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ON A GRAPH LABELLING CONJECTURE INVOLVING COLOURED LABELS

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Abstract

In this work, we investigate a recent conjecture by Baudon, Bensmail, Davot, Hocquard, Przybyło, Senhaji, Sopena and Woźniak, which states that graphs, in general, can be edge-labelled with red labels $1, 2$ and blue labels $1, 2$ so that every two adjacent vertices are distinguished accordingly to either the sums of their incident red labels or the sums of their incident blue labels. To date, this was verified for several classes of graphs. Also, it is known how to design several labelling schemes that are very close to what is desired.

In this work, we adapt two important proofs of the field, leading to some progress towards that conjecture. We first prove that graphs can be labelled with red labels $1, 2, 3$ and blue labels $1, 2$ so that every two adjacent vertices are distinguished as required. We then verify the conjecture for graphs with chromatic number at most 4.

Keywords: proper labelling, coloured label, Weak $(2, 2)$-Conjecture, 1-2-3 Conjecture.

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1. Introduction

We deal with undirected simple graphs only. Given a graph G, a *k-labelling* $\ell : E(G) \to \{1, \dots, k\}$ usually refers to an assignment of *labels* from the set $\{1, \dots, k\}$ to the edges of G. Labellings of graphs form an attractive field of research of graph theory, due, notably, to numerous real-life applications. There actually exist many types of such graphs labellings, as reported, for instance, in the dynamic survey [9] maintained by Gallian.

In this work, we are more particularly interested in *distinguishing labellings*, which are a type of graph labellings where one aims at making some vertices distinguishable accordingly to some aggregate induced by the edge labels. Even more specifically, we are here interested in *proper labellings*, which are defined as follows. Given a labelling ℓ of a graph G, one can compute, for every vertex v of G, the *sum* $\sigma(v)$ of labels assigned to the edges incident to v. We say that ℓ is *proper* if σ is a proper vertex-colouring of G, or, in other words, if no two adjacent vertices u and v of G verify $\sigma(u) = \sigma(v)$.

Proper labellings of graphs are objects of interest for various reasons, such as their connection with proper vertex-colourings [3] and graph regularity [4]. In connection with an approach first considered by Chartrand *et al.* [8], one important question related to proper labellings, the 1-2-3 *Conjecture*, deals with the smallest k such that a given graph admits proper k-labellings. Before being more precise, let us mention that the only connected graph that does not admit proper labellings at all is K_2. This can easily be shown e.g. by induction. Thus, the question of investigating the smallest k for which proper k-labellings do exist only makes sense for graphs that do not admit K_2 as a connected component. In this field, such graphs are said *nice*, and, for a nice graph G, the parameter $\chi_\Sigma(G)$, which is the smallest k such that G admits proper k-labellings, is properly defined.

In 2004, Karoński, Łuczak and Thomason conjectured in [14] that $\chi_\Sigma(G)$ should be rather small, for every nice graph G.

1-2-3 Conjecture. *For every nice graph G, we have $\chi_\Sigma(G) \leq 3$.*

The 1-2-3 Conjecture has been attracting a lot of attention since its introduction, as illustrated e.g. by the survey [17] of Seamone dedicated to this topic. We refer the interested reader to that reference for more details. Regarding our investigations in this paper, let us mention that there do exist graphs G with $\chi_\Sigma(G) = 3$, such as complete graphs, odd-length cycles, etc. To date, the conjecture was mainly shown for complete graphs, 3-colourable graphs, and various classes of graphs of less importance. The best-known result towards the 1-2-3 Conjecture is that $\chi_\Sigma(G) \leq 5$ holds for every nice graph G, as proved by Kalkowski, Karoński and Pfender in [12]. The proof of this result is mostly an adaptation of a nice algorithm originally designed by Kalkowski in [11] for proving a result towards a total variant of the 1-2-3 Conjecture.

Speaking of variants of the 1-2-3 Conjecture, it is important to mention that a quite investigated line of research in this field actually deals with slight modifications of the original conjecture. In particular, in [1] the authors introduced a *multiset version* of the 1-2-3 Conjecture, in which the goal is to design labellings ℓ where adjacent vertices are distinguished accordingly to their incident *multisets* of labels. Note that two vertices being distinguished by their incident sums of labels are distinguished by their incident multisets of labels. Mostly for that reason, it is legitimate to believe that the multiset version of the 1-2-3 Conjecture should be easier to prove than its sum counterpart. This is something that was proven true recently, as Vučković provided a proof of that multiset version [18].

Among other variants of the 1-2-3 Conjecture, let us briefly mention that there exists one in [4], related to so-called *locally irregular decompositions*, that deals with a decompositional approach to the conjecture.

Our results in this work deal with another conjecture that is somewhat related to the 1-2-3 Conjecture, that was introduced in [2] by Baudon, Bensmail, Davot, Hocquard, Przybyło, Senhaji, Sopena and Woźniak. In that work, the authors introduced some general labelling terminology and notions that cover several aspects of the field, resulting in a general context encapsulating several related problems. The formal details, which can be defined through "coloured labellings", are as follows. Let G be a graph, and $\alpha, \beta \geq 1$ be two integers. An (α, β)-*labelling* ℓ of G is an assignment $\ell : E(G) \to \{1, \dots, \alpha\} \times \{1, \dots, \beta\}$ where every edge uv is assigned a *coloured label* $\ell(uv) = (x, y)$ with colour x and value y. For every $v \in V(G)$ and every $i \in \{1, \dots, \alpha\}$, one can compute the i-*sum* $\sigma_i(v)$ of v, being the sum of the values of the labels with colour i assigned to the edges incident to v.

