

SET-SEQUENTIAL LABELINGS OF ODD TREES

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Abstract

A tree T on 2^n vertices is called set-sequential if the elements in $V(T) \cup E(T)$ can be labeled with distinct nonzero $(n+1)$ -dimensional 01-vectors such that the vector labeling each edge is the component-wise sum modulo 2 of the labels of the endpoints. It has been conjectured that all trees on 2^n vertices with only odd degree are set-sequential (the “Odd Tree Conjecture”), and in this paper, we present progress toward that conjecture. We show that certain kinds of caterpillars (with restrictions on the degrees of the vertices, but no restrictions on the diameter) are set-sequential. Additionally, we introduce some constructions of new set-sequential graphs from smaller set-sequential bipartite graphs (not necessarily odd trees). We also make a conjecture about pairings of the elements of \mathbb{F}_2^n in a particular way; in the process, we provide a substantial clarification of a proof of a theorem that partitions \mathbb{F}_2^n from a paper [P.N. Balister, E. Györi and R.H. Schelp, *Coloring vertices and edges of a graph by nonempty subsets of a set*, European J. Combin. 32 (2011) 533–537]. Finally, we put forward a result on bipartite graphs that is a modification of a theorem in the aforementioned paper.

Keywords: trees, coloring graphs by sets, caterpillars.

2020 Mathematics Subject Classification: 05C78.

1. INTRODUCTION

In 1985, in [1], Acharya and Hegde introduced the notion of a set-sequential graph, which they defined as a graph G for which it is possible to assign distinct nonempty subsets of a set X to the edges and vertices of the graph in such a way that for each $e \in E(G)$, the label of $e = uv$ is the symmetric difference of the labels of $u, v \in V(G)$. Notice that each set X of size n has 2^n subsets, and that we can represent these subsets by n -dimensional 01-vectors: a 1 in the i -th position of the vector indicates membership of the i -th element of X in the subset. Under this representation, the symmetric difference becomes addition modulo 2, leading to the definition presented in the abstract. The definition of set-sequential in the abstract fits better with our methods of proof throughout this paper, so we have opted to use it throughout.

It is easy to understand what is meant by the term “set-sequential” (in other literature, “strongly set colorable”), but it is difficult to provide an exhaustive list of all graphs or classes of graphs that have this property. In fact, in the more than thirty-five years since Acharya and Hegde’s initial introduction of the problem of classifying set-sequential graphs, this problem remains open. Certain classes of graphs are known to be set-sequential, however, including stars on 2^n vertices (it is an easy exercise to check this). Note here that these stars contain only vertices of odd degree.

Another broad class of set-sequential graphs is the set of paths on 2^n vertices for $n = 1$ and $n \geq 4$. This result was proved initially by Mehta and Vijayakumar in [7].

Theorem 1.1 (Mehta and Vijayakumar [7]). *For any integer $n \geq 2$, \mathbb{G}^n is sequentially ternary if and only if n is neither 3 nor 4.*

Here, Mehta and Vijayakumar define \mathbb{G}^n in the same way as \mathbb{F}_2^n , and their definition of “sequentially ternary” is equivalent to our definition of “set-sequential”. This result was also proved by Balister *et al.* in [4], using terminology more similar to what is used in the remainder of this paper.

Theorem 1.2 (Balister *et al.* [4]). *The paths P_4 and P_8 are not strongly set colorable while all other paths of the form $P_{2^{n-1}}$ are strongly set colorable.*

This raises the following question. Why are the paths P_4 and P_8 not set-sequential? This question is answered in [4], so we do not provide a rigorous explanation, but rather give the following sketch. Consider the path P_4 . There are two vertices of even degree here, which means that for two of the vertices, the vectors v and u labeling them appear three times in the overall sum, which must be zero. Since we consider addition modulo 2, this leads us to the conclusion that $v = u$, which contradicts our definition of set-sequential. So P_4 is not

set-sequential. A similar but more complex argument gives that P_8 is not set-sequential. Not all graphs, then, are set-sequential, and what is more, it seems to be the vertices of even degree that introduce some measure of uncertainty regarding whether a graph is set-sequential or not. In a 2009 paper [6], Hegde proved that there must indeed be restrictions on vertices of even degree.

Theorem 1.3 (Hegde [6]). *If a graph G ($p > 2$, here $p = |V(G)|$) has:*

1. *exactly one or two vertices of even degree or*
2. *exactly three vertices of even degree, say, v_1, v_2, v_3 , and any two of these vertices are adjacent or*
3. *exactly four vertices of even degree, say, v_1, v_2, v_3, v_4 such that v_1v_2 and v_3v_4 are edges in G , then G is not strongly-set colorable.*

This theorem prompts the following conjecture (which has been put forth before by others, among them Golowich and Kim in [5], though the precise origin is unknown).

Conjecture 1.4 (Odd Tree Conjecture). *Any tree on 2^n vertices with only vertices of odd degree is set-sequential.*

In particular, the class of caterpillars with vertices of only odd degree has been of interest to some, among them Golowich and Kim [5], Abhishek [2] and Agustine (together with Abhishek) [3]. A 2012 paper by Abhishek and Agustine [3] presented results for some classes of graphs of diameter 4 — in particular, caterpillars with vertices of certain odd degrees. Following that, in 2013, Abhishek [2] extended those results to certain caterpillars of diameter 5. Most recently, in a 2020 paper [5], Golowich and Kim set forth results that show that several classes of graphs are set-sequential, including odd caterpillars of diameter at most 18. We were able to show that another larger class of caterpillars is set-sequential, without making the assumption that the caterpillars are “small enough”.

Further results on set-sequential trees can be found in a 2011 paper by Balister *et al.* [4]. In addition to the result that all paths except P_4 and P_8 are set-sequential, the authors proved that bipartite graphs can be connected in such a way that produces larger set-sequential graphs. Balister *et al.* also put forth a conjecture that partitions \mathbb{F}_2^n (the field of n -dimensional 01-vectors under addition modulo 2) in a specific way and provided a proof of one case of that conjecture. We as well prove some results for bipartite graphs and address another case of their partitioning conjecture, noting that this conjecture can be used to aid us in our goal of proving the Odd Tree Conjecture. The vertex and edge labels in a set-sequential labeling are in fact elements of \mathbb{F}_2^n .

In addition to these results, we note that the proof of Theorem 4 given in [4] treats only one of several cases of that theorem and so is incomplete. In this

paper, we give a more explicit generalization of the technique of that proof and so provide a rigorous proof of the theorem.

2. ODD TREE CONJECTURE

One approach to attempting to prove the Odd Tree Conjecture is to try to find a labeling for each odd tree on 2^n vertices for all n , perhaps with the aid of a computer program. This idea does have merit, and we used it, along with the fact stated in the introduction that stars are set-sequential, to show that all odd trees on 8 vertices are set-sequential.

Lemma 2.1. *The Odd Tree Conjecture is true for $n \leq 3$.*

Proof. This is trivial for $n = 1$. For $n = 2$ and $n = 3$, it is sufficient to exhibit a set-sequential labeling of the edges and vertices for each odd tree on 4 and 8 vertices. For $n = 2$, there is only one odd tree on 4 vertices, and we may label it as in Figure 1. For $n = 3$, there are 3 odd trees on 8 vertices, and we may label those as in Figure 2. ■

As n grows large (greater than 3 in fact), however, both the number of odd trees on 2^n vertices and the number of possible labelings become too great to check exhaustively due to limits in computational speed. We do know that at least some odd trees on 16 vertices are set-sequential, but this is not due to a computer program but rather to methods set forth in the remainder of this section. We have studied in earnest three main operations which we present here: constructing caterpillars, splicing smaller graphs together, and using a method introduced to us in [4].

