Discussiones Mathematicae
Graph Theory 42 (2022) 1383–1388
https://doi.org/10.7151/dmgt.2437

A NOTE ON PACKING OF UNIFORM HYPERGRAPHS

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Abstract

We say that two n-vertex hypergraphs H_1 and H_2 pack if they can be found as edge-disjoint subhypergraphs of the complete hypergraph K_n. Whilst the problem of packing of graphs (i.e., 2-uniform hypergraphs) has been studied extensively since seventies, much less is known about packing of k-uniform hypergraphs for $k \geq 3$. Naroski [*Packing of nonuniform hypergraphs - product and sum of sizes conditions*, Discuss. Math. Graph Theory 29 (2009) 651–656] defined the parameter $m_k(n)$ to be the smallest number m such that there exist two n-vertex k-uniform hypergraphs with total number of edges equal to m which do not pack, and conjectured that $m_k(n) = \Theta(n^{k-1})$. In this note we show that this conjecture is far from being truth. Namely, we prove that the growth rate of $m_k(n)$ is of order $n^{k/2}$ exactly for even k's and asymptotically for odd k's.

Keywords: packing, hypergraphs.

2010 Mathematics Subject Classification: 05C35.

1. Introduction

By a *hypergraph* H we mean a pair $(V(H), E(H))$ where $V(H)$ is a finite set (elements of $V(H)$ are called *vertices*) and $E(H)$ is a family of subsets of $V(H)$ (members of $E(H)$ are called *edges*). A hypergraph is *complete* if its set of edges consists of all subsets of $V(H)$. The complete hypergraph on n vertices is denoted K_n. Furthermore, we use the term *k-uniform hypergraph* to refer to hypergraphs with all edges consisting of exactly k vertices. Let H_1 and H_2 be two hypergraphs such that $|V(H_1)| = |V(H_2)| = n$.

We say that H_1 and H_2 *pack into* K_n (in short pack) if H_1 and H_2 can be found as edge-disjoint subhypergraphs in K_n. Clearly, the more edges H_1 and H_2 have, the harder is to pack them. Hence a natural question is to determine the bound on the common number of edges that guarantees the packing property of any two hypergraphs satisfying the bound. To this end let us define $m_k(n)$ to be the least integer m such that there exist two n-vertex k-uniform hypergraphs H_1 and H_2 with $|E(H_1)| + |E(H_2)| = m$ which do not pack. It is trivial that

$$m_1(n) = n + 1$$

and it was shown by Sauer and Spencer [9] that

$$m_2(n) = \left\lceil \frac{3}{2} n \right\rceil - 1,$$

(see also, among others, [1, 2, 4, 8] for generalizations). For $k \geq 3$, Naroski [5] proved that

$$m_k(n) \geq 2\sqrt{\binom{n}{k}}, \tag{1}$$

and conjectured that $m_k(n) = \Theta(n^{k-1})$ (earlier Pilśniak and Woźniak [7] conjectured the same for $k = 3$). It turns out that this is far from being truth. In this note we establish the growth order of $m_k(n)$ exactly for even k's, and asymptotically for odd k's by proving the following theorem.

Theorem 1. *If k is even then*

$$m_k(n) \leq \binom{n - k/2}{k/2} + \frac{\binom{n + k^{k/2} - 1}{k/2}}{\binom{k}{k/2}}.$$

If k is odd then, as n tends to infinity,

$$m_k(n) \leq \left(\frac{k^{k-1} + 1}{(k-1)!} + o(1) \right) n^{(k^2 - k - 1)/(2k-3)}.$$

Note that for $k = 2$, the bound agrees with the result of Sauer and Spencer.

2. Proof of Theorem 1

For any positive integer n let $[n] = \{1, \ldots, n\}$. In the proof we will use the following famous results, one very old and one relatively new.

Theorem 2 (Chinese Remainder Theorem [6])**.** *Let* a_i, b_i, $i = 1, \dots, r$, *be any integers. If* $a_i \equiv a_j \mod \gcd(b_i, b_j)$, *then there exists exactly one* $0 \le x < \mathrm{lcm}(b_1, \dots, b_r)$ *satisfying*

$$x \equiv a_i \mod b_i \quad i = 1, \dots, r.$$

A t-(n, k, λ)-design on a set X of size n is a collection T of k-element subsets of X such that every t elements of X are contained in exactly λ sets of T. Recently Keevash [3] proved the following deep existence theorem.

Theorem 3 ([3])**.** *A* t-(n, k, λ)*-design on a set* X *exists if and only if for every* $0 \le i \le t - 1$

$$\binom{k-i}{t-i} \quad \text{divides} \quad \lambda\binom{n-i}{t-i}, \tag{2}$$

apart from a finite number of exceptional n *given fixed* k, t, λ.

In the sequel we will identify the design with the hypergraph (X, T). Note that if (X, T) is a t-(n, k, λ)-design, then

$$|T| = \frac{\lambda\binom{n}{t}}{\binom{k}{t}} \tag{3}$$

since each t-subset of X is contained in exactly λ edges (i.e., elements of T), and each such edge is counted $\binom{k}{t}$ many times.

Now we are ready to give a proof of Theorem 1.

***Proof of Theorem* 1.** We are going to construct two n-vertex hypergraphs H_1 and H_2 which do not pack and have few edges. Let $V(H_1) = V(H_2) = [n]$.

