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EDGE INTERSECTION HYPERGRAPHS

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Abstract

If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph, its *edge intersection hypergraph* $EI(\mathcal{H}) = (V, \mathcal{E}^{EI})$ has the edge set $\mathcal{E}^{EI} = \{e_1 \cap e_2 \mid e_1, e_2 \in \mathcal{E} \ \wedge \ e_1 \neq e_2 \ \wedge \ |e_1 \cap e_2| \geq 2\}$. Besides investigating several structural properties of edge intersection hypergraphs, we prove that all trees but seven exceptional ones are edge intersection hypergraphs of 3-uniform hypergraphs.

Using the so-called *clique-fusion*, as a conclusion we obtain that nearly all cacti are edge intersection hypergraphs of 3-uniform hypergraphs, too.

Keywords: edge intersection hypergraph, tree, cactus.

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1. Introduction and Basic Definitions

All hypergraphs $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$ and (undirected) graphs $G = (V(G), E(G))$ considered in the following may have isolated vertices but no multiple edges or loops. A hypergraph $\mathcal{H} = (V, \mathcal{E})$ is *k-uniform* if all hyperedges $e \in \mathcal{E}$ have the cardinality k. Trivially, any 2-uniform hypergraph $\mathcal{H}$ is a graph. The *degree* $d(v)$ (or $d_{\mathcal{H}}(v)$) of a vertex $v \in V$ is the number of hyperedges $e \in \mathcal{E}$ being incident to the vertex v. $\mathcal{H}$ is *linear* if any two distinct hyperedges $e, e' \in \mathcal{E}$ have at most one vertex in common. In standard terminology we follow Berge [1].

If $\mathcal{H} = (V, \mathcal{E})$ is a hypergraph, its *edge intersection hypergraph* $EI(\mathcal{H}) = (V, \mathcal{E}^{EI})$ has the same vertex set V as the original hypergraph $\mathcal{H}$ and the edge set $\mathcal{E}^{EI} = \{e_1 \cap e_2 \mid e_1, e_2 \in \mathcal{E} \ \wedge \ e_1 \neq e_2 \ \wedge \ |e_1 \cap e_2| \geq 2\}$. Our motivation for defining $EI(\mathcal{H})$ with $V(EI(\mathcal{H})) = V = V(\mathcal{H})$ results from certain applications; as an example a communication system will be described below. (Obviously, another natural approach would be to delete those vertices $v \in V$ which become isolated in the edge intersection hypergraph.)

For $k \geq 1$, the k-th iteration of the EI-operator is defined to be $EI^k(\mathcal{H}) := EI(EI^{k-1}(\mathcal{H}))$, where $EI^0(\mathcal{H}) := \mathcal{H}$. Moreover, the *EI-number* $k^{EI}(\mathcal{H})$ is the smallest $k \in \mathbb{N}$ such that $\mathcal{E}(EI^k(\mathcal{H})) = \emptyset$.

Let $e = \{v_1, v_2, \ldots, v_l\} \in \mathcal{E}^{EI}$ be a hyperedge in $EI(\mathcal{H})$. By definition, in $\mathcal{H}$ there exist (at least) two hyperedges $e_1, e_2 \in \mathcal{E}(\mathcal{H})$ both containing all the vertices $v_1, v_2, \ldots, v_l$, more precisely $\{v_1, v_2, \ldots, v_l\} = e_1 \cap e_2$. In this sense, the hyperedges of $EI(\mathcal{H})$ describe sets $\{v_1, v_2, \ldots, v_l\}$ of vertices having a certain,"strong" neighborhood relation in the original hypergraph $\mathcal{H}$.

As an application, we consider a hypergraph $\mathcal{H} = (V, \mathcal{E})$ representing a communication system. The vertices $v_1, v_2, \ldots, v_n \in V$ and the hyperedges $e_1, e_2, \ldots, e_m \in \mathcal{E}$ correspond to n people and to m (independent) communication channels, respectively. A group $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq V$ of people can communicate in a conference call if and only if their members use one and the same communication channel, i.e., there is a hyperedge $e \in \mathcal{E}$ such that $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq e$. If we ask whether or not $v_{i_1}, v_{i_2}, \ldots, v_{i_k}$ can continue to communicate in a conference call after the breakdown of an arbitrarily chosen communication channel, then this question is equivalent to the problem of the existence of a hyperedge $e^{EI} \in \mathcal{E}^{EI}$ in the edge intersection hypergraph $EI(\mathcal{H})$ containing all these vertices, i.e., $\{v_{i_1}, v_{i_2}, \ldots, v_{i_k}\} \subseteq e^{EI}$.

Note that our notion differs significantly from the well-known notions of the *intersection graph* (cf. [5]) or *edge intersection graph* (cf. [8]) $G = (V(G), E(G))$ *of linear hypergraphs* $\mathcal{H} = (V(\mathcal{H}), \mathcal{E}(\mathcal{H}))$, since there we have $V(G) = \mathcal{E}(\mathcal{H})$.

In [2, 4] and [3] the same notation is used for so-called *edge intersection graphs of paths*, but there the authors consider paths in a given graph G and the vertices of the resulting edge intersection graph correspond to these paths in the original graph G.

Obviously, for certain hypergraphs $\mathcal{H}$ the edge intersection hypergraph $EI(\mathcal{H})$ can be 2-uniform; in this case $EI(\mathcal{H})$ is a simple, undirected graph G. In order to delimit our notion from the intersection graphs or edge intersection graphs mentioned above, we consistently use our notion "edge intersection hypergraph" also when this hypergraph is 2-uniform.

First of all, in Section 2 we investigate structural properties of edge intersection hypergraphs.

To answer the question, which hypergraphs are edge intersection hypergraphs, seems to be very difficult. As a first step, in Sections 3 and 5 we consider some classes of 2-uniform edge intersection hypergraphs, i.e., graphs.

In Section 3, we will show that all but a few trees are edge intersection hypergraphs of 3-uniform hypergraphs; the exceptional graphs have at most 6 vertices. Whereas the proofs for paths and stars are simple, in the case of arbitrary trees we will make use of a special kind of induction. In the subsequent sections, the characterization of the trees being edge intersection hypergraphs (see Theorem 6) will be used as a basis to enlarge our investigations to the class of cacti. A simple, connected graph $G = (V, E)$ is referred to as a *cactus* if and only if every edge $e \in E$ is contained in at most one cycle of G.

In Section 4 we introduce a powerful tool for the construction of edge intersection hypergraphs, the so-called *clique-fusion*.

At the beginning of Section 5, we describe a special decomposition of cacti into trees and cycles. Note that Corollary 4 (see Section 2) and Theorem 6 characterize the cycles and the cycle-free cacti, respectively, being edge intersection hypergraphs of 3-uniform hypergraphs.

The *circumference* $ci(G)$ of a graph G is the length of a longest cycle in G. Using the clique-fusion, we prove that cacti, having either a circumference of at least 5 or containing (in the decomposition mentioned above) a tree T being none of the "forbidden trees" given in Theorem 6, are edge intersection hypergraphs of 3-uniform hypergraphs.

At the end of the introduction, let us mention a tool, which is useful for the investigation of small examples. For this end let $G = (V, E)$ and $\mathcal{H} = (V, \mathcal{E})$ be a graph and a hypergraph, respectively, having one and the same vertex set V. The verification of $\mathcal{E}(EI(\mathcal{H})) = E(G)$ can be done by hand or by computer, e.g., using the computer algebra system MATHEMATICA® [10] with the function

$$EEI[eh\_] := Complement[Select[Union[Flatten[Outer[\\ Intersection, eh, eh, 1], 1]], Length[\#] > 1\&], eh],$$

where the argument eh has to be the list of the hyperedges of $\mathcal{H}$ in the form $\{\{a, b, c\}, \ldots, \{x, y, z\}\}$. Then $EEI[eh]$ provides the list of the hyperedges of $EI(\mathcal{H})$.

2. Some Structural Properties

Theorem 1. (i) *For each linear hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $V \notin \mathcal{E}$ there is a hypergraph $\mathcal{H}' = (V, \mathcal{E}')$ with $EI(\mathcal{H}') = \mathcal{H}$.*

(ii) *Let $\mathcal{H} = (V, \mathcal{E})$ be a hypergraph containing $e_1, e_2 \in \mathcal{E}$ with $|e_1 \cap e_2| \geq 2$, $e_1 \not\subseteq e_2$, $e_2 \not\subseteq e_1$ and $\mathcal{H}' = (V, \mathcal{E}')$ be a hypergraph with $\mathcal{H} = EI(\mathcal{H}')$. Then there is an $\tilde{e} \in \mathcal{E} \setminus \{e_1, e_2\}$ with $e_1 \cap e_2 \subseteq \tilde{e}$.*

(iii) *Not every hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $V \notin \mathcal{E}$ is an edge intersection hypergraph of some hypergraph $\mathcal{H}' = (V, \mathcal{E}')$.*

Proof. (i) Choosing $\mathcal{E}' = \mathcal{E} \cup \{V\}$ we have $\mathcal{H} = EI(\mathcal{H}')$.

(ii) There are vertices $v_1 \in e_1 \setminus e_2$ and $v_2 \in e_2 \setminus e_1$ and edges $e_1', e_1'', e_2', e_2'' \in \mathcal{E}'$ with $e_1' \cap e_1'' = e_1$ and $e_2' \cap e_2'' = e_2$. Clearly

$$\exists\, e^1 \in \{e_1', e_1''\} : v_2 \notin e^1 \wedge \exists e^2 \in \{e_2', e_2''\} : v_1 \notin e^2.$$

Without loss of generality let $e^1 = e_1'$, $e^2 = e_2'$. Then

$$\tilde{e} = e_1' \cap e_2' \supseteq (e_1' \cap e_1'') \cap (e_2' \cap e_2'') = e_1 \cap e_2.$$

Hence $\tilde{e} \in \mathcal{E}$ and $e_1 \not\subseteq \tilde{e}$, $e_2 \not\subseteq \tilde{e}$.

(iii) This follows from (ii); a minimal example is $\mathcal{H} = (V, \mathcal{E})$ with $V = \{1, 2, 3, 4\}$, $\mathcal{E} = \{\{1, 2, 3\}, \{2, 3, 4\}\}$. ■

In the introduction, we shortly discussed in which way the edge intersection hypergraph $EI(\mathcal{H})$ mirrors certain neighborhood relations between vertices of the original hypergraph $\mathcal{H}$. In this context, it is interesting that the Helly property of the hypergraph $\mathcal{H}$ is hereditary if we go over to $EI(\mathcal{H})$.

A hypergraph $\mathcal{H} = (V, \mathcal{E})$ has the *Helly property* (Berge [1]) if

$$\forall \mathcal{E}' \subseteq \mathcal{E} : (\forall e_1, e_2 \in \mathcal{E}' : e_1 \cap e_2 \neq \emptyset) \rightarrow \bigcap_{e' \in \mathcal{E}'} e' \neq \emptyset.$$

Theorem 2. *If $\mathcal{H} = (V, \mathcal{E})$ has the Helly property, then $EI(\mathcal{H}) = (V, \mathcal{E}^{EI})$ has this property, too.*

Proof. Let $\mathcal{E}_S^{EI} = \{e_1, \ldots, e_t\} \subseteq \mathcal{E}^{EI}$ with $t \geq 1$ and $e_i \cap e_j \neq \emptyset$ for $i, j \in \{1, \ldots, t\}$. Clearly, for all $i \in \{1, \ldots, t\}$ there exists $e_i', e_i'' \in \mathcal{E}$ such that $e_i = e_i' \cap e_i''$ and $e_i' \neq e_i''$. Let $\mathcal{E}_S = \{e_1', \ldots, e_t', e_1'', \ldots, e_t''\}$. By $e_i \cap e_j \neq \emptyset$, for $i, j \in \{1, \ldots, t\}$, we have $\bar{e} \cap \bar{\bar{e}} \neq \emptyset$ for arbitrary $\bar{e}, \bar{\bar{e}} \in \mathcal{E}_S$.