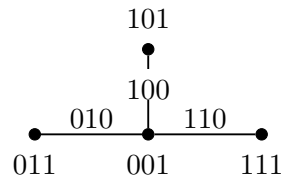


Figure 1. Set-sequential labeling of the only odd tree on 4 vertices.

2.1. Constructing Caterpillars

As mentioned in the introduction, a 2020 paper by Golowich and Kim [5] presents some results regarding the set-sequentialness of caterpillars. Let C be an odd caterpillar of diameter k . One result in [5] is that C is set-sequential if k is at most 18, and another is that C is set-sequential if $2^{k-1} \leq |V(C)|$. We add to

these results that an even larger class of odd caterpillars is set-sequential. Our result places some restrictions on the possible degrees of vertices along the path that serves as the “bone” of the caterpillar but has no limitation on the diameter.

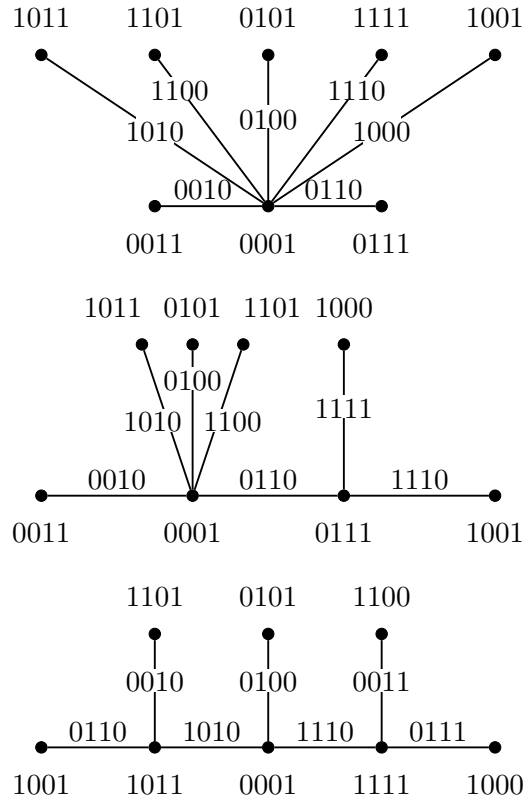


Figure 2. Set-sequential labelings of the three odd trees on 8 vertices.

Recall from the introduction the result by Balister *et al.* in [4] that all paths of the form P_{2^n} except for P_4 and P_8 are set-sequential. Start with a path P_{2^k} , with $k > 4$. From this we will construct a larger caterpillar with vertices of certain degrees that is also set-sequential. We allow the path P_{2^k} to become the “bone” of the caterpillar and add pendent edges in such a manner that each new vertex and the edge attaching it to the path correspond to the preceding or subsequent vertex and edge in the path.

Lemma 2.2. *Let $C_{k,3}$ denote the caterpillar on k vertices with only vertices of degrees 1 and 3. For $n = 2, 3$ and $n > 4$, $C_{2^n,3}$ is set-sequential.*

Proof. Lemma 2.1 gives that $C_{2^2,3}$ and $C_{2^3,3}$ are set-sequential. To show the cases where $n > 4$, we utilize Theorem 1 from [4]. Paths of the form P_{2^m} for $m \geq 4$ have a labeling.

Observe that we may write $C_{2^n,3}$ as the path P_{2^n-1} with one edge connecting each interior vertex to a single additional vertex and two edges connecting two vertices to the last vertex in the path. Suppose $n > 4$. Take some labeling $w_1, f_1, w_2, f_2, \dots, f_{2^{n-1}-1}, w_{2^n-1}$ with n -dimensional vectors for the path P_{2^n-1} (denoted in blue). Append a 0 to each vector to form vectors of dimension $(n+1)$, noting that this does not change the validity of the labeling. Label the pendent edges and outer vertices with $(n+1)$ -dimensional vectors $v_1, e_1, \dots, v_{2^{n-1}-1}, e_{2^{n-1}-1}, v_{2^n-1}, e_{2^n-1}$, as shown in Figure 3.

Since $1+1=0$ under addition modulo 2, $w_i 1 + f_i 1 = w_i 0 + f_i 0 = w_{i+1} 0$. So let $v_i = w_i 1$ and $e_i = f_i 1$. This is a good labeling for v_i and e_i for $i = 1, \dots, 2^{n-1} - 1$. What remains is to label v_{2^n-1} and e_{2^n-1} . Observe that we did not use the vector $w_{2^n-1} 1$ or the $(n+1)$ -dimensional vector $0 \cdots 01$ for any labelings so far. Since $w_{2^n-1} 0 + w_{2^n-1} 1 = 0 \cdots 01$, let $v_{2^n-1} = w_{2^n-1} 1$ and $e_{2^n-1} = 0 \cdots 01$. ■

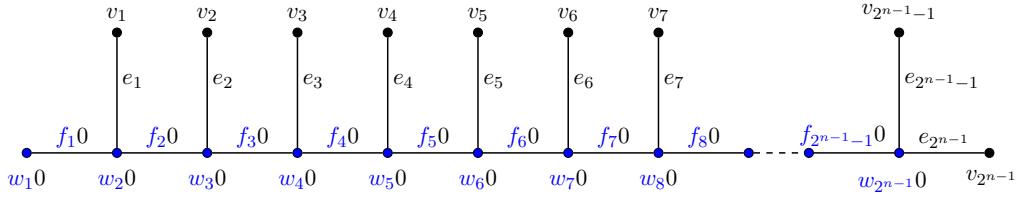


Figure 3. Labeling of the pendent edges and vertices.

Another more general result may be obtained in the same manner.

Theorem 2.3. Take $n, k > 4$ with $n \geq k$, and consider the path P_{2^k-1} . Join $2^{n-k+1} - 1$ pendent edges and vertices to each of the $2^{k-1} - 2$ interior vertices of the path, and join an additional $2 \cdot (2^{n-k+1} - 1)$ pendent edges and vertices to any one of the vertices in P_{2^k-1} . The caterpillar on 2^n vertices constructed in this way is set-sequential.

Note that choosing $k = n$ and choosing to join the pendent edges and vertices to the last vertex in the path gives Lemma 2.2.

Proof. As in the proof of Lemma 2.2, we use that paths of the form P_{2^m} have a labeling for $m \geq 4$. Note here that in the construction we propose, we have

$$2^{k-1} + (2^{k-1} - 2) \cdot (2^{n-k+1} - 1) + 2 \cdot (2^{n-k+1} - 1) = 2^n$$

vertices, so this construction does in fact yield a caterpillar on 2^n vertices. Suppose $n, k > 4$ and $n \geq k$. Take some labeling $w_1, f_1, w_2, f_2, \dots, f_{2^{k-1}-1}, w_{2^k-1}$ with k -dimensional vectors for the path P_{2^k-1} . Append $n - k + 1$ zeros (denoted

in general by 0^{n-k+1} , but here more succinctly as $\vec{0}$) to each vector to form vectors of dimension $(n+1)$, noting that this does not change the validity of the labeling. We will prove the theorem in two cases.