Note that, by (α, β)-labellings, vertices get incident to several coloured sums, and there are thus many ways for considering that two adjacent vertices are distinguished by such a labelling. In [2], the authors observed that by considering particular distinction conditions, it is possible to make all the previous notions related to the 1-2-3 Conjecture fall into the realm of (α, β)-labellings. By playing further with the possibilities, they also ran into new problems of interest, including the main one investigated in this paper. This problem is the weakest among all those considered in [2]. It deals with the following notion. For a graph G and an (α, β)-labelling ℓ of G, we consider that ℓ is *proper* if, for every two adjacent vertices u and v, there is an $i \in \{1, \dots, \alpha\}$ such that $\sigma_i(u) \neq \sigma_i(v)$. In other words, two vertices are considered distinguished if there is one colour for which the corresponding coloured sums are different.

Note, in particular, that a proper $(1, \beta)$-labelling is nothing but a proper β-labelling. Also, a proper $(\alpha, 1)$-labelling is an α-labelling distinguishing adjacent vertices by their incident multisets. Thus, an alternative way for stating the 1-2-3 Conjecture is that nice graphs should admit proper $(1, 3)$-labellings. The proof of the multiset version of the 1-2-3 Conjecture by Vučković implies that all nice graphs admit proper $(3, 1)$-labellings. Furthermore, it has been known for long that there exist graphs with no proper $(1, 2)$-labellings, and there exist graphs with no proper $(2, 1)$-labellings. A natural question, then, is whether graphs, in general, admit proper $(2, 2)$-labellings. This resulted in the following *Weak $(2, 2)$-Conjecture* [2].

Weak $(2, 2)$-Conjecture. *Nice graphs admit proper $(2, 2)$-labellings.*

By arguments above, the Weak $(2,2)$-Conjecture holds for every graph G admitting 2-labellings distinguishing adjacent vertices by sums or multisets. In [2], the conjecture was further verified for complete graphs and bipartite graphs. In [15], Przybyło also verified the conjecture for graphs with large enough minimum degree. Additionally, the authors of [2] proved that all nice graphs admit proper $(2,3)$-labellings.

One additional motivation behind the Weak $(2,2)$-Conjecture is that it is weaker than the original 1-2-3 Conjecture, in the following sense.

Observation 1. *The* 1-2-3 *Conjecture, if true, would imply the Weak* $(2,2)$*-Conjecture.*

Proof. Let G be any nice graph. If G admits a proper k-labelling ℓ for $k \in \{1,2\}$, then turning to 1 the colour of all labels assigned by ℓ (and keeping their value) results in a proper $(1,k)$-labelling of G, as pointed out earlier. Now, for a proper 3-labelling ℓ of G assigning label 3 at least once, note that turning to 1 the colour of all 1's and 2's assigned by ℓ (and keeping their value), and replacing all 3's assigned by ℓ by a label with value 1 and colour 2, results in a proper $(2,2)$-labelling ℓ' of G. Indeed, it can be checked that if by ℓ' we had both $\sigma_1(u) = \sigma_1(v)$ and $\sigma_2(u) = \sigma_2(v)$ for an edge uv of G, then by ℓ we would have $\sigma(u) = \sigma(v)$, contradicting that ℓ is proper. Any proper 3-labelling can thus be turned into a proper $(2,2)$-labelling, and the claim follows. ∎

In this work, we provide further support towards the Weak $(2,2)$-Conjecture. We start in Section 2 by improving the last result mentioned above, by showing that every nice graph admits proper labellings assigning red labels $1,2,3$ and blue labels $1,2$. We continue in Section 3 by showing that the Weak $(2,2)$-Conjecture holds for nice graphs with chromatic number at most 4 (thereby going beyond the "4-chromatic barrier" that currently holds for the 1-2-3 Conjecture – recall that this is significant due to the connection established in Observation 1). An important aspect lies in the proof schemes we develop, which are non-trivial improvements and modifications of two of the most important tools in the field.

2. Proper Labellings with Red Labels $1,2,3$ and Blue Labels $1,2$

Since, throughout this work, we focus on (sometimes restrictions of) $(2,\beta)$-labellings, to ease the reading we regard these labellings as assigning two types of labels, red ones and blues ones. For a vertex v, we consequently denote by $\sigma_{\rm r}(v)$ its incident sum of red labels, while we denote by $\sigma_{\rm b}(v)$ its incident sum of blue labels.

The next result is obtained by modifications and refinements of Kalkowski's Algorithm, which was introduced in [11] to deal with the total version of the

1-2-3 Conjecture. To date, it stands as one of the best tools in the context for designing proper labellings with small number of different labels. In particular, straight modifications of it led to the best-known result towards the 1-2-3 Conjecture to date [12]. More or less complicated improvements can be found e.g. in [2, 5, 7, 10, 13, 16], and led to proving results of various importance. Hence, one meaningful point behind our proof lies in the new mechanisms we enhance Kalkowski's Algorithm with.

