LOWER BOUNDARY INDEPENDENT BROADCASTS IN TREES

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Abstract

A broadcast on a connected graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \operatorname{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$ if $|V| \geq 2$, and $f(v) = 1$ if $V = \{v\}$. The cost of f is $\sigma(f) = \sum_{v \in V} f(v)$. Let $V_f^+ = \{v \in V : f(v) > 0\}$. A vertex u hears f from $v \in V_f^+$ if the distance $d(u, v) \leq f(v)$. When f is a broadcast such that every vertex x that hears f from more than one vertex in V_f^+ also satisfies $d(x, u) \geq f(u)$ for all $u \in V_f^+$, we say that the broadcast only overlaps in boundaries. A broadcast f is boundary independent if it overlaps only in boundaries. Denote by $i_{\mathrm{bn}}(G)$ the minimum cost of a maximal boundary independent broadcast.

We obtain a characterization of maximal boundary independent broadcasts, show that $i_{\mathrm{bn}}(T') \leq i_{\mathrm{bn}}(T)$ for any subtree T' of a tree T, and determine an upper bound for $i_{\mathrm{bn}}(T)$ in terms of the broadcast domination number of T. We show that this bound is sharp for an infinite class of trees.

Keywords: broadcast domination, broadcast independence, boundary independence.

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1. INTRODUCTION

If X is an independent set of vertices of a graph G, then no edge of G is incident with, or covered by, more than one vertex in X. Neilson [19] used this property to generalize independent sets to independent broadcasts, calling them boundary independent broadcasts (see Definition 2). Here we continue the investigation of the parameter $i_{\mathrm{bn}}(G)$, the minimum cost of a maximal boundary independent broadcast on G. We obtain conditions for a boundary independent broadcast to be maximal boundary independent, prove that if T' is a subtree of a tree T, then $i_{\mathrm{bn}}(T') \leq i_{\mathrm{bn}}(T)$, and improve Neilson's upper bound for $i_{\mathrm{bn}}(G)$ (Theorem 4) in the case of trees, showing that it is sharp for infinitely many trees.

We begin by giving definitions, background and known results in Section 2. Section 3 contains the characterization of maximal boundary independent broadcasts, as well as a corollary for later reference. We prove the upper bound (Theorem 5) in Section 4.1. A class of trees that demonstrates the sharpness of the bound is given in Theorem 6, which is proved in Section 4.4. This is preceded by preparatory results in Section 4.2 and an informal outline of the proof in Section 4.3. We close by mentioning open problems in Section 5.

We refer the reader to [7] for general graph theory concepts and notation and to [12] for domination related concepts. For a recent survey of broadcasts in graphs, see the chapter by Henning, MacGillivray and Yang [14]. We denote domination number of a graph G by $\gamma(G)$, the independence number by $\alpha(G)$, and the minimum cardinality of a maximal independent set (the *independent domination number* of G) by $i(G)$. If P is a path in G, we denote the length of P by $\ell(P)$; a *diametrical path* of G is a path P such that $\ell(P) = \operatorname{diam}(G)$ (the diameter of G).

2. BROADCAST DEFINITIONS AND KNOWN RESULTS

A *broadcast* on a connected graph $G = (V, E)$ is a function $f : V \to \{0, 1, \ldots, \operatorname{diam}(G)\}$ such that $f(v) \leq e(v)$ (the eccentricity of v) for all $v \in V$ if $|V| \geq 2$, and $f(v) = 1$ if $V = \{v\}$. When G is disconnected, we define a broadcast on G as the union of broadcasts on its components. Let $V_f^+ = \{v \in V : f(v) > 0\}$. A vertex in V_f^+ is called a *broadcasting vertex*. A vertex u *hears* f from $v \in V_f^+$, and v *f-dominates* u, if the distance $d(u, v) \leq f(v)$. If $d(u, v) < f(v)$, we also say that say that v *overdominates* u. A broadcast f is *dominating* if every vertex of G hears f from at least one vertex in V_f^+. The *cost* of f is $\sigma(f) = \sum_{v \in V} f(v)$, and the *broadcast domination number* (or simply the *broadcast number*) of G is $\gamma_b(G) = \min\{\sigma(f) : f \text{ is a dominating broadcast of } G\}$. A dominating broadcast f such that $\sigma(f) = \gamma_b(G)$ is called a *γ_b-broadcast*. Following [17], for a broadcast