Case 1. The additional $2 \cdot (2^{n-k+1} - 1)$ pendent edges and vertices are added to one of the two end vertices of $P_{2^{k-1}}$. Without loss of generality, suppose the pendent edges and vertices are added to the final vertex in the path. Let $i = 2^{n-k+1} - 1, j = 2^{k-1} - 2$, and label these pendent edges and vertices with $(n+1)$ -dimensional vectors $v_1, e_1, \dots, v_{j \cdot i + 2 \cdot (2^{n-k+1} - 1)}, e_{j \cdot i + 2 \cdot (2^{n-k+1} - 1)}$ as in Figure 4. Since $1+1=0$ under addition modulo 2, for any 01-vector x of dimension $n-k+1$, we have $w_\ell x + f_\ell x = w_{\ell+1} \vec{0}$. There are $2^{n-k+1} - 1$ nonzero vectors we may construct using $n-k+1$ symbols. Denote them by z_a , for $a = 1, \dots, 2^{n-k+1} - 1$. So we may express each vector $w_\ell \vec{0}$, $\ell \geq 2$ a total of $2^{n-k+1} - 1$ ways using the sums $w_{\ell-1} z_a + f_{\ell-1} z_a = w_\ell \vec{0}$. Now for each of the interior vertices $w_\ell \vec{0}$ of $P_{2^{k-1}}$, let their pendent edges e_a and vertices v_a be given by $e_a = f_{\ell-1} z_a$, $v_a = w_{\ell-1} z_a$. We now have a set-sequential labeling using $(n+1)$ -dimensional vectors for $w_1, f_1, \dots, f_{2^{k-1}-1}, w_{2^{k-1}}$ and for $v_1, e_1, \dots, v_{(j+1) \cdot i}, e_{(j+1) \cdot i}$. What remains is to label $v_{(j+1) \cdot i+1}, e_{(j+1) \cdot i+1}, \dots, v_{(j+2) \cdot i}, e_{(j+2) \cdot i}$.

So far, we have not used the vectors $w_{2^{k-1}} z_a$ or the vector $0^k z_a$, for $a = 1, \dots, 2^{n-k+1} - 1$. Let each of the remaining unlabeled pendent edges e_a and vertices v_a of $w_{2^{k-1}} \vec{0}$ be given by $e_a = 0^k z_a$, $v_a = w_{2^{k-1}} z_a$. Then for each v_b, e_b attached to $w_{2^{k-1}} \vec{0}$, we have $v_b + e_b = w_{2^{k-1}} \vec{0}$, finishing the required labeling for this caterpillar.

Case 2. The additional $2 \cdot (2^{n-k+1} - 1)$ pendent edges and vertices are added to one of the $2^{k-1} - 2$ interior vertices of $P_{2^{k-1}}$, say, $w_h \vec{0}$, for $2 \leq h \leq 2^{k-1} - 1$. Let $i = 2^{n-k+1} - 1, j = 2^{k-1} - 2$, and label these pendent edges and vertices with $(n+1)$ -dimensional vectors $v_1, e_1, \dots, v_{j \cdot i + 2 \cdot (2^{n-k+1} - 1)}, e_{j \cdot i + 2 \cdot (2^{n-k+1} - 1)}$ as in Figure 5. Since $1+1=0$ under addition modulo 2, for any 01-vector x of dimension $n-k+1$, we have $w_\ell x + f_\ell x = w_{\ell+1} \vec{0}$. There are $2^{n-k+1} - 1$ nonzero vectors we may construct using $n-k+1$ symbols. Denote them by z_a , for $a = 1, \dots, 2^{n-k+1} - 1$. So we may express each vector $w_\ell \vec{0}$, $2 \leq \ell \leq 2^{k-1} - 1$, a total of $2^{n-k+1} - 1$ ways using either $w_{\ell-1} z_a + f_{\ell-1} z_a = w_\ell \vec{0}$ or $w_{\ell+1} z_a + f_\ell z_a = w_\ell \vec{0}$. Now for each of the interior vertices $w_\ell \vec{0}$ of $P_{2^{k-1}}$ with $\ell \leq h-1$, let their pendent edges e_a and vertices v_a be given by $e_a = f_{\ell-1} z_a$, $v_a = w_{\ell-1} z_a$. For each of the interior vertices $w_\ell \vec{0}$ with $\ell \geq h+1$, let their pendent edges e_a and vertices v_a be given by $e_a = f_\ell z_a$, $v_a = w_{\ell+1} z_a$. We now have a good labeling using $(n+1)$ -dimensional vectors for $w_1, f_1, \dots, f_{2^{k-1}-1}, w_{2^{k-1}}$ and for $v_1, e_1, \dots, v_{(h-2) \cdot i}, e_{(h-2) \cdot i}, v_{(h-1) \cdot i+1}, e_{(h-1) \cdot i+2}, \dots, v_{j \cdot i}, e_{j \cdot i}$. What remains is to appropriately label $v_{(h-2) \cdot i+1}, e_{(h-2) \cdot i+1}, \dots, v_{(h-1) \cdot i}, e_{(h-1) \cdot i}$ and $v_{j \cdot i+1}, e_{j \cdot i+1}, \dots, v_{(j+2) \cdot i}, e_{(j+2) \cdot i}$.

So far, the vectors $w_{h-1} z_a, f_{h-1} z_a, w_h z_a, f_h z_a, w_{h+1} z_a$ or $0^k z_a$, for $a =$

$1, \dots, 2^{n-k+1} - 1$, have not been used. Since $w_{h-1}z_a + f_{h-1}z_a = w_{h+1}z_a + f_h z_a = w_h z_a + 0^k z_a = w_h \vec{0}$, let each pair of vectors $(v_{(h-2)\cdot i+1}, e_{(h-2)\cdot i+1}), \dots, (v_{(h-1)\cdot i}, e_{(h-1)\cdot i})$ be equal to a pair $(w_{h-1}z_a, f_{h-1}z_a)$, let each pair of vectors $(v_{j\cdot i+1}, e_{j\cdot i+1}), \dots, (v_{(j+1)\cdot i}, e_{(j+1)\cdot i})$ be equal to a pair $(w_{h+1}z_a, f_h z_a)$, and finally let each pair of vectors $(v_{(j+1)\cdot i+1}, e_{(j+1)\cdot i+1}), \dots, (v_{(j+2)\cdot i}, e_{(j+2)\cdot i})$ be equal to a pair $(w_h z_a, 0^k z_a)$. Then for each v_b, e_b attached to $w_h \vec{0}$, we have $v_b + e_b = w_h \vec{0}$, finishing the required labeling for this caterpillar. ■

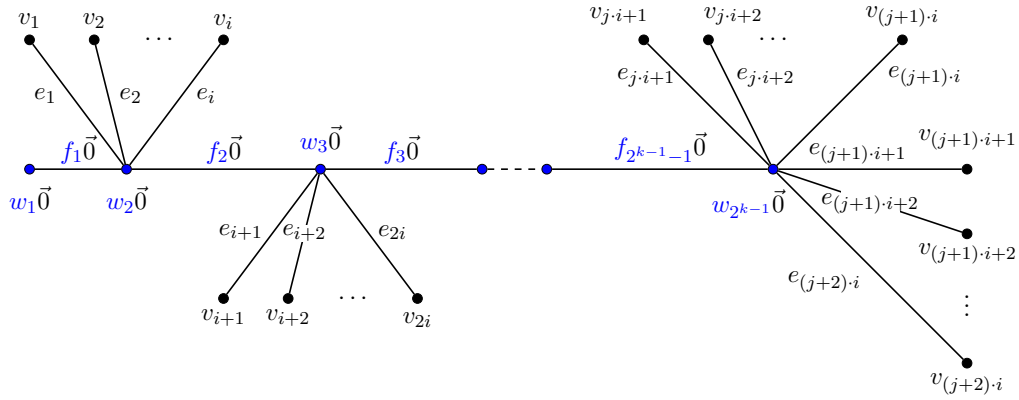


Figure 4. Labeling of the caterpillar in Case 1.