The Helly property of $\mathcal{H}$ yields

$$\emptyset \neq \bigcap_{\bar{e} \in \mathcal{E}_S} \bar{e} = e_1' \cap \cdots \cap e_t' \cap e_1'' \cap \cdots \cap e_t'' = (e_1' \cap e_1'') \cap \cdots \cap (e_t' \cap e_t'') = e_1 \cap \cdots \cap e_t = \bigcap_{e \in \mathcal{E}_S^{EI}} e,$$

i.e., $EI(\mathcal{H})$ has the Helly property. ■

From the definition of edge intersection hypergraphs it follows immediately that for $k \geq 1$

$$\max\left\{|e| \;\middle|\; e \in \mathcal{E}\big(EI^k(\mathcal{H})\big)\right\} < \max\left\{|e| \;\middle|\; e \in \mathcal{E}\big(EI^{k-1}(\mathcal{H})\big)\right\}.$$

Hence the EI-number $k^{EI}(\mathcal{H})$ is well defined. In the following we determine the edge intersection hypergraph and the EI-number k^{EI} for some special classes of hypergraphs. The strong *d-uniform hypercycle* $\hat{\mathcal{C}}_n^d$ and the strong *d-uniform hyperpath* $\hat{\mathcal{P}}_n^d$ both have the vertex set $\{v_1, \ldots, v_n\}$ and the edge sets

$$\mathcal{E}(\hat{\mathcal{C}}_n^d) = \{e_i = \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \,\big|\, i = 1, \ldots, n\} \quad \text{(indices taken modulo n)}$$

and

$$\mathcal{E}(\hat{\mathcal{P}}_n^d) = \{e_i = \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \,\big|\, i = 1, \ldots, n-d+1\}.$$

We consider only those strong d-uniform hypercycles $\hat{\mathcal{C}}_n^d$ with $n \geq 2d-1$. This condition implies that for different edges $e_i, e_j \in \mathcal{E}(\hat{\mathcal{C}}_n^d)$ the intersection is empty or contains only vertices being consecutive on the cycle, i.e., $e_i \cap e_j = \{v_s, v_{s+1}, \ldots, v_{s+t}\}$ for $s = 1, \ldots, n$ and $t = 0, \ldots, d-2$ (indices taken modulo n). For "small" cycles $\hat{\mathcal{C}}_n^d$ with $n < 2d-1$ the edge intersection hypergraph is confusing in the sense that it contains edges of other types, too, and the following structural results, which are partly contained in the Bachelor Thesis [7] of a student of the second author, are not true.

Theorem 3. *Let $\hat{\mathcal{C}}_n^d$ and $\hat{\mathcal{P}}_n^d$ be a strong d-uniform hypercycle and a strong d-uniform hyperpath, respectively.*

(i) $EI^k(\hat{\mathcal{C}}_n^d) = \hat{\mathcal{C}}_n^{d-k} \cup \hat{\mathcal{C}}_n^{d-k-1} \cup \cdots \cup \hat{\mathcal{C}}_n^2$ *for* $d \geq 3$, $n \geq 2d-1$ *and* $k = 1, \ldots, d-2$.

(ii) $k^{EI}(\hat{\mathcal{C}}_n^d) = d-1$ *for* $d \geq 2$ *and* $n \geq 2d-1$.

(iii) $k^{EI}(\hat{\mathcal{P}}_n^d) = \begin{cases} d-1 & \text{for } d \geq 2 \text{ and } n \geq 2d-1, \\ n-d+1 & \text{for } d \geq 2 \text{ and } n < 2d-1. \end{cases}$

Proof. (i) In strong d-uniform hypercycles $\hat{\mathcal{C}}_n^d$ with $n \geq 2d-1$ there are intersections of cardinalities at least two between the edges $e_i, e_{i+1}, \ldots, e_{i+d-2}$; $i = 1, \ldots, n$ (indices taken modulo n). Hence $EI(\hat{\mathcal{C}}_n^d)$ contains the following edges (see Figure 1).

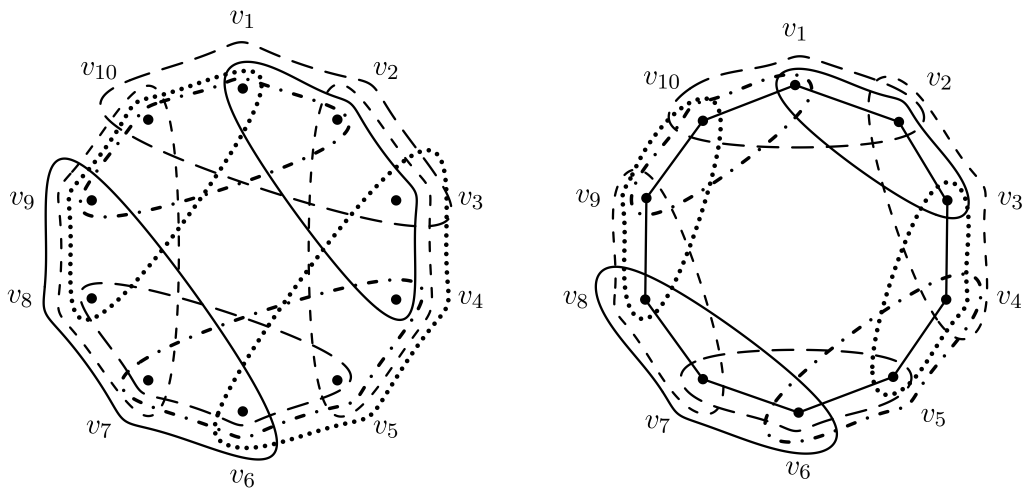
$$e_{j,i} = e_i \cap e_{i+j} = \{v_{i+j}, \ldots, v_{i+d-1}\},$$

with $i = 1, \ldots, n$ and $j = 1, \ldots, d-2$ (indices taken modulo n). This yields $EI(\hat{\mathcal{C}}_n^d) = \hat{\mathcal{C}}_n^{d-1} \cup \hat{\mathcal{C}}_n^{d-2} \cup \cdots \cup \hat{\mathcal{C}}_n^2$, i.e., by using the EI-operator the maximum edge cardinality decreases by one. For the k-th iteration we obtain

$$EI^k(\hat{\mathcal{C}}_n^d) = \hat{\mathcal{C}}_n^{d-k} \cup \cdots \cup \hat{\mathcal{C}}_n^2, \quad k = 1, \ldots, d-2.$$

(ii) The case $d = 2$ is trivial; for $d \geq 3$ it follows with (i) that $EI^{d-2}(\hat{\mathcal{C}}_n^d) = \hat{\mathcal{C}}_n^2 = C_n$ and hence $k^{EI}(\hat{\mathcal{C}}_n^d) = d-1$.

(iii) The result is trivial for $d = 2$ in both cases; in the following we assume $d \geq 3$.



$\hat{\mathcal{C}}^4_{10}$ $\qquad$ $EI^1(\hat{\mathcal{C}}^4_{10}) = \hat{\mathcal{C}}^3_{10} \cup \hat{\mathcal{C}}^2_{10}$

Figure 1. The strong 4-uniform hypercycle $\hat{\mathcal{C}}^4_{10}$ and the edge intersection hypergraph $EI^1(\hat{\mathcal{C}}^4_{10})$. Obviously, $EI^2(\hat{\mathcal{C}}^4_{10}) = \hat{\mathcal{C}}^2_{10} = C_{10}$ and $\mathcal{E}(EI^3(\hat{\mathcal{C}}^4_{10})) = \emptyset$.

For $n \geq 2d-1$ we have $|e_1 \cap e_{n-d+1}| \leq 1$, i.e., the intersection of the first edge and of the last edge of $\hat{\mathcal{P}}^d_n$ does not generate an edge in $EI(\hat{\mathcal{P}}^d_n)$. The edges of $EI(\hat{\mathcal{P}}^d_n)$ are generated by the following intersections (see Figure 2).

$$\begin{aligned}
e_{1,i} = e_i \cap e_{i+1} &= \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+1}, v_{i+2}, \ldots, v_{i+d}\} \\
&= \{v_{i+1}, \ldots, v_{i+d-1}\} \qquad\qquad \text{for } \; i = 1, \ldots, n-d, \\
e_{2,i} = e_i \cap e_{i+2} &= \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+2}, v_{i+3}, \ldots, v_{i+d+1}\} \\
&= \{v_{i+2}, \ldots, v_{i+d-1}\} \qquad\qquad \text{for } \; i = 1, \ldots, n-d-1, \\
&\;\vdots \\
e_{d-2,i} = e_i \cap e_{i+d-2} &= \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+d-2}, v_{i+d-1}, \ldots, v_{i+d+1}\} \\
&= \{v_{i+d-2}, v_{i+d-1}\} \qquad\qquad \text{for } \; i = 1, \ldots, n-2d+3.
\end{aligned}$$

Hence $EI(\hat{\mathcal{P}}^d_n)$ has edges of cardinalities $d-1, d-2, \ldots, 2$ and the edge set

$$\begin{aligned}
&\mathcal{E}(EI(\hat{\mathcal{P}}^d_n)) \\
&= \{e_{1,1}, \ldots, e_{1,n-d}, e_{2,i}, \ldots, e_{2,n-d-1}, e_{d-2,1}, \ldots, e_{d-2,n-2d+3}\} \\
&= \mathcal{E}(\hat{\mathcal{P}}^{d-1}_n) \setminus \{\{v_1, \ldots, v_{d-1}\}, \{v_{n-d+2}, \ldots, v_n\}\} \\
&\cup \mathcal{E}(\hat{\mathcal{P}}^{d-2}_n) \setminus \{\{v_1, \ldots, v_{d-2}\}, \{v_2, \ldots, v_{d-1}\}, \{v_{n-d+2}, \ldots, v_{n-1}\}, \{v_{n-d+3}, \ldots, v_n\}\} \\
&\cup \cdots \cup \mathcal{E}(\hat{\mathcal{P}}^2_n) \setminus \{\{v_1, v_2\}, \ldots, \{v_{d-2}, v_{d-1}\}, \{v_{n-d+2}, v_{n-d+3}\}, \ldots, \{v_{n-1}, v_n\}\}.
\end{aligned}$$

The reapplication of the EI-operator yields a hypergraph without the edges of maximum cardinality $(d-1)$, while all other edges of $EI(\hat{\mathcal{P}}^d_n)$ remain (because they are contained in the edges of cardinality $(d-1)$). After $(d-2)$ iterations we

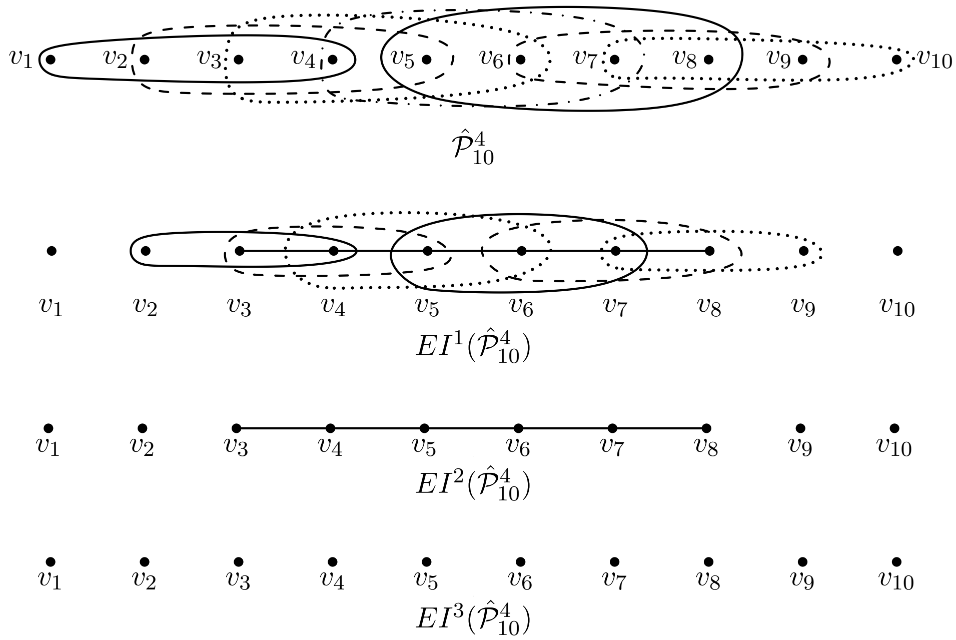


Figure 2. The strong 4-uniform hyperpath $\hat{\mathcal{P}}_{10}^4$ and the corresponding edge intersection hypergraphs.

obtain $EI^{d-2}(\hat{\mathcal{P}}_n^d) = P_{n-2d+4} \cup I_{2d-4}$, where I_t denotes a set of t isolated vertices; hence $k^{EI}(\hat{\mathcal{P}}_n^d) = d-1$.