Theorem 2. *Nice graphs admit proper labellings with red labels* $1, 2, 3$ *and blue labels* $1, 2$.

Proof. Let G be a nice graph. We may assume that G is connected. Our aim is to construct a labelling ℓ of G assigning red labels $1, 2, 3$ and blue labels $1, 2$, so that $\sigma_{\mathrm{r}}(u) \neq \sigma_{\mathrm{r}}(v)$ or $\sigma_{\mathrm{b}}(u) \neq \sigma_{\mathrm{b}}(v)$ holds for every two adjacent vertices u and v.

Let $S \subset V(G)$ be any maximal independent set of G, and set $R = V(G) \setminus S$. By maximality of S, note that every vertex of R has neighbours in S. For every $v \in R$, we call its incident edges going to S the *private edges* of v.

In rough words, the construction of ℓ will follow two main steps. During the first step, we will label the edges of $G[R]$ and most edges of the cut (S, R) so that every two adjacent vertices of $G[R]$ are distinguished by either σ_{r} or σ_{b}. At the end of this first step, the only edges that will not be labelled yet will be those incident to the isolated vertices in $G[R]$. In the second step, we will handle such edges, making sure that no conflicts remain. By that, we mean, in particular, labelling the remaining edges so that adjacent vertices in S and R are distinguished by σ_{r} or σ_{b}.

Step 1. Labelling the edges of $G[R]$ and most edges of (S, R).

Let us denote by $\mathcal{H}$ the (connected) components of $G[R]$ which have at least two vertices (thus at least one edge). We deal with the components of $\mathcal{H}$ independently. Let us focus on one such component H. Consider an arbitrary ordering $v_1, \ldots, v_n$ over the vertices of H, where $n \geq 2$. From the point of view of a vertex v_i, an incident edge $v_i v_j$ is called *backward* if $j < i$, while it is called *forward* otherwise, i.e., if $j > i$. The notions of *backward neighbours* and *forward neighbours* of any vertex v_i are defined in the obvious way. From now on, we assume that the v_i's are ordered so that v_n is the only vertex with no forward neighbour. This is an assumption that can be made, as such an ordering $v_1, \ldots, v_n$ can easily be obtained e.g. by reversing the ordering in which the vertices of H are encountered during a Breadth-First Search algorithm performed from any root vertex as v_n.

Let us start by assigning an initial label to all edges (in G) incident to vertices in H:

- to every edge $v_i v_j \in E(H)$, we assign red label 2;

- to every edge v_iu with $v_i \in V(H)$ and $u \in S$, we assign blue label 1.

Our goal now is to process the v_i's one after another, from first (v_1) to last (v_n), without ever coming back, and apply local label modifications to get rid of all possible conflicts between a considered vertex and its backward neighbours. More precisely, whenever considering a such v_i, we will modify labels of private edges and backward edges of v_i so that, for every backward neighbour v_j, either

1. $\sigma_{\mathrm{b}}(v_i)$ and $\sigma_{\mathrm{b}}(v_j)$ have different parities (and thus $\sigma_{\mathrm{b}}(v_i) \neq \sigma_{\mathrm{b}}(v_j)$), and/or
2. $\sigma_{\mathrm{r}}(v_i) \neq \sigma_{\mathrm{r}}(v_j)$.

An important point to mention right away is that, by applying label modifications to private edges and backward edges only, whenever first considering a new vertex v_i we are sure that all its backward edges are currently assigned red label 2. Furthermore, in the upcoming process, the value of $\sigma_{\mathrm{b}}(v_i)$ will never be altered again once v_i has been treated.

In order to ensure the second condition above, we will sometimes have to alter the red label of a backward edge v_iv_j, from 2 to either 1 or 3. A problem is that such a modification alters $\sigma_{\mathrm{r}}(v_j)$ as well, which might raise new conflicts. To make sure this does not happen, whenever considering a v_i with forward edges, we will define two allowed "safe" values $\phi(v_i)$ and $\phi(v_i)+1$ as $\sigma_{\mathrm{r}}(v_i)$, such that, as soon as they are defined, $\sigma_{\mathrm{r}}(v_i)$ must, at any further step of the process, lie in $\Phi(v_i) = \{\phi(v_i), \phi(v_i)+1\}$. This way, whenever dealing with a v_i with a backward edge v_iv_j (thus currently assigned red label 2, by earlier arguments), we know that we can switch the label of v_iv_j to either red label 1 or red label 3 with keeping $\sigma_{\mathrm{r}}(v_j)$ in $\Phi(v_j)$. That is, if currently $\sigma_{\mathrm{r}}(v_j) = \phi(v_j)$, then we can assign red label 3 to v_iv_j (resulting in $\sigma_{\mathrm{r}}(v_j) = \phi(v_j)+1$); while, if currently $\sigma_{\mathrm{r}}(v_j) = \phi(v_j)+1$, then we can assign red label 1 to v_iv_j (resulting in $\sigma_{\mathrm{r}}(v_j) = \phi(v_j)$). When treating v_i and considering a backward edge v_iv_j, we call the correct of these two label modifications the *valid modification* for v_iv_j.

