leq d(u) + d(v) = (d^+(u) + d^+(v)) + (d^-(u) + d^-(v)) \leq d^+(u) + d^+(v) + 2a - 2$. Hence, $d^+(u) + d^+(v) \geq a + 1$. Analogously, $d^-(u) + d^-(v) \geq a + 1$. ■

Lemma 27. *If D contains a cycle of length $2a - 2$, then it contains cycles of all lengths $2, 4, \ldots, 2a - 2$.*

Proof. Let $Q = x_1y_1x_2y_2 \cdots x_{a-1}y_{a-1}x_1$ be a cycle of length $2a - 2$ in D and $\{x, y\} = V(D) \setminus V(Q)$, where $x, x_i \in V_1$ and $y, y_i \in V_2$. If $d_Q(x) \geq a$ or $d_Q(y) \geq a$, then, by Lemma 25, D contains cycles of all lengths $2, 4, \ldots, 2a - 2$. Now assume that $d_Q(x) \leq a - 1$ and $d_Q(y) \leq a - 1$. These together with (3.2) imply that $d_Q(x) = d_Q(y) = a - 1$ and $d(x) = d(y) = a + 1$. Let $x_i \in V(Q) \cap V_1$ be arbitrary. By Lemma 26, $\{x, x_i\}$ forms a good pair. Then

$$d(x_i) \geq 3a - 1 - d(x) \geq 2a - 2. \tag{3.3}$$

If $d_Q^+(x_i) = a - 1$ or $d_Q^-(x_i) = a - 1$, then $D[V(Q)]$ contains cycles of all lengths $2, 4, \ldots, 2a - 2$. So assume that $d_Q^-(x_i) \leq a - 2$ and $d_Q^+(x_i) \leq a - 2$. By (3.3), $2a - 2 \leq d(x_i) = d_Q(x_i) + d_{\{y\}}(x_i) \leq d_Q(x_i) + 2 \leq 2a - 4 + 2 = 2a - 2$, which implies $x_i \leftrightarrow y$. Hence $d_Q(y) = 2a - 2$, a contradiction to $d_Q(y) \leq a - 1$. ■

Now, we will denote the two partite sets of D by V_1 and V_2, with elements $\{x_1, \ldots, x_a\}$ and $\{y_1, \ldots, y_a\}$ respectively, ordered so that C is the cycle $x_1y_1 \cdots x_ay_ax_1$. The subscripts are taken modulo a.

Like [4] and [13], we will associate with D a new digraph, D^*, constructed as follows. Set $V(D^*) = \{v_1, \ldots, v_a\}$, and $v_iv_j \in A(D^*)$ whenever $y_ix_j \in A(D)$, for $i, j \in \{1, \ldots, a\}, i \neq j$. For every $1 \leq i \leq a$, we have

$$d^+_{D^*}(v_i) = \begin{cases} d^+_D(y_i), & \text{if } y_i \nrightarrow x_i; \\ d^+_D(y_i) - 1, & \text{if } y_i \to x_i; \end{cases}$$

$$d^-_{D^*}(v_i) = \begin{cases} d^-_D(x_i), & \text{if } y_i \nrightarrow x_i; \\ d^-_D(x_i) - 1, & \text{if } y_i \to x_i. \end{cases} \tag{3.4}$$

Theorem 28 [13]**.** *If D^* contains a cycle of length k, then there is a cycle of length $2k$ in D.*

Now we prove the main result of this section.

***Proof of Theorem* 16.** In the proof, subscripts are taken modulo a. By Lemma 27 and Theorem 8, it suffices to show that D contains a cycle of length $2a-2$. By way of contradiction, suppose that D contains no a cycle of length $2a-2$. In particular, by Theorem 28, D^* contains no a cycle of length $a-1$. Recall that $C = x_1y_1 \cdots x_ay_ax_1$ is a hamiltonian cycle of D. Clearly, $x_i \nrightarrow y_{i+1}$ and $y_i \nrightarrow x_{i+2}$, for any $i \in \{1, 2, \ldots, a\}$. First we prove the following claim.

Claim 13. *For any two non-adjacent vertices u, v in D^*, we have $d_{D^*}(u) + d_{D^*}(v) \geq 2a$.*

Proof. Suppose, on the contrary, that there exist two non-adjacent vertices v_i, v_j in $V(D^*)$ such that $d_{D^*}(v_i) + d_{D^*}(v_j) \leq 2a-1$. Consider the corresponding vertices x_i, y_i and x_j, y_j of D. By Lemma 26, we have $d_D(x_i) + d_D(x_j) \geq 3a-1$ and $d_D(y_i) + d_D(y_j) \geq 3a-1$. Without loss of generality, assume that $j = 1$ and so $3 \leq i \leq a-1$. It follows that

$$
\begin{aligned}
6a-2 &\leq d_D(x_i) + d_D(x_1) + d_D(y_i) + d_D(y_1) \\
&= d^-_D(x_i) + d^+_D(y_i) + d^-_D(x_1) + d^+_D(y_1) \\
&\quad + d^+_D(x_i) + d^-_D(y_i) + d^+_D(x_1) + d^-_D(y_1) \\
&\leq d_{D^*}(v_i) + d_{D^*}(v_1) + 4 + d^+_D(x_i) + d^-_D(y_i) + d^+_D(x_1) + d^-_D(y_1) \\
&\leq 2a + 3 + d^+_D(x_i) + d^-_D(y_i) + d^+_D(x_1) + d^-_D(y_1).
\end{aligned} \tag{3.5}
$$

By (3.1) and (3.5),

$$4a-5 \leq d^+_D(x_i) + d^-_D(y_i) + d^+_D(x_1) + d^-_D(y_1) \leq 4a-4. \tag{3.6}$$

From these, we know that at least three of $d^+_D(x_i)$, $d^-_D(y_i)$, $d^+_D(x_1)$ and $d^-_D(y_1)$ are equal to $a-1$, say $d^-_D(y_i) = a-1$, $d^+_D(x_i) = a-1$ and $d^+_D(x_1) = a-1$, furthermore, $d^-_D(y_1) \geq a-2$. Then $x_1 \to V_2 \setminus \{y_2\}$, $x_i \to V_2 \setminus \{y_{i+1}\}$ and $V_1 \setminus \{x_{i-1}\} \to y_i$ since D contains no cycle of length $2a-2$.