f on G and $v \in V_f^+$, we define the

- *f-neighbourhood* of v by $N_f(v) = \{u \in V : d(u,v) \leq f(v)\}$,
- *f-boundary* of v by $B_f(v) = \{u \in V : d(u,v) = f(v)\}$,
- *f-private neighbourhood* of v by $\mathrm{PN}_f(v) = \{u \in N_f(v) : u \notin N_f(w) \text{ for all } w \in V_f^+ \setminus \{v\}\}$,
- *f-private boundary* of v by $\mathrm{PB}_f(v) = \{u \in N_f(v) : u \text{ is not dominated by } (f - \{(v, f(v))\}) \cup \{(v, f(v) - 1)\}\}$.

Note that if $f(u) = 1$ and u does not hear f from any vertex $v \in V_f^+ \setminus \{u\}$, then $u \in \mathrm{PB}_f(u)$ (but $u \notin B_f(u)$), and if $f(u) \geq 2$, then $\mathrm{PB}_f(u) = B_f(u) \cap \mathrm{PN}_f(u)$. If f is a broadcast such that every vertex x that hears more than one broadcasting vertex also satisfies $d(x,u) \geq f(u)$ for all $u \in V_f^+$, then the *broadcast only overlaps in boundaries*. When $xy \in E(G)$ and $x, y \in N_f(u)$ for some $u \in V_f^+$ such that at least one of x and y does not belong to $B_f(u)$, we say that the edge xy is *covered* in f by u. When xy is not covered by any $u \in V_f^+$, we say that xy is *uncovered by f* or *f-uncovered*. We denote the set of f-uncovered edges by U_f^E.

Erwin [10, 11] was the first to consider the broadcast domination problem and noted that $\gamma_b(G) \leq \min\{\mathrm{rad}(G), \gamma(G)\}$. A natural question follows from this bound: which graphs G satisfy $\gamma_b(G) = \mathrm{rad}(G)$? Such a graph G is called *radial*; the problem of characterizing radial trees was addressed by Dunbar, Erwin, Haynes, Hedetniemi and Hedetniemi [8] and Dunbar, Hedetniemi and Hedetniemi [9], and solved completely by Herke and Mynhardt [16]. The characterization in [16] involves the concept of a split-set, which is important for the present paper as well. Definition 1 is illustrated in Figure 1.
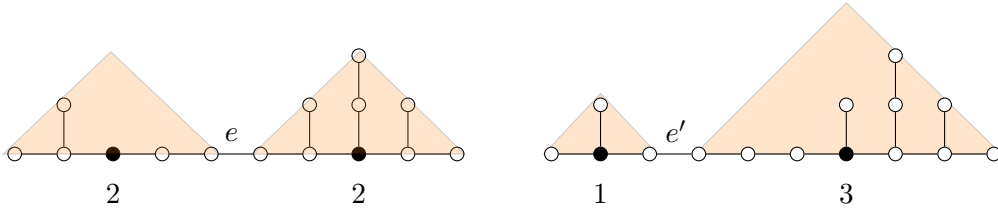

Figure 1. A tree with two maximum split-sets $\{e\}$ and $\{e'\}$; a central vertex and the radius of each subtree are shown.

Definition 1. Let T be a tree with diametrical path P, and $M \subseteq E(P)$ a set of cardinality $m \geq 1$. Let $T_1, \ldots, T_{m+1}$ be the components of $T - M$, and P_i the subpath of P in T_i.