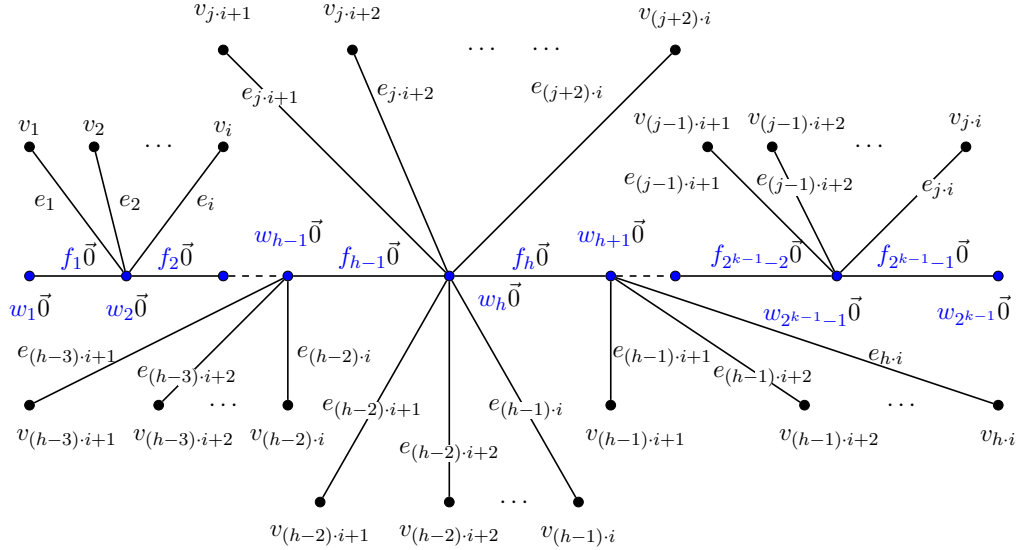


Figure 5. Labeling of the caterpillar in Case 2.

To allow for a more solid understanding of this proof, we present the following example.

Example 2.4. Take $n = 6$, $k = 5$, and consider a caterpillar on 64 vertices, constructed in the manner of the above proof. We wish to show that it is set-sequential; that is, we wish to label it with the 127 nonzero 01-vectors of dimension 7. The “bone” of the caterpillar is the path P_{16} . We will add to each interior vertex of this path 3 pendent edges and vertices, and we will add to some other vertex in the path 6 additional pendent edges and vertices. This gives 16 vertices in the path, 3 extra vertices attached to 14 of the path vertices, and 6 extra vertices attached to some path vertex: in total, there are $16 + 3 \cdot 14 + 6 = 64$ vertices, with 63 edges. Take a labeling of P_{16} using the 5-dimensional vectors $w_1, f_2, \dots, f_{15}, w_{16}$. Figures 6 and 7 show labelings for the caterpillar constructions in Cases 1 and 2 of the previous proof, using 7-dimensional 01-vectors.

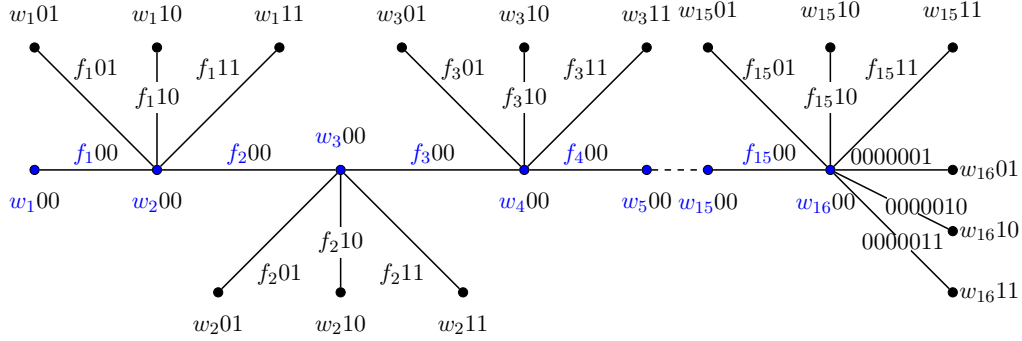


Figure 6. Labeling of the caterpillar in Case 1.

2.2. Splicing

Another technique that aids our goal of proving the Odd Tree Conjecture is one that constructs a large set-sequential tree from four smaller ones with an equal number of vertices. It is important that we start with four small trees rather than only two. Suppose we start with two copies T_1 and T_2 of a set-sequential tree on 2^n vertices. From them we want to construct a tree on 2^{n+1} vertices that is also set-sequential. In order to do this, we must take the vectors of dimension $n + 1$ and extend them by both a zero and a one in order to get all vectors of dimension $n + 2$ (except the vector with $n + 1$ zeros followed by a 1, denoted $\vec{0}1$). Express the trees as bipartite graphs with color classes X_1, Y_1 and X_2, Y_2 and edge sets E_1 and E_2 , respectively. We have by the set-sequential nature of T_1 and T_2 that sums of the vectors labeling the vertices in X_i and the vectors labeling the vertices in Y_i are equal to the vectors labeling the edges in E_i . Any extensions of these vector labels must preserve this, but if we only increase by one dimension, we may only extend with a zero or a one. It is not possible to do this. Extending the vectors labeling the vertices in X_1 and Y_1 and the edges in

E_1 by zero leaves that we must extend the vertices in X_2 and Y_2 and the edges in E_2 by one, which produces not sums of zero but sums of $\vec{0}1$. The only other option is, without loss of generality, to extend the vectors labeling the vertices in X_1 and Y_1 by one and the vectors labeling the edges in E_1 by zero. This satisfies the sum condition for T_1 , but for T_2 it gives again sums of $\vec{0}1$. In order to avoid this, we consider constructing a tree on 2^n vertices from four smaller trees on 2^{n-2} vertices.

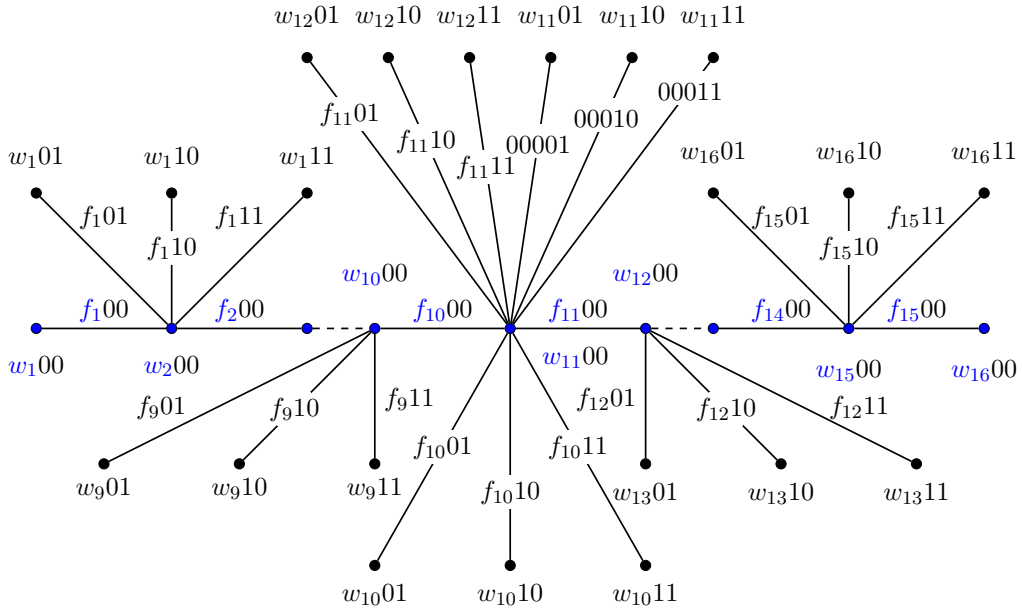


Figure 7. Labeling of the caterpillar in Case 2.