For $n < 2d-1$ we have $|e_1 \cap e_{n-d+1}| \geq 2$, i.e., the intersection of the first and the last edge of $\hat{\mathcal{P}}_n^d$ generates in $EI(\hat{\mathcal{P}}_n^d)$ the edge of minimum cardinality $(2d-n)$. All edges of $EI(\hat{\mathcal{P}}_n^d)$ are generated by the following intersections (see Figure 3).

$$\begin{aligned}
e_{1,i} = e_i \cap e_{i+1} &= \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+1}, v_{i+2}, \ldots, v_{i+d}\} \\
&= \{v_{i+1}, \ldots, v_{i+d-1}\} \qquad \text{for} \quad i = 1, \ldots, n-d, \\
e_{2,i} = e_i \cap e_{i+2} &= \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+2}, v_{i+3}, \ldots, v_{i+d+1}\} \\
&= \{v_{i+2}, \ldots, v_{i+d-1}\} \qquad \text{for} \quad i = 1, \ldots, n-d-1, \\
&\ \vdots \\
e_{n-d-1,i} = e_i \cap e_{i+n-d-1} &= \{v_i, v_{i+1}, \ldots, v_{i+d-1}\} \cap \{v_{i+n-d-1}, \ldots, v_{i+n-2}\} \\
&= \{v_{i+n-d-1}, \ldots, v_{i+d-1}\} \qquad \text{for} \quad i = 1, 2, \\
e_{n-d,1} = e_1 \cap e_{n-d+1} &= \{v_1, v_2, \ldots, v_d\} \cap \{v_{n-d+1}, \ldots, v_n\} \\
&= \{v_{n-d+1}, \ldots, v_d\}.
\end{aligned}$$

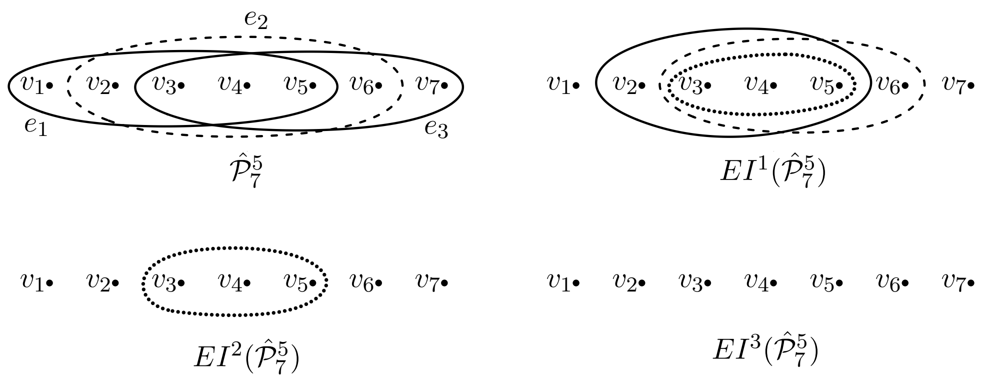


Figure 3. The strong 5-uniform hyperpath $\hat{\mathcal{P}}_7^5$ and the corresponding edge intersection hypergraphs.

Hence $EI(\hat{\mathcal{P}}_n^d)$ has edges of cardinalities $d-1, d-2, \ldots, 2d-n$ and the edge set

$$\begin{aligned}\mathcal{E}(EI(\hat{\mathcal{P}}_n^d)) &= \{e_{1,1}, \ldots, e_{1,n-d}, e_{2,1}, \ldots, e_{2,n-d-1}, e_{n-d-1,1}, e_{n-d-1,2}, e_{n-d,1}\}\\ &= \mathcal{E}(\hat{\mathcal{P}}_n^{d-1}) \setminus \{\{v_1, \ldots, v_{d-1}\}, \{v_{n-d+2}, \ldots, v_n\}\}\\ &\cup \mathcal{E}(\hat{\mathcal{P}}_n^{d-2}) \setminus \{\{v_1, \ldots, v_{d-2}\}, \{v_2, \ldots, v_{d-1}\}, \{v_{n-d+2}, \ldots, v_{n-1}\},\\ &\quad \{v_{n-d+3}, \ldots, v_n\}\}\\ &\cup \cdots \cup\ \mathcal{E}(\hat{\mathcal{P}}_n^{2d-n}) \setminus \{\{v_1, \ldots, v_{2d-n}\}, \ldots, \{v_{n-d}, \ldots, v_{d-1}\},\\ &\quad \{v_{n-d+2}, \ldots, v_{d+1}\}, \ldots, \{v_{2(n-d)+1}, \ldots, v_n\}\}.\end{aligned}$$

Again, the reapplication of the EI-operator yields a hypergraph without the edges of maximum cardinality $(d-1)$, while all the other edges remain. After $(n-d)$ iterations we obtain $EI^{n-d}(\hat{\mathcal{P}}_n^d) = (V(\hat{\mathcal{P}}_n^d), \{\tilde{e}\})$, where $\tilde{e} = \{v_{n-d+1}, \ldots, v_d\}$ with cardinality $|\tilde{e}| = 2d-n$ is the only hyperedge in $EI^{n-d}(\hat{\mathcal{P}}_n^d)$. Hence $k^{EI}(\hat{\mathcal{P}}_n^d) = n-d+1$. ∎

For $n \geq 5$, $d=3$ and $k=1$, Theorem 3(i) provides the following.

Corollary 4. *For $n \geq 5$ the cycle C_n is an edge intersection hypergraph of a 3-uniform hypergraph, namely $C_n = EI(\hat{\mathcal{C}}_n^3)$.*

Berge [1] generalized the complete graph K_n by the definition of the *complete d-uniform hypergraph* $\mathcal{K}_n^d$ as follows

$$V(\mathcal{K}_n^d) = \{v_1, \ldots, v_n\}, \quad \mathcal{E}(\mathcal{K}_n^d) = \{T \subseteq V(\mathcal{K}_n^d) \mid |T| = d\}.$$

Theorem 5. *Let $\mathcal{K}_n^d$ be a complete d-uniform hypergraph with $n-1 \geq d \geq 3$.*

(i) $EI^k(\mathcal{K}_n^d) = \mathcal{K}_n^{d-k} \cup \mathcal{K}_n^{d-k-1} \cup \cdots \cup \mathcal{K}_n^{t_k}$ *for $1 \leq k \leq d-2$, where $t_k = \max\{2, 2^k(d-n)+n\}$ for $1 \leq k \leq d-2$.*

(ii) $k^{EI}(\mathcal{K}_n^d) = d-1$ *for $d \geq 2$.*

Proof. (i) The intersections (with cardinality of a least two) of edges in $\mathcal{K}_n^d$ are all subsets $T \subseteq V(\mathcal{K})_n^d$ with cardinalities in the range $d-1 \geq |T| \geq t_1 = \max\{2, 2d-n\}$; hence

$$EI(\mathcal{K}_n^d) = \mathcal{K}_n^{d-1} \cup \mathcal{K}_n^{d-2} \cup \cdots \cup \mathcal{K}_n^{t_1}.$$

Using induction, the reapplication of the EI-operator to $EI^k(\mathcal{K}_n^d)$ yields all subsets $T \subseteq V(\mathcal{K}_n^d)$ of cardinalities in the range

$$d-(k+1) \geq |T| \geq \max\{2, 2(2^k(d-n)+n)-n\} = \max\{2, 2^{k+1}(d-n)+n\}.$$

(ii) From (i) we know that $EI^{d-2}(\mathcal{K}_n^d) = \mathcal{K}_n^2 = K_n$, hence $k^{EI}(\mathcal{K}_n^d) = d-1$. ■

3. TREES

In the following, for the trees up to 8 vertices we often use the notations $T1, T2, \ldots, T48$ corresponding to [6]. Moreover, for brevity we will conveniently write ij instead of $\{i,j\}$ and ijk instead of $\{i,j,k\}$ for edges and hyperedges, respectively.

The main result of the section is that all but seven exceptional trees are edge intersection hypergraphs of 3-uniform hypergraphs. The exceptional trees have at most 6 vertices, namely the paths P_n with $n \in \{2,3,4,5,6\}$ vertices and the trees $T7$ and $T12$ with 5 and 6 vertices, respectively. In detail, $T7 = (V = \{v_1, v_2, \ldots, v_5\}, E = \{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_3,v_5\}\})$ and $T12 = (V = \{v_1, v_2, \ldots, v_6\}, E = \{\{v_1,v_2\},\{v_2,v_3\},\{v_3,v_4\},\{v_2,v_5\},\{v_5,v_6\}\})$ (see Figure 4).

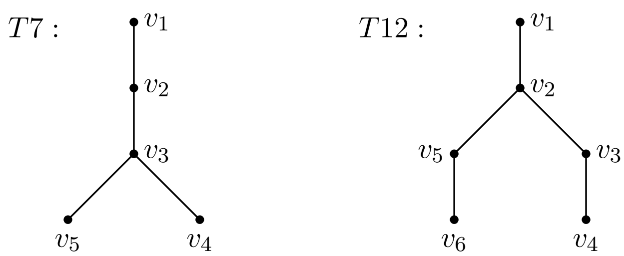


Figure 4. Two exceptional trees.

Theorem 6. *All trees but* $T2 = P_2$, $T3 = P_3$, $T5 = P_4$, $T8 = P_5$, $T14 = P_6$, $T7$ *and* $T12$ *are edge intersection hypergraphs of a* 3*-uniform hypergraph* $\mathcal{H}$*.*

The proof will be done by induction. The induction basis includes the investigation of all 48 trees having at most 8 vertices (see Lemmas 7–9 below). Note that this set of trees contains the seven exceptional cases mentioned above.

The remaining part of the section consists of the inductive step of the proof. In the inductive step we will make use of the deletion of a (shortest) so-called *leg* in a tree.

***Proof of Theorem* 6.**

<u>*Induction basis.*</u>

Above all, for two simple classes of trees, namely paths and stars, we easily obtain a first result.

Lemma 7. (i) *For $n \geq 3$, the star $K_{1,n}$ is an edge intersection hypergraph of a* 3*-uniform hypergraph.*

(ii) *For $n = 1$ and for $n \geq 7$, the path P_n is an edge intersection hypergraph of a* 3*-uniform hypergraph.*

Proof. (i) Let $K_{1,n} = (V, E)$ with $V = \{1, 2, \ldots, n, n+1\}$, $E = \{\{1, 2\}, \{1, 3\}, \ldots, \{1, n\}, \{1, n+1\}\}$ and $\mathcal{H} = (V, \mathcal{E})$ with $\mathcal{E} = \{\{1, 2, 3\}, \{1, 3, 4\}, \ldots, \{1, n, n+1\}, \{1, n+1, 2\}\}$. Then $K_{1,n} = EI(\mathcal{H})$.

(ii) Let $n \geq 7$ and $P_n = \hat{\mathcal{P}}_n^2 = (V, E)$; for simplicity we identify the vertices $v_i \in V$ with their indices: $v_i = i$. With $\mathcal{H} = (V, \mathcal{E})$, where $\mathcal{E} = \{\{1, 2, 3\}, \{2, 3, 4\}, \ldots, \{n-2, n-1, n\}, \{1, 2, n-2\}, \{n-1, n, 3\}\}$, we have $P_n = EI(\mathcal{H})$. $\square$

Now we discuss the seven exceptional trees.

Lemma 8. $T2 = P_2, T3 = P_3, T5 = P_4, T8 = P_5, T14 = P_6, T7$ *and* $T12$ *are not edge intersection hypergraphs of a* 3*-uniform hypergraph.*

Proof. In the following, we give the most effortful proofs for P_6, $T7$ and $T12$, respectively. All other cases can be shown in a similar, but easier way.

(P_6) Note that we have $E(P_6) = \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 5\}, \{5, 6\}\}$. For any graph G, to generate the edges of $G = EI(\mathcal{H})$, in $\mathcal{H}$ we need only hyperedges e which contain at least two of the adjacent vertices of G. Such hyperedges will be called *useful hyperedges.*

In the present case, we have $G = P_6$ and a useful hyperedge e has to fulfil $\{i, i+1\} \subset e$, where $i \in \{1, 2, \ldots, 5\}$. Therefore, the useful hyperedges which may occur in $\mathcal{H} = (V, \mathcal{E})$ are the following.

123, 124, 125, 126 — to generate 12 in P_6;
234, 235, 236 — to generate 23 (should the occasion arise, in connection with 123);
134, 345, 346 — to generate 34 (... with 234);
145, 245, 456 — to generate 45 (... with 345);
156, 256, 356 — to generate 56 (... with 456).

Clearly, each of the edges $\{i, i+1\}$ in P_6 is contained in exactly four useful hyperedges and at least two of these hyperedges have to appear in $\mathcal{H}$ to generate $\{i, i+1\}$ in $EI(\mathcal{H}) = P_6$.

By case distinction, we discuss the possible combinations of useful hyperedges in $\mathcal{H}$ and will obtain a contradiction (abbreviated by the symbol $\lightning$) in every case. We have six possibilities to generate $12 \in \mathcal{E}(EI(\mathcal{H})) = E(P_6)$.

