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GRAPH GRABBING GAME ON GRAPHS WITH FORBIDDEN SUBGRAPHS

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Abstract

The *graph grabbing game* is a two-player game on a connected graph with a weight function. In the game, they alternately remove a non-cut vertex from the graph (i.e., the resulting graph remains connected) and get the weight assigned to the vertex. Each player's aim is to maximize his or her outcome, when all vertices have been taken. Seacrest and Seacrest proved that if a given graph G is a tree with even order, then the first player can win the game for every weight function on G, and conjectured that the same statement holds if G is a connected bipartite graph with even order [D.E. Seacrest and T. Seacrest, *Grabbing the gold*, Discrete Math. 312 (2012) 1804–1806]. In this paper, we introduce a conjecture which is stated in terms of forbidden subgraphs and includes the above conjecture, and give two partial solutions to the conjecture.

Keywords: graph grabbing game, forbidden subgraph, corona product.

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1. Introduction

In this paper, we deal only with finite simple undirected graphs. For a graph G and a subset S of $V(G)$, the subgraph $G[S]$ induced by S in G is defined by $V(G[S]) = S$ and $E(G[S]) = \{e \in E(G) : e$ joins two vertices in $S\}$, and $G - S$ denotes the subgraph induced by $V(G) \setminus S$ (if S consists of a single vertex v, then we write $G - v$ instead of $G - \{v\}$). If $G - S$ (respectively, $G - v$) is not connected, then S is called a *cutset* (respectively, v is called a *cutvertex*). A graph is *even* (respectively, *odd*) if the number of vertices is even (respectively, odd). We mainly deal with a graph G with a weight function $w : V(G) \to \mathbb{R}^+ \cup \{0\}$, where $\mathbb{R}^+$ denotes the set of positive real numbers. We remark that when we consider a fixed weight function w, we often argue without referring to w explicitly. For basic terminology in graph theory, we refer the reader to [5].

In this paper, we consider a game on a graph G with a weight function, called the *graph grabbing game* (in this paper, we often refer to the graph grabbing game on G simply as the *game* on G).

Graph grabbing game

There are two players: Alice and Bob. Starting with Alice, they take the vertices alternately one by one and collect their weights. The vertices taken are removed from the graph. The choice of a vertex to be played in each move is restricted by the rule that after each move the remaining vertices form a connected graph (that is, each player is prohibited from taking a cutvertex). Both players' aim is to maximize their outcomes at the end of the game, when all vertices have been taken. Alice wins the game if she gets at least half of the total weight of the graph, and otherwise, Bob wins.

In the literature, the graph grabbing game is introduced in Winkler's puzzle book [19]. He showed that if G is an even path, then for every weight function, Alice can win the game on G. Moreover, he observed that there exists an odd path with a weight function for which Alice cannot win the game; for example, consider an odd path with three vertices and a weight function which assigns a positive weight to the middle vertex and zero to the other two vertices (the fact is that for every odd path of order at least three, there exists a weight function for which Alice cannot win the game; see the last paragraph of Section 4). Concerning cycles, as in the case of paths, if G is an even cycle, then for every weight function, Alice can win the game on G, and there exists an odd cycle with a weight function for which Alice can obtain at most $4/9$ of the total weight. It is also known that if G is an odd cycle, then for every weight function on G, Alice can obtain at least $4/9$ of the total weight ([4, 12]). For variants of the graph grabbing game, see [1, 9, 16].

Considering the advantage of the first player, it is easy to show that for every connected graph G, there exists a weight function on G for which Alice can win the game. However, for a given graph G with a weight function, to decide which player wins the game on G is PSPACE-complete ([3]). Thus we seek for a class of graphs such that as long as a given graph G is in the class, Alice can win the game for every weight function on G. Micek and Walczak [15] proved that if T is an even tree, then Alice can obtain at least $1/4$ of the total weight for every weight function on T, and conjectured that Alice can win the game for every weight function on T. After that, Seacrest and Seacrest [17] verified the conjecture by an elegant proof, and they conjectured the following (a partial solution to the conjecture is given in [7]).

Conjecture 1 [17]**.** *Let G be a connected bipartite even graph. Then for every weight function Alice can win the game on G.*

In this paper, we consider the graph grabbing game in connection with forbidden subgraphs. Let C_n $(n \geq 3)$ denote the cycle of order n, and K_m $(m \geq 1)$ denote the complete graph of order m. Let H and R be graphs, and write $V(H) = \{h_1, h_2, \ldots, h_{|V(H)|}\}$. The *corona product* of H and R, denoted by $H \odot R$, is defined by

$$V(H \odot R) = V(H) \cup V(R^1) \cup V(R^2) \cup \cdots \cup V\left(R^{|V(H)|}\right), \text{ and}$$

$$E(H \odot R) = E(H) \cup \bigcup_{i=1}^{|V(H)|} \left\{ E(R^i) \cup \bigcup_{v \in V(R^i)} \{h_i v\} \right\},$$

where R^i is a copy of R for each $i \in \{1, 2, \ldots, |V(H)|\}$. The corona product $C_n \odot K_1$ is referred to as the C_n*-corona*. We let $\mathcal{C}_{odd}$ denote the set of the C_{2k+1}-coronas as k ranges over all positive integers. Let G be a graph. For a graph H, G is said to be H*-free* if G does not contain a copy of H as an induced subgraph. For a set $\mathcal{H}$ of graphs, G is said to be $\mathcal{H}$*-free* if G is H-free for all $H \in \mathcal{H}$ (in this context, members of $\mathcal{H}$ are referred to as *forbidden subgraphs*).

Let $k \geq 1$ be an integer. Cibulka *et al.* [3] reported that there exists a weight function on the C_{2k+1}-corona $C_{2k+1} \odot K_1$ for which Alice cannot win the game. In fact, the same statement holds for $C_{2k+1} \odot H$ for any odd graph H (see the paragraph following the proof of Theorem 40 in Section 4). Note that $C_{2k+1} \odot H$ contains a copy of the C_{2k+1}-corona as an induced subgraph. The following conjecture has recently been made by Eoh and Choi in [8], which includes Conjecture 1.

Conjecture 2 [8]**.** *Let G be a connected even graph. If G is $\mathcal{C}_{odd}$-free, then for every weight function Alice can win the game on G.*

In [8], a weaker version of Conjecture 2 in which w is assumed to take only two values 0 and 1 is also proposed, and even this weaker version remains open.

In [8], it is also conjectured that the converse of Conjecture 2 holds. However, as we shall show in Section 4 (see Theorem 40), there exists a graph which shows that the converse of Conjecture 2 does not hold.

The main purpose of this paper is to give two partial solutions to Conjecture 2. We need some more definitions. For $n \geq 2$, let P_n denote the path of order n. A graph isomorphic to the C_3-corona is called a *net*. A *bull* is a graph obtained from a net by removing one vertex of degree 1. A graph isomorphic to the complete bipartite graph $K_{1,3}$ having bipartite sets of cardinalities 1 and 3 is called a *claw*. A connected graph is said to be *nonseparable* if it has no cutvertex. A *block* B of a connected graph G is a *maximal* nonseparable subgraph of G. A block which contains at most one cutvertex of G is called an *end block*. A *block path* is a graph consisting of a sequence of blocks $B_0, B_1, \ldots, B_m$ ($m \geq 0$) for each such that $i \in \{0, 1, \ldots, m-1\}$, $|V(B_i) \cap V(B_{i+1})| = 1$ and we have $V(B_i) \cap V(B_j) = \emptyset$ for any $j \notin \{i+1, i-1\}$. Under this notation, B_0 and B_m are the end blocks (it is possible that $B_0 = B_m$, i.e., $m = 0$). A special kind of block path in which each block is a clique, i.e., a complete subgraph, is called a *clique path*. A subgraph H of a graph G *dominates* G if each vertex in $V(G) \setminus V(H)$ is adjacent to a vertex in $V(H)$.

Path-free graphs have extensively been studied because they have nice properties. For example, a graph is P_2-free if and only if it has no edge, and a graph is P_3-free if and only if it is a disjoint union of complete graphs. It is also known that a connected graph G is P_4-free if and only if each connected induced subgraph G has either a dominating K_1 or a dominating induced C_4 ([10]), and that a connected graph G is P_5-free if and only if each connected induced subgraph of G has either a dominating clique or a dominating induced C_5 ([13]). As for the graph grabbing game on path-free graphs, we can prove the following partial solution to Conjecture 2 rather easily (see the paragraph following the proof of Claim 29 in Section 3).

Proposition 1. *Let G be a connected P_4-free even graph. Then for every weight function Alice can win the game on G.*

In this paper, we prove the following stronger result.

Theorem 2. *Let G be a connected $\{P_5, bull\}$-free even graph. Then for every weight function Alice can win the game on G.*

Note that every P_4-free graph is $\{P_5, bull\}$-free, and every $\{P_5, bull\}$-free graph is $\mathcal{C}_{odd}$-free. Thus Theorem 2 is stronger than Proposition 1, and is a partial solution to Conjecture 2.

Along a different line of research, claw-free graphs have been studied in connection with the conjecture that every 4-connected claw-free graph has a Hamiltonian cycle, which was made in [14]. This conjecture is still open, but it has been proved that every 2-connected $\{claw, net\}$-free graph has a Hamiltonian cycle, and that every connected $\{claw, net\}$-free graph has a Hamiltonian path. It is also known that every connected $\{claw, net\}$-free graph is a block path (see [6, 11, 18]). Having in mind the fact that not every block path is $\{claw, net\}$-free, we here confine ourselves to clique paths. Note that every clique path is $\{claw, net\}$-free, and every $\{claw, net\}$-free graph is $\mathcal{C}_{odd}$-free. Thus the following theorem provides another partial solution to Conjecture 2.

Theorem 3. *Let G be a clique path of even order. Then for every weight function Alice can win the game on G.*

The rest of the paper is organized as follows. In the next section, we prove a structural result concerning $\{P_5, bull\}$-free graphs, which we use in Section 3. In Section 3, we prove Theorems 2 and 3. In Section 4, we give several graphs on which Alice can/cannot win the game.

2. Characterization of $\{P_5, bull\}$-Free Graphs

In this section, we give a structural result concerning $\{P_5, bull\}$-free graphs. The main result is Theorem 18. For a graph G and a set $S \subseteq V(G)$, we denote by $N_G(S)$ the set of vertices adjacent to a vertex in S. If $S = \{v\}$, then we write $N_G(v)$ for $N_G(S)$.