We therefore present an operation on four odd set-sequential trees on 2^{k-2} vertices with equal bipartitions (that is, color classes must be of equal size, but edge sets may be different). We claim that this operation can be used to construct an odd set-sequential tree on 2^k vertices. This operation utilizes the following definition, which we have named “splicing”.

Definition 2.5. Let G be a graph with some vertex v_1 and some edge with endpoints u_1, u_2 . We define *splicing v_1 into $(u_1 u_2)$* to be the operation that removes the edge (u_1, u_2) and adds the edges (v_1, u_1) and (v_1, u_2) .

Consider three sets of splicing operations applied to four bipartite graphs with equal bipartitions (that is, the cardinalities of the color classes must be the same but the edge sets may be different).

Set 1

- (i) *Splice $v_i \in X_i$ into $(u_{1,j}u_{2,j})$, where $u_{1,j} \in X_j$, $u_{2,j} \in Y_j$, and $j \neq i$.*
- (ii) *Splice $v_i \in X_i$ into $(u_{1,k}u_{2,k})$, where $u_{1,k} \in X_k$, $u_{2,k} \in Y_k$, and $k \notin \{i, j\}$.*
- (iii) *Splice $v_i \in X_i$ into $(u_{1,\ell}u_{2,\ell})$, where $u_{1,\ell} \in X_\ell$, $u_{2,\ell} \in Y_\ell$, and $\ell \notin \{i, j, k\}$.*

Set 2

- (i) *Splice $v_i \in X_i$ into $(u_{1,j}u_{2,j})$, where $u_{1,j} \in X_j$, $u_{2,j} \in Y_j$, and $j \neq i$.*
- (ii) *Splice $v_i \in X_i$ into $(u_{1,k}u_{2,k})$, where $u_{1,k} \in X_k$, $u_{2,k} \in Y_k$, and $k \notin \{i, j\}$.*
- (iii) *Splice $v_m \in Y_m$ into $(u_{1,\ell}u_{2,\ell})$, where $u_{1,\ell} \in X_\ell$, $u_{2,\ell} \in Y_\ell$, $\ell \notin \{i, j, k\}$, and $m \in \{j, k\}$.*

Set 3

- (i) *Splice $x_i \in X_i$ into $(u_{1,j}u_{2,j})$, where $u_{1,j} \in X_j$ and $u_{2,j} \in Y_j$.*
- (ii) *Splice $v_j \in Y_j$ into $(u_{1,k}u_{2,k})$, where $u_{1,k} \in X_k$, $u_{2,k} \in Y_k$, and $k \notin \{i, j\}$.*
- (iii) *Splice $v_k \in Y_k$ into $(u_{1,\ell}u_{2,\ell})$, where $u_{1,\ell} \in X_\ell$, $u_{2,\ell} \in Y_\ell$, and $\ell \notin \{i, j, k\}$.*

We claim that if the four bipartite graphs were odd set-sequential trees then the graph resulting from any of the sets of splicing operations is also an odd set-sequential tree.

Theorem 2.6. *Take four odd set-sequential trees T_i on 2^{k-2} vertices with color classes X_i and Y_i , for $i = 1, 2, 3, 4$. Suppose that $|X_i| = |X_j| = \ell$ and $|Y_i| = |Y_j| = m$ for $i \neq j$. If there exists a labeling of each T_i such that $\{v_1, v_2, \dots, v_\ell\}$ labels each X_i and $\{v_{\ell+1}, v_{\ell+2}, \dots, v_{\ell+m}\}$ labels each Y_i , then we may perform one of the sets of splicing operations given above to construct a set-sequential tree on 2^k vertices.*

Proof. The graphs resulting from the three splicing operations defined above are given in Figures 8–10, with the removed edge indicated by dashed line. By examining the extensions of the color classes and the indicated edges connecting them, the graphs are seen to be set-sequential. ■

As an example of this splicing operation, we present in Figures 11–13 some constructions of odd set-sequential trees on 32 vertices from four odd set-sequential trees on 8 vertices. Note here that these graphs are not caterpillars, nor can they be produced by the technique we will describe in the following section, so this splicing operation is indeed useful.

What follows now is an example showing that in Theorem 2.6 the four smaller bipartite graphs must have the same labeling set for each upper and lower color class in order to extend a set-sequential labeling via the splicing operations defined.

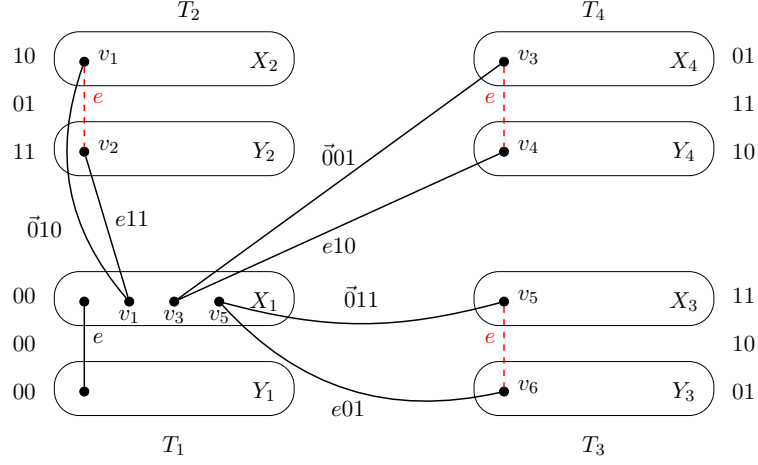


Figure 8. Set 1 splicing operations.

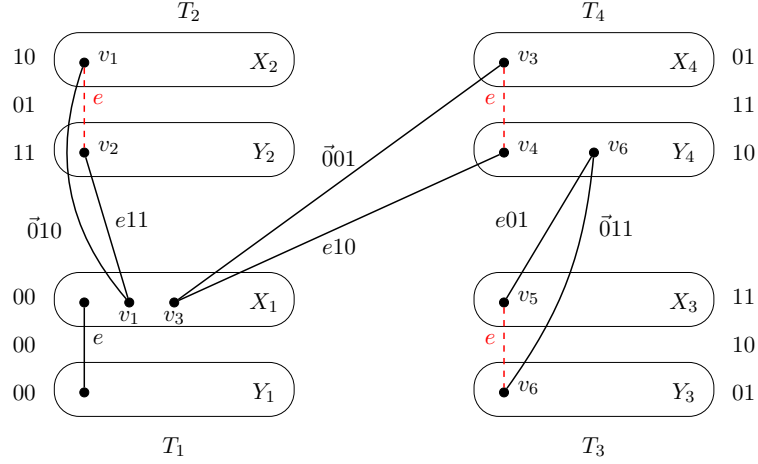


Figure 9. Set 2 splicing operations.

Example 2.7. Let R and S be the trees given in Figure 14. Label R and S using the labelings given in Lemma 2.1, noting that the vertices in the upper class of R are labeled with the vectors $\{0001, 1000, 1001\}$ and the vertices in the upper class of S are labeled with the vectors $\{0101, 1011, 1111\}$. Let $T_1 = T_2 = T_3 = R$ and $T_4 = S$. Before attempting any splicing, we must extend the vectors in each T_i by 00, 01, 10, or 11. Doing so, however, does not produce entirely distinct vectors. Extend the color classes X_1, Y_1 of T_1 by 00, X_2, Y_4 by 01, X_3, Y_2 by 10, and X_4, Y_3 by 11 (or some similar arrangement). Then E_1 is extended by 00, E_2 by 11, E_3 by 01, and E_4 by 10. Extending in this way, we find 0111 extended by 00, 10, and 11, since it is in Y_1, Y_2, Y_3 . But in T_4 , 0111 is an edge, so it is extended by 10. This method of extension, then, gives that the vector 011110 is

used twice in the larger tree on 32 vertices, which prevents that tree from being set-sequential. It is thus not enough that the bipartitions of each T_i are equal — we must assume that their upper and lower color classes have the same labeling set.