(a) If $123, 124 \in \mathcal{E}$, then $134 \notin \mathcal{E}$ (otherwise $13 \in E(P_6)$ $\lightning$) and $234 \notin \mathcal{E}$ (otherwise $24 \in E(P_6)$ $\lightning$). Therefore, $345, 346 \in \mathcal{E}$ in order to generate $34 \in E(P_6)$. On the other hand, $145 \notin \mathcal{E}$ (otherwise $14 \in E(P_6)$ $\lightning$) and $245 \notin \mathcal{E}$ (otherwise $24 \in E(P_6)$ $\lightning$). Hence, $456 \in \mathcal{E}$ in order to generate $45 \in E(P_6)$. This includes $346 \cap 456 = 46 \in \mathcal{E}(EI(\mathcal{H})) = E(P_6)$ $\lightning$.

(b) If $123, 125 \in \mathcal{E}$, then $145 \notin \mathcal{E}$ (otherwise $15 \in E(P_6)$ $\lightning$) and $245 \notin \mathcal{E}$ (otherwise $25 \in E(P_6)$ $\lightning$). Therefore, $345, 456 \in \mathcal{E}$ in order to generate $45 \in E(P_6)$. On the other hand, $156 \notin \mathcal{E}$ (otherwise $15 \in E(P_6)$ $\lightning$) and $256 \notin \mathcal{E}$ (otherwise $25 \in E(P_6)$ $\lightning$). Hence, $356 \in \mathcal{E}$ in order to generate $56 \in E(P_6)$. This includes $345 \cap 356 = 35 \in \mathcal{E}(EI(\mathcal{H})) = E(P_6)$ $\lightning$.

(c) If $123, 126 \in \mathcal{E}$, then $156 \notin \mathcal{E}$ (otherwise $16 \in E(P_6)$ $\lightning$) and $256 \notin \mathcal{E}$ (otherwise $26 \in E(P_6)$ $\lightning$). Therefore, $356, 456 \in \mathcal{E}$ in order to generate $56 \in E(P_6)$. On the other hand, $134 \notin \mathcal{E}$ (otherwise $13 \in E(P_6)$ $\lightning$), $345 \notin \mathcal{E}$ (otherwise $35 \in E(P_6)$ $\lightning$) and $346 \notin \mathcal{E}$ (otherwise $46 \in E(P_6)$ $\lightning$). Hence, $34 \notin \mathcal{E}(EI(\mathcal{H})) = E(P_6)$ $\lightning$.

(d) If $124, 125 \in \mathcal{E}$, then $123 \notin \mathcal{E}$ (because of (a), (b)), $234 \notin \mathcal{E}$ (otherwise $24 \in E(P_6)$ $\lightning$) and $235 \notin \mathcal{E}$ (otherwise $25 \in E(P_6)$ $\lightning$). Consequently, $23 \notin \mathcal{E}(EI(\mathcal{H})) = E(P_6)$ $\lightning$.

(e) If $124, 126 \in \mathcal{E}$, then $123 \notin \mathcal{E}$ (because of (a), (c)), $234 \notin \mathcal{E}$ (otherwise $24 \in E(P_6)$ $\lightning$) and $236 \notin \mathcal{E}$ (otherwise $26 \in E(P_6)$ $\lightning$). Consequently, $23 \notin \mathcal{E}(EI(\mathcal{H})) = E(P_6)$ $\lightning$.

(f) If $125, 126 \in \mathcal{E}$, then $123 \notin \mathcal{E}$ (because of (b), (c)), $235 \notin \mathcal{E}$ (otherwise $25 \in E(P_6)$ $\lightning$) and $236 \notin \mathcal{E}$ (otherwise $26 \in E(P_6)$ $\lightning$). Consequently, $23 \notin \mathcal{E}(EI(\mathcal{H})) = E(P_6)$ $\lightning$.

This implies that 12 cannot be generated in $EI(\mathcal{H}) = P_6$ $\lightning$. Therefore, P_6 is not an edge intersection hypergraph of a 3-uniform hypergraph.

($T7$) We investigate the graph $T7 = (V, E)$ with $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{3,5\}\}$. Here are the possible useful hyperedges in $\mathcal{H} = (V, \mathcal{E})$.

123, 124, 125 — to generate 12 in $T7$;
234, 235 — to generate 23 in $T7$ (should the occasion arise, in connection with 123);
134, 345 — to generate 34 in $T7$ (... with 234);
135 — to generate 35 in $T7$ (... with 235 or 345).

Clearly, each of the edges $\{i, j\}$ in $T7$ is contained in exactly three useful hyperedges and at least two of these hyperedges have to appear in $\mathcal{H}$ to generate $\{i, j\}$ in $EI(\mathcal{H}) = T7$. We have three possibilities to generate $12 \in \mathcal{E}(EI(\mathcal{H})) = E(T7)$.

(a) If $123, 124 \in \mathcal{E}$, then $134 \notin \mathcal{E}$ (otherwise $13 \in \mathcal{E}(EI(\mathcal{H})) = E(T7) \lightning$) and $234 \notin \mathcal{E}$ (otherwise $24 \in \mathcal{E}(EI(\mathcal{H})) = E(T7) \lightning$). This includes $34 \notin \mathcal{E}(EI(\mathcal{H})) = E(T7) \lightning$.

(b) If $123, 125 \in \mathcal{E}$, then $135 \notin \mathcal{E}$ (otherwise $13 \in \mathcal{E}(EI(\mathcal{H})) = E(T7) \lightning$) and $235 \notin \mathcal{E}$ (otherwise $25 \in \mathcal{E}(EI(\mathcal{H})) = E(T7) \lightning$). This includes $35 \notin \mathcal{E}(EI(\mathcal{H})) = E(T7) \lightning$.

(c) If $124, 125 \in \mathcal{E}$, then $123 \notin \mathcal{E}$ (because of (a)) and $234 \notin \mathcal{E}$ (otherwise $24 \in \mathcal{E}(EI(\mathcal{H})) = E(T7) \lightning$). This includes $23 \notin \mathcal{E}(EI(\mathcal{H})) = E(T7) \lightning$.

Thats why, 12 cannot be generated in $EI(\mathcal{H}) = T7 \lightning$. Therefore, $T7$ is not an edge intersection hypergraph of a 3-uniform hypergraph.

($T12$) The most laborious case is the graph $T12 = (V, E)$ with $E = \{\{1,2\}, \{2,3\}, \{3,4\}, \{2,5\}, \{5,6\}\}$. The useful hyperedges which may occur in $\mathcal{H} = (V, \mathcal{E})$ are the following.

123, 124, 125, 126 — to generate 12 in $T12$;
234, 235, 236 — to generate 23 (should the occasion arise, in connection with 123);
134, 345, 346 — to generate 34 (... with 234);
245, 256 — to generate 25 (... with 125 or 235);
156, 356, 456 — to generate 56 (... with 256).

Again, we discuss the combinations of useful hyperedges in $\mathcal{H}$ and will obtain a contradiction in every case. Obviously, we have six possibilities to generate $12 \in \mathcal{E}(EI(\mathcal{H})) = E(T12)$ and in some cases several subcases have to be investigated.

(a) If $123, 124 \in \mathcal{E}$, then $134 \notin \mathcal{E}$ (otherwise $13 \in E(T12) \lightning$) and $234 \notin \mathcal{E}$ (otherwise $24 \in E(T12) \lightning$). Therefore, $345, 346 \in \mathcal{E}$ in order to generate $34 \in E(T12)$. Hence, $235 \notin \mathcal{E}$ (otherwise $35 \in E(T12) \lightning$). In order to generate $23 \in E(T12)$ it follows $236 \in \mathcal{E}$. This includes $236 \cap 346 = 36 \in \mathcal{E}(EI(\mathcal{H})) = E(T12) \lightning$.

(b) If $123, 125 \in \mathcal{E}$, then $134 \notin \mathcal{E}$ (otherwise $13 \in E(T12) \lightning$) and $156 \notin \mathcal{E}$ (otherwise $15 \in E(T12) \lightning$). In order to generate the edge $23 \in E(T12)$, the three subcases (b1), (b2) and (b3) have to be considered.

(b1) If $234 \in \mathcal{E}$, then for $34 \in E(T12)$ we need $345 \in \mathcal{E}$ or $346 \in \mathcal{E}$. Assume, $345 \in \mathcal{E}$. Then $356 \notin \mathcal{E}$ (otherwise $35 \in E(T12) \lightning$) and $456 \notin \mathcal{E}$ (otherwise $45 \in E(T12) \lightning$). Together with $156 \notin \mathcal{E}$ (see (b) above), we obtain $56 \notin E(T12) \lightning$. So assume $346 \in \mathcal{E}$. Then $356 \notin \mathcal{E}$ (otherwise $36 \in E(T12) \lightning$) and $456 \notin \mathcal{E}$ (otherwise $46 \in E(T12) \lightning$). As above, $56 \notin E(T12) \lightning$. Consequently, (b1) cannot occur.

(b2) If $235 \in \mathcal{E}$, then $345 \notin \mathcal{E}$ follows (otherwise $35 \in E(T12) \lightning$). Since $134 \notin \mathcal{E}$ (see (b)) and $234 \notin \mathcal{E}$ (see (b1)), we easily get $34 \notin E(T12) \lightning$.

So the only possibility in case (b) would be the next one.

(b3) Let $236 \in \mathcal{E}$. We have $256 \notin \mathcal{E}$ (otherwise $26 \in E(T12)$ $\lightning$) and, additionally, since (b2) is impossible, also $235 \notin \mathcal{E}$. Hence for $25 \in E(T12)$ we need $245 \in \mathcal{E}$. Since $156 \notin E(T12)$ (see at the beginning of (b)) for $56 \in E(T12)$ necessarily $356, 456 \in \mathcal{E}$. This provides $245 \cap 456 = 45 \in \mathcal{E}(EI(\mathcal{H})) = E(T12)$ $\lightning$.

Thus case (b) cannot occur.

(c) If $123, 126 \in \mathcal{E}$, then $125 \notin \mathcal{E}$ (because of (b)), $156 \notin \mathcal{E}$ (otherwise $16 \in E(T12)$ $\lightning$) and $256 \notin \mathcal{E}$ (otherwise $26 \in E(T12)$ $\lightning$). Therefore, it follows $235, 245 \in \mathcal{E}$ in order to generate $25 \in E(T12)$ as well as $356, 456 \in \mathcal{E}$ in order to generate $56 \in E(T12)$. But then $235 \cap 356 = 35 \in \mathcal{E}(EI(\mathcal{H})) = E(T12)$ $\lightning$.

(d) If $124, 125 \in \mathcal{E}$, then $123 \notin \mathcal{E}$ (because of (a), (b)), $134 \notin \mathcal{E}$ (otherwise $14 \in E(T12)$ $\lightning$) and $234 \notin \mathcal{E}$ (otherwise $24 \in E(T12)$ $\lightning$). Consequently, we need $345, 346 \in \mathcal{E}$ in order to generate $34 \in E(T12)$. Thus $236 \notin \mathcal{E}$ (otherwise $36 \in E(T12)$ $\lightning$). Together with $123, 234 \notin \mathcal{E}$ we obtain $23 \notin \mathcal{E}(EI(\mathcal{H})) = E(T12)$ $\lightning$.

(e) If $124, 126 \in \mathcal{E}$, then $123 \notin \mathcal{E}$ (because of (a), (c)), $234 \notin \mathcal{E}$ (otherwise $24 \in E(T12)$ $\lightning$) and $236 \notin \mathcal{E}$ (otherwise $26 \in E(T12)$ $\lightning$). Consequently, $23 \notin \mathcal{E}(EI(\mathcal{H})) = E(T12)$ $\lightning$.

(f) If $125, 126 \in \mathcal{E}$, then $123 \notin \mathcal{E}$ (because of (b), (c)) and $236 \notin \mathcal{E}$ (otherwise $26 \in E(T12)$ $\lightning$). So we need $234, 235 \in \mathcal{E}$ in order to generate $23 \in E(T12)$. On the other hand, $156 \notin \mathcal{E}$ (otherwise $15 \in E(T12)$ $\lightning$) and $256 \notin \mathcal{E}$ (otherwise $26 \in E(T12)$ $\lightning$). Hence necessarily $356, 456 \in \mathcal{E}$ in order to generate $56 \in E(T12)$. This leads to $235 \cap 356 = 35 \in \mathcal{E}(EI(\mathcal{H})) = E(T12)$ $\lightning$.

This implies that 12 cannot be generated in $EI(\mathcal{H}) = T12$ $\lightning$. Therefore, $T12$ is not an edge intersection hypergraph of a 3-uniform hypergraph and Lemma 8 is proved. □

Lemma 9. *All trees with at most eight vertices are edge intersection hypergraphs of a 3-uniform hypergraph, but* $T2 = P_2, T3 = P_3, T5 = P_4, T8 = P_5, T14 = P_6, T7$ *and* $T12$.