Before proceeding to the concrete arguments for achieving such a label modification process, it is important to emphasise that the two values of any $\Phi(v_i)$ are consecutive ($\phi(v_i)$ and $\phi(v_i)+1$). Another technical point that will be important for both the first and the second steps, is that we need each $\phi(v_i)$ to be even with value at least 2.

Let us now describe the process in details. We consider the v_i's one after another, starting from v_1. Vertex v_1 has $b = 0$ backward neighbours, $f \geq 1$ forward neighbours (since $n \geq 2$), and $p \geq 1$ private edges. Thus, at the moment, we have $\sigma_{\mathrm{r}}(v_1) = 2f$ and $\sigma_{\mathrm{b}}(v_1) = p$. Here, we do not modify labels around v_1, and just set $\phi(v_1) = 2f$ (thus defining $\Phi(v_1)$). Thus, at the moment, v_1 is not in conflict with any backward neighbour, all forward edges of v_1 remain assigned red label 2, we currently have $\sigma_{\mathrm{r}}(v_1) \in \Phi(v_1)$, and $\phi(v_1) = 2f \geq 2$ is even.

Let us now focus on the general case of a vertex v_i with $i > 1$. We can assume that v_i has backward neighbours, as otherwise we could just apply the previous arguments used to deal with v_1. Thus v_i has $b \geq 1$ backward neighbours $u_1, \dots, u_b$, where each of $v_i u_1, \dots, v_i u_b$ is currently assigned red label 2 by assumption. Furthermore, v_i has $f \geq 0$ forward neighbours and $p \geq 1$ private edges. Thus, currently, $\sigma_{\mathrm{r}}(v_i) = 2(b+f)$ and $\sigma_{\mathrm{b}}(v_i) = p$. We consider two main cases, depending on the value of f.

• First assume that $f \geq 1$. Recall that, for every backward neighbour u_j of v_i, the pair $\Phi(u_j)$ was defined, and we currently have $\sigma_{\mathrm{r}}(u_j) \in \Phi(u_j)$. Furthermore, $v_i u_j$ is currently assigned red label 2, and there is one valid label modification that can be applied to $v_i u_j$ that preserves $\sigma_{\mathrm{r}}(u_j) \in \Phi(u_j)$. This modification is either a decrement or an increment. Let us assume that $s \geq 0$ of the valid modifications to the $v_i u_j$'s are decrements, while $t \geq 0$ of them are increments. Then $s + t = b$.

First off, note that either at most $\lfloor b/2 \rfloor$ of the u_j's have $\sigma_{\mathrm{b}}(u_j)$ being even, or at most $\lfloor b/2 \rfloor$ of the u_j's have $\sigma_{\mathrm{b}}(u_j)$ being odd. In the first case, if necessary, we change the blue label of a private edge of v_i from 1 to 2 so that $\sigma_{\mathrm{b}}(v_i)$ gets even. In the second case, if needed we change the blue label of a private edge so that $\sigma_{\mathrm{b}}(v_i)$ gets odd. This way, $\sigma_{\mathrm{b}}(v_i)$ permits to distinguish v_i from all but at most $\lfloor b/2 \rfloor$ of the u_j's.

We now need to define $\Phi(v_i)$ and apply valid modifications to the $v_i u_j$'s so that $\sigma_{\mathrm{r}}(v_i)$ distinguishes v_i and the remaining at most $\lfloor b/2 \rfloor$ u_j's. By performing valid modifications backwards, note that we can make the red sum of v_i take any value in

$$\{\sigma_{\mathrm{r}}(v_i) - s, \dots, \sigma_{\mathrm{r}}(v_i) - 1, \sigma_{\mathrm{r}}(v_i), \sigma_{\mathrm{r}}(v_i) + 1, \dots, \sigma_{\mathrm{r}}(v_i) + t\},$$

which is a set of $s + t + 1 = b + 1$ distinct values.

Assume first that $\sigma_{\mathrm{r}}(v_i) - s$ is even. On the one hand, if $\sigma_{\mathrm{r}}(v_i) + t$ is even, then note that all values in $\{\sigma_{\mathrm{r}}(v_i) - s, \sigma_{\mathrm{r}}(v_i) - s + 2, \dots, \sigma_{\mathrm{r}}(v_i) + t\}$ are candidates as $\phi(v_i)$ (recall that we want to define $\phi(v_i)$ so that it is even with value at least 2). On the other hand, if $\sigma_{\mathrm{r}}(v_i) + t$ is odd, then note that all values in $\{\sigma_{\mathrm{r}}(v_i) - s, \sigma_{\mathrm{r}}(v_i) - s + 2, \dots, \sigma_{\mathrm{r}}(v_i) + t - 1\}$ are candidates. In both cases, since the set of candidates includes at least $\lfloor b/2 \rfloor + 1$ distinct values, at least one of them, say r, is different from the value $\phi(u_j)$ of all of the at most $\lfloor b/2 \rfloor$ u_j's that might have the same blue sum as v_i.