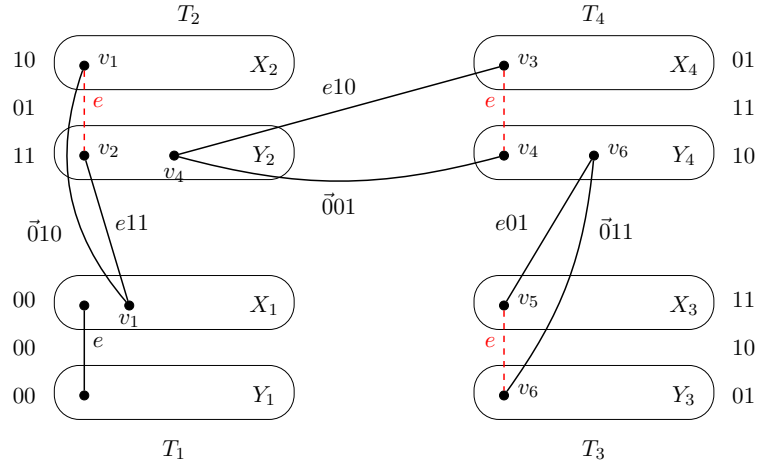


Figure 10. Set 3 splicing operations.

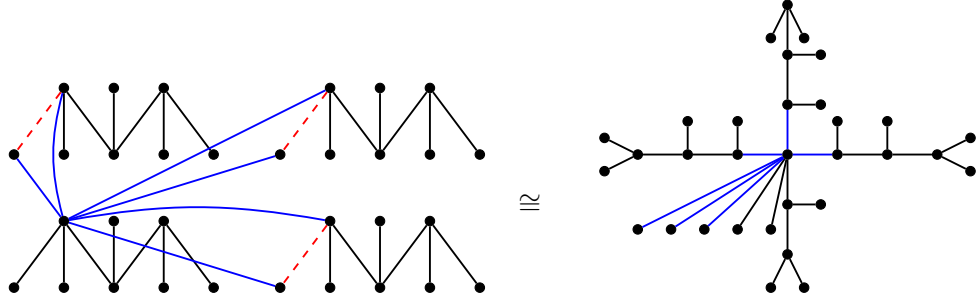


Figure 11. Set 1 splicing operations.

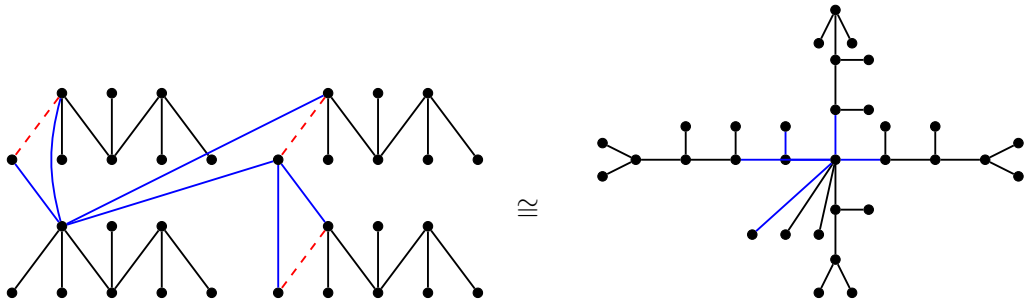


Figure 12. Set 2 splicing operations.

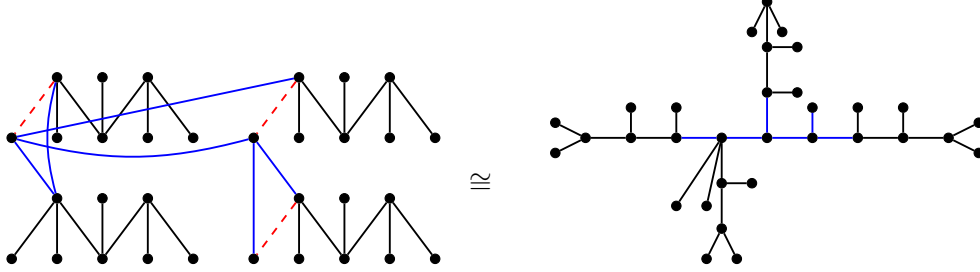


Figure 13. Set 3 splicing operations.

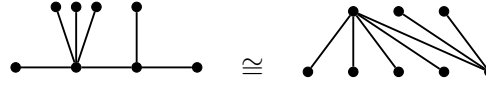
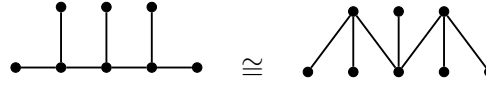
(a) Tree R (b) Tree S

Figure 14. Trees used in Example 2.7.

Though we can use this splicing technique to construct many odd trees, we cannot use it to produce all odd trees.

Counterexample 2.8. Consider an odd tree on n vertices with one vertex of degree of at least $\frac{n}{2}$. If we take four small trees and perform one of the sets of splicing operations, the maximum degree in each small tree will be $\frac{n}{4}$. We are adding only 6 edges to the graph, however. This means that we cannot produce a vertex of degree $\frac{n}{2}$ when n is large.

How, then, can we produce labelings for graphs like the one in the above counterexample or for other graphs that cannot be obtained via splicing? We introduce the following section on partitions of \mathbb{F}_2^n , motivated by work done in [4], to provide additional ways to construct odd set-sequential trees.

2.3. Partitioning \mathbb{F}_2^n

One conjecture that is of significant interest with respect to the Odd Tree Conjecture is Conjecture 1 in [4].

Conjecture 2.9 (Balister *et al.* [4]). Given 2^{n-1} non-zero (not necessarily distinct) vectors $v_1, \dots, v_{2^{n-1}} \in \mathbb{F}_2^n$, $n \geq 2$, with $\sum_{i=1}^{2^{n-1}} v_i = 0$, there exists a

partition of \mathbb{F}_2^n into pairs of vectors $\{p_i, q_i\}$, $i = 1, \dots, 2^{n-1}$ such that for all i , $v_i = p_i - q_i$.

Some cases of this conjecture have been proven, in both [4] and [5] (one result of particular note is that the conjecture is true for $n \leq 5$), though the general statement is yet unproven. This conjecture is of interest to our work because it gives us a way to begin with some odd set-sequential tree on 2^n vertices and add 2^n more vertices in such a way that the resulting tree on 2^{n+1} vertices is still both odd and set-sequential.

Example 2.10. Consider the tree T defined in Figure 15(a). This graph is not a caterpillar, nor can it be obtained via splicing, but we can use a special case of Conjecture 1 to prove that it is set-sequential. Note first that this T is one of the odd trees on 8 vertices with 4 vertices to which two additional vertices are joined. The four vertices v_1, v_2, v_3, v_4 are indicated in Figure 15(b).

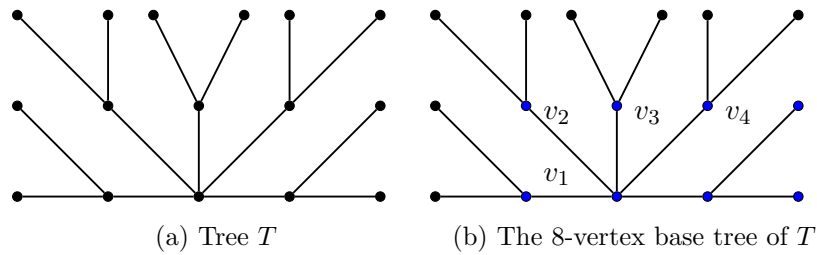


Figure 15. Trees used in Example 2.10.