Proof. Because of Lemma 7 and Lemma 8 for the trees $T1 - T9$, $T12$, $T14$, $T15$, $T25$, $T26$ and $T48$ there is nothing to show.

For the remaining 33 trees $Tn = (V_n, E_n)$ in each case we give the edge set E_n and the set of hyperedges $\mathcal{E}_n$ of a 3-uniform hypergraph $\mathcal{H}_n = (V_n, \mathcal{E}_n)$ with $Tn = EI(\mathcal{H}_n)$. The verification of $\mathcal{E}(EI(\mathcal{H}_n)) = E(Tn)$ can be done by hand for all n or by computer, e.g. using the MATHEMATICA®-function $EEI[eh\_]$ given at the end of the introduction.

$n = 6$ vertices:

$E_{10} = \{12, 23, 34, 35, 36\}$, $\mathcal{E}_{10} = \{123, 124, 235, 236, 345, 346\}$.
$E_{11} = \{12, 23, 24, 45, 46\}$, $\mathcal{E}_{11} = \{123, 124, 234, 245, 246, 456\}$.
$E_{13} = \{12, 23, 34, 45, 46\}$, $\mathcal{E}_{13} = \{123, 125, 234, 345, 346, 456\}$.

$n = 7$ vertices:

$E_{16} = \{12, 23, 34, 35, 36, 37\}$, $\mathcal{E}_{16} = \{123, 125, 234, 237, 345, 356, 367\}$.
$E_{17} = \{12, 23, 24, 25, 56, 57\}$, $\mathcal{E}_{17} = \{123, 124, 234, 256, 257, 567\}$.
$E_{18} = \{12, 23, 24, 25, 56, 67\}$, $\mathcal{E}_{18} = \{123, 124, 167, 234, 245, 256, 567\}$.
$E_{19} = \{12, 23, 34, 35, 36, 67\}$, $\mathcal{E}_{19} = \{123, 124, 235, 236, 345, 346, 367, 567\}$.
$E_{20} = \{12, 23, 24, 45, 56, 57\}$, $\mathcal{E}_{20} = \{123, 124, 234, 456, 457, 567\}$.
$E_{21} = \{12, 23, 34, 36, 45, 47\}$, $\mathcal{E}_{21} = \{123, 125, 234, 236, 345, 346, 347, 457\}$.
$E_{22} = \{12, 23, 24, 45, 56, 67\}$, $\mathcal{E}_{22} = \{123, 124, 167, 234, 245, 456, 567\}$.
$E_{23} = \{12, 23, 34, 45, 46, 57\}$, $\mathcal{E}_{23} = \{123, 126, 157, 234, 345, 346, 456, 457\}$.
$E_{24} = \{12, 23, 34, 36, 45, 67\}$, $\mathcal{E}_{24} = \{123, 127, 145, 234, 236, 345, 367, 567\}$.

$n = 8$ vertices:

$E_{27} = \{17, 18, 27, 37, 47, 57, 67\}$, $\mathcal{E}_{27} = \{127, 138, 167, 178, 237, 347, 457, 567\}$.
$E_{28} = \{15, 25, 35, 45, 56, 67, 68\}$, $\mathcal{E}_{28} = \{125, 145, 235, 345, 567, 568, 678\}$.
$E_{29} = \{14, 24, 34, 45, 56, 57, 58\}$, $\mathcal{E}_{29} = \{124, 145, 234, 345, 567, 568, 578\}$.
$E_{30} = \{16, 17, 26, 36, 46, 56, 78\}$, $\mathcal{E}_{30} = \{126, 156, 167, 178, 236, 278, 346, 456\}$.
$E_{31} = \{16, 17, 26, 28, 36, 46, 56\}$, $\mathcal{E}_{31} = \{126, 137, 156, 167, 236, 248, 268, 346, 456\}$.
$E_{32} = \{15, 16, 25, 35, 45, 67, 68\}$, $\mathcal{E}_{32} = \{125, 145, 167, 168, 235, 345, 678\}$.
$E_{33} = \{15, 16, 17, 25, 35, 45, 78\}$, $\mathcal{E}_{33} = \{156, 157, 167, 178, 235, 245, 278, 345\}$.
$E_{34} = \{12, 18, 23, 24, 25, 56, 57\}$, $\mathcal{E}_{34} = \{123, 124, 168, 178, 234, 256, 257, 567\}$.
$E_{35} = \{12, 23, 24, 45, 48, 56, 57\}$, $\mathcal{E}_{35} = \{123, 124, 234, 248, 456, 457, 458, 567\}$.
$E_{36} = \{12, 23, 24, 28, 45, 56, 67\}$, $\mathcal{E}_{36} = \{123, 124, 128, 167, 234, 238, 245, 456, 567\}$.
$E_{37} = \{12, 23, 24, 25, 38, 56, 67\}$, $\mathcal{E}_{37} = \{123, 124, 167, 234, 238, 245, 256, 378, 567\}$.
$E_{38} = \{12, 23, 34, 36, 38, 45, 67\}$, $\mathcal{E}_{38} = \{123, 127, 145, 234, 236, 345, 348, 367, 368, 567\}$.
$E_{39} = \{12, 23, 24, 45, 48, 56, 67\}$, $\mathcal{E}_{39} = \{123, 124, 167, 234, 245, 248, 456, 458, 567\}$.
$E_{40} = \{12, 23, 24, 45, 56, 67, 68\}$, $\mathcal{E}_{40} = \{123, 124, 167, 234, 245, 456, 567, 568, 678\}$.
$E_{41} = \{12, 23, 24, 45, 56, 57, 68\}$, $\mathcal{E}_{41} = \{123, 124, 168, 234, 456, 457, 567, 568\}$.
$E_{42} = \{12, 23, 34, 38, 45, 46, 57\}$, $\mathcal{E}_{42} = \{123, 126, 157, 234, 238, 345, 346, 348, 456, 457\}$.
$E_{43} = \{12, 23, 28, 34, 36, 45, 67\}$, $\mathcal{E}_{43} = \{123, 127, 128, 145, 234, 236, 238, 345, 367, 567\}$.
$E_{44} = \{12, 23, 24, 45, 56, 67, 78\}$, $\mathcal{E}_{44} = \{123, 124, 178, 234, 245, 456, 567, 678\}$.
$E_{45} = \{12, 23, 34, 36, 45, 67, 78\}$, $\mathcal{E}_{45} = \{123, 127, 145, 234, 236, 345, 367, 578, 678\}$.
$E_{46} = \{12, 23, 24, 38, 45, 56, 67\}$, $\mathcal{E}_{46} = \{123, 124, 167, 234, 238, 245, 378, 456, 567\}$.
$E_{47} = \{12, 23, 34, 45, 46, 57, 78\}$, $\mathcal{E}_{47} = \{123, 126, 178, 234, 345, 346, 456, 457, 578\}$.

□

This completes the induction basis and we have to define some notions in order to prepare the inductive step.

Let $G = (V, E)$ be a graph and $s \geq 1$.

Definition. A path $L = (v_0, e_1, v_1, \ldots, v_{s-1}, e_s, v_s)$ is referred to as a *leg* (of *length* s) in G if and only if

(i) $V(L) = \{v_0, \ldots, v_s\} \subseteq V$;

(ii) $E(L) = \{e_1, \ldots, e_s\} \subseteq E$;

(iii) $d_G(v_0) \geq 3$, $d_G(v_1) = \cdots = d_G(v_{s-1}) = 2$, $d_G(v_s) = 1$.

The vertex v_0 is the *joint* or *joint vertex* and v_s is the *end vertex* of L. Clearly, every graph G with minimum degree $\delta(G) = 1$ and maximum degree $\Delta(G) \geq 3$ has a leg. Moreover, each tree T being not a path P_n $(n \geq 1)$ has at least three legs.

Definition. The graph $G \ominus L = (V', E')$ results from $G = (V, E)$ by *deleting the leg* $L = (v_0, e_1, v_1, \ldots, v_{s-1}, e_s, v_s)$ if and only if $V' = V \setminus \{v_1, \ldots, v_s\}$ and $E' = E \setminus \{e_1, \ldots, e_s\}$.

Obviously, $G \ominus L$ is connected if and only if G is connected, since the joint vertex v_0 is not deleted by the deletion of the leg L in G.

Inductive step.

Note that all trees $T = (V, E)$ with 7 or 8 vertices are edge intersection hypergraphs of a 3-uniform hypergraph (cf. Lemma 9).

Induction hypothesis. Every tree $T = (V, E)$ with $7 \leq |V| \leq n$ is an edge intersection hypergraph of a 3-uniform hypergraph.

Let $n \geq 8$, $T' = (V', E')$ be a tree with $|V'| = n + 1$ vertices; because of Lemma 7 we can exclude stars and paths from our considerations. Therefore T' has at least three end vertices and also at least three legs. Let v_0 and v_s be the joint vertex and the end vertex of a shortest leg $L = (v_0, v_1, \ldots, v_s)$, respectively.

We delete the leg L in T' and obtain $T = (V, E) = T' \ominus L$. Obviously, $v_0 \in V$ and $v_1, \ldots, v_s \in V' \setminus V$. According to the length s of the leg L we consider two cases.

Case 1. $s = 1$. Because of $d_T(v_0) \geq 2$ there are at least two neighbors $u \neq u'$ of v_0 in the tree T. Moreover, we have $|V| = n \geq 8$ and the induction basis implies the existence of a hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $T = EI(\mathcal{H})$ and $\mathcal{H}$ is 3-uniform.

Consider $\mathcal{E}' = \mathcal{E} \cup \{\{u, v_0, v_1\}, \{u', v_0, v_1\}\}$ and the 3-uniform hypergraph $\mathcal{H}' = (V', \mathcal{E}')$. Then $\{v_0, v_1\} = \{u, v_0, v_1\} \cap \{u', v_0, v_1\}$.

Clearly, $\{u, v_0, v_1\} \cap V = \{u, v_0\}$ and $\{u', v_0, v_1\} \cap V = \{u', v_0\}$. Taking an arbitrarily chosen hyperedge $e \in \mathcal{E}' \setminus \{\{u, v_0, v_1\}, \{u', v_0, v_1\}\} = \mathcal{E}$, the only edge which can result from the intersection $\{u, v_0, v_1\} \cap e$ and $\{u', v_0, v_1\} \cap e$ in $EI(\mathcal{H}')$ is the edge $\{u, v_0\} \in E(T)$ and $\{u', v_0\} \in E(T)$, respectively. Consequently, the hypergraph $\mathcal{H}'$ has the edge intersection hypergraph $EI(\mathcal{H}') = T'$.

Case 2. $s \geq 2$. Let L, L', L'' be three legs in $T' = (V', E')$. Deleting the legs L, L', L'' in T', we would obtain a new tree with at least one vertex. Since $L = (v_0, v_1, \ldots, v_s)$ is a shortest leg and $|V'| = n + 1$ is valid, the leg L contains at most $\frac{n}{3}$ vertices. Because of $n \geq 8$, the deletion of the leg L corresponds to the deletion of the vertices $v_1, \ldots, v_s$ in T' and leads to the tree $T = (V, E)$ with $|V| = n + 1 - s \geq n + 1 - \frac{n}{3} = \frac{2}{3}n + 1 \geq \frac{19}{3} > 6$. Therefore, T has at least 7 vertices; we apply the induction basis and obtain the existence of a 3-uniform hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $T = EI(\mathcal{H})$.

Now we construct the hypergraph $\mathcal{H}' = (V', \mathcal{E}')$ from $\mathcal{H} = (V, \mathcal{E})$.

In comparison to T, in T' we find the additional edges $\{v_0, v_1\}, \{v_1, v_2\}, \ldots, \{v_{s-1}, v_s\}$ which have to be generated by certain hyperedges of $\mathcal{H}'$. We add the following three types of hyperedges to the hypergraph $\mathcal{H}$.

• The first one is the hyperedge $\{u, v_0, v_1\}$, where $u \in V$ is a neighbor of the vertex v_0 in the tree T. Because of $v_1 \notin V(T)$, the only edge being induced by this hyperedge and the hyperedges of $\mathcal{E}(\mathcal{H})$ in the edge intersection hypergraph of $\mathcal{H}_0 = (V \cup \{v_1\}, \mathcal{E} \cup \{\{u, v_0, v_1\}\})$ is the edge $\{u, v_0\} \in E(T)$.