Assume now that $\sigma_{\mathrm{r}}(v_i) - s$ is odd. On the one hand, if $\sigma_{\mathrm{r}}(v_i) + t$ is even, then note that all values in $\{\sigma_{\mathrm{r}}(v_i) - s - 1, \sigma_{\mathrm{r}}(v_i) - s + 1, \dots, \sigma_{\mathrm{r}}(v_i) + t\}$ are candidates as $\phi(v_i)$. In particular, note that the even value $\sigma_{\mathrm{r}}(v_i) - s - 1$ is eligible as $\phi(v_i)$ since it has value at least 2 due to the fact that $f \geq 1$ and $b \geq 1$. This last fact implies that, through valid label modifications backwards, the red sum of v_i cannot get strictly smaller than 2 (which is one of the requirements that $\phi(v_i)$ must fulfill). In other words, in this case, the "smallest" pair $(\sigma_{\mathrm{r}}(v_i) - s - 1, \sigma_{\mathrm{r}}(v_i) - s)$ that we might have to consider as $\Phi(v_i) = (\phi(v_i), \phi(v_i) + 1)$ is $(2, 3)$. On the other hand, if $\sigma_{\mathrm{r}}(v_i) + t$ is odd, then note that all values in $\{\sigma_{\mathrm{r}}(v_i) - s - 1, \sigma_{\mathrm{r}}(v_i) - s + 1 \ldots, \sigma_{\mathrm{r}}(v_i) + t - 1\}$ are candidates, by the same arguments. In both cases, since the set of candidates includes at least $\lfloor b/2 \rfloor + 1$ distinct values, at least one r of them is different from the value $\phi(u_j)$ of all of at most $\lfloor b/2 \rfloor$ u_j's that might have the same blue sum as v_i.

In all cases, we set $\phi(v_i) = r$, define $\Phi(v_i)$ accordingly, and apply valid label modifications backwards so that $\sigma_{\mathrm{r}}(v_i) \in \Phi(v_i)$. As a consequence, v_i cannot be in conflict with any of its backward neighbours, all forward edges of v_i remain assigned red label 2, and we have $\sigma_{\mathrm{r}}(v_i) \in \Phi(v_i)$ with $\phi(v_i)$ being of even value at least 2.

• Now assume that $f = 0$. Recall that only v_n has this property, by the choice of the ordering $v_1, \ldots, v_n$. Also, because v_n is the last vertex of H to be processed, we here do not have to follow all the rules related to the sets Φ, and, instead, we just need to apply label modifications around v_n so that no conflict with its backward neighbours remains, and none of the backward neighbours get involved in new conflicts.

Just as in the previous case, we first modify, if needed, the blue label (from 1 to 2) of a private edge of v_n so that $\sigma_{\mathrm{b}}(v_n)$ gets its parity meeting that of at most $\lfloor b/2 \rfloor$ of the u_j's. Note that if $b = 1$, then we are already done with getting rid of all conflicts between v_n and the u_j's. So assume $b \geq 2$. We now need to modify $\sigma_{\mathrm{r}}(v_n)$ to make sure it is different from the value of $\sigma_{\mathrm{r}}(u_j)$ of these at most $\lfloor b/2 \rfloor$ other u_j's. To make sure these u_j's do not get involved in new conflicts, we will only perform valid modifications to the backward edges of v_n. Again, $s \geq 0$ of the valid modifications backwards are decrements while $t \geq 0$ of them are increments; thus $s + t = b$.

Start by performing the s valid modifications that are decrements. This way, all backward edges of v_n get assigned red label 1 or 2, and, for every backward neighbour u_j, we have $\sigma_{\mathrm{r}}(u_j) = \phi(u_j)$ which is an even value. If $\sigma_{\mathrm{r}}(v_n)$ is currently odd, then we are done. So assume $\sigma_{\mathrm{r}}(v_n)$ is even, and that some of the u_j's have the same value as $\sigma_{\mathrm{r}}(v_n)$ by σ_{r} (as otherwise we would be done as well). Actually, if there exists u_j such that we currently have $\sigma_{\mathrm{r}}(u_j) \neq \sigma_{\mathrm{r}}(v_n)$, then we are done when resetting the red label of $v_n u_j$ to 2 (as $\sigma_{\mathrm{r}}(v_n)$ gets odd, and only $\sigma_{\mathrm{r}}(u_j)$ is odd but with different value $\phi(u_j)+1$). So we may lastly assume that we actually currently have $\sigma_{\mathrm{r}}(u_j) = \sigma_{\mathrm{r}}(v_n)$ for every $j \in \{1, \ldots, b\}$. Here, we are done when resetting the red label of both $v_n u_1$ and $v_n u_2$ to 2, as $\sigma_{\mathrm{r}}(u_1)$ and $\sigma_{\mathrm{r}}(u_2)$ get odd, while we get $\sigma_{\mathrm{r}}(u_j) = \sigma_{\mathrm{r}}(v_n) - 2$ for every $j \in \{3, \ldots, b\}$ and $\sigma_{\mathrm{r}}(v_n)$ is even.

In all cases, it is important to note that $\sigma_{\mathrm{r}}(v_n) \geq 2$. This is clear when $b \geq 2$, since all backward edges remain assigned red labels. When $b = 1$, this is because the label of $v_n u_1$ is not modified, and the edge remains assigned red label 2.