In the larger 16 vertex tree, label v_1, \dots, v_8 using the labeling of that odd tree on 8 vertices (denoted k_1, \dots, k_8) extended by 0. Even though Conjecture 1 in [4] is yet unproven, it is known to be true for $n \leq 5$. So we may partition \mathbb{F}_2^4 into quadruples of vectors (p_i, q_i, ℓ_i, m_i) such that $k_i = p_i + q_i = \ell_i + m_i$, for $i = 1, 2, 3, 4$. (This is accomplished by letting $k_1 = k_2$, $k_3 = k_4$, $k_5 = k_6$, and $k_7 = k_8$ in Conjecture 1 of [4].) Without loss of generality, label the 8 appended vertices with $p_i 1$ and $\ell_i 1$ and the edges with $q_i 1$ and $m_i 1$. Then since $k_i = p_i + q_i = \ell_i + m_i$, we have $k_i 0 = p_i 1 + q_i 1 = \ell_i 1 + m_i 1$. We know that the sum condition for the 8-vertex subgraph is satisfied as well since we have merely extended it by 0. Therefore T is set-sequential.

To construct an explicit labeling of the T in the proof above, we start from the labeling of the 8-vertex subgraph as given in the proof of Lemma 2.1. Then, using the vertex labels given in the above proof, we have $k_1 = 0011$, $k_2 = 1011$, $k_3 = 0101$, and $k_4 = 1101$. Even though the construction leading to Conjecture 1 in [4] was not given, through some trial and error, and following in some ways the style of the proof of Conjecture 2 in the same work, we can obtain that $0011 = 0100 + 0111 = 0101 + 0110$, $1011 = 0000 + 1011 = 0001 + 1010$, $0101 =$

$1000 + 1101 = 1001 + 1100$, and $1101 = 0010 + 1111 = 0011 + 1110$. It is an easy exercise to verify that the vectors in the sums form a partition of \mathbb{F}_2^4 . So label the appended vertices with one of the summands in each of the 8 sums and the edge with the other, both extended by 1.

This special case of Conjecture 2.9 given above seems to be of significant use, so we endeavor to prove it for all n (we know now only that it is true for $n \leq 5$). Note here that we chose this particular case in order to ensure oddness is preserved. Since applying the method shown above adds in all cases an even number of pendent edges to a vertex of presently odd degree, the resulting larger tree still contains vertices of only odd degree.

Conjecture 2.11 (Pairing Conjecture). *Take nonzero vectors $v_1, v_2, \dots, v_{2^{n-1}} \in \mathbb{F}_2^n$, where $n \geq 2$, and $v_{2i+1} = v_{2i+2}$, for $i = 0, \dots, 2^{n-2} - 1$. We may partition \mathbb{F}_2^n into pairs (p_i, q_i) such that $v_i = p_i + q_i$ for all i .*

A useful case of the Pairing Conjecture (which is also a case of Conjecture 1 in [4]) is the following, which appeared as Theorem 4 in [4].

Theorem 2.12 (Balister, Györi, Schelp [4]). *Given 2^{n-1} non-zero vectors $v_1, \dots, v_{2^{n-1}} \in \mathbb{F}_2^n$, $n \geq 2$, with $v_1 = v_2 = \dots = v_{2^{n-2}}$ and $v_{2i+1} = v_{2i+2}$ for all $i = 0, \dots, 2^{n-2} - 1$, there exists a partition of \mathbb{F}_2^n into pairs of vectors $\{p_i, q_i\}$, $i = 1, \dots, 2^{n-1}$ such that for all i , $v_i = p_i - q_i$.*

The proof of this theorem that appeared in [4] only dealt with a single (rather neat) case and did not provide details as to how one might alter the proof to account for the other cases. We present here a proof of Theorem 4 that uses the same basic idea as the proof in [4], but we do so more rigorously in an effort to eliminate any confusion regarding its validity.

Proof. This proof contains similar ideas to those found in the proof in [4], though it allows for v_1 to be chosen to be some vector other than $\vec{0}$. First partition \mathbb{F}_2^n into 2^{n-1} pairs $\{r_m, s_m\}$ so that $v_1 = r_m + s_m$ for $m = 1, \dots, 2^{n-1}$. (If $n = 2$, set $p_1 = r_1, q_1 = s_1, p_2 = r_2, q_2 = s_2$, and we are done.) Observe that for $j = 2^{n-2}$, we may express v_{j+1} as 2^{n-1} distinct sums, so set $p_{j+1}, q_{j+1}, p_{j+2}, q_{j+2}$ according to the following cases.

1. $v_{j+1} = r_a + r_b, a \neq b$. Then we have

$$v_{j+2} = v_{j+1} = r_a + r_b = (v_1 + s_a) + (v_1 + s_b) = s_a + s_b,$$

so let $p_{j+1} = r_a, q_{j+1} = r_b, p_{j+2} = s_a, q_{j+2} = s_b$.

2. $v_{j+1} = s_a + s_b, a \neq b$. Then, similarly to the previous case, we have

$$v_{j+2} = v_{j+1} = s_a + s_b = (v_1 + r_a) + (v_1 + r_b) = r_a + r_b,$$

so let $p_{j+1} = s_a, q_{j+1} = s_b, p_{j+2} = r_a, q_{j+2} = r_b$.

3. $v_{j+1} = r_a + s_b, a \neq b$. Then we have

$$v_{j+2} = v_{j+1} = r_a + s_b = (v_1 + s_a) + (v_1 + r_b) = s_a + r_b,$$

so let $p_{j+1} = r_a, q_{j+1} = s_b, p_{j+2} = s_a, q_{j+2} = r_b$.

4. $v_{j+1} = r_a + s_a$. Then, for some $b \neq a$, we have

$$v_{j+2} = v_{j+1} = r_a + s_a = r_b + s_b,$$

so let $p_{j+1} = r_a, q_{j+1} = s_a, p_{j+2} = r_b, q_{j+2} = s_b$.

In all cases, then, we have $\{p_{j+1}, q_{j+1}, p_{j+2}, q_{j+2}\} = \{r_a, s_a, r_b, s_b\}, a \neq b$. (If $n = 3$, we are done.)

Consider now the next pair of equal vectors, v_{j+3}, v_{j+4} . As in the case of $v_{j+1} = v_{j+2}$, we may express v_{j+3} as 2^{n-1} distinct sums. Here, as many as four sums may contain one of the vectors r_a, s_a, r_b, s_b , so since $2^{n-1} - 4 > 0$, for $n \geq 4$, by the same process as above, we may assign $p_{j+3}, q_{j+3}, p_{j+4}, q_{j+4}$ so that $\{p_{j+3}, q_{j+3}, p_{j+4}, q_{j+4}\} = \{r_c, s_c, r_d, s_d\}$, where c and d are distinct and are not equal to a or b . (If $n = 4$, we are done.)

The next pair of equal vectors, v_{j+5}, v_{j+6} can be again expressed as 2^{n-1} distinct sums, but this time as many as 8 sums may contain one of $r_a, s_a, r_b, s_b, r_c, s_c, r_d, s_d$. In a similar manner to the above, since $2^{n-1} - 8 > 0$ for $n \geq 5$, we may assign $p_{j+5}, q_{j+5}, p_{j+6}, q_{j+6}$ so that $\{p_{j+5}, q_{j+5}, p_{j+6}, q_{j+6}\} = \{r_e, s_e, r_f, s_f\}$ for e and f distinct and not equal to a, b, c or d .