• The second set of new hyperedges consists of $\{v_0, v_1, v_2\}, \{v_1, v_2, v_3\}, \ldots, \{v_{s-2}, v_{s-1}, v_s\}$. Adding these hyperedges (and the vertices $v_2, \ldots, v_s$) to $\mathcal{H}_0$ we obtain a hypergraph $\mathcal{H}_1$; in the corresponding edge intersection hypergraph $EI(\mathcal{H}_1)$ we find the new edges $\{v_0, v_1\} = \{u, v_0, v_1\} \cap \{v_0, v_1, v_2\}, \{v_1, v_2\} = \{v_0, v_1, v_2\} \cap \{v_1, v_2, v_3\}, \ldots, \{v_{s-2}, v_{s-1}\} = \{v_{s-3}, v_{s-2}, v_{s-1}\} \cap \{v_{s-2}, v_{s-1}, v_s\}$ and not more (because of $v_1, \ldots, v_s \notin V(T)$).

• To obtain the last edge needed in $T' = EI(\mathcal{H}')$, we choose a vertex $w \in V(T) \setminus \{v_0\}$ being not a neighbor of v_0. The existence of such a vertex w becomes clear since all legs in T' have to have a length of at least 2. Therefore, for w we can choose an end vertex of an arbitrary leg $L' \neq L$ in T. Considering $\{w, v_{s-1}, v_s\}$, we see that $\{w, v_{s-1}, v_s\} \cap \{v_{s-2}, v_{s-1}, v_s\} = \{v_{s-1}, v_s\}$.

We add the new hyperedge $\{w, v_{s-1}, v_s\}$ to the hypergraph $\mathcal{H}_1$ and obtain the hypergraph $\mathcal{H}'$. For two reasons, $\{v_{s-1}, v_s\}$ is the only edge being generated by the hyperedge $\{w, v_{s-1}, v_s\}$ in $EI(\mathcal{H}')$.

(i) $|V(T) \cap \{w, v_{s-1}, v_s\}| = 1$, therefore the intersection of $\{w, v_{s-1}, v_s\}$ with any hyperedge of the original hypergraph $\mathcal{H} = (V, \mathcal{E}) = (V(T), \mathcal{E})$ cannot lead to an additional edge in $EI(\mathcal{H}')$.

(ii) Because of $w \in V \setminus \{u\}$, the intersection of $\{w, v_{s-1}, v_s\}$ with one of the "new" hyperedges $\{u, v_0, v_1\}, \{v_0, v_1, v_2\}, \{v_1, v_2, v_3\}, \ldots, \{v_{s-2}, v_{s-1}, v_s\}$ is always a subset of $\{v_{s-1}, v_s\}$. Hence the edge $\{v_{s-1}, v_s\} \in E(T')$ is the only edge being induced by $\{w, v_{s-1}, v_s\}$ in $EI(\mathcal{H}')$.

This completes the proof of Theorem 6. ■

4. The Clique-Fusion

Let $r \geq 2$ and $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_r = (V_r, E_r)$ be graphs. Moreover, let $k \geq 1$, $V' = \{v_1, \ldots, v_k\} = \{v \mid \exists i, j \in \{1, \ldots, r\} : i \neq j \wedge v \in V_i \cap V_j\}$ and $E' = \{\{v, v'\} \mid v, v' \in V' \wedge \exists i \in \{1, \ldots, r\} : \{v, v'\} \in E_i\}$.

Incidentally, if for each $i \in \{1, \ldots, r\}$ the graph G_i is connected and $V_i \cap V' \neq \emptyset$, then the union $G_1 \cup \cdots \cup G_r = (V_1 \cup \cdots \cup V_r, E_1 \cup \cdots \cup E_r)$ is connected, too.

Consider the case that $E' = \{\{v_i, v_j\} \mid 1 \leq i < j \leq k\}$, i.e., the subgraph $\langle V' \rangle_{G_1 \cup \cdots \cup G_r}$ induced by the vertices of V' in $G_1 \cup \cdots \cup G_r$ is a k-clique. Then we refer to the union $G_1 \cup \cdots \cup G_r$ as the *clique-fusion* or *k-fusion* of the graphs $G_1 \cup \cdots \cup G_r$ and write $G_1 \oplus \cdots \oplus G_r = G_1 \cup \cdots \cup G_r$.

For an example, consider three graphs $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ and $G_3 = (V_3, E_3)$, where $\{x, y\} \in E_1$, $\{y, z\} \in E_2$ and $\{x, z\} \in E_3$ are edges as well as $V_1 \cap V_2 = \{y\}$, $V_2 \cap V_3 = \{z\}$ and $V_1 \cap V_3 = \{x\}$ hold. Then $V' = \{x, y, z\}$ induces a 3-clique $\langle \{x, y, z\} \rangle_{G_1 \cup G_2 \cup G_3}$ in $G_1 \cup G_2 \cup G_3$ and $G_1 \oplus G_2 \oplus G_3 = G_1 \cup G_2 \cup G_3$ is the 3-fusion of the graphs G_1, G_2 and G_3 (see Figure 5).

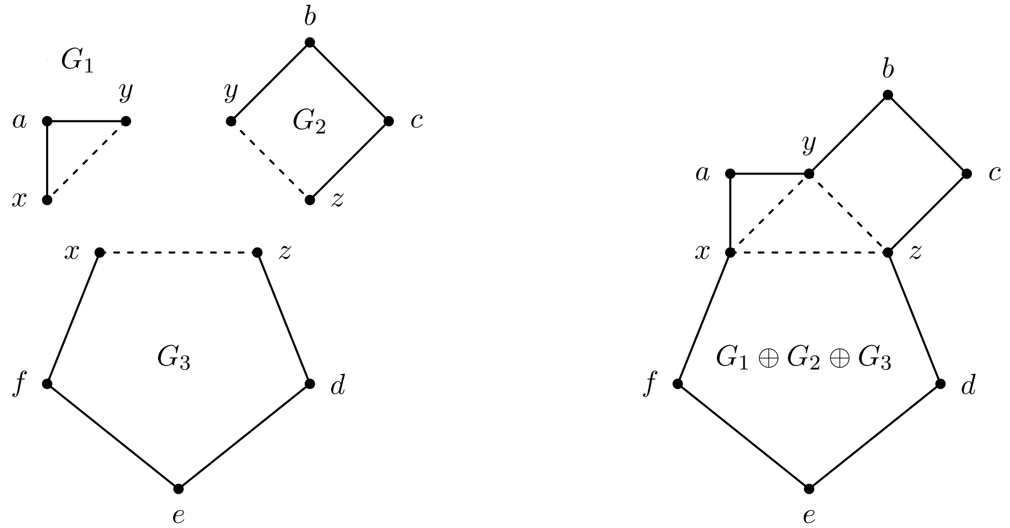


Figure 5. Three graphs and their clique-fusion.

Note that we obtain the same 3-fusion taking the modified graphs $G_1' = (V_1 \cup \{z\}, E_1 \cup \{\{x, z\}, \{y, z\}\})$ and $G_3' = (V_3, E_3 \setminus \{\{x, z\}\})$ instead of G_1 and G_3, i.e., we have $G_1 \oplus G_2 \oplus G_3 = G_1' \oplus G_2 \oplus G_3'$.

Using the above notations we have a look at a special situation.

Special Case 1. For all $i, j \in \{1, \ldots, r\} : V' = V_i \cap V_j$ and $\langle V' \rangle_{G_i}$ is a k-clique.

In this case, all graphs G_i, G_j $(i \neq j)$ have all the vertices $v_1, \ldots, v_k$ (and only these vertices) in common. Additionally, in each G_i $(i \in \{1, \ldots, k\})$ (as well as in $\langle V' \rangle_{G_1 \cup \cdots \cup G_r}$) the vertices $v_1, \ldots, v_k$ induce one and the same k-clique.

Let us mention two further special cases; the first one corresponds to $k = 1$ and the second one to $r = 2$, respectively.

Special Case 2. For all $i, j \in \{1, \ldots, r\} : V' = \{v\} = V_i \cap V_j$, where v is a uniquely determined vertex.

Special Case 3. $r = 2$, i.e., we consider the clique-fusion $G_1 \oplus G_2$ of two graphs.

Investigating cacti, we only need Special Case 2 and Special Case 3 in combination, i.e., we have $k = 1$ as well as $r = 2$. Only for proof-technical reasons, in very few exceptions we use the 2-fusion (see $G_1 \oplus K_{1,3}$ in the part (b) of the proof of Theorem 12).

Remark 10. The clique-fusion can be easily generalized to pairwise vertex-disjoint graphs $G_1 = (V_1, E_1), G_2 = (V_2, E_2), \ldots, G_r = (V_r, E_r)$ by identifying certain vertices $v^i \in V_i$ and $v^j \in V_j$: $v^i \equiv v^j$, where $1 \leq i < j \leq r$. Obviously, also the edges $\{v_1^i, v_2^i\} \in E_i$ and $\{v_1^j, v_2^j\} \in E_j$ have to be identified $\big(\{v_1^i, v_2^i\} \equiv \{v_1^j, v_2^j\}\big)$, if the corresponding vertices have been identified $\big(v_1^i \equiv v_1^j$ and $v_2^i \equiv v_2^j\big)$.

In the following, we will only make use of the clique-fusion in its original form, not in the generalized sense described in Remark 10.

Now we prove that the clique-fusion of graphs which are edge intersection hypergraphs of 3-uniform hypergraphs is an edge intersection hypergraph of a 3-uniform hypergraph, too.

Theorem 11. *Let $G = (V, E)$ be the clique-fusion $G_1 \oplus \cdots \oplus G_r$ of graphs $G_1 = (V_1, E_1), \ldots, G_r = (V_r, E_r)$, where $G_1 = EI(\mathcal{H}_1), \ldots, G_r = EI(\mathcal{H}_r)$ are edge intersection hypergraphs of the* 3*-uniform hypergraphs $\mathcal{H}_1 = (V_1, \mathcal{E}_1), \ldots, \mathcal{H}_r = (V_r, \mathcal{E}_r)$. Then $\mathcal{H} = \mathcal{H}_1 \cup \cdots \cup \mathcal{H}_r$ is* 3*-uniform and $G = EI(\mathcal{H})$.*

Proof. The 3-uniformity of $\mathcal{H} = (V, \mathcal{E})$ is trivial because of $V = V_1 \cup \cdots \cup V_r$ and $\mathcal{E} = \mathcal{E}_1 \cup \cdots \cup \mathcal{E}_r$. Owing to $G = G_1 \oplus \cdots \oplus G_r = G_1 \cup \cdots \cup G_r = EI(\mathcal{H}_1) \cup \cdots \cup EI(\mathcal{H}_r)$ in addition to $V = V_1 \cup \cdots \cup V_r$ we have $E = E_1 \cup \cdots \cup E_r = \mathcal{E}(EI(\mathcal{H}_1)) \cup \cdots \cup \mathcal{E}(EI(\mathcal{H}_r))$ and it suffices to show $\mathcal{E}(EI(\mathcal{H})) = \mathcal{E}(EI(\mathcal{H}_1)) \cup \cdots \cup \mathcal{E}(EI(\mathcal{H}_r))$.

Part 1. $\mathcal{E}(EI(\mathcal{H})) \supseteq \mathcal{E}(EI(\mathcal{H}_1)) \cup \cdots \cup \mathcal{E}(EI(\mathcal{H}_r))$.

We consider an arbitrarily chosen edge $\{x, y\} \in \mathcal{E}(EI(\mathcal{H}_i))$, where $i \in \{1, \ldots, r\}$. Then there are hyperedges $e_i, e_i' \in \mathcal{E}(\mathcal{H}_i) \subseteq \mathcal{E}(\mathcal{H})$ with $e_i \cap e_i' = \{x, y\}$. This implies $\{x, y\} \in \mathcal{E}(EI(\mathcal{H}))$.

Part 2. $\mathcal{E}(EI(\mathcal{H})) \subseteq \mathcal{E}(EI(\mathcal{H}_1)) \cup \cdots \cup \mathcal{E}(EI(\mathcal{H}_r))$.

Let $e, e' \in \mathcal{E}(\mathcal{H})$ with $e \cap e' \in \mathcal{E}(EI(\mathcal{H}))$. The 3-uniformity of $\mathcal{H}$ includes $|e \cap e'| = 2$.

Assume, there exists an $i \in \{1, \ldots, r\}$ such that $e, e' \in \mathcal{E}(\mathcal{H}_i)$. Then $e \cap e' \in \mathcal{E}(EI(\mathcal{H}_i)) \subseteq \mathcal{E}(EI(\mathcal{H}_1)) \cup \cdots \cup \mathcal{E}(EI(\mathcal{H}_r))$.