Once the whole process ends, note that every two adjacent vertices v_i and v_j of H get distinguished, either because $\sigma_{\mathrm{b}}(v_i)$ and $\sigma_{\mathrm{b}}(v_j)$ have distinct parities, or because $\sigma_{\mathrm{r}}(v_i)$ and $\sigma_{\mathrm{r}}(v_j)$ are different (in most cases, due to how $\Phi(v_i)$ and $\Phi(v_j)$ were chosen). Also, it is important to recall that we have $\sigma_{\mathrm{r}}(v_i) \geq 2$ for every $i \in \{1, \dots, n\}$. For $i \neq n$, this is because we have always chosen $\phi(v_i)$ so that it is even with value at least 2. For $i = n$, this is by arguments above.

In what follows, we assume that the previous process was performed for every $H \in \mathcal{H}$.

Step 2. Labelling the remaining edges.

Note that, so far, every vertex in S has its edges being either assigned a blue label (edges going to $\mathcal{H}$) or not labelled (otherwise). All edges that are not labelled yet join a vertex of S and a vertex of $\mathcal{I}$, where $\mathcal{I} = G[R \setminus V(\mathcal{H})]$. By definition of $\mathcal{H}$, all components of $\mathcal{I}$ have order 1, i.e., are isolated vertices. Note that, no matter how we label the remaining edges, no conflict can involve two adjacent vertices u and v with $u \in S$ and $v \in V(\mathcal{H})$, as long as we maintain $\sigma_{\mathrm{r}}(u) \leq 1$ (as v verifies $\sigma_{\mathrm{r}}(v) \geq 2$, as explained earlier).

Let us denote by $\mathcal{U}$ the subgraph of G induced by the edges incident to the vertices in $V(\mathcal{I})$. Note that $\mathcal{U}$ contains exactly the edges that remain to be labelled. Also $\mathcal{U}$ might contain several components, no two of which can have adjacent vertices as otherwise they would form a bigger component. Hence, we can treat the components of $\mathcal{U}$ independently.

Assume first that $U \in \mathcal{U}$ is just an edge uv with $u \in S$ and $v \in R$. This implies that v has degree 1 in G. Because G is nice, u must have other incident edges, all of which go to vertices in $V(\mathcal{H})$ that are thus assigned blue labels. We are here done with U by assigning e.g. red label 1 to its unique edge. No conflict involving u or v arises, since $\sigma_{\mathrm{b}}(v) = 0 < 1 \leq \sigma_{\mathrm{b}}(u)$, and $\sigma_{\mathrm{r}}(u) = 1$ while all its neighbours v' in $V(\mathcal{H})$ verify $\sigma_{\mathrm{r}}(v') \geq 2$.

Let us now consider the general case of a component $U \in \mathcal{U}$ with at least two edges. If no vertex of $V(U) \cap R$ has degree at least 2, then U is a star with center u in S and at least two leaves $v_1, \dots, v_k$ in R, in which case we are done by assigning blue label 1 to all edges of U. Indeed, we get $\sigma_{\mathrm{b}}(u) \geq 2$ while $\sigma_{\mathrm{b}}(v_i) = 1$ for every $i \in \{1, \dots, k\}$, and u remains distinguished from its neighbours in $V(\mathcal{H})$ because $\sigma_{\mathrm{r}}(u) = 0$.

Assume now that U has a vertex $v^* \in V(U) \cap R$ with degree at least 2. We assign blue label 2 to all edges of U, and apply the following process.

- For every vertex $u \in V(U) \cap S$ such that $\sigma_{\mathrm{b}}(u)$ is currently odd, we choose a path P from u to v^* in U, and, as going from u to v^* along P, we relabel with blue label 1 every traversed edge with blue label 2, and *vice versa*.

• For every vertex $v \in V(U) \cap R \setminus \{v^*\}$, we choose a path P from v to v^* in U, and, as going from v to v^* along P, we relabel with blue label 1 every traversed edge with blue label 2, and *vice versa*.

As a result, note that all vertices u in $V(U) \cap S$ have $\sigma_{\mathrm{b}}(u)$ even, while all vertices v in $v \in V(U) \cap R \setminus \{v^*\}$ have $\sigma_{\mathrm{b}}(v)$ odd. If also $\sigma_{\mathrm{b}}(v^*)$ is odd, then no conflict involving adjacent vertices of U remains, and we are done (in particular, note that, also, every vertex u in $V(U) \cap S$ verifies $\sigma_{\mathrm{r}}(u) = 0$, and thus cannot be in conflict with any neighbour in $V(\mathcal{H})$). Thus assume that $\sigma_{\mathrm{b}}(v^*)$ is even. In that case, we consider two distinct neighbours u_1, u_2 of v^* in U (thus $u_1, u_2 \in V(U) \cap S$), and assign red label 1 to both v^*u_1 and v^*u_2. As a result, we get $\sigma_{\mathrm{r}}(v^*) = 2$ while none of its neighbours verifies this. We also get $\sigma_{\mathrm{r}}(u_1) = \sigma_{\mathrm{r}}(u_2) = 1$, while none of their neighbours (in particular those in $V(\mathcal{H})$) verifies this. Thus, again, no vertex of U remains involved in a conflict.