After each assignment of the pairs $\{p_k, q_k\}$ and $\{p_{k+1}, q_{k+1}\}$, we have 4 fewer “available” sums for the remaining pairs of vectors. The last pair of vectors for which we need to assign p_k, q_k is $v_{2^{n-1}-1} = v_{2^{n-1}}$. By this time, we have $2^{n-1} - 4 \cdot (2^{n-3} - 1)$ possible sums not containing vectors used in a previous partitioning step, since each of the 2^{n-3} pairs after the first reduces the number of “available” sums by 4. Now $2^{n-1} - 4 \cdot (2^{n-3} - 1) = 2^{n-1} - 2^{n-1} + 4 > 0$, so it is still possible to set $p_{2^{n-1}-1}, q_{2^{n-1}-1}, p_{2^{n-1}}, q_{2^{n-1}}$ so that $\{p_{2^{n-1}-1}, q_{2^{n-1}-1}, p_{2^{n-1}}, q_{2^{n-1}}\} = \{r_y, s_y, r_z, s_z\}$ for y and z distinct and not equal to any of the a, b, c, d, e, f, \dots used previously.

There remain 2^{n-2} pairs $\{r_k, s_k\}$ such that $v_1 = r_k + s_k$ that have not yet been assigned as some pair $\{p_\ell, q_\ell\}$. So let each pair $\{p_m, q_m\}$ for $m = 1, \dots, 2^{n-2}$ be given by one of the pairs $\{r_k, s_k\}$. We have now partitioned \mathbb{F}_2^n as desired. ■

The collection of techniques and constructions given here allows us to make substantial progress on the Odd Tree Conjecture, but most graphs cannot be obtained using one of the operations or constructions we have given. For example, the graph in Figure 16 is not a caterpillar, nor can it be obtained via either the splicing method we proposed or the method that arises from partitioning \mathbb{F}_2^n in any way.

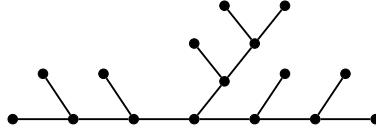


Figure 16. A tree that cannot be obtained via the preceding methods.

3. ADDITIONAL RESULTS INVOLVING BIPARTITE GRAPHS

In addition to pursuing the proof of the Odd Tree Conjecture, we also studied a technique presented in [4] that involves joining four copies of a set-sequential bipartite graph (not necessarily an odd tree) in such a way as to produce a larger graph that is set-sequential. This raises the following question. Given four copies of a set-sequential bipartite graph, how can we join them by three single edges so that the resulting graph is set-sequential?

Lemma 3.1. *Let G be a set-sequential graph on n vertices that is either a caterpillar or a bipartite graph with at least two vertices of degree 1 in each color class. Let G_1, G_2, G_3, G_4 be copies of G . In each G_i , choose two pendent vertices v_i and u_i , $i = 1, 2, 3, 4$, that are in the same color class. The graph constructed by adding the edges (v_1, v_2) , (u_2, u_3) , and (v_3, v_4) is set-sequential.*

Proof. Note that caterpillars are bipartite graphs, so we may express G as a bipartite graph with color classes X and Y . Take four copies G_1, G_2, G_3, G_4 of G with color classes $X_1, Y_1, X_2, Y_2, X_3, Y_3, X_4, Y_4$, respectively. We must extend the $(n+1)$ -dimensional vectors labeling G by two digits so they are of the proper dimension to form a set-sequential labeling for the larger graph consisting of G_1, G_2, G_3, G_4 with three edges added. Extend the vectors in X_1 and Y_1 by 00, the vectors in X_2 and Y_4 by 11, the vectors in X_3 and Y_2 by 10, and the vectors in X_4 and Y_3 by 01. Note that these extensions produce all nonzero vectors of dimension $(n+3)$ with the exceptions of $\vec{0}01$, $\vec{0}10$, and $\vec{0}11$. The edge (v_1, v_2) will be labeled with $\vec{0}11$, the edge (u_2, u_3) will be labeled with $\vec{0}01$, but the edge (v_3, v_4) will also be labeled with $\vec{0}11$, which does not fulfill the requirements for a set-sequential labeling. We may alter this labeling to be set-sequential, however. The vector labeling v_3 ends in 10, and the vector labeling its pendent edge ends in 11. Since v_3 is a leaf, we may “swap” the last two digits of these vectors with each other without violating the sum condition that is necessary for a set-sequential labeling. Now then the edge (v_3, v_4) is labeled with $\vec{0}10$, so the graph consisting of G_1, G_2, G_3, G_4 together with the three indicated edges is has a labeling and so is set-sequential. ■

We give now a modification of Theorem 3 in [4], imposing two extra conditions

that allow us to present a more complete proof of the theorem than was originally given in [4].

Theorem 3.2. *Let G be a strongly set colorable bipartite graph with color classes X, Y and edge set E . Let G_1, G_2, G_3, G_4 be four disjoint copies of G with color classes $X_1, Y_1, X_2, Y_2, X_3, Y_3, X_4, Y_4$ and edge sets E_1, E_2, E_3, E_4 , respectively. Let G_0 denote the graph obtained from the disjoint union of the graphs G_1, G_2, G_3, G_4 by adding edges e_1, e_2, e_3 with the following four properties:*

1. *each e_i joins two copies of the same vertex;*
2. *one of the following three possibilities occurs:*
 - (a) *the edges join X_1 and X_2 , X_2 and X_3 , X_3 and X_4 , respectively; or*
 - (b) *the edges join X_1 and Y_2 , X_2 and Y_3 , X_3 and Y_4 , respectively; or*
 - (c) *the edges join X_1 and X_2 , Y_2 and Y_4 , Y_1 and Y_3 , respectively;*
3. *there are two pendent vertices in G , namely $u_1, u_2, u_3, u_4, v_1, v_2, v_3, v_4$ in corresponding G_1, G_2, G_3, G_4 ;*
4. *all the edges are joining leaves in partite sets.*

Then G_0 is strongly set colorable.

Proof. For Case (a), consider extensions 00 to X_1, Y_1 , 11 on X_2, Y_4 , 10 on Y_2, Y_3 and 01 on X_3, X_4 . This is a set-sequential labeling for all vertices and edges except for the edge from X_3 to X_4 . As in the proof of Lemma 3.1, switch the extension of the vector labeling the vertex with connecting edges to X_3 in X_4 with the extension of the vector labeling its pendent edge in X_4 .

For Case (b), consider extensions 00 to X_1, Y_1 , 11 on X_2, Y_4 , 10 on Y_2, Y_3 and 01 on X_3, X_4 . This is a set-sequential labeling for all vertices and edges except for the edge from X_3 to Y_4 . As before, switch the extension of the vector labeling the vertex with connecting edges to X_3 in Y_4 with the extension of the vector labeling its pendent edge in Y_4 .

For Case (c), consider extensions 00 to X_1, Y_1 , 11 on X_2, Y_4 , 10 on Y_2, Y_3 and 01 on X_3, X_4 . This is a set-sequential labeling for all vertices and edges except the edge from Y_2 to Y_4 . Again, switch the extension of the vector labeling the vertex with connecting edges to Y_2 in Y_4 with the extension of the vector labeling its edge pendent in Y_2 . ■

4. CONCLUSION

One object of immediate interest is resolving the Pairing Conjecture. With that proof in hand, we would then have found a large class of odd trees that is set-sequential. This combined with our results on caterpillars and splicing small trees together represents substantial progress toward the Odd Tree Conjecture.

As explained at the end of Section 2, though, this set of constructions does not prove the Odd Tree Conjecture in its entirety. Further study may produce additional constructions and techniques that would allow us to join smaller trees together in ways that preserve both oddness and set-sequentialness. This may still fall short of proving the Odd Tree Conjecture as presently stated, but it is thought that larger classes of set-sequential trees may be obtained in this manner.

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