Otherwise, for all $i, j \in \{1, \ldots, r\}$ from $e \in \mathcal{E}(\mathcal{H}_i)$ and $e' \in \mathcal{E}(\mathcal{H}_j)$ we get $i \neq j$. In this case we have $e \cap e' = \{x, y\}$, with $\{x, y\} \subseteq V_i \cap V_j$ and $x \neq y$. Since G is the clique-fusion $G_1 \oplus \cdots \oplus G_r$, the vertices x and y have to be adjacent in G and, therefore, $e \cap e' = \{x, y\} \in E(G) = E = E_1 \cup \cdots \cup E_r = \mathcal{E}(EI(\mathcal{H}_1)) \cup \cdots \cup \mathcal{E}(EI(\mathcal{H}_r))$.

(Note that because of $e \cap e' = \{x, y\} \in E_1 \cup \cdots \cup E_r$ there exist an $l \in \{1, \ldots, r\}$ and hyperedges $e_l, e'_l \in \mathcal{E}(\mathcal{H}_l)$ with $e_l \cap e'_l = \{x, y\}$. Therefore, we even get $e \cap e' = e_l \cap e'_l \in \mathcal{E}(EI(\mathcal{H}_l)) = E_l$.) ■

Theorem 11 is a very useful tool for constructing graphs being edge intersection hypergraphs of 3-uniform hypergraphs. But in order to verify our main result for cacti (cf. Theorem 16 in Section 5), we need also an analog result for the 1-fusion of graphs G (which are edge intersection hypergraphs of 3-uniform hypergraphs) with arbitrary trees and cycles, respectively (cf. Theorems 12 and 14).

The attempt to extend Theorems 12 and 14 also for k-fusions, where $k > 1$, would lead to an extensive additional effort in the proofs of the corresponding results. This becomes clear if we look at the verification of Theorems 12 and 14. For every k, each of the nine exceptional trees and cycles being not the edge intersection hypergraph of a 3-uniform hypergraph ($P_2, P_3, P_4, P_5, P_6, T7, T12, C_3, C_4$) would require a separate case which has to be considered.

Theorem 12. *Let the graph $G_1 = (V_1, E_1)$ be the edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$, $v \in V_1$ with $d_{G_1}(v) \geq 2$ and $T = (V_2, E_2)$ be a tree with $V_2 = \{v_1, v_2, \ldots, v_n\}$, where $n \geq 2$. Moreover, let $V_1 \cap V_2 = \{v\}$. Then the 1-fusion $G_1 \oplus T$ is an edge intersection hypergraph of a 3-uniform hypergraph.*

Proof. If T is an edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$, owing to Theorem 11 there is nothing to show. So we have to consider only the exceptional trees $P_2, P_3, P_4, P_5, P_6, T7$ and $T12$ from Theorem 6. Note that the one and only case where the condition $d_{G_1}(v) \geq 2$ will be needed is the path P_2. At first we investigate the paths $P_n = (V_2, E_2)$ with $E_2 = \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}\}$, where $n \in \{2, \ldots, 6\}$.

In advance, we mention that $|V_1| \geq 4$ is valid, since G_1 is an edge intersection hypergraph of a 3-uniform hypergraph containing the non-isolated vertex $v \in V_1$.

(a) $T = P_2$. Let $v = v_1$ and $u, w \in V_1$ be two neighbors of v_1 in G_1. Then $G_1 \oplus P_2 = EI(\mathcal{H})$ with $\mathcal{H} = (V_1 \cup \{v_2\}, \mathcal{E}_1 \cup \{\{u, v_1, v_2\}, \{w, v_1, v_2\}\})$.

(b) $T \in \{P_3, P_4, P_5, P_6\}$. For $3 \leq n \leq 6$, let $i \leq n$, $v = v_i$ and $u, w \in V_1$, where u is a neighbor of v_i in G_1.

First we consider the situation $n = 3$ and $v = v_2$, i.e., v is the inner vertex of P_3. Then the 1-fusion $G_1 \oplus P_3$ is nothing else than the 2-fusion $G_1 \oplus K_{1,3}$, where

$K_{1,3} = (V_2 \cup \{u\}, E_2 \cup \{\{v_2, u\}\})$. Theorem 11 implies that this 2-fusion of G_1 and $K_{1,3}$ has the required properties.

Next we investigate the 1-fusion $G_1 \oplus P_n$ in the end vertex $v = v_1$ of P_n. For this end, we consider the hypergraph $\mathcal{H} = (V_1 \cup \{v_2, \ldots, v_n\}, \mathcal{E}_1 \cup \{\{u, v_1, v_2\}, \{v_1, v_2, v_3\}, \{v_2, v_3, v_4\}, \ldots, \{v_{n-2}, v_{n-1}, v_n\}, \{w, v_{n-1}, v_n\}\})$ and obtain $G_1 \oplus P_n = EI(\mathcal{H})$.

The remaining cases ($n \geq 4$ and the vertex $v_i \in V_1 \cap V_2$ is an inner vertex of the path P_n, i.e., $1 < i < n$) can be obtained from the results above in two steps.

Let $P'_n = (V'_2 = \{v_1, v_2, \ldots, v_i\}, E'_2 = \{\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{i-1}, v_i\}\})$ and $P''_n = (V''_2 = \{v_i, v_{i+1}, \ldots, v_n\}, E'_2 = \{\{v_i, v_{i+1}\}, \{v_{i+1}, v_{i+2}\}, \ldots, \{v_{n-1}, v_n\}\})$. Assume, $|V'_2| \geq |V''_2|$. Then P'_n contains at least three vertices and the assumption of case (b) is fulfilled. In a first step, $G_1 \oplus P'_n$ and, in a second step, $(G_1 \oplus P'_n) \oplus P''_n$ is an edge intersection hypergraph of a 3-uniform hypergraph, respectively. Only in the second step, when $n - i = 1$ holds (i.e., P''_n contains exactly two vertices) we have to make use of part (a). In this case, the vertex v_i has minimum degree 2 in $G_1 \oplus P'_n$; hence part (a) is applicable.

Now we come to the 1-fusion of G_1 and the exceptional trees $T7$ and $T12$.

(c) $T \in \{T7, T12\}$. Trivially, every vertex in $V_2 = V(T)$ is included in a path of length 3 in T. So let P_4 be such a path in T containing the vertex $v \in V_1 \cap V_2$. Obviously, T is a 1-fusion of P_4 and a second path P_t, such that $T = P_4 \oplus P_t$ and $V(P_4) \cap V(P_t) = \{v'\}$ for a certain vertex $v' \in V_2$ with $d_T(v') = 3$. Clearly, for $T = T7$ we have $t = 2$ and for $T = T12$ we get $t = 3$.

Part (b) provides that the 1-fusion $G_1 \oplus P_4$ is an edge intersection hypergraph of a 3-uniform hypergraph. In order to obtain the final 1-fusion $G_1 \oplus T$ from $G_1 \oplus P_4$ it suffices to add the path P_t, i.e., $G_1 \oplus T = (G_1 \oplus P_4) \oplus P_t$. In dependence on t, part (a) and part (b), respectively, provides that $G_1 \oplus T$ is an edge intersection hypergraph of a 3-uniform hypergraph.

Note that the assumption for (a) is fulfilled, since $d_{G_1 \oplus P_4}(v') \geq d_T(v') - 1 = 2$, i.e., the vertex v' has at least two neighbours in $G_1 \oplus P_4$. ■

Remark 13. Because in the above proof the condition $d_{G_1}(v) \geq 2$ is needed only for $T = P_2$, this condition in Theorem 12 can be weakened to $d_{G_1}(v) \geq 1$ if we restrict ourselves to trees with at least 3 vertices.

Theorem 14. *Let the graph $G_1 = (V_1, E_1)$ be the edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}_1 = (V_1, \mathcal{E}_1)$, $v \in V_1$ with $d_{G_1}(v) \geq 1$ and $C_n = (V_2, E_2)$ be the cycle of length $n \geq 3$. Moreover, let $V_1 \cap V_2 = \{v\}$ and, for $n = 4$, the number of vertices in G_1 be at least 5. Then the 1-fusion $G_1 \oplus C_n$ is an edge intersection hypergraph of a 3-uniform hypergraph.*

Proof. For $n \geq 5$, Corollary 4 provides that $C_n = (V_2, E_2)$ is an edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}_2 = (V_2, \mathcal{E}_2)$. Therefore, owing to Theorem 11 there is nothing to show in this case.

So we have to investigate only the cycles C_3 and C_4. For this end let $V_2 = \{v_1, v_2, \ldots, v_n\}$, $E_2 = \{\{v_1, v_2\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}\}$ and $v_1 = v \in V_1 \cap V_2$. Moreover, let $u \in V_1$ be a neighbor of v_1 in the graph G_1. As mentioned at the beginning of the proof of Theorem 12, we have $|V_1| \geq 4$ and so we can choose a vertex $x \in V_1 \setminus \{v_1, u\}$.

(a) $n = 3$. We consider $\mathcal{H} = (V_1 \cup \{v_2, v_3\}, \mathcal{E}_1 \cup \{\{v_1, v_2, v_3\}, \{u, v_1, v_2\}, \{u, v_1, v_3\}, \{x, v_2, v_3\}\})$.

In comparison with G_1, the new hyperedges $\{v_1, v_2, v_3\}, \{u, v_1, v_2\}, \{u, v_1, v_3\}, \{x, v_2, v_3\}$ induce no additional edges in $(G_1 \oplus C_3) \setminus \{v_2, v_3\}$. Clearly, $\{u, v_1, v_2\} \cap \{u, v_1, v_3\} = \{u, v_1\}$ is always an edge in G_1, namely $\{u, v_1\} = \{u, v\} \in E_1$. Moreover, the hyperedge $\{v_1, v_2, v_3\}$ and $\{x, v_2, v_3\}$ has only one vertex with G_1 in common, namely the vertex v_1 and x, respectively.

Hence, we obtain $G_1 \oplus C_3 = EI(\mathcal{H})$.

(b) $n = 4$. Now we have $|V_1| \geq 5$ and there are two additional vertices $y, z \in V_1 \setminus \{v_1, u, x\}$ such that v_1, u, x, y, z are pairwise distinct. We use the hypergraph $\mathcal{H} = (V_1 \cup \{v_2, v_3, v_4\}, \mathcal{E}(\mathcal{H}))$ with $\mathcal{E}(\mathcal{H})) = \mathcal{E}_1 \cup \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{u, v_1, v_4\}, \{x, v_2, v_3\}, \{y, v_3, v_4\}, \{z, v_3, v_4\}\}$.

Analogously to the previous case, it can be verified that $G_1 \oplus C_4 = EI(\mathcal{H})$ holds. ∎

5. Cacti

The basic idea is to decompose a given cactus G into cycles and (in a certain sense maximal) trees. After that we begin with a cycle C_n of length $n \geq 5$ and a tree $T \notin \{P_1, P_2, \ldots, P_6, T7, T12\}$, respectively, (which is an edge intersection hypergraph of a 3-uniform hypergraph) and reconstruct the original cactus step by step using Theorem 12 and Theorem 14. For this purpose, in each step we build a 1-fusion of a (connected) subgraph G_i (which is an edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}_i$) of the original cactus and one of the cycles or trees described above.

We begin with the decomposition. First of all, let $G = (V, E)$ be a cactus and $V' = \{v_1, \ldots, v_s\} \subseteq V$ be the set of all vertices having a degree of at least 3 and being contained in a cycle of G. Evidently, the vertices in V' are articulation vertices of G. We refer to these vertices as the *decomposition vertices* or shortly *d-vertices* of G. It is easy to see that $V' = \emptyset$ is equivalent to the case that the cactus G is a cycle or a tree, respectively. In this case Theorem 6 and Corollary 4 include the corresponding characterizations. So in the following assume $V' \neq \emptyset$.

The so-called *tree-cycle-decomposition* of the cactus G will be carried out in two consecutive steps.

Step 1. In each cycle of G, we delete all edges and all vertices of degree 2. Thus we obtain a forest consisting of pairwise vertex-disjoint trees $T^1, T^2, \ldots, T^k$, the so-called *limbs* of G. Clearly, every limb contains at least one d-vertex. If the tree T^j is a single vertex, i.e., $V(T^j) = \{v_j\}$, then v_j is an articulation vertex that connects $\frac{1}{2}d_G(v_j)$ cycles in G. Note that the limbs are the "(in a certain sense maximal) trees" mentioned at the beginning of the section.

Step 2. We start again with the original cactus G and delete all edges $e \in E(T^1), \dots, E(T^k)$. Besides several isolated vertices, this leads to a system of Eulerian graphs which can be uniquely decomposed into a system $C^1, \dots, C^l$ of pairwise edge-disjoint cycles. We denote $C^1, \dots, C^l$ as *the cycles of* G.

The next remark is a collection of some simple, but useful properties of the tree-cycle-decomposition.