Once all $U \in \mathcal{U}$ have been dealt with, the resulting labelling ℓ is as desired. ■

3. The Weak (2, 2)-Conjecture for 4-Colourable Graphs

Recall that a graph G is *k-colourable* if it admits a *proper k-vertex-colouring*, meaning a partition $(V_1, \ldots, V_k)$ of its vertex set $V(G)$ where each V_i is independent, i.e., no two of its vertices are joined by an edge. The *chromatic number* $\chi(G)$ of G is the smallest k such that G is k-colourable, and we say that G is *$\chi(G)$-chromatic*.

The next proof is an adaptation of the proof scheme developed by Vučković in [18] to prove the multiset version of the 1-2-3 Conjecture. Just as Kalkowski's Algorithm, this tool showed up to be important in this field for deducing results. For instance, Vučković's ideas were modified in [6] to get progress towards a product version of the 1-2-3 Conjecture.

Theorem 3. *The Weak $(2,2)$-Conjecture holds for nice 4-colourable graphs.*

Proof. Let G be a nice 4-colourable graph. We can assume that G is connected. Our goal is to produce a proper labelling ℓ of G with red labels $1, 2$ and blue labels $1, 2$. As mentioned in the introductory section, the Weak $(2,2)$-Conjecture was verified for bipartite graphs in [2]. Hence, we can assume that $\chi(G) \in \{3, 4\}$. In what follows, we focus on the most intricate case, which is when $\chi(G) = 4$. At the end of the proof, we will explain how to adapt the argument for the case $\chi(G) = 3$.

Let (V_1, V_2, V_3, V_4) be a proper 4-vertex-colouring of G. For a vertex $v \in V_i$ with $i > 1$, an incident edge vu going to some $u \in V_j$ with $j < i$ is called an *upward edge*. Conversely, for $v \in V_i$ with $i < 4$, an incident edge vu going to some $u \in V_j$ with $j > i$ is called a *downward edge*. Free to move vertices from parts to parts, we can assume that (V_1, V_2, V_3, V_4) has the property that for every $v \in V_i$, vertex v has upward edges to each of $V_1, \dots, V_{i-1}$. Indeed, if there is a $j < i$ such that v has no upward edge to V_j, then note that, when moving v to V_j, we obtain another proper 4-vertex-colouring of G. By repeating this process as long as necessary, we necessarily end up with a proper 4-vertex-colouring with the desired property, since vertices are only moved to parts with lower index. Also, none of the parts can become empty, since G is 4-chromatic.

We design ℓ through two main steps. During the first step, we will consider the vertices of V_4, V_3 one after another, and label their upward edges so that particular values by σ_r and σ_b are obtained, guaranteeing that no two adjacent of these vertices are in conflict. During the second step, we will label the remaining edges, those in the cut (V_1, V_2), so that no conflict remains.

Step 1. Labelling the upward edges of V_4, V_3.

We aim at producing the following values by σ_r and σ_b for the vertices in V_4 and V_3.

- $v \in V_3$: $\sigma_\mathrm{r}(v)$ even at least 2, $\sigma_\mathrm{b}(v)$ at least 2.
- $v \in V_4$: $\sigma_\mathrm{r}(v)$ odd at least 1, $\sigma_\mathrm{b}(v)$ odd at least 3.

It should be clear that no conflict involving two adjacent vertices in V_4 and V_3 can hold as soon as these properties are met (due to values of distinct parity by σ_r). To achieve this, we consider the vertices in V_4, V_3 following this ordering, and, for each vertex considered that way, we label all its upward edges so that the desired properties are met. Note that, in the course of this process, whenever considering a vertex $v \in V_3$, it can then be assumed that all downward edges of v have already been labelled.

- We start with vertices $v \in V_4$. We assign blue label 1 to an upward edge to V_3, and blue label 2 to all other upward edges to V_3 and V_2. This way, we get that $\sigma_\mathrm{b}(v)$ is odd with value at least 3. We then consider all remaining upward edges, which all go to V_1, and assign red labels to them so that $\sigma_\mathrm{r}(v) \geq 1$ is odd.
- Next, we consider vertices $v \in V_3$. At this point, the downward edges of v are all assigned blue labels. We assign blue label 2 to all upward edges to V_2, so that $\sigma_\mathrm{b}(v)$ has value at least 2. We then consider all remaining upward edges of v, which all go to V_1, and assign red label 2 to them so that $\sigma_\mathrm{r}(v) \geq 2$ is even.

Remark that, this far, all downward edges of the vertices in V_2 are assigned blue label 2, while all downward edges to V_3, V_4 of the vertices in V_1 are assigned red labels. Also, note that, this far, no vertex v verifies $\sigma_\mathrm{b}(v) = 1$.

Step 2. Labelling the edges of (V_1, V_2).