Lemma 4. *Let G be a connected P_5-free graph. Then any two distinct cutvertices are adjacent.*

Proof. Let u, v be two cutvertices of G ($u \neq v$). Let F be a connected component of $G - u$ not containing v, and let F' be a connected component of $G - v$ not containing u. Now suppose that $uv \notin E(G)$, and let P be an induced u-v path in G. Then $|V(P)| \geq 3$. Let u' be a vertex in F adjacent to u, and let v' be a vertex in F' adjacent to v. Since $V(P) \subseteq V(G - V(F) - V(F'))$, it follows that $V(P) \cup \{u', v'\}$ induces a path of order $|V(P)| + 2 \geq 5$, a contradiction. ■

Lemma 5. *Let G be a connected $\{P_5, bull\}$-free graph, and suppose that G has two cutvertices u, v ($u \neq v$). Then the following statements hold.*

(i) *We have $uv \in E(G)$, $N_G(u) \cap N_G(v) = \emptyset$ and $N_G(u) \cup N_G(v) = V(G)$.*

(ii) *There exists no cutvertex of G other than u, v.*

(iii) *If F is a block of G, then one of the following holds.*

  (a) *$uv \in E(F)$;*

  (b) *$u \in V(F), v \notin V(F)$ and $\{u\} \cup N_F(u) = V(F)$; or*

  (c) *$u \notin V(F), v \in V(F)$ and $\{v\} \cup N_F(v) = V(F)$.*

Proof. Let u', v' be as in the proof of Lemma 4.

(i) By Lemma 4, $uv \in E(G)$. If there exists $x \in N_G(u) \cap N_G(v)$, then $\{x, u, v, u', v'\}$ induces a bull, a contradiction. Thus $N_G(u) \cap N_G(v) = \emptyset$. Suppose that $N_G(u) \cup N_G(v) \neq V(G)$. There exist $x \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$ and $x' \in V(G) \setminus (N_G(u) \cup N_G(v))$ such that $xx' \in E(G)$. We may assume that $x \in N_G(u) \setminus \{v\}$. Since $N_G(u) \cap N_G(v) = \emptyset$, $xv \notin E(G)$. Hence $\{x', x, u, v, v'\}$ induces a copy of P_5, a contradiction. Thus $N_G(u) \cup N_G(v) = V(G)$.

(ii) Note that (i) holds for any two distinct cutvertices. Thus if there exists a third cutvertex p, then by the first assertion of (i), $pu, pv \in E(G)$, which contradicts the second assertion of (i), a contradiction.

(iii) Let F be a block of G. Since u, v are the only cutvertices, it follows that if F is not an end block, then $\{u, v\} \subseteq V(F)$, and that if F is an end block, then $\{u, v\} \cap V(F) = \{u\}$ or $\{v\}$. Hence if F is not an end block, then (a) holds. Thus we may assume that F is an end block. By symmetry, we may assume that $\{u, v\} \cap V(F) = \{u\}$. Then $N_G(v) \cap (V(F) \setminus \{u\}) = \emptyset$. By the third assertion of (i), this implies that $V(F) \setminus \{u\} \subseteq N_G(u)$, i.e., $\{u\} \cup N_F(u) = V(F)$. ■

Let G be a connected graph. For two disjoint subsets S_1, S_2 of $V(G)$, let $E_G(S_1, S_2)$ be the set of edges joining a vertex in S_1 and a vertex in S_2.

A partition $(X_1, X_2, \ldots, X_{2p})$ of $V(G)$ ($X_i \neq \emptyset$ for each i) is *admissible* if the following holds: for i, j with $1 \leq i, j \leq p$, if $i \leq j$, then each vertex in X_{2j} is adjacent to all vertices in X_{2i-1}, and if $j < i$, then $E_G(X_{2i-1}, X_{2j}) = E_G(X_{2i}, X_{2j}) = E_G(X_{2i-1}, X_{2j-1}) = \emptyset$ (for example, see Figure 1).

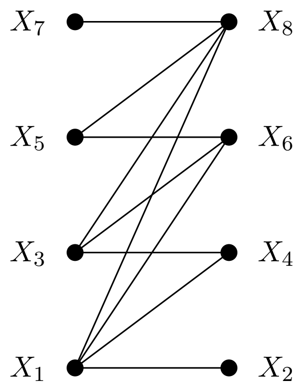

Figure 1. An admissible partition $(X_1, X_2, \ldots, X_8)$ where $|X_i| = 1$ for each $i \in \{1, 2, \ldots, 8\}$.

We prove the following key proposition for our structural result about $\{P_5, bull\}$-free graphs.

Proposition 6. *Let G be a connected $\{P_5, bull\}$-free graph, and suppose that G has two cutvertices u, v ($u \neq v$). Let $A_1, \ldots, A_s$ be the blocks of G such that*

$u \notin V(A_i)$ and $v \in V(A_i)$ $(1 \leq i \leq s)$, and let $B_1, \ldots, B_t$ be the blocks of G such that $u \in V(B_j)$ and $v \notin V(B_j)$ $(1 \leq j \leq t)$. Then G has an admissible partition $(X_1, X_2, \ldots, X_{2p})$ $(p \geq 2)$ with $X_1 = \{u\}$, $X_2 = (V(B_1) \cup \cdots \cup V(B_t)) \setminus \{u\}$, $X_{2p-1} = (V(A_1) \cup \cdots \cup V(A_s)) \setminus \{v\}$ and $X_{2p} = \{v\}$.

Proof. Let H be the block of G such that $uv \in E(H)$. If $|V(H)| = 2$, then the desired conclusion holds with $p = 2$. Thus we may assume that $|V(H)| \geq 3$. Set $A = N_H(v) \setminus \{u\}$, $B = N_H(u) \setminus \{v\}$.

Claim 7. *$A \cap B = \emptyset$ and $A \cup B = V(H) \setminus \{u, v\}$.*

Proof. This immediately follows from Lemma 5(i). □

Let $H_1, H_2, \ldots, H_r$ be the connected components of $H - \{u, v\}$.

Claim 8. *For each $1 \leq i \leq r$, $V(H_i) \cap A \neq \emptyset$ and $V(H_i) \cap B \neq \emptyset$.*

Proof. If $V(H_i) \cap A = \emptyset$, then by the definition of A, u is a cutvertex of H, a contradiction. Thus $V(H_i) \cap A \neq \emptyset$ and, similarly, we have $V(H_i) \cap B \neq \emptyset$. □

It follows from Claim 8 that $A \neq \emptyset$ and $B \neq \emptyset$.

Claim 9. *We have $r = 1$, i.e., $H - \{u, v\}$ is connected.*

Proof. Suppose that $r \geq 2$. For $i = 1, 2$, it follows from Claim 8 that there exists $x_i y_i \in E(H_i)$ with $x_i \in V(H_i) \cap A$ and $y_i \in V(H_i) \cap B$. Then $x_1 y_1 u y_2 x_2$ is an induced path of order 5, a contradiction. □

Define an equivalence relation $\sim$ on A by letting $x \sim x'$ if and only if $N_H(x) \cap B = N_H(x') \cap B$ $(x, x' \in A)$. Let $X_3, X_5, \ldots, X_{2p-3}$ be the equivalence classes with respect to $\sim$, and let $X_1 = \{u\}$ and $X_{2p-1} = (V(A_1) \cup \cdots \cup V(A_s)) \setminus \{v\}$.

Define an equivalence relation $\approx$ on B by letting $y \approx y'$ if and only if $N_H(y) \cap A = N_H(y') \cap A$ $(y, y' \in B)$. Let $X_4, X_6, \ldots, X_{2q-2}$ be the equivalence classes with respect to $\approx$, and let $X_2 = (V(B_1) \cup \cdots \cup V(B_t)) \setminus \{u\}$ and $X_{2q} = \{v\}$. By the definition of A and B and by Claim 7, we have the following.

Claim 10.

(i)–(a) *For each $1 \leq i \leq p - 1$, $E_G(X_{2i-1}, X_{2p-1}) = \emptyset$.*

–(b) *For each $2 \leq i \leq p - 1$, $E_G(X_1, X_{2i-1}) = \emptyset$.*

(ii)–(a) *For each $2 \leq j \leq q$, $E_G(X_2, X_{2j}) = \emptyset$.*

–(b) *For each $2 \leq j \leq q - 1$, $E_G(X_{2j}, X_{2q}) = \emptyset$.*

Claim 11. (i) *$E_G(X_{2i-1}, X_{2i'-1}) = \emptyset$ for any i, i' $(1 \leq i, i' \leq p)$ with $i \neq i'$.*

(ii) *$E_G(X_{2j}, X_{2j'}) = \emptyset$ for any j, j' $(1 \leq j, j' \leq q)$ with $j \neq j'$.*

Proof. We prove (i) (we can prove (ii) in a similar way). By Claim 10, we may assume that $2 \leq i, i' \leq p-1$. Suppose that there exist $x \in X_{2i-1}$ and $x' \in X_{2i'-1}$ such that $xx' \in E(G)$. By the definition of $\sim$, there exists $y \in B$ such that $|N_H(y) \cap \{x, x'\}| = 1$. Take $z \in X_{2p-1}$. Then $G[\{v, x, x', z, y\}]$ is isomorphic to a bull, a contradiction. $\square$

Claim 12. (i) *$N_H(x) \cap B \neq \emptyset$ for every $x \in A$.*
(ii) *$N_H(y) \cap A \neq \emptyset$ for every $y \in B$.*

Proof. We prove (i) (we can prove (ii) in a similar way). If there exists $a \in A$ such that $N_H(a) \cap B = \emptyset$, then it follows from Claim 11(i) that v is a cutvertex of H, a contradiction. $\square$

Claim 13. (i) *Let $1 \leq i, i' \leq p$ with $i \neq i'$. Then $N_G(X_{2i-1}) \cap B \subseteq N_G(X_{2i'-1}) \cap B$ or $N_G(X_{2i-1}) \cap B \supseteq N_G(X_{2i'-1}) \cap B$.*
(ii) *Let $1 \leq j, j' \leq q$ with $j \neq j'$. Then $N_G(X_{2j}) \cap A \subseteq N_G(X_{2j'}) \cap A$ or $N_G(X_{2j}) \cap A \supseteq N_G(X_{2j'}) \cap A$.*

Proof. We prove (i) (we can prove (ii) in a similar way). By the definition of X_1 and B, $N_G(X_1) \cap B = B$. By the definition of X_{2p-1}, we also have $N_G(X_{2p-1}) \cap B = \emptyset$. Thus we may assume that $2 \leq i, i' \leq p-1$.