- We say that M is a *split-set* of T if each P_i has even, positive length and is a diametrical path of T_i.
- An edge of a split-set is called a *split-edge* of T.

- A split-set is a *maximum split-set* if it is a split-set of T of maximum cardinality.

- When M is a maximum split-set of T the components of $T - M$ are called the *radial subtrees* of T.

As shown in [16], the cardinality of a maximum split-set is independent of the chosen diametrical path of the tree. Herke and Mynhardt [16] used Definition 1 to characterize radial trees and determine the broadcast number of a given tree.

Theorem 1 [16]**.**
(i) *A tree T is radial if and only if it does not possess a split-set.*
(ii) *If T is a radial, then $\gamma_b(T) = \left\lceil \frac{1}{2}\mathrm{diam}(T) \right\rceil = \mathrm{rad}(T)$. If T is nonradial and M is a maximum split-set of T, then $\gamma_b(T) = \frac{1}{2}(\mathrm{diam}(T) - |M|)$.*

Dunbar, Hedetniemi and Hedetniemi [9] showed that the broadcast number of a tree is bounded below by that of any of its subtrees.

Proposition 2 [9]**.** *If T' is a subtree of a tree T, then $\gamma_b(T') \leq \gamma_b(T)$.*

Erwin [10] defined a broadcast f to be *independent* if no vertex $u \in V_f^+$ hears f from any other vertex $v \in V_f^+$; that is, broadcasting vertices only hear themselves. This version of broadcast independence, which we call *hearing independence*, was also considered by, among others, Ahmane, Bouchemakh and Sopena [1], Bessy and Rautenbach [2, 3], Bouchemakh and Zemir [4], and Dunbar *et al.* [8]. Consider the tree obtained from the star $K_{1,r}$, $r \geq 3$, by subdividing each edge once. Broadcasting from each leaf with a strength of 3 gives a hearing independent broadcast of cost $3r$. Each non-pendant vertex and edge hear the broadcast from r vertices. Suppose, instead, that we want as many edges as possible to hear (or be covered by) the broadcast, but without signals overlapping on edges because such interference is undesirable. To accomplish this, Mynhardt and Neilson [18] and Neilson [19] pointed out that another way to generalize independent sets to independent broadcasts is by ensuring that each edge belongs to the f-neighbourhood of at most one broadcasting vertex; using this approach they defined *boundary independent broadcasts* as an alternative to hearing independent broadcasts.

Definition 2 [18, 19]**.** Let f be a broadcast on a graph G. We say that
- f is *boundary independent*, abbreviated *bn-independent* if it overlaps only in boundaries, and
- *maximal bn-independent* if, in addition, there is no bn-independent broadcast g on G such that $g > f$.
- We denote the minimum weight of a maximal bn-independent broadcast f on G by $i_{\mathrm{bn}}(G)$, and call f an i_{bn}-*broadcast*.

Erwin similarly denoted the minimum weight of a maximal hearing independent broadcast f on G by $i_b(G)$. Neilson [19] and Mynhardt and Neilson [18] characterized bn-independent broadcasts that are maximal bn-independent.

Proposition 3 [18, 19]. *A bn-independent broadcast f is maximal bn-independent if and only if it is dominating, and either $V_f^+ = \{v\}$ or $B_f(v) \setminus \mathrm{PB}_f(v) \neq \emptyset$ for each $v \in V_f^+$.*
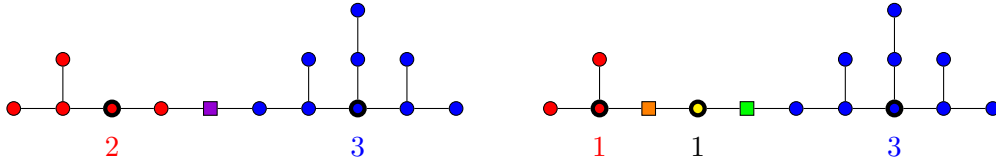

Figure 2. Two maximal bn-independent broadcasts f and g on the tree in Figure 1.