It remains to label the edges in (V_1, V_2). We do so by considering every vertex $v \in V_2$, and labelling its upward edges (to V_1) with red labels so that

$\sigma_{\mathrm{r}}(v)$ gets odd. Note that this results in all vertices $v \in V_1$ having $\sigma_{\mathrm{b}}(v) = 0$. A vertex $v \in V_2$ can be of two types: either $\sigma_{\mathrm{b}}(v) = 0$ and $\sigma_{\mathrm{r}}(v) \geq 1$ is odd, or $\sigma_{\mathrm{b}}(v) \geq 2$ is even and $\sigma_{\mathrm{r}}(v) \geq 1$ is odd. The first case corresponds to when v has no downward edges to V_3, V_4, while the second case is when it has. In the second case, note also that, at the moment, v cannot be in conflict with any of its neighbours, due to the constraints we have maintained: v cannot be in conflict with its neighbours in V_4 because $\sigma_{\mathrm{b}}(v)$ is even, v cannot be in conflict with its neighbours in V_3 because $\sigma_{\mathrm{r}}(v)$ is odd, and v cannot be in conflict with its neighbours in V_1 because $\sigma_{\mathrm{b}}(v) > 0$. So, the only possible conflicts are when v has upward edges only.

Let us consider the subgraph $\mathcal{H}$ of G induced by the upward edges (being all assigned red labels) of the vertices $v \in V_2$ with $\sigma_{\mathrm{b}}(v) = 0$. Note that $\mathcal{H}$ is a bipartite graph with (possibly) several (connected) components containing all vertices in conflict. A component of $\mathcal{H}$ containing conflicting vertices is said to be a *conflicting component*. Note that no two components of $\mathcal{H}$ have adjacent vertices, as otherwise they would, altogether, form a bigger component. From this, we can treat the conflicting components of $\mathcal{H}$ independently.

Observe that no conflicting component $H \in \mathcal{H}$ can contain just an edge uv with $u \in V_1$ and $v \in V_2$. Indeed, by definition of $\mathcal{H}$, this would mean that $\sigma_{\mathrm{b}}(v) = 0$, thus that v has no downward edges, and that vu is the only upward edge of v. In other words, $d_G(v) = 1$. Since G is nice, we have $d_G(u) \geq 2$, and, because all downward edges of u are assigned red labels, that $\sigma_{\mathrm{r}}(u) > \ell(uv) = 1 = \sigma_{\mathrm{r}}(v)$. Thus H is not conflicting, a contradiction.

So every conflicting component $H \in \mathcal{H}$ has at least two edges. If no vertex of $V(H) \cap V_2$ has degree at least 2, then H is a star with center u in V_1 all of whose incident edges are assigned red label 1, in which case as well u cannot have the same value by σ_{r} as its neighbours in H, which are all of degree 1 in G (and thus have value 1 by σ_{r}). This is again a contradiction to the fact that H is conflicting.

Thus, every conflicting component $H \in \mathcal{H}$ has at least two edges and a vertex $v^* \in V(H) \cap V_2$ with degree at least 2. In that case, we relabel the edges of H accordingly to the following procedure.

1. For every vertex $v \in V(H) \cap V_1$ with $\sigma_{\mathrm{r}}(v)$ even, we choose a path P from v to v^* in H, and, as going from v to v^* along P, we relabel with red label 1 every traversed edge with red label 2, and *vice versa*.
2. For every vertex $v \in V(H) \cap V_2 \setminus \{v^*\}$ with $\sigma_{\mathrm{r}}(v)$ odd, we choose a path P from v to v^* in H, and, as going from v to v^* along P, we relabel with red label 1 every traversed edge with red label 2, and *vice versa*.

As a result, all vertices v in $V(H) \cap V_1$ get $\sigma_{\mathrm{r}}(v) \geq 1$ odd, while all vertices v in $V(H) \cap V_2 \setminus \{v^*\}$ get $\sigma_{\mathrm{r}}(v) \geq 1$ even. If also $\sigma_{\mathrm{r}}(v^*)$ is even, then no conflict remains and we are done. Otherwise, we choose any two neighbours u_1, u_2 of v^*

in H, and assign blue label 1 to both v^*u_1 and v^*u_2. As a result, $\sigma_{\mathrm{b}}(v^*) = 2$ while none of its neighbours verifies this property (recall, in particular, that v^* has no downward edges). Also, we have $\sigma_{\mathrm{b}}(u_1) = \sigma_{\mathrm{b}}(u_2) = 1$ while, as mentioned earlier, none of their neighbours verifies this. Finally, all other adjacent vertices of H remain distinguished due to their values by σ_{r} being of different parities. Thus, no conflict involving vertices of H remains or arises.

Once all conflicting components $H \in \mathcal{H}$ have been treated this way, ℓ is as desired.

Let us conclude by mentioning that the exact same arguments work for a proper 3-vertex-colouring (V_1, V_2, V_3), by omitting all details dealing with V_4 above. Said differently, the exact same proof as above with $V_4 = \emptyset$ works. Indeed, a key point is that, in all arguments above, the fact that we can obtain particular red sums and blue sums for the vertices in V_1, V_2, V_3 does not depend on whether they are incident to downward edges to V_4. Thus, one can produce a proper $(2,2)$-labelling of any nice 3-chromatic graph G in a similar way as above, by 1) first considering a proper 3-vertex-colouring (V_1, V_2, V_3) where the vertices in V_3 have neighbours in both V_2 and V_1, and the vertices in V_2 have neighbours in V_1, by 2) then labelling the upward edges incident to the vertices in V_3 as described above, so that these vertices have even red sum at least 2 and blue sum at least 2, and by 3) labelling the edges of (V_1, V_2) as described above so that no conflict remains. Thus, the similar proof also holds for 3-chromatic graphs. ■

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