Suppose to the contrary that $(N_G(X_{2i-1}) \setminus N_G(X_{2i'-1})) \cap B \neq \emptyset$ and $(N_G(X_{2i'-1}) \setminus N_G(X_{2i-1})) \cap B \neq \emptyset$. Take $y \in (N_G(X_{2i-1}) \setminus N_G(X_{2i'-1})) \cap B$ and $y' \in (N_G(X_{2i'-1}) \setminus N_G(X_{2i-1})) \cap B$. If $N_G(X_{2i-1}) \cap N_G(X_{2i'-1}) \cap B \neq \emptyset$, then letting $z \in N_G(X_{2i-1}) \cap N_G(X_{2i'-1}) \cap B$, $x \in X_{2i-1}$ and $x' \in X_{2i'-1}$, we see that $x \not\sim x'$ and $y \not\approx z \not\approx y' \not\approx y$, and hence, it follows from Claim 11 that $yxzx'y'$ is an induced P_5, a contradiction. Consequently, $N_G(X_{2i-1}) \cap N_G(X_{2i'-1}) \cap B = \emptyset$.

By Claim 9, $H - u - v$ is connected. Let $z_1 z_2 \cdots z_\ell$ be a shortest $(X_{2i-1}, X_{2i'-1})$-path in $H - u - v$. It follows from Claim 11(i) that $z_2, z_{\ell-1} \in B$. Thus $z_2 \not\approx z_{\ell-1}$. By Claim 11(ii), $d_{H-u-v}(z_2, z_{\ell-1}) \geq 2$. This implies $\ell \geq 5$, which contradicts the assumption that G is P_5-free. Therefore $N_G(X_{2i-1}) \cap B \subseteq N_G(X_{2i'-1}) \cap B$ or $N_G(X_{2i-1}) \cap B \supseteq N_G(X_{2i'-1}) \cap B$. $\square$

In view of Claim 13, relabeling $X_3, \ldots, X_{2p-3}$ if necessary, we may assume that we have $N_G(X_{2i-1}) \cap B \supseteq N_G(X_{2i'-1}) \cap B$ for any i, i' with $1 \leq i < i' \leq p$. For each $2 \leq i \leq p$, set $J_{2i-1} = \{2j : 2 \leq j \leq q-1, X_{2j} \subseteq N_G(X_{2i-1}) \cap B\}$. Then $J_3 \supseteq J_5 \supseteq \cdots \supseteq J_{2p-1} = \emptyset$.

Claim 14. $J_3 = \{4, 6, \ldots, 2q-2\}$.

Proof. If $J_3 \neq \{4, 6, \ldots, 2q-2\}$, then letting $2j \in \{4, 6, \ldots, 2q-2\} \setminus J_3$ and $y \in X_{2j}$, we get $N_G(y) \cap A = \emptyset$, which contradicts Claim 12(ii). $\square$

Claim 15. *For each $2 \leq i \leq p-1$, $|J_{2i-1} \setminus J_{2i+1}| = 1$.*

Proof. By the definition of $\sim$ and $\approx$, $J_{2i-1} \neq J_{2i+1}$. Suppose to the contrary that $|J_{2i-1} \setminus J_{2i+1}| \geq 2$. Take $2j, 2j' \in J_{2i-1} \setminus J_{2i+1}$ with $j \neq j'$, and let $y \in X_{2j}$ and $y' \in X_{2j'}$. Then $N_G(y) \cap A = X_1 \cup X_3 \cup \cdots \cup X_{2i-1} = N_G(y') \cap A$, and hence $y \approx y'$, which contradicts the fact that $j \neq j'$. Thus $|J_{2i-1} \setminus J_{2i+1}| = 1$. □

It follows from Claims 14 and 15 that $q = p$ and, relabeling $X_4, \ldots, X_{2q-2}$ if necessary, we may assume that $J_{2i-1} = \{2i, 2i+2, \ldots, 2p-2\}$ for each $2 \leq i \leq p$. By the definition of J_{2i-1}, this implies that $(X_1, X_2, \ldots, X_{2p})$ is an admissible partition of G. This completes the proof of Proposition 6. ■

We prepare two more lemmas and then prove our main theorem in this section.

Lemma 16. *Let G be a connected P_5-free graph, and suppose that G has precisely one cutvertex u. Then G has at most one block H such that $V(H) \neq \{u\} \cup N_H(u)$.*

Proof. Otherwise, we can easily find an induced P_5, a contradiction. ■

Lemma 17. *Let H be a connected $\{P_5, bull\}$-free graph. Let $u \in V(H)$, and suppose that $V(H) \neq \{u\} \cup N_H(u)$. Then there exists $v \in N_H(u)$ such that $H - (N_H(u) \setminus \{v\})$ is connected, and v is a cutvertex of $H - (N_H(u) \setminus \{v\})$.*

Proof. Take $v \in N_H(u)$ so that $|N_H(v) \setminus (\{u\} \cup N_H(u))|$ is as large as possible. Since $V(H) \neq \{u\} \cup N_H(u)$, we have $N_H(v) \setminus (\{u\} \cup N_H(u)) \neq \emptyset$. Suppose that $H - (N_H(u) \setminus \{v\})$ is not connected. Let F be a connected component of $H - (N_H(u) \setminus \{v\})$ which does not contain v. Then $N_H(V(F)) \cap (N_H(u) \setminus \{v\}) \neq \emptyset$. Take $ry \in E(H)$ with $r \in V(F)$ and $y \in N_H(u) \setminus \{v\}$. By the choice of F, we have $r \notin \{u\} \cup N_H(u)$ and $rv \notin E(H)$. Hence $r \in (N_H(y) \setminus N_H(v)) \setminus (\{u\} \cup N_H(u))$. In view of the maximality of $|N_H(v) \setminus (\{u\} \cup N_H(u))|$, we can take $x \in (N_H(v) \setminus N_H(y)) \setminus (\{u\} \cup N_H(u))$. Again by the choice of F, we have $rx \notin E(H)$. Now, according as $yv \in E(H)$ or $yv \notin E(H)$, $\{r, y, u, v, x\}$ induces a bull or a copy of P_5, a contradiction. Thus $H - (N_H(u) \setminus \{v\})$ is connected. Since $N_{H-(N_H(u) \setminus \{v\})}(u) = \{v\}$ and $N_{H-(N_H(u) \setminus \{v\})}(v) \setminus \{u\} = N_H(v) \setminus (\{u\} \cup N_H(u)) \neq \emptyset$, it follows that v is a cutvertex of $H - (N_H(u) \setminus \{v\})$. ■

Theorem 18. *Let G be a connected $\{P_5, bull\}$-free graph, and suppose that G has a cutvertex u. Then G has an admissible partition $(X_1, X_2, \ldots, X_{2p})$ $(p \geq 1)$ with $X_1 = \{u\}$.*

Proof. In view of Proposition 6, we may assume that u is the only cutvertex of G. We may also assume that $V(G) \neq \{u\} \cup N_G(u)$; for otherwise, (X_1, X_2) with $X_2 = V(G) \setminus \{u\}$ is an admissible partition. Having Lemma 16 in mind, let H be the unique block of G such that $V(H) \neq \{u\} \cup N_H(u)$, and let $B_1, \ldots, B_t$ be all other blocks of G. Let $v \in N_H(u)$ be as in Lemma 17, and let Z be a minimal subset of $N_H(u) \setminus \{v\}$ such that v is a cutvertex of $H - Z$. Set $G' = G - Z$ and $H' = H - Z$, and let $A_1, \ldots, A_s$ be the blocks of G' such that $v \in V(A_i)$ and $u \notin V(A_i)$ (i.e., $(V(A_i) \setminus \{v\}) \cap N_{H'}(u) = \emptyset$).

Claim 19. *Let $z \in Z$. Then $xz \in E(G)$ for every $x \in (V(A_1) \cup \cdots \cup V(A_s)) \setminus \{v\}$.*

Proof. Suppose that there exists $x \in (V(A_1) \cup \cdots \cup V(A_s)) \setminus \{v\}$ such that $xz \notin E(G)$. We may assume that $x \in V(A_1) \setminus \{v\}$. By the minimality of Z, there exists $x' \in V(A_1) \setminus \{v\}$ such that $x'z \in E(G)$. By choosing $x, x' \in V(A_1) \setminus \{v\}$ with $xz \notin E(G)$ and $x'z \in E(G)$ so that $d_{A_1 - v}(x, x')$ is as small as possible, we may assume that $xx' \in E(G)$. Take $y \in V(B_1)$. Then $xx'zuy$ is an induced P_5, a contradiction. □

Claim 20. *The blocks $B_1, \ldots, B_t$ are the only blocks F of G' which satisfy $u \in V(F)$ and $v \notin V(F)$.*

Proof. Suppose that G' has a block B_{t+1} ($\notin \{B_1, \ldots, B_t\}$) such that $u \in V(B_{t+1})$ and $v \notin V(B_{t+1})$. Since B_{t+1} is not a block of G, $V(B_{t+1}) \subseteq V(H)$, and there exist $z \in Z$ and $y \in V(B_{t+1}) \setminus \{u\}$ such that $yz \in E(G)$. Take $x \in V(A_1) \setminus \{v\}$ and $y' \in V(B_1) \setminus \{u\}$. Then by Claim 19, $G[\{z, u, y, x, y'\}]$ is isomorphic to a bull, a contradiction. □

In view of Lemma 5, it follows from Claim 20 that $H'' = H' - ((V(A_1) \cup \cdots \cup V(A_s)) \setminus \{v\})$ is the block of G' such that $uv \in E(H'')$. By Proposition 6, G' has an admissible partition $(X'_1, X'_2, \ldots, X'_{2p})$ such that $X'_1 = \{u\}$, $X'_2 = (V(B_1) \cup \cdots \cup V(B_t)) \setminus \{u\}$, $X'_{2p-1} = (V(A_1) \cup \cdots \cup V(A_s)) \setminus \{v\}$ and $X'_{2p} = \{v\}$. Set $A = N_{H''}(v) \setminus \{u\}$ and $B = N_{H''}(u) \setminus \{v\}$. Then $A = X'_3 \cup X'_5 \cup \cdots \cup X'_{2p-3}$ and $B = X'_4 \cup X'_6 \cup \cdots \cup X'_{2p-2}$.