Two maximal bn-independent broadcasts f and g on the tree in Figure 1 are shown in Figure 2. The broadcasting vertices are shown in bold-outlined primary colours, the shared boundaries as squares in the corresponding secondary colours, and the rest of the f- and g-neighbourhoods in the same primary colours as the broadcasting vertices.

Proposition 3 implies that $\gamma_b(G) \leq i_{\mathrm{bn}}(G)$ for every graph G. Moreover, broadcasting with a strength of $\mathrm{rad}(G)$ from a central vertex of G produces a maximal bn-independent broadcast, hence $i_{\mathrm{bn}}(G) \leq \mathrm{rad}(G)$ for all graphs G. In particular, if G is radial, then $\gamma_b(G) = i_{\mathrm{bn}}(G)$. On the other hand, $i_{\mathrm{bn}}(G)$ is not comparable with the independent domination number $i(G)$. For example, $i(P_6) = 2$ and $i_{\mathrm{bn}}(P_6) = 3$ [19, Theorem 3.1.4], whereas $i(K_{3,3}) = 3$ and $i_{\mathrm{bn}}(K_{3,3}) = \mathrm{rad}(K_{3,3}) = 2$.

Neilson [19] bounded $i_{\mathrm{bn}}(G)$ in terms of $\gamma_b(G)$ as follows.

Theorem 4 [19, Corollary 2.2.10]. *For any graph G, $i_{\mathrm{bn}}(G) \leq \left\lceil \frac{4\gamma_b(G)}{3} \right\rceil$.*

Our goal is to improve the bound in Theorem 4 for trees as follows, and to show that the bound is achieved by certain trees whose maximum split-sets have cardinality 1 or 2.

Theorem 5. *Let T be a non-radial tree having a maximum split-set M. Then $i_{\mathrm{bn}}(T) \leq \gamma_b(T) + \left\lceil \frac{|M|+1}{3} \right\rceil$.*

Theorem 6. *Let T be a tree whose maximum split-set(s) have cardinality $m \in \{1,2\}$. Suppose that for each maximum split-set M, every component of $T - M$ is uniquely radial. Then $i_{\mathrm{bn}}(T) = \gamma_b(T) + 1$.*

3.   Maximal bn-Independent Broadcasts

Before turning our attention to trees, we provide a characterization, different from the one in Proposition 3, of maximal bn-independent broadcasts having at least two broadcasting vertices. This characterization is of general interest and it and its corollary will be used frequently in the proofs that follow.

Proposition 7. *Let f be a bn-independent broadcast on a connected graph G such that $|V_f^+| \geq 2$. Then f is maximal bn-independent if and only if each component of $G - U_f^E$ contains at least two broadcasting vertices.*

Proof. Assume that f is maximal bn-independent. By Proposition 3, f is dominating. Suppose that some component H of $G - U_f^E$ contains fewer than two broadcasting vertices. Since f is dominating, $|V_f^+ \cap V(H)| = 1$; say $V_f^+ \cap V(H) = \{v\}$. Since G is connected, v is not isolated in G, and, since v is broadcasting, all edges incident with v also belong to H. Therefore $|V(H)| \geq 2$. Since H is a component of $G - U_f^E$, $B_f(v) \neq \emptyset$ and $B_f(v) \subseteq \mathrm{PB}_f(v)$. But then $B_f(v) \setminus \mathrm{PB}_f(v) = \emptyset$, contradicting Proposition 3.