Now we shall show $y_1 \to x_1$ and $y_i \to x_i$. If $y_1 \nrightarrow x_1$, then $d^+_D(y_1) = d^+_{D^*}(v_1)$ and $d^-_D(x_1) = d^-_{D^*}(v_1)$. Then $d^-_D(x_i) + d^+_D(y_i) + d^-_D(x_1) + d^+_D(y_1) \leq d^*_D(v_i) + d^*_D(v_1) + 2 \leq 2a+1$. By (3.5), $d^+_D(x_i) + d^-_D(y_i) + d^+_D(x_1) + d^-_D(y_1) \geq (6a-2)-(2a+1) = 4a-3$, a contradiction to (3.6). Hence $y_1 \to x_1$. Analogously, $y_i \to x_i$.

Now we show that $i \geq a-2$ and $i = 3$. For any $t \in \{i+1, \ldots, a-1\}$, $x_t \nrightarrow y_1$, otherwise $x_ty_1Cx_iy_{t+1}Cx_1y_iCx_t$ is a cycle of length $2a-2$ in D, a contradiction. Combining this with $x_a \nrightarrow y_1$, we have $\{x_{i+1}, \ldots, x_a\} \nrightarrow y_1$. By $d^-_D(y_1) \geq a-2$, we have $|\{x_{i+1},\ldots,x_a\}|\le 2$, that is $i\ge a-2$. For any $p\in\{2,\ldots,i-2\}$, $x_p\nrightarrow y_i$, otherwise $x_py_iCx_1y_{p+1}Cx_iy_1Cx_p$ is a cycle of length $2a-2$ in D, a contradiction. Combining this with $x_{i-1}\nrightarrow y_i$, we have $\{x_2,\ldots,x_{i-1}\}\nrightarrow y_i$. By $d^-_D(y_i)=a-1$, we have $|\{x_2,\ldots,x_{i-1}\}|=1$, that is $i=3$. This together with $i\ge a-2$ implies $a\le 5$.

Note that $y_1\nrightarrow x_4$, otherwise $y_1x_4Cx_1y_3x_3y_1$ is a cycle of length $2a-2$ in D, a contradiction. Combining this with $y_1\nrightarrow x_3$, we have $d^+_D(y_1)\le a-2$. In addition $y_a\nrightarrow x_3$, otherwise $y_ax_3y_1x_1y_3Cy_a$ is a cycle of length $2a-2$, and $y_a\nrightarrow x_4$, otherwise $y_ax_4Cx_ay_3x_3y_1x_1y_a$ is a cycle of length $2a-2$, a contradiction. Combining this with $y_a\nrightarrow x_2$, we have $d^+_D(y_a)\le a-3$. From these with Lemma 26, $a+1\le d^+_D(y_1)+d^+_D(y_a)\le 2a-5$, a contradiction to $a\le 5$. Hence, for any two non-adjacent vertices u,v in D^*, we have $d_{D^*}(u)+d_{D^*}(v)\ge 2a$. $\square$

Now we return to the proof of this theorem. Clearly, D^* is strong. According to Claim 13 and Theorem 24, D^* is either pancyclic, or a tournament, or n is even and D^* is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$. If D^* is pancyclic, then we are done. If D^* is a tournament, then D^* contains a cycle of length $a-1$. Now assume that a is even and D^* is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$. Since D^* contains a hamiltonian cycle $v_1\cdots v_av_1$, the two partite sets must be precisely $\{v_1,v_3,\ldots,v_{a-1}\}$ and $\{v_2,v_4,\ldots,v_a\}$. Moreover, we have $d^+_{D^*}(v_i)=\frac{a}{2}$, for every v_i in D^*. Hence, by (3.4), $d^+_D(y_i)\le\frac{a}{2}+1$, for all $1\le i\le a$. By Lemma 26, for $y_i,y_j\in V_2$, $3a-1\le d_D(y_i)+d_D(y_j)=d^+_D(y_i)+d^+_D(y_j)+d^-_D(y_i)+d^-_D(y_j)\le 2(\frac{a}{2}+1)+d^-_D(y_i)+d^-_D(y_j)$, which means $d^-_D(y_i)+d^-_D(y_j)\ge 2a-3$. This means that for every pair of vertices y_i,y_j, $d^-(y_i)\ge a-1$ or $d^-(y_j)\ge a-1$. This together with (3.1) implies that $d^-(y_i)=a-1$ for all i, except at most one. Let $d^-_D(y_3)=a-1$. We have that $x_4\to y_3$ since $x_2y_3\notin A(D)$. Since D^* is isomorphic to $K^*_{\frac{n}{2},\frac{n}{2}}$, we obtain $y_p\to x_{p-1}$ with $p\in\{1,\ldots,a\}$. However $x_ay_ax_{a-1}y_{a-1}\cdots x_5y_5x_4y_3x_2y_2x_1y_1x_a$ is a cycle of length $2a-2$ in D, a contradiction. ■

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