Claim 21. *Let $z \in Z$. Then $yz \notin E(G)$ for every $y \in B$.*

Proof. If there exists $y \in B$ such that $yz \in E(G)$, then letting $x \in X'_{2p-1}$ and $y' \in X'_2$, we see from Claim 19 that $G[\{z, u, y, x, y'\}]$ is isomorphic to a bull, a contradiction. □

Claim 22. *Let $z \in Z$. Then $xz \in E(G)$ for every $x \in A$.*

Proof. Suppose that there exists $x \in A$ such that $xz \notin E(G)$. Let $2i - 1$ ($2 \leq i \leq p - 1$) be the index such that $x \in X'_{2i-1}$. Take $y \in X'_{2i}$. Then $xy \in E(G)$. Take $x' \in X'_{2p-1}$. By Claims 19 and 21, $xyuzx'$ is an induced path of order 5, a contradiction. □

It follows from Claims 19, 21 and 22 that $(X'_1, X'_2, \ldots, X'_{2p-1}, X'_{2p} \cup Z)$ is an admissible partition of G. This completes the proof of Theorem 18. ■

We here remark that if a graph G has an admissible partition $(X_1, \ldots, X_{2p})$ such that $G[X_i]$ is $\{P_5, bull\}$-free for every $1 \leq i \leq 2p$, then G is $\{P_5, bull\}$-free. Thus Theorem 18 gives a characterization of connected $\{P_5, bull\}$-free graphs which have a cutvertex.

3. Partial Solutions to Conjecture 2

In this section, we prove Theorems 2 and 3.

3.1. $\{P_5, bull\}$-free graphs

In this subsection, by using the characterization of $\{P_5, bull\}$-free graphs which we proved in the preceding section, we prove Theorem 2. In order that induction arguments may work, we actually prove the following proposition (the proof of Theorem 2 is included at the end of this subsection).

Proposition 23. *Let G be a connected even graph having an admissible partition $(X_1, \ldots, X_{2p})$. Let w be a fixed weight function on G, and suppose that*

(1) *Alice can win the game on every connected induced even subgraph G' with $|V(G')| < |V(G)|$ for the weight function w.*

Then Alice can win the game on G.

Proof. We first make the following observations.

Observation 24. *For each $1 \leq i \leq 2p$ and each $X \subsetneq X_i$, $(X_1, \ldots, X_i - X, \ldots, X_{2p})$ is an admissible partition of $G - X$.*

Observation 25. *If $p \geq 2$, then for each $2 \leq i \leq 2p-1$, $(X_1, \ldots, X_{i-1} \cup X_{i+1}, \ldots, X_{2p})$ is an admissible partition of $G - X_i$.*

Observation 26. *If $p \geq 2$, then for each $1 \leq i \leq 2p-2$, $(X_1, \ldots, X_{i-1}, X_{i+2}, \ldots, X_{2p})$ is an admissible partition of $G - X_i - X_{i+1}$.*

Throughout the rest of the proof of the proposition, we let $w_k = \max\{w(x) : x \in X_k\}$ for each $1 \leq k \leq 2p$. Also for each $1 \leq k \leq 2p$ with $|X_k| = 1$, we let x_k denote the unique vertex in X_k (thus $w_k = w(x_k)$). We start by proving three claims concerning the case where $p = 1$.

Claim 27. *Suppose that $p = 1$ and $|X_1| = 1$. Then Alice can win the game at least by $w_1 - w_2$.*

Proof. We proceed by induction on $|V(G)|$. If $|V(G)| = 2$, then the claim clearly holds. Thus let $|V(G)| \geq 4$, and assume that the claim holds for smaller graphs. If $G[X_2]$ is connected, then if follows from (1) that Alice can win the game at least by $w_1 - w_2$ by taking x_1. Thus we may assume that $G[X_2]$ is disconnected.

Write $X_2 = \{y_1, y_2, \dots, y_{|X_2|}\}$ so that $w_2 = w(y_1) \geq w(y_2) \geq \cdots \geq w(y_{|X_2|})$. First we consider the case where $G[X_2 \setminus \{y_1\}]$ is disconnected. In this case, Alice takes y_1. Let b be the vertex taken by Bob in his first turn. Since $G[X_2 \setminus \{y_1\}]$ is disconnected, $b \neq x_1$, and hence $w(y_1) \geq w(b)$. By the induction hypothesis, Alice can win the game on $G - y_1 - b$ at least by $w_1 - \max\{w(y) : y \in X_2 \setminus \{y_1, b\}\}$. Since $w(y_1) \geq w(b)$ and $w_2 \geq \max\{w(y) : y \in X_2 \setminus \{y_1, b\}\}$, this implies the desired conclusion.

Next we consider the case where $G[X_2 \setminus \{y_1\}]$ is connected. In this case, y_1 is an isolated vertex of $G[X_2]$, and hence $G[X_2 \setminus \{y_2\}]$ is disconnected. Alice takes y_2. Let b be the vertex taken by Bob in his first turn. Since $G[X_2 \setminus \{y_2\}]$ is disconnected, $b \neq x_1$. Assume for the moment that $b \neq y_1$. Then $w(y_2) \geq w(b)$. Since it follows from the induction hypothesis that Alice can win the game on $G - y_2 - b$ at least by $w_1 - w_2$, this implies the desired conclusion. Thus we may assume that $b = y_1$. Then by the induction hypothesis, Alice can win the game on $G - y_1 - y_2$ at least by $w_1 - w(y_3)$. Since $w(y_2) \geq w(y_3)$, this implies that Alice can win the game on G at least by $(w(y_2) - w(y_1)) + (w_1 - w(y_3)) \geq w_1 - w_2$. □

Claim 28. *Suppose that $p = 1$ and $|X_2| = 1$, and let $x_1 \in X_1$ be a vertex such that $w(x_1) = w_1$. Then Alice can win the game at least by $\min\{w_1 - w_2, 0\}$ by taking x_1.*

Proof. Let b be the vertex taken by Bob in his first turn. By (1), Alice can win the game on $G - x_1 - b$. Since $w(x_1) - w(b) \geq \min\{w_1 - w_2, 0\}$ by the choice of x_1, this implies the desired conclusion (note that this argument works in the case where $|V(G)| = 2$ as well). □

Claim 29. *Suppose that $p = 1$. Then Alice can win the game.*

Proof. By symmetry, we may assume that $w_1 \geq w_2$. Choose $x_1 \in X_1$ so that $w(x_1) = w_1$. If $G - x_1$ is connected, then it follows from (1) that Alice can win the game by taking x_1. Thus we may assume that x_1 is a cutvertex. Then $X_1 = \{x_1\}$ (and $|X_2| \geq 3$). Hence the desired conclusion follows from Claim 27. □

In passing, we mention that Proposition 1 follows from Claim 29. Returning to the proof of Proposition 23, we prove the following technical claim.

Claim 30. *Suppose that $|X_k| = 1$ for every $1 \leq k \leq 2p$, and there exists k_0 such that $w_1 \geq \cdots \geq w_{k_0-1} \geq w_{k_0} \leq w_{k_0+1} \leq \cdots \leq w_{2p}$. Then Alice can win the game at least by $\sum_{i=1}^{p} w_{2i-1} - \sum_{i=1}^{p} w_{2i}$ by taking w_{2p-1}.*

Proof. Let $\alpha = \sum_{i=1}^{p} w_{2i-1} - \sum_{i=1}^{p} w_{2i}$. We proceed by induction on $|V(G)|$. If $|V(G)| = 2$, then the claim clearly holds. Thus let $|V(G)| \geq 4$, and assume that the claim holds for smaller graphs. Let v_ℓ be the vertex taken by Bob in his first turn.

Case 1. $\ell = 2p$ or $\ell = 2p-2$. Set $G' = G - x_{2p-1} - x_\ell$, and let $x'_{2p-2} = x_{2p-2}$ or $x'_{2p-2} = x_{2p}$ according as $\ell = 2p$ or $\ell = 2p-2$. By Observation 26, $(\{x_1\}, \ldots, \{x_{2p-3}\}, \{x'_{2p-2}\})$ is an admissible partition of G', and G' satisfies the assumption of the claim. Hence by the induction hypothesis, Alice can win the game on G' at least by $\sum_{i=1}^{p-1} w_{2i-1} - (\sum_{i=1}^{p-2} w_{2i} + w(x'_{2p-2})) = \alpha - (w_{2p-1} - w(x_\ell))$, which implies the desired conclusion.

Case 2. $\ell \leq 2p-3$. Note that $\ell \neq 1$ because $G - x_{2p-1} - x_1$ is disconnected. This implies $p \geq 3$. If $\ell < k_0$, then let $a = x_{\ell-1}$ and $\ell' = \ell - 1$; if $\ell \geq k_0$, then let $a = x_{\ell+1}$ and $\ell' = \ell$; By the assumption of the claim, we have $w(a) \geq w(x_\ell)$. Set $G' = G - x_\ell - a$. By Observation 26, $(\{x_1\}, \ldots, \{x_{\ell'-1}\}, \{x_{\ell'+2}\}, \ldots, \{x_{2p}\})$ is an admissible partition of G', and G' satisfies the assumption of the claim (with k_0 replaced by $k_0 - 2$ in the case where $\ell < k_0$, and possibly by $k_0 - 1$ in the case where $\ell = k_0$). By the induction hypothesis, by taking x_{2p-1}, Alice can win the game on G' at least by $\alpha - w(x_{\ell'}) + w(x_{\ell'+1})$ if ℓ' is odd, and at least by $\alpha - w(x_{\ell'+1}) + w(x_{\ell'})$ if ℓ' is even. Since $\{x_\ell, a\} = \{x_{\ell'}, x_{\ell'+1}\}$ and $w(a) \geq w(x_\ell)$, it follows that Alice can win the game on G' at least by $\alpha - w(a) + w(x_\ell)$ by taking x_{2p-1}. Since $G' - x_{2p-1} = G - x_{2p-1} - x_\ell - a$, we now see that Alice can win the game on G at least by α by taking a in her second turn. $\square$

We here proceed to prove the proposition by distinguishing two cases (Claims 31 and 33).