Conversely, assume that each component of $G - U_f^E$ contains at least two broadcasting vertices. Since $G - U_f^E$ is a spanning subgraph of G, f is dominating. Let v be any vertex of G. First assume that $f(v) = 0$. Since f is dominating, it is clear that increasing the strength of the broadcast from v results in a broadcast that is not bn-independent. Assume therefore that $v \in V_f^+$ and say v belongs to the component H of $G - U_f^E$. Let h be the restriction of f to H; clearly, h is bn-independent. Since h covers all edges of H and $|V_h^+| \geq 2$, there exists a vertex $u \in V_h^+ \setminus \{v\}$ such that some vertex w of H hears the broadcast from u as well as from v. Since h is bn-independent, $w \in B_h(u) \cap B_h(v)$. Thus $w \notin \mathrm{PB}_h(v)$, so $B_h(v) \setminus \mathrm{PB}_h(v) \neq \emptyset$. This implies that $B_f(v) \setminus \mathrm{PB}_f(v) \neq \emptyset$. The result follows from Proposition 3. ∎

Corollary 8. *Let f be a maximal bn-independent broadcast on a connected graph G with $|V_f^+| \geq 2$ and let $X \subseteq U_f^E$. For each component H of $G - X$, denote the restriction of f to H by f_H. Then f_H is a maximal bn-independent broadcast on H.*

Proof. By Proposition 7, each component of $G - U_f^E$ has at least two broadcasting vertices. Since $X \subseteq U_f^E$, each component of $G - X$, including H, therefore has at least two broadcasting vertices.

Let $e = xy$ be an f_H-uncovered edge of H. Then e is also an f-uncovered edge of G, otherwise there exists a vertex $v \in V_f^+ \setminus V(H)$ that broadcasts to x and y. But then $G - U_f^E$ contains a path from v to x, hence so does $G - X$. Since $x \in V(H)$ and $v \notin V(H)$, this contradicts H being a component of $G - X$.

Therefore each f_H-uncovered edge of H is an f-uncovered edge of G. It follows that each component of $H - U_{f_H}^E$ is a component of $G - U_f^E$, so each component of $H - U_{f_H}^E$ has at least two broadcasting vertices. By Proposition 7, f_H is maximal bn-independent. ∎

4. Bn-Independent Broadcasts on Trees

Our focus in this section is the proofs of Theorem 5 in Section 4.1 and Theorem 6 in Section 4.4. The latter proof requires substantial preparation, which is given in Section 4.2. Some of these results, such as Lemma 9 and Theorem 10, are also of wider interest.

First we need a few more definitions. It is well known that the centre of every connected graph G lies in a single block of G — see e.g. [7, Theorem 2.9]. Therefore, the centre of a tree T consists of either a single vertex, in which case T is called a *central* tree, or of two adjacent vertices, in which case T is a *bicentral* tree. Two vertices u and v of a connected graph G with $d(u, v) = \operatorname{diam}(G)$ are called *antipodal* vertices.

4.1. Proof of Theorem 5

We prove Theorem 5 using the same approach as Neilson [19, Proposition 2.2.9], as well as the well-known fact that $i(P_n) = \lceil n/3 \rceil$ [12, Theorem 6.1]. We restate the theorem for convenience.

Theorem 5. *Let T be a non-radial tree having a maximum split-set M. Then $i_{\mathrm{bn}}(T) \le \gamma_b(T) + \left\lceil \frac{|M|+1}{3} \right\rceil$.*

Proof. Say $m = |M|$ and consider the $m + 1$ radial subtrees $T_1, \ldots, T_{m+1}$ of $T - M$. By Definition 1, each T_i has even diameter, hence is a central tree. Let v_i be the unique central vertex of T_i. We know that $\gamma_b(T_i) = \operatorname{rad}(T_i)$ for each i. Theorem 1(ii) implies that $\gamma_b(T) = \sum_{i=1}^{m+1} \operatorname{rad}(T_i)$. Hence the broadcast f on T defined by $f(v_i) = \operatorname{rad}(T_i)$ for $i = 1, \ldots, m + 1$ and $f(v) = 0$ otherwise is a bn-independent γ_b-broadcast on T. Observe that $M = U_f^E$ and that each component of $T - U_f^E$ contains exactly one broadcasting vertex. By Proposition 7, f is not maximal bn-independent.