Remark 15. (i) The d-vertices $v_1, \dots, v_s$, the limbs $T^1, \dots, T^k$ as well as the cycles $C^1, \dots, C^l$ are uniquely determined.

(ii) The limbs $T^1, \dots, T^k$ are pairwise vertex-disjoint.

(iii) Any d-vertex is contained in at least one of the cycles of G.

(iv) Any d-vertex is contained in at most one of the limbs of G.

(v) If $V' \neq \emptyset$, then every cycle and every limb includes at least one d-vertex.

Now we are ready to formulate our main theorem.

Theorem 16. *Let $G = (V, E)$ be a cactus with*

(a) *circumference $ci(G) \geq 5$ or*

(b) *G contains a limb $T \notin \{P_1, P_2, \dots, P_6, T7, T12\}$.*

Then G is an edge intersection hypergraph of a 3-uniform hypergraph.

Proof. Let $\{G_1, G_2, \dots, G_{k+l}\} = \{C^1, C^2, \dots, C^l, T^1, T^2, \dots, T^k\}$, where $C^1, C^2, \dots, C^l$ and $T^1, T^2, \dots, T^k$ are the cycles and the limbs of G, respectively.

Case (a). $ci(G) \geq 5$. Let the indices of the subgraphs $G_1, \dots, G_{k+l}$ be chosen so that $G_1 = C^1$ is a cycle of a length of at least 5 and, for every $p \in \{1, 2, \dots, k+l-1\}$, the subgraph $G_1 \cup \dots \cup G_p$ of the cactus G has a vertex v_p in common with the subgraph G_{p+1}. Obviously, $\big(V(G_1) \cup \dots \cup V(G_p)\big) \cap V(G_{p+1}) = \{v_p\} \subseteq V'$, where V' is the set of the d-vertices of G.

Trivially, $G = G_1 \cup \dots \cup G_{k+l}$ and, for any $p \in \{1, 2, \dots, k+l-1\}$, from $|V(G_{p+1})| = 1$ (this is the case if and only if G_{p+1} is a trivial tree containing only one vertex, i.e., $V(G_{p+1}) = \{v_p\}$) it follows $(G_1 \cup \dots \cup G_p) \cup G_{p+1} = G_1 \cup \dots \cup G_p$. Owing to our indexing of $G_1, G_2, \dots, G_{k+l}$ this is equivalent to $(\cdots((G_1 \oplus G_2) \oplus G_3) \cdots \oplus G_p) \oplus G_{p+1} = (\cdots((G_1 \oplus G_2) \oplus G_3) \cdots \oplus G_p)$. Note that all the clique-fusions in this expression are 1-fusions and, additionally, all these clique-fusions are connected.

To mention the most trivial case, the 1-fusion of any graph $G' = (V(G'), E(G'))$ with a graph $G'' = (V(G'') = \{v'\}, E(G'') = \emptyset)$, with $v' \in V(G')$, is the original graph G' itself. Therefore, if G' is an edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}$, then also $G' \oplus G'' = EI(\mathcal{H})$ is valid.

According to the assumption, G_1 is an edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}_1$ containing at least 5 vertices. Therefore G_1 fulfills also the assumptions of Theorems 12 and 14, and $G_1 \oplus G_2$ is also an edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}_2$. Since G_1 is a cycle, all of its vertices have degree 2 and it plays no role whether or not G_2 is a cycle or a limb.

With two little additional arguments we can argue in the same manner for arbitrarily chosen $p \in \{2, \ldots, k+l-1\}$ considering $(\cdots((G_1 \oplus G_2) \oplus G_3) \cdots \oplus G_p) \oplus G_{p+1}$.

First, if $(\cdots((G_1 \oplus G_2) \oplus G_3) \cdots \oplus G_p)$ is an edge intersection hypergraph of a 3-uniform hypergraph $\mathcal{H}_p$ then it contains at least 5 vertices (since $|V(G_1)| \geq 5$ holds). This implies that the clique-fusion $(\cdots((G_1 \oplus G_2) \oplus G_3) \cdots \oplus G_p) \oplus G_{p+1}$ also in the case $G_{p+1} = C_4$ is an edge intersection hypergraph of a 3-uniform hypergraph. Clearly, also for $G_{p+1} = C_3$ no problem occurs.

Secondly, in case of $G_{p+1} = P_2$, we have to ensure that the vertex $v_p \in \big(V(G_1) \cup \cdots \cup V(G_p)\big) \cap V(G_{p+1})$ has a degree $d_{G_1 \cup \cdots \cup G_p}(v_p) \geq 2$ (cf. Theorem 12 and Remark 13). This yields from the definition of the d-vertices and the limbs as well as from the construction of our tree-cycle-decomposition of the cactus G in the following way. In case of $G_{p+1} = P_2$ the graph G_{p+1} is a limb. Remark 15(iv) includes that the d-vertex $v_p \in V(G_{p+1})$ cannot be contained in another limb of G. Therefore v_p is included in a cycle of $(\cdots((G_1 \oplus G_2) \oplus G_3) \cdots \oplus G_p) = G_1 \cup \cdots \cup G_p$ and has a degree $d_{G_1 \cup \cdots \cup G_p}(v_p) \geq 2$.

This completes the proof of Case (a).

Case (b). $ci(G) \leq 4$. We use nearly the same argumentation as in Case (a), the only modification is that we have to start with $G_1 = T^1$ instead of $G_1 = C^1$, where $T^1 \notin \{P_1, P_2, \ldots, P_6, T7, T12\}$ is a limb of G. If $G = T^1$ holds then there is nothing to show.

So assume $k+l > 1$ and choose the indices of $G_2, G_3, \ldots, G_{k+l}$ in the same way as in Case (a). Thus the graph $G_1 = T^1$ is again an edge intersection hypergraph of a 3-uniform hypergraph. Moreover, $G_2 = C_n$ $(n \geq 3)$ and it suffices to prove that any fusion $G_1 \oplus G_2 = T^1 \oplus C_n$ is an edge intersection hypergraph of a 3-uniform hypergraph. The rest of the argumentation can be taking over word-for-word from Case (a). So let us consider $T^1 \oplus C_n$.

Since T^1 is an edge intersection hypergraph of a 3-uniform hypergraph, we can apply Theorem 14. The only exception is the case $|V(T^1)| = 4$ and $n = 4$. Obviously, this corresponds to $T^1 = K_{1,3}$ and we have to investigate the two possible 1-fusions of $K_{1,3}$ and C_4.

For this end, let $K_{1,3} = (V_1 = \{v_1, v_2, v_3, v_4\}, E_1)$, $C_4 = (V_2 = \{v_4, v_5, v_6, v_7\}, E_2)$, where $E_2 = \{\{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_7\}, \{v_7, v_4\}\}$, and look at $K_{1,3} \oplus C_4$. In the following, we discuss both 1-fusions.

The first 1-fusion we have to investigate is the situation that v_4 is not the center of the star; so without loss of generality let v_1 be the center, i.e., $E_1 = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}\}$. Then the hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $\mathcal{E} = \{\{v_1, v_2, v_3\}, \{v_1, v_2, v_4\}, \{v_1, v_3, v_4\}, \{v_1, v_4, v_7\}, \{v_1, v_5, v_6\}, \{v_2, v_6, v_7\}, \{v_3, v_6, v_7\}, \{v_4, v_5, v_6\}, \{v_4, v_5, v_7\}\}$ provides the 1-fusion $EI(\mathcal{H}) = K_{1,3} \oplus C_4 = (V, E)$ with $V = V_1 \cup V_2$ and $E = \{\{v_1, v_2\}, \{v_1, v_3\}, \{v_1, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_7\}, \{v_7, v_4\}\}$ (see the first picture in Figure 6).

Now let v_4 be the center of the star. Thus we have $E_1 = \{\{v_1, v_4\}, \{v_2, v_4\}, \{v_3, v_4\}\}$. We consider the hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $\mathcal{E} = \{\{v_1, v_3, v_4\}, \{v_1, v_4, v_7\}, \{v_2, v_3, v_4\}, \{v_2, v_4, v_5\}, \{v_2, v_4, v_7\}, \{v_3, v_6, v_7\}, \{v_4, v_5, v_6\}, \{v_5, v_6, v_7\}\}$. $\mathcal{H}$ has the edge intersection hypergraph $EI(\mathcal{H}) = K_{1,3} \oplus C_4 = (V, E)$ with $V = V_1 \cup V_2$ and $E = \{\{v_1, v_4\}, \{v_2, v_4\}, \{v_3, v_4\}, \{v_4, v_5\}, \{v_5, v_6\}, \{v_6, v_7\}, \{v_7, v_4\}\}$. This is the second possible 1-fusion of $K_{1,3}$ and C_4 (see the second picture in Figure 6). ■

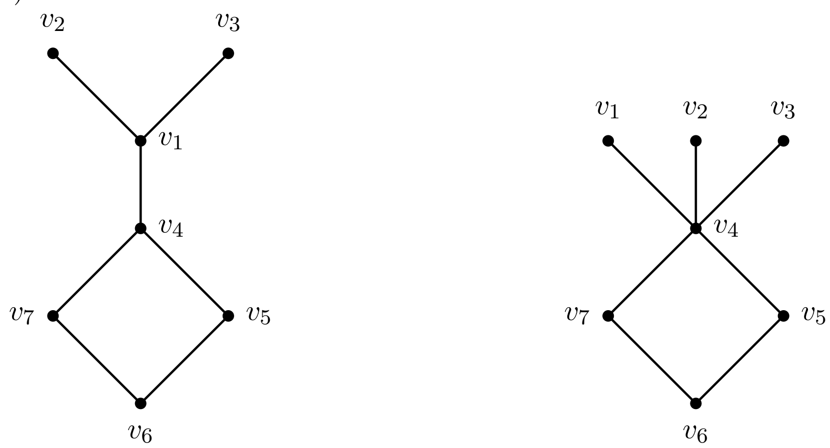


Figure 6. The two 1-fusions of $K_{1,3}$ and C_4.

6. Concluding Remarks

In Corollary 4, Theorem 6 and Theorem 16 we characterized those cycles, trees and cacti which are edge intersection hypergraphs of 3-uniform hypergraphs. In connection with these results several interesting problems occur.

Problem 17. Find more classes of graphs being edge intersection hypergraphs of r-uniform ($r \geq 3$) or non-uniform hypergraphs.

In Section 5 we made use of very special clique-fusions, namely the (iterated) 1-fusion of edge intersection hypergraphs of 3-uniform hypergraphs. Since cacti can be decomposed into limbs and cycles using the d-vertices, the 1-fusion had been proved to be a suitable tool for our investigations on cacti.

Besides cycles, trees and cacti, also other classes of graphs are known to be edge intersection hypergraphs of 3-uniform hypergraphs, e.g. as wheels and complete graphs with at least 5 and 4 vertices, respectively. Having this in mind, the clique-fusion (k-fusion, for $k \geq 1$) may be an appropriate tool to construct other classes of graphs being edge intersection hypergraphs of 3-uniform hypergraphs.

Remember that in Section 5 we restricted ourselves on the *iterated* clique-fusion in the sense, that in each step we applied the 1-fusion only to two graphs (see the proof of Theorem 16 where we dealt with the clique fusion in the form $(\cdots((G_1 \oplus G_2) \oplus G_3) \cdots \oplus G_p) \oplus G_{p+1})$. The usage of the k-fusion in a more common sense, namely as $G_1 \oplus \cdots \oplus G_r$, where $r > 2$ holds (see the original definition at the beginning of Section 4) could be interesting for further investigations. We conjecture that — using the clique-fusion in this common sense — the construction of corresponding classes of edge intersection hypergraphs of 3-uniform hypergraphs may be much more complicated.

An interesting question is to have a look at the number of hyperedges being necessary to generate certain graphs as edge intersection hypergraphs.

Problem 18. Let $\mathcal{G}$ be a class of graphs, $r \geq 3$, $n_0 \in \mathbb{N}^+$, $n \geq n_0$ and $G_n \in \mathcal{G}$ a graph with n vertices. What is the minimum cardinality $|\mathcal{E}|$ of the edge set of an r-uniform hypergraph $\mathcal{H}_n = (V, \mathcal{E})$ with $EI(\mathcal{H}_n) = G_n$?

We conjecture that the solution of Problem 18 becomes difficult for $r > 3$. A breadcrumb for this can be found in [9], where for $n \geq 24$ we prove that there is a 3-regular (and, if n is even, 6-uniform) hypergraph $\mathcal{H} = (V, \mathcal{E})$ with $\lceil \frac{n}{2} \rceil$ hyperedges and $EI(\mathcal{H}) = C_n$.

Another direction for further investigations may be to drop the restriction onto the class of graphs and search for hypergraphs which are edge intersection hypergraphs.

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