Claim 31. *Suppose that there exists k_0 such that $|X_k| = 1$ for every $k \neq k_0$, and $w_1 \geq \cdots \geq w_{k_0-1} \geq w_{k_0} \leq w_{k_0+1} \leq \cdots \leq w_{2p}$. Then Alice can win the game.*

Proof. We first prove the following subclaim, which is a modification of Claim 30.

Subclaim 3.1. *Suppose that $p \geq 2$ and k_0 is even. Then Alice can win the game at least by $\sum_{i=1}^{p} w_{2i-1} - \sum_{i=1}^{p} w_{2i}$ by taking x_{2p-1}.*

Proof. Let $\alpha = \sum_{i=1}^{p} w_{2i-1} - \sum_{i=1}^{p} w_{2i}$. We proceed by induction on $|V(G)|$. If $|V(G)| = 4$, then $|X_{k_0}| = 1$, and hence the desired conclusion follows from Claim 30. Thus let $|V(G)| \geq 6$, and assume that the subclaim holds for smaller graphs. In view of Claim 30, we may assume that $|X_{k_0}| \geq 2$. Since $|V(G)|$ is even, it follows that $|X_{k_0}| \geq 3$. Let b be the vertex taken by Bob.

Case 1. $b \in X_{k_0}$. Choose $a \in X_{k_0} \setminus \{b\}$ so that $w(a) = \max\{w(x) : x \in X_{k_0} \setminus \{b\}\}$, and set $G' = G - b - a$, $X'_{k_0} = X_{k_0} \setminus \{b, a\}$ and $w'_{k_0} = \max\{w(x) : x \in X'_{k_0}\}$. By Observation 24, $(\{x_1\}, \ldots, \{x_{k_0-1}\}, X'_{k_0}, \{x_{k_0+1}\}, \ldots, \{x_{2p}\})$ is an admissible partition of G', and G' satisfies the assumption of the subclaim. We have $w(b) \leq w_{k_0}$ and $w(a) \geq w'_{k_0}$. By the induction hypothesis, Alice can win the game on G' at least by $\alpha + w_{k_0} - w'_{k_0}$. Since $G' - x_{2p-1} = G - x_{2p-1} - b - a$, and $w_{k_0} - w'_{k_0} \geq w(b) - w(a)$, this implies that Alice can win the game on G at least by α by taking a in her second turn.

Case 2. $b \notin X_{k_0}$. Write $b = x_\ell$. Then $\ell \neq k_0$.

Subcase 2.1. $\ell = 2p$ or $\ell = 2p - 2$. Set $G' = G - x_{2p-1} - x_\ell$, and let $X'_{2p-2} = X_{2p-2}$ or $X'_{2p-2} = X_{2p}$ according as $\ell = 2p$ or $\ell = 2p - 2$. By Observation 26, $(X_1, \ldots, X_{2p-3}, X'_{2p-2})$ is an admissible partition of G' and, in the case where $p \geq 3$, G' satisfies the assumption of the subclaim. Hence by Claim 27 or the induction hypothesis according as $p = 2$ or $p \geq 3$, Alice can win the game on G' at least by $\alpha - w_{2p-1} + w_\ell$, which implies the desired conclusion.

Subcase 2.2. $\ell \leq 2p - 3$. Note that $\ell \neq 1$ because $G - x_{2p-1} - x_1$ is disconnected. This implies $p \geq 3$. If $\ell < k_0$, then let $a = x_{\ell-1}$ and $\ell' = \ell - 1$; if $\ell > k_0$, then let $a = x_{\ell+1}$ and $\ell' = \ell$. By assumption, we have $w(a) \geq w(b)$. Set $G' = G - b - a$. By Observation 26, $(X_1, \ldots, X_{\ell'-1}, X_{\ell'+2}, \ldots, X_{2p})$ is an admissible partition of G', and G' satisfies the assumption of the subclaim. Arguing as in Claim 30, we see from the induction hypothesis that Alice can win the game on G' at least by $\alpha - w(a) + w(b)$ by taking x_{2p-1}, which implies that Alice can win the game on G at least by α by taking a in her second turn. □

We also need the following subclaim.

Subclaim 3.2. *Suppose that k_0 is odd and $|X_{k_0}| \geq 3$, and let $a \in X_{k_0}$ be a vertex such that $w(a) = w_{k_0}$. Then Alice can win the game at least by* $\min\{\sum_{i=1}^p w_{2i-1} - \sum_{i=1}^p w_{2i}, 0\}$ *by taking a.*

Proof. Let $\alpha = \sum_{i=1}^p w_{2i-1} - \sum_{i=1}^p w_{2i}$. We proceed by induction on $|V(G)|$. If $|V(G)| = 4$, then the desired conclusion follows from Claim 28. Thus let $|V(G)| \geq 6$, and assume that the subclaim holds for smaller graphs. In view of Claim 28, we may assume that $p \geq 2$. Let b be the vertex taken by Bob. First assume that $b \in X_{k_0}$. By (1), Alice can win the game on $G - a - b$. Since $w(a) \geq w(b)$ by the choice of a, this implies the desired conclusion.

Next assume that $b \notin X_{k_0}$, and write $b = x_\ell$. Then $\ell \neq k_0$. If $\ell = 1$ or $\ell = 2p$, then $G - a - b$ is disconnected, which contradicts the fact that b is a feasible move for Bob (note that if $\ell = 1$, then $k_0 \neq 1$). Thus $\ell \neq 1$ and $\ell \neq 2p$. If $\ell < k_0$, then let $a' = x_{\ell-1}$ and $\ell' = \ell - 1$; if $\ell > k_0$, then let $a' = x_{\ell+1}$ and $\ell' = \ell$. By assumption, we have $w(a') \geq w(b)$. Set $G' = G - b - a'$. By Observation 26, $(X_1, \ldots, X_{\ell'-1}, X_{\ell'+2}, \ldots, X_{2p})$ is an admissible partition of G', and G' satisfies the assumption of the subclaim. Arguing as in Claim 30, we see from the induction hypothesis that Alice can win the game on G' at least by $\min\{\alpha - w(a') + w(b), 0\}$ by taking a. Since $G' - a = G - a - b - a'$ and $w(a') \geq w(b)$, this implies that Alice can win the game on G at least by $\min\{\alpha, 0\}$ by taking a' in her second turn. □

We can now complete the proof of Claim 31. Considering the partition $(X_{2p}, \dots, X_2, X_1)$ in place of $(X_1, X_2, \dots, X_{2p})$ if necessary, we may assume that $\sum_{i=1}^{p} w_{2i-1} \geq \sum_{i=1}^{p} w_{2i}$. Now if k_0 is even, then the desired conclusion follows from Claim 27 or Subclaim 3.1, according as $p = 1$ or $p \geq 2$; if k_0 is odd, then the desired conclusion follows from Claim 30 or Subclaim 3.2, according as $|X_{k_0}| = 1$ or $|X_{k_0}| \geq 3$. □

Claim 32. *Suppose that the assumption of Claim* 31 *does not hold.*

(i) (a) *There exists k such that $|X_k| \geq 2$, or $k \leq 2p-1$ and $|X_k| = 1$ and $w_k < w_{k+1}$.*

(b) *There exists k such that $|X_k| \geq 2$, or $k \geq 2$ and $|X_k| = 1$ and $w_k < w_{k-1}$.*

(ii) *Let $k_1 = \min\{k : |X_k| \geq 2$, or $k \leq 2p-1$ and $|X_k| = 1$ and $w_k < w_{k+1}\}$, and let $k_2 = \max\{k : |X_k| \geq 2$, or $k \geq 2$ and $|X_k| = 1$ and $w_k < w_{k-1}\}$. Then $k_1 < k_2$.*

(iii) *Set $X = \bigcup_{k_1 \leq k \leq k_2} X_k$, and choose $a \in X$ so that $w(a) = \max\{w(x) : x \in X\}$.*

(a) *If $|X_{k_1}| = 1$, then $a \neq x_{k_1}$.*

(b) *If $|X_{k_2}| = 1$, then $a \neq x_{k_2}$.*

(iv) *The graph $G - a$ is connected.*

(v) (a) *If $k_1 > 1$, then $G - a - x_1$ is disconnected.*

(b) *If $k_2 < 2p$, then $G - a - x_{2p}$ is disconnected.*

Proof. (i) If (a) does not hold, then the assumption of Claim 31 holds with $k_0 = 2p$. Thus (a) holds, and (b) can be shown in a similar way.

(ii) First assume that there exists ℓ such that $|X_\ell| \geq 2$. Then $k_1 \leq \ell \leq k_2$. If $k_1 = \ell = k_2$, then it follows from the definition of k_1 and k_2 that the assumption of Claim 31 holds with $k_0 = \ell$, a contradiction. Thus $k_1 < k_2$. Next assume that $|X_k| = 1$ for all k. Suppose that $k_1 \geq k_2$. Then $w_1 \geq w_2 \geq \cdots \geq w_{k_2} \geq \cdots \geq w_{k_1}$ by the definition of k_1, and $w_{k_2} \leq \cdots \leq w_{k_1} \leq w_{k_1+1} \cdots \leq w_{2p}$ by the definition of k_2. Hence $w_1 \geq \cdots \geq w_{k_2} = \cdots = w_{k_1} \leq \cdots \leq w_{2p}$. Consequently, the assumption of Claim 31 holds with $k_0 = k_1$, a contradiction. Thus $k_1 < k_2$.

(iii) We prove (a) ((b) can similarly be proved). Assume that $|X_{k_1}| = 1$. Then we have $w(x_{k_1}) = w_{k_1} < w_{k+1}$ by the definition of k_1. Since $w(a) \geq \max\{w(x) : x \in X_{k_1+1}\} = w_{k+1}$ by the choice of a, this implies $a \neq x_k$.

(iv) Note that if there is a cutvertex c of G, then it follows from the definition of an admissible partition that $p \geq 2$, and either $|X_1| = 1$ and $c = x_1$ or $|X_{2p}| = 1$ and $c = x_{2p}$. Hence the desired conclusion follows from (iii).