Let $P \cong P_{m+1}$ be the path whose vertices and edges are the trees T_i and the edges in M, respectively. Since $i(P) = \lceil (m + 1)/3 \rceil$, there exists an independent dominating set X of P with $|X| = \lceil (m + 1)/3 \rceil$. Let $\mathcal{X}$ denote the collection of trees T_i that correspond to a vertex in X, and $\mathcal{Y}$ the rest of the trees T_i. Define the broadcast g on T by $g(v_i) = f(v_i) + 1$ if $T_i \in \mathcal{X}$, and $g(v) = f(v)$ otherwise. Then $\sigma(g) = \sigma(f) + \lceil (m + 1)/3 \rceil$.

The independence of X ensures that each edge in M (and hence each edge of T) is covered by at most one vertex in V_g^+, while the fact that X is dominating ensures that each $T_j \in \mathcal{Y}$ belongs to the same component of $T - U_g$ as some $T_i \in \mathcal{X}$. Conversely, since f is a dominating broadcast, each $T_i \in \mathcal{X}$ belongs to the same component of $T - U_g$ as some $T_j \in \mathcal{Y}$. Hence each component of $T - U_g$ contains at least two vertices belonging to V_g^+. By Proposition 7, g is maximal bn-independent. Therefore $i_{\mathrm{bn}}(T) \leq \sigma(g) = \gamma_b(T) + \lceil (m+1)/3 \rceil$. ∎

Observe that the bound in Theorem 5 is better than the one in Theorem 4 for all values of $|M|$ when $\gamma_b(T) \geq |M| + 3$, that is, when $\sum_{i=1}^{|M|+1} \mathrm{rad}(T_i) \geq |M| + 3$. For example, it is shown below that for the tree T_1 in Figure 3, $i_{\mathrm{bn}}(T_1) = \gamma_b(T_1) = 6$. The bound in Theorem 4 gives $i_{\mathrm{bn}}(T_1) \leq 8$ whereas the bound in Theorem 5 gives $i_{\mathrm{bn}}(T_1) \leq 7$.

As examples of trees whose radial subtrees are not uniquely radial, consider the trees in Figure 3. The tree T_1 has a unique maximum split-set $\{e, e'\}$. Each of its radial subtrees has a dominating broadcast of cost 2 from a central vertex, as well as the broadcast shown. The uncovered edges are shown in cyan. By Proposition 7, the broadcast on T_1 is maximal bn-independent. Hence $i_{\mathrm{bn}}(T_1) = \gamma_b(T_1) = 6$. The tree T_2 has a unique maximum split-set $\{e\}$. By Proposition 7, the broadcast on T_2 is not maximal bn-independent. It is easy to check that the two broadcasts shown are the only possible γ_b-broadcasts on the (isomorphic) radial subtrees of T_2, and no combination of them produces a maximal bn-independent broadcast on T_2. By Theorem 5, $i_{\mathrm{bn}}(T_2) = \gamma_b(T_2) + 1 = 7$.
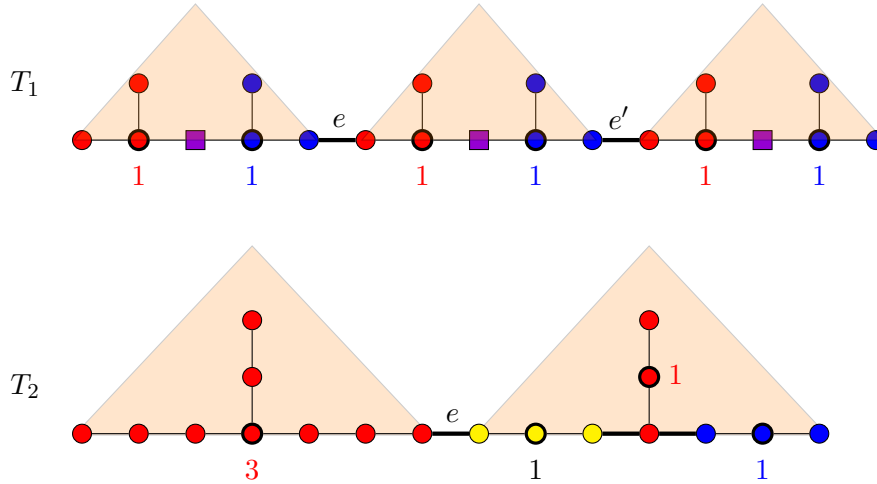