Consider first the case when k is even, i.e., $k = 2t$. Then let

$$E(H_1) = \left\{ e \in \binom{[n]}{k} : [t] \subset e \right\}.$$

If n satisfies the divisibility conditions (2), then let

$$H_2 \text{ be a } t\text{-}(n, k, 1)\text{-design.}$$

Since $[t]$ forms an edge with every t-subset of $V(H_1) \setminus [t]$ and every t-subset of $V(H_2)$ is contained in exactly one edge of H_2, it is not possible to find edge disjoint copies of H_1 and H_2 in K_n. Hence H_1 and H_2 do not pack. In the case when (2) is not satisfied, we have to modify a bit the construction of H_2. First we will find the smallest possible number x such that $n+x$ satisfies (2) with n replaced by $n+x$. Note that

$$\binom{k-i}{t-i} \Big| \binom{n+x-i}{t-i} \Leftrightarrow \frac{(n+x-i)!}{(t-i)!(n+x-t)!} = p\frac{(k-i)!}{(t-i)!(k-t)!}$$
$$\Leftrightarrow (n+x-i)\cdots(n+x-t+1) = p(k-i)\cdots(k-t+1)$$

for some integer p. Hence, in order to assure that $n+x$ satisfies (2) it suffices that

$$k-i|n+x-i \text{ for } i=0,\ldots,t-1. \tag{4}$$

In other words

$$x \equiv i-n \mod (k-i) \text{ for } i=0,\ldots,t-1. \tag{5}$$

Note that if $g|(k-i)$ and $g|(k-j)$, then $g|(i-j)$. Thus, $i-n \equiv j-n \mod g$. Hence, by the Chinese Remainder Theorem, there exist

$$x \le k^t - 1 \tag{6}$$

that satisfies (4). Thus, by Theorem 3, as H_2' we can take a t-$(n+x,k,1)$-design on the set $[n+x]$. In order to construct H_2 we replace each k-subset e' of H_2' such that $e' \not\subset [n]$ by a k-set $e = (e' \cap [n]) \cup f$ where f is an arbitrary $|e' \setminus [n]|$-subset of $[n] \setminus e'$. Thus

$$|E(H_2)| = |E(H_2')| \text{ and} \tag{7}$$
$$\text{each } t\text{-subset of } V(H_2) \text{ is contained in at least one edge of } H_2. \tag{8}$$

Hence, H_1 and H_2 do not pack. Furthermore, by (7), (6) and (3), we have

$$|E(H_1)| + |E(H_2)| \le \binom{n-k/2}{k/2} + \frac{\binom{n+k^{k/2}-1}{k/2}}{\binom{k}{k/2}},$$

which proves the statement in the case when k is even.

In the case when k is odd one can obtain the bound $m_k(n) = O(n^{(k+1)/2})$ in a similar way. However, we can do better via a different construction. Let

$$s = \lfloor n^{(k-2)/(2k-3)} \rfloor \tag{9}$$

and a, b be non-negative integers satisfying

$$n = a\lfloor n/s \rfloor + b\lceil n/s \rceil, \quad a+b=s. \tag{10}$$

Define

$$E(H_1) = \left\{ e \in \binom{[n]}{k} : |e \cap [s(k-2)+1]| \geq k-1 \right\}.$$

By a similar reasoning as in the construction of H_2, we can deduce the existence of a $(k-1)$-$(\lceil n/s \rceil + x, k, 1)$-design with $x < k^{k-1}$. We identify the design with a hypergraph H' on $\lceil n/s \rceil + x$ vertices and then modify it (again in the same way as before) to get a $\lfloor n/s \rfloor$-vertex hypergraph $H_{\lfloor n/s \rfloor}$ and a $\lceil n/s \rceil$-vertex hypergraph $H_{\lceil n/s \rceil}$ satisfying

(11) $|E(H)| = |E(H')|$ and

(12) each $(k-1)$-subset of $V(H)$ is contained in at least one edge of H,

where $H \in \left\{ H_{\lfloor n/s \rfloor}, H_{\lceil n/s \rceil} \right\}$. Now let

$$H_2 = aH_{\lfloor n/s \rfloor} + bH_{\lceil n/s \rceil},$$

that means H_2 is a disjoint union of a hypergraphs $H_{\lfloor n/s \rfloor}$ and b hypergraphs $H_{\lceil n/s \rceil}$. By a Pigeonhole Principle, the subset $[s(k-2)+1]$ of the vertex set of every copy of H_1 in K_n intersects the vertex set of some $H_{\lfloor n/s \rfloor}$ or $H_{\lceil n/s \rceil}$ in every copy of H_2 in K_n in at least $k-1$ vertices. Let I be such an intersection for fixed copies of H_1 and H_2 in K_n. By the construction of H_1, every k-subset containing I is an edge of H_1, while by (11), at least one k-subset containing I is an edge of H_2. Hence, the copies have at least one common edge. Thus, H_1 and H_2 do not pack. Furthermore, by (9), (3) and (6),

$$\begin{aligned}
|E(H_1)| + |E(H_2)| &\leq \binom{s(k-2)+1}{k-1}(n - s(k-2) - 1) + s\binom{\lceil n/s \rceil + x - 1}{k-1} \\
&\leq \frac{(ks)^{k-1}}{(k-1)!} n + s \frac{(n/s + x)^{k-1}}{(k-1)!} \\
&= \frac{k^{k-1}+1}{(k-1)!} n^{(k^2-k-1)/(2k-3)} + o\left(n^{(k^2-k-1)/(2k-3)}\right).
\end{aligned}$$

This proves the statement in the case when k is odd. ■

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Received 24 June 2021
Revised 2 November 2021
Accepted 2 November 2021