(v) Assume that $k_1 > 1$. Then $p \geq 2$ and $|X_1| = 1$, and hence, x_1 is a cutvertex of G separating X_2 and $X_3 \cup \cdots \cup X_{2p}$. Suppose that $G - a - x_1$ is connected. Since $|X_3 \cup \cdots \cup X_{2p}| \geq 2p-2 \geq 2$, we get $X_2 = \{a\}$. Since $k_1 > 1$, this forces $k_1 = 2$, which contradicts (iii). Consequently $G - x_1 - a$ is disconnected. Thus (a) is proved, and (b) can be verified in a similar way. □

Claim 33. *Suppose that G does not satisfy the assumption of Claim* 31*, and let G, k_1, k_2, X, a be as in Claim* 32*. Then Alice can win the game by taking a.*

Proof. We proceed by induction on $|V(G)|$. When $|V(G)| = 2$, the claim holds in the sense that the assumption of the claim cannot be satisfied. Thus let $|V(G)| \geq 4$, and assume that the claim holds for smaller graphs. Note that Claim 32 (iv) implies that Alice can take a in her first turn. Let b be the vertex taken by Bob in his first turn. First assume that $b \in X$. By (1), Alice can win the game on $G - a - b$. Since $w(a) \geq w(b)$, this implies the desired conclusion.

Next assume that $b \notin X$. Note that by Claim 32 (ii), we have $p \geq 2$. Write $b = x_\ell$. Since $G - a - b$ is connected, it follows from Claim 32 (v) that $\ell \neq 1$ and $\ell \neq 2p$. If $\ell < k_1$, then let $a' = x_{\ell-1}$ and $\ell' = \ell - 1$; if $\ell > k_2$, then let $a' = x_{\ell+1}$ and $\ell' = \ell$. By the definition of k_1 and k_2, we have $w(a') \geq w(b)$. Set $G' = G - b - a'$. By Observation 26, $(X_1, \ldots, X_{\ell'-1}, X_{\ell'+2}, \ldots, X_{2p})$ is an admissible partition of G'. It follows that G' does not satisfy the assumption of Claim 31, and that if we define k_1', k_2' and X' for G' as we defined k_1, k_2 and X for G in Claim 32, then we have $k_1' = k_1 - 2$ and $k_2' = k_2 - 2$ or $k_1' = k_1$ and $k_2' = k_2$ according as $\ell < k_1$ or $\ell > k_2$, and $X' = X$. Hence $w(a) = \max\{w(x) : x \in X'\}$. Applying Claim 32 (iv) to G', we see that $G - a - b - a'$ is connected, and hence Alice can take a' in her second turn. Moreover, by the induction hypothesis, Alice can win the game on G' by taking a in her first turn. Since $G' - a = G - a - b - a'$, this implies that Alice can win the game on G by taking a' in her second turn. □

Now the conclusion of the proposition follows from Claims 31 and 33. This completes the proof of Proposition 23. ■

***Proof of Theorem* 2.** We proceed by induction on $|V(G)|$. If $|V(G)| = 2$, then the theorem clearly holds. Thus let $|V(G)| \geq 4$, and assume that the theorem holds for smaller graphs. Fix an arbitrary weight function on G. If G has no cutvertex, then it follows from the induction hypothesis that Alice can win the game by taking a vertex which has the maximum weight. Thus we may assume that G has a cutvertex. By Theorem 18, G has an admissible partition. The induction hypothesis implies that (1) holds. Therefore it follows from Proposition 23 that Alice can win the game on G. ■

3.2. Clique paths

In this subsection, we prove Theorem 3. We here make some definitions. Let G

be a connected graph with a fixed weight function w. The first (respectively, the second) player in the game is denoted by 1 (respectively, 2), and the player who makes the last move (respectively, the last move but one) is denoted by -1 (respectively, -2). Note that if G is even then $1 = -2$ and $2 = -1$, and if G is odd then $1 = -1$ and $2 = -2$. As is introduced in [7] and [17], for $k \in \{1, 2, -1, -2\}$, $N(G, k)$ denotes the gain of player k when both players play optimally. We have $N(G, 1) + N(G, 2) = \sum_{x \in V(G)} w(x)$. Also note that if $x \in V(G)$ is a feasible move for player 1 in the game on G, i.e., if $G - x$ is connected, then we have $N(G, 1) \geq w(x) + N(G - x, 2)$, where equality holds if and only if x is an optimal move for player 1.

Now let α be a real number (we allow α to be negative). When $N(G, 1) - N(G, 2) \geq \alpha$, we say that Alice can win the game on G at least by α. Similarly, for a vertex $x \in V(G)$ such that $G - x$ is connected, when $(w(x) + N(G - x, 2)) - N(G - x, 1) \geq \alpha$, we say that Alice can win the game on G at least by α by taking x.

Following [7] and [17], we introduce the notion of a *rooted game* (in those papers, rooted games on disconnected graphs are also considered, but we here consider rooted games on connected graphs only). Let G be a connected graph with a weight function, and let S be a nonempty subset of $V(G)$, called a *root set*, such that $G[S]$ is a complete graph. In the rooted game on G with root set S, the players' aim is the same as that in the usual graph grabbing game, and each move is restricted by the same rule, and it is further restricted by the rule that except for the last move, at least one vertex in S is to remain untaken after the move. Thus if S consists of a single vertex v, then v can be taken only in the last move. The rooted game on G with root set S is often referred to as the rooted game on $G_{R(S)}$. Recall that for $k \in \{1, 2, -1, -2\}$, $N(G, \alpha)$ denotes the gain of player α in the game on G when both players play optimally. As in [7], we similarly let $N_{R(S)}(G, \alpha)$ denote the gain of player α in the rooted game on $G_{R(S)}$ when both players play optimally (when S consists of a single vertex v, we write $G_{R(v)}$ and $N_{R(v)}(G, \alpha)$ for $G_{R(S)}$ and $N_{R(S)}(G, \alpha)$). We state three observations.

Observation 34. *If x is a feasible move for player 1 in the game on G, then*

$$N(G, 2) \leq N(G - x, 1),$$

with equality if and only if x is an optimal move for player 1.

Observation 35. *If S is a nonempty subset of $V(G)$ such that $G[S]$ is a complete graph, and x is a feasible move for player 1 in the rooted game on $G_{R(S)}$, then*

$$N_{R(S)}(G, 2) \leq N_{R(S \setminus \{x\})}(G - x, 1),$$

with equality if and only if x is an optimal move for player 1.

Observation 36. *If $v \in V(G)$ is a vertex such that $G[N_G(v)]$ is a complete graph, then $G - v$ is connected and*

$$N_{R(v)}(G, -2) = N_{R(N_G(v))}(G - v, -1).$$

Throughout the rest of this subsection, we let G be a clique path with a weight function, and let B_0 be an end block of G. We simultaneously prove the following two claims by induction.

Claim 37. *Let S be a subset of $V(B_0)$ such that either $S = V(B_0)$ or $|S| = 1$ and the unique vertex in S is not a cutvertex of G. Then*

$$N(G, -2) \geq N_{R(S)}(G, -2),$$

i.e.,

$$N(G, -1) \leq N_{R(S)}(G, -1).$$

Claim 38. *Suppose that G is even, and let S be a subset of $V(B_0)$ such that either $S = V(B_0)$ or $|S| = 1$ and the unique vertex in S is not a cutvertex of G. Then*

$$N(G, 1) \geq N_{R(S)}(G, 2).$$

It is easy to see that both claims hold if $|V(G)| \leq 2$. In the following proofs of Claims 37 and 38, we let $|V(G)| \geq 3$, and assume that both claims hold for smaller graphs. Also in the case where $B_0 \neq G$, we let t_0 be the unique cutvertex of G which belongs to B_0, and let B_1 be the unique block other than B_0 which contains t_0. We observe that if $x \in V(G)$ and $G' = G - x$ is connected and the set $S' = S \setminus \{x\}$ is nonempty, then in G', S' satisfies the assumption of the claims (with B_0 replaced by B_1 in the case where $B_0 \neq G$ and $S = V(B_0) = \{x, t_0\}$). We also observe that if $x \in V(B_0)$ and x is not a cutvertex of G, then in the graph $G' = G - x$, the set $S' = N_G(x)$ $(= V(B_0) \setminus \{x\})$ satisfies the assumption of the claims. These two observations will be used implicitly throughout the proof.

***Proof of Claim* 37**. First assume that G is even. Let a be an optimal move for player 1 in the rooted game on $G_{R(S)}$. Then

$$\begin{aligned}
N_{R(S)}(G, -1) &= N_{R(S)}(G, 2) \\
&= N_{R(S\setminus\{a\})}(G - a, 1) \quad \text{(by Observation 35)} \\
&= N_{R(S\setminus\{a\})}(G - a, -1) \\
&\geq N(G - a, -1) \quad \text{(by induction for Claim 37)} \\
&= N(G - a, 1) \\
&\geq N(G, 2) \quad \text{(by Observation 34)} \\
&= N(G, -1).
\end{aligned}$$

Next assume that G is odd. Let b be an optimal move for player 1 in the game on G. If b is also feasible in the rooted game on $G_{R(S)}$, then

$$\begin{aligned} N(G,-2) &= N(G,2) \\ &= N(G-b,1) \quad \text{(by Observation 34)} \\ &= N(G-b,-2) \\ &\geq N_{R(S\setminus\{b\})}(G-b,-2) \quad \text{(by induction for Claim 37)} \\ &= N_{R(S\setminus\{b\})}(G-b,1) \\ &\geq N_{R(S)}(G,2) \quad \text{(by Observation 35)} \\ &= N_{R(S)}(G,-2). \end{aligned}$$

Thus we may assume that b is not feasible in the rooted game on $G_{R(S)}$. Then $S=\{b\}$, and b is not a cutvertex. Hence

$$\begin{aligned} N(G,-2) &= N(G,2) \\ &= N(G-b,1) \quad \text{(by Observation 34)} \\ &\geq N_{R(N_G(b))}(G-b,2) \quad \text{(by induction for Claim 38)} \\ &= N_{R(N_G(b))}(G-b,-1) \\ &= N_{R(S)}(G,-2) \quad \text{(by Observation 36)}. \end{aligned}$$ □

***Proof of Claim* 38**. We consider the following two cases.

Case 1. $S=V(B_0)$. If $B_0=G$, then the rooted game on $G_{R(S)}$ is the same as the game on G and, since $B_0=G$ is a complete graph, Alice can win the game on G, which implies that $N(G,1)\geq N(G,2)=N_{R(S)}(G,2)$. Thus we may assume that $B_0\neq G$.

Let H and H' be the connected components of $G-t_0$. Exactly one of the components, say H', has even order. Let $H''=G[V(H')\cup\{t_0\}]$. Then H'' is odd. Let $X=N_G(t_0)\cap V(H)$. Then H and X satisfy the assumption of the claims except the parity of the order of graphs (in the case where $H=G-V(B_0)$, this can be verified by applying to $G-(V(B_0-t_0))$ the second observation made in the paragraph following the statement of Claim 38).

Subclaim 3.3. $N_{R(S)}(G,2)\leq N_{R(X)}(H,2)+N_{R(t_0)}(H'',1)$.

Proof. The desired inequality is equivalent to

$$N_{R(S)}(G,1)\geq N_{R(X)}(H,1)+N_{R(t_0)}(H'',2).$$

Alice and Bob play the rooted game on $G_{R(S)}$. Alice plays by following an optimal strategy of player $1=-1$ in the rooted game on $H_{R(X)}$, and an optimal strategy of player $2=-2$ in the rooted game on $H''_{R(t_0)}$. Note that in H, Bob has to follow the rule of the rooted game on $H_{R(X)}$ until H or H'' is completely taken away. This is clear if $H = B_0 - t_0$ or $H = G - V(B_0)$ and $G = B_0 \cup B_1$ (because $X = V(H)$); if $H = G - V(B_0)$ and $G \neq B_0 \cup B_1$, then this follows from the fact that $X = V(B_1) \setminus \{t_0\}$ is a cutset of G separating B_0 and $G - V(B_0 \cup B_1)$. Similarly, in H'', Bob has to follow the rule of the rooted game on $H''_{R(t_0)}$ until H or H'' is completely taken away (because t_0 is a cutvertex of G).

First we consider the case where H is completely taken away before H'' is taken away. Let α be the sum of the weights of those vertices of H'' which are taken by Alice before H is taken away, and let H_0 denote the subgraph induced by those vertices of H'' which remain untaken when H is taken away. Then

$$\alpha + N_{R(t_0)}(H_0, -2) \geq N_{R(t_0)}(H'', -2) = N_{R(t_0)}(H'', 2).$$

Note that H_0 is odd. After H is taken away, if $H'' = G - V(B_0 - t_0)$, then Alice can keep following the optimal strategy of player -2 in the rooted game on $H''_{R(t_0)}$ (because t_0 is the only vertex in S which remains untaken) and, if $H'' = B_0$, then Alice can follow an optimal strategy of player -2 in the game on H_0 (because $S \cap V(H_0) = V(H_0)$). Since $N(H_0, -2) \geq N_{R(t_0)}(H_0, -2)$ by the induction hypothesis (for Claim 37), it follows that regardless of whether $H'' = G - V(B_0 - t_0)$ or B_0, Alice's gain on H_0 is at least $N_{R(t_0)}(H_0, -2)$. Hence Alice's gain on H'' is at least $\alpha + N_{R(t_0)}(H_0, -2) \geq N_{R(t_0)}(H'', 2)$ (note that this argument works even if $|V(H_0)| = 1$). Since Alice's gain on H is at least $N_{R(X)}(H, 1)$, we see that the total gain of Alice is at least $N_{R(X)}(H, 1) + N_{R(t_0)}(H'', 2)$.

Next we consider the case where H'' is taken away before H is taken away. In this case, $H'' = G - V(B_0 - t_0)$ (for if $H'' = B_0$, then $V(H'') = S$, and hence H'' cannot be taken away when some other vertices remain untaken). Let α be the sum of the weights of those vertices of H which are taken by Alice before H'' is taken away, and let H_0 denote the subgraph induced by those vertices of H which remain untaken when H'' is taken away. Then

$$\alpha + N_{R(V(H_0))}(H_0, -1) \geq N_{R(X)}(H, -1) = N_{R(X)}(H, 1).$$

Note that H_0 is even. After H'' is taken away, Alice follows an optimal strategy for player 1 in the game on H_0. Since $N(H_0, 1) \geq N_{R(V(H_0))}(H_0, 2)$ by the induction hypothesis (for Claim 38), Alice's gain on H_0 is at least $N_{R(V(H_0))}(H_0, 2) = N_{R(V(H_0))}(H_0, -1)$. Hence Alice's game on H is at least $\alpha + N_{R(V(H_0))}(H_0, -1) \geq N_{R(X)}(H, 1)$. Since Alice's game on H'' is at least $N_{R(t_0)}(H'', 2)$, we see that the total gain of Alice is at least $N_{R(X)}(H, 1) + N_{R(t_0)}(H'', 2)$.

Thus in either case, Alice's gain is at least $N_{R(X)}(H, 1) + N_{R(t_0)}(H'', 2)$. Therefore $N_{R(S)}(G, 1) \geq N_{R(X)}(H, 1) + N_{R(t_0)}(H'', 2)$, as desired. $\square$

Subclaim 3.4. $N(G, 1) \geq N_{R(X)}(H, 2) + N_{R(t_0)}(H'', 1)$.

Proof. Alice and Bob play the game on G. Alice plays by following an optimal strategy of player 2 in the rooted game on $H_{R(X)}$, and an optimal strategy of player 1 in the rooted game on $H''_{R(t_0)}$. As in Subclaim 3.3, in H (respectively, H''), Bob has to follow the rule of the rooted game on $H_{R(X)}$ (respectively, $H''_{R(t_0)}$) until H or H'' is taken away. First we consider the case where H'' is taken away before H is taken away. Define α and H_0 as in the second case of the proof of Subclaim 3.3. Then

$$\alpha + N_{R(X\cap V(H_0))}(H_0, -2) \geq N_{R(X)}(H, -2) = N_{R(X)}(H, 2).$$

After H'' is taken away, Alice follows an optimal strategy of player -2 in the game on H_0. Since $N(H_0, -2) \geq N_{R(X\cap V(H_0))}(H_0, -2)$ by the induction hypothesis (for Claim 37), it follows that Alice's gain on H is at least $\alpha + N_{R(X\cap V(H_0))}(H_0, -2) \geq N_{R(X)}(H, 2)$. Since Alice's gain on H'' is at least $N_{R(t_0)}(H'', 1)$, we see that the total gain of Alice is at least $N_{R(X)}(H, 2) + N_{R(t_0)}(H'', 1)$.

Next we consider the case where H is taken away before H'' is taken away. Define α and H_0 as in the first case of the proof of Subclaim 3.3. Then

$$\alpha + N_{R(t_0)}(H_0, 2) = \alpha + N_{R(t_0)}(H_0, -1) \geq N_{R(t_0)}(H'', -1) = N_{R(t_0)}(H'', 1).$$

After H is taken away, Alice follows an optimal strategy of player 1 in the game on H_0. Since $N(H_0, 1) \geq N_{R(t_0)}(H_0, 2)$ by the induction hypothesis (for Claim 38), it follows that Alice's gain on H'' is at least $\alpha + N_{R(t_0)}(H_0, 2) \geq N_{R(t_0)}(H'', 1)$. Since Alice's gain on H is at least $N_{R(X)}(H, 2)$, we see that the total gain of Alice is at least $N_{R(X)}(H, 2) + N_{R(t_0)}(H'', 1)$.

Thus in either case, Alice's gain is at least $N_{R(X)}(H, 2) + N_{R(t_0)}(H'', 1)$, which implies the desired inequality. □

It follows from Subclaims 3.3 and 3.4 that $N(G, 1) \geq N_{R(X)}(H, 2) + N_{R(t_0)}(H'', 1) \geq N_{R(S)}(G, 2)$. This concludes the discussion for Case 1.

Case 2. $|S| = 1$. We show that $N_{R(S)}(G, 1) \geq N(G, 2)$. Write $S = \{u\}$. By the assumption of the claim, u is not a cutvertex. Hence

$$\begin{aligned} N_{R(u)}(G, 1) &= N_{R(u)}(G, -2) \\ &= N_{R(N_G(u))}(G - u, -1) \quad \text{(by Observation 36)} \\ &\geq N(G - u, -1) \quad \text{(by induction for Claim 37)} \\ &= N(G - u, 1) \\ &\geq N(G, 2) \quad \text{(by Observation 34)}, \end{aligned}$$

as desired. This completes the proof of Claim 38. □

***Proof of Theorem* 3.** Fix a weight function on G. Let S be the vertex set of an end block of G. Then by Claims 37 and 38,

$$N(G,1) \geq N_{R(S)}(G,2) = N_{R(S)}(G,-1) \geq N(G,-1) = N(G,2),$$

which implies the desired conclusion. ■

4. Examples

In this section, we construct graphs on which Alice can/cannot win the game and show that the converse of Conjecture 2 does not hold in general. We start with the definition of the toughness of a graph. As in [2], the *toughness* $\tau(G)$ of a noncomplete connected graph G is defined by

$$\tau(G) = \min\left\{\frac{|S|}{\omega(G-S)} : S \subseteq V(G) \text{ and } \omega(G-S) > 1\right\},$$

where $\omega(G-S)$ is the number of connected components of $G-S$ (the toughness of a complete graph is defined to be ∞). Note that if G has toughness t, then G is $\lceil 2t \rceil$-connected.

A connected graph G is *good* (for Alice) if Alice can win the game for every weight function on G; otherwise, it is said to be *bad*. It is easy to construct a bad even graph with arbitrarily small toughness. Let $k \geq 1$ be an integer. As is mentioned in the Introduction, $C_{2k+1} \odot H$ is bad for any odd graph H. Letting $H = I_{2\ell+1}$ ($\ell \geq 0$), where $I_{2\ell+1}$ is the null graph of order $2\ell+1$, i.e., the graph consisting of $2\ell+1$ isolated vertices, we see that $G = C_{2k+1} \odot I_{2\ell+1}$ is a bad even graph and satisfies $\tau(G) = \frac{1}{2\ell+2}$.

In [3], it is shown that there exists a bad even graph with arbitrarily large connectivity. As a refinement of this result, we here show that there exists a bad even graph with arbitrarily large toughness.

Let $k \geq 2$ be an integer. Our construction is a modification of the k-connected graph $G'_{n,k}$ constructed in [3], where $n = 2^m > k$. The graph $G'_{n,k}$ consists of a complete graph with n vertices $a_1, a_2, \ldots, a_n$ and $\binom{n}{k}$ vertices b_S where S ranges over all k-element subsets of $\{1, 2, \ldots, n\}$, and each b_S is adjacent only to the k vertices a_i with $i \in S$. Note that $|V(G'_{n,k})| = n + \binom{n}{k}$ is even. Let ℓ be an odd integer with $\ell \geq \binom{n}{k}$. For each $i \in \{1, 2, \ldots, n\}$, we replace a_i with a complete graph Q_i such that $|V(Q_i)| = \ell$ and, for all i, j with $1 \leq i < j \leq n$, join each vertex of Q_i to all vertices of Q_j, and join each b_S to all vertices in $\bigcup_{i \in S} V(Q_i)$. The resulting graph is denoted by $G'_{n,k,\ell}$. Note that $\bigcup_{i \in \{1,\ldots,n\}} V(Q_i)$ induces a complete subgraph in $G'_{n,k,\ell}$, and $|V(G'_{n,k,\ell})| = n\ell + \binom{n}{k}$ is even. Further $\tau(G'_{n,k,\ell}) = n\ell / \binom{n}{k} \geq n$.

Now in $G'_{n,k}$, assign weight 1 to each a_i, and weight 0 to each b_S; in $G'_{n,k,\ell}$, assign weight 1 to each vertex in $\bigcup_{i\in\{1,\dots,n\}} V(Q_i)$, and weight 0 to each b_S. In [3], it is shown that Alice can take at most $\lfloor k/2\rfloor + 1$ vertices of weight 1 in the game on $G'_{n,k}$ (if Bob plays optimally). Arguing as in [3] (we omit the details), we can show that in the game on $G'_{n,k,\ell}$, Bob can play according to the following strategy, i.e., for Bob, whenever the move described in 1 is not possible, the move described in 2 is possible, and that if Bob plays according to the strategy, then Alice can take at most $n\lfloor \ell/2\rfloor + \lfloor k/2\rfloor + 1$ vertices of weight 1:

1. if possible, Bob takes a vertex of weight 1;
2. otherwise he takes a vertex of weight 0 which is not a unique leaf neighbor of a vertex of weight 1 in the remaining graph.

We now construct an example which shows that the converse of Conjecture 2 does not hold. We first make the following observation, which follows from Theorem 2 because a graph of order four is clearly $\{P_5, bull\}$-free.

Observation 39. *Let G be a connected graph of order four. Then for every weight function Alice can win the game on G.*

Let H be a C_3-corona, and write $V(H) = \{u_1, u_2, u_3, v_1, v_2, v_3\}$, where $\{u_1, u_2, u_3\}$ forms a 3-cycle and v_i is adjacent only to u_i. Let G be the graph obtained from H by adding two adjacent vertices x, x' and joining them to all vertices of H (see Figure 2).

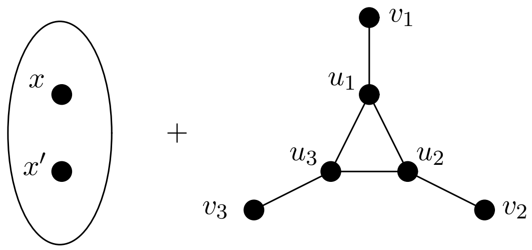

Figure 2. The graph G where the plus sign means the join of $\{x, x'\}$ and H.

Theorem 40. *Let G be as above. Then for every weight function Alice can win the game on the graph G.*

Proof. Let w be a weight function on G. We prove several claims.

Claim 41. *Let H' be an induced subgraph of order six of G with $H' \neq H$. Then Alice can win the game on H'.*

Proof. If $\{u_1, u_2, u_3\} \subseteq V(H')$, then H' is 2-connected, and hence it follows from Observation 39 that Alice can win the game on H' by taking a heaviest vertex. If

$\{u_1, u_2, u_3\} \not\subseteq V(H')$, then H' is $\{P_5, bull\}$-free, and hence the desired conclusion follows from Theorem 2. $\square$

Let $p_{\max}$ be a heaviest vertex of G.

Claim 42. *If $p_{\max} \notin \{x, x'\}$, then Alice can win the game on G.*

Proof. Alice takes $p_{\max}$. Let b be the vertex taken by Bob. Then $G - \{p_{\max}, b\} \neq H$, and hence by Claim 41, Alice can win the game on $G - \{p_{\max}, b\}$. Since $w(p_{\max}) \geq w(b)$, this implies the desired conclusion. $\square$

In view of Claim 42, we may assume that $p_{\max} \in \{x, x'\}$. By symmetry, we may assume that $p_{\max} = x$.

Claim 43. *If there exists $j \in \{1, 2, 3\}$ such that $w(u_j) \leq w(v_j)$, then Alice can win the game on G.*

Proof. Alice takes x. If Bob takes a vertex other than x', then it follows from Claim 41 that Alice can win the game on G. Hence we may assume that Bob takes x'. Thus it suffices to show that Alice can win the game on H.

Case 1. For all $i \in \{1, 2, 3\}$, $w(u_i) \leq w(v_i)$. It follows from Observation 39 that by taking a heaviest vertex among the v_i's, Alice can win the game on H.

Case 2. Otherwise. We may assume that $w(v_1) - w(u_1) \geq w(v_2) - w(u_2) \geq w(v_3) - w(u_3)$. Then $w(v_1) \geq w(u_1)$ and $w(v_3) < w(u_3)$. Alice takes v_1. Let b be the vertex taken by Bob. In view of Observation 39, we may assume that $b \neq u_1$. Then $b \in \{v_2, v_3\}$. Write $b = v_k$ and $\{2, 3\} = \{k, \ell\}$. Alice takes u_k in her second turn (in the game on H). The remaining graph is a path of order three having u_ℓ as its center. Hence Alice can take u_ℓ in her final turn. Since $\{k, \ell\} = \{2, 3\}$, Alice's gain is $w(v_1) + w(u_2) + w(u_3)$, and Bob's gain is $w(u_1) + w(v_2) + w(v_3)$. Since $w(v_1) - w(u_1) \geq w(v_2) - w(u_2)$ and $w(v_3) < w(u_3)$, it follows that Alice can win the game on H. $\square$

By Claim 43, we may assume that $w(u_i) > w(v_i)$ for all $i \in \{1, 2, 3\}$. We may assume that $w(u_1) \geq w(u_i)$ for each $i \in \{2, 3\}$. Set $\alpha = w(x) + w(v_1) + w(v_2) + w(u_3)$ and $\beta = w(x') + w(u_1) + w(u_2) + w(v_3)$. We distinguish two cases.

Case 1. $\alpha \geq \beta$. Alice takes x. In view of Claim 41, we may assume that Bob takes x'. Alice takes v_1 in her second turn. Let b be the vertex taken by Bob in his second turn. First assume that $b = u_1$. Alice takes v_2 in her third turn. Since the remaining graph is a path of order three having u_3 as its center, Alice can take u_3 in her final turn. Thus Alice's gain is α and Bob's gain is β, which means that Alice can win the game. Next assume that $b \in \{v_2, v_3\}$. Write $b = v_k$ and $\{2, 3\} = \{k, \ell\}$. Alice takes u_k in her third turn. As above, Alice can take u_ℓ in her final turn. Since $\{k, \ell\} = \{2, 3\}$, Alice's gain is $\alpha + (w(u_2) - w(v_2))$ and

Bob's gain is $\beta-(w(u_2)-w(v_2))$. Since $w(u_2)-w(v_2)>0$, this means that Alice can win the game.

Case 2. $\alpha<\beta$. Alice takes u_1. Let b be the vertex taken by Bob. If $b \in V(H-u_1)$, then $w(u_1)>w(b)$, and hence the desired conclusion follows from Claim 41. Thus we may assume that $b \in \{x,x'\}$. Write $\{x,x'\}=\{b,b'\}$. Alice takes u_2 in her second turn. Note that b' is a cutvertex of the remaining graph. If Bob takes u_3, Alice takes v_3; otherwise, Alice takes u_3. As above, Alice can take b' in her final turn. Thus Alice's gain is $\beta+(w(b')-w(x'))$ or $\beta+(w(b')-w(x'))+(w(u_3)-w(v_3))$, and Bob's gain is $\alpha+(w(b)-w(x))$ or $\alpha+(w(b)-w(x))-(w(u_3)-w(v_3))$. Since $\{b,b'\}=\{x,x'\}$, $w(x)-w(x')\geq 0$ and $w(u_3)-w(v_3)>0$, this means that Alice can win the game. This completes the proof of Theorem 40. ∎

We have seen that the condition that G is $\mathcal{C}_{odd}$-free is not a necessary condition for a connected graph G to be good. We conclude this paper by exhibiting simple necessary conditions.

Arguing as in the fifth paragraph of this section, we can show that the condition that the set S of cutvertices of G induces a bipartite graph is a necessary condition for a connected even graph G to be good (this is not a sufficient condition because, as is described in the fourth and the fifth paragraphs of this section, there exists a 2-connected bad even graph). To see this, by way of contradiction, suppose that there exists a subset X of S such that $G[X]$ is an odd cycle. Assign weight 1 to all vertices in X, and weight 0 to all other vertices. Bob plays according to the following strategy:

1. if possible, Bob takes a vertex of weight 1;
2. in the case where Bob cannot make the move described in 1, if possible, he takes a vertex v of weight 0 such that the removal of v does not create a feasible (for Alice) vertex of weight 1 (unlike the case of $G'_{n,k,\ell}$, Bob is sometimes forced to take a vertex of weight 0 whose removal creates a feasible vertex of weight 1).

Then we can verify that the first vertex x of weight 1 taken by a player is taken by Bob. Note that when Bob has taken x, $X\setminus\{x\}$ induces an even path in the remaining graph. Based on this, we can show that after x is taken by Bob, Bob's gain is strictly greater than Alice's gain at any stage of the game (since Bob can take at least half of vertices of $X\setminus\{x\}$).

Similarly, the condition that G has no cutvertex is a necessary condition for a connected odd graph to be good (this is not a sufficient condition because, as is mentioned in the third paragraph of the Introduction, there exists a 2-connected bad odd graph). To see this, by way of contradiction, suppose that G has a cutvertex x. Assigning weight 1 to x, and weight 0 to all other vertices. We can show that if Bob plays according to the strategy described in the preceding paragraph, then Bob can take x.

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