Figure 3. Trees T_1 and T_2 whose radial subtrees are not uniquely radial, with a γ_b- and i_{bn}-broadcast on T_1, and a γ_b-broadcast on T_2 that is not maximal bn-independent.

4.2.   Preparatory results for the proof of Theorem 6

We begin our preparation for the proof of Theorem 6 with a lemma that gives a lower bound on the cost of a broadcast f on a tree T in terms of the number of edges of a path P that are covered by f. Each vertex $v \in V_f^+$ covers at most $2f(v)$ edges on P. Therefore, if f covers q edges on P, then $\sigma(f) \geq \lceil \frac{q}{2} \rceil$. In fact, a stronger result holds, as we show in Lemma 9 below.

For any vertex v of T, let $d(v, P)$ denote the distance from v to the vertex on P nearest to v; note that $d(v, P) = 0$ if and only if $v \in V(P)$. With respect to P, the set V_f^+ of broadcasting vertices of f are of three types: those that lie on P, those that do not lie on P but broadcast to at least one vertex of P, and those that do not broadcast to any vertex of P at all. Denote the first and second types by $\mathrm{Touch}(P)$, that is, $\mathrm{Touch}(P) = \{v \in V_f^+ : N_f(v) \cap V(P) \neq \emptyset\}$, and the third type by $\mathrm{Off}(P)$, that is, $\mathrm{Off}(P) = \{v \in V_f^+ : N_f(v) \cap V(P) = \emptyset\}$. The next lemma holds for any broadcast on a tree, not only bn-independent broadcasts.

Lemma 9. *Let P be a path of a tree T and let f be a broadcast on T. Suppose*

$$\sigma_1 = \sum_{v \in \mathrm{Touch}(P)} d(v, P) \text{ and } \sigma_2 = \sum_{v \in \mathrm{Off}(P)} f(v).$$

Then

(i) *f covers at most $2\big(\sum_{v \in \mathrm{Touch}(P)} f(v) - \sigma_1\big)$ edges of P;*

(ii) *if f covers b edges of P, then $\sigma(f) \geq \lceil \frac{b}{2} \rceil + \sigma_1 + \sigma_2$.*

Proof. Consider any vertex $v \in \mathrm{Touch}(P)$ and let u be the unique vertex of P at distance $d(v, P)$ from v — note that $u = v$ when $v \in V(P)$. Since u hears the broadcast from v, $f(v)$ covers the same number of edges of P as a broadcast with strength $f(v) - d(v, P)$ from u does, namely at most $2(f(v) - d(v, P))$. Summing over $\mathrm{Touch}(P)$, we obtain (i). If f covers b edges of P, then (i) implies that $b \leq 2\big(\sum_{v \in \mathrm{Touch}(P)} f(v) - \sigma_1\big)$, from which it follows that

$$\sum_{v \in \mathrm{Touch}(P)} f(v) \geq \left\lceil \frac{b}{2} \right\rceil + \sigma_1.$$

Since the vertices in $\mathrm{Off}(P)$ do not broadcast to P at all, (ii) follows.   ∎

We next prove a result similar to Proposition 2 for bn-independence in subtrees of a tree.

Theorem 10. *If T' is a subtree of a tree T, then $i_{\mathrm{bn}}(T') \leq i_{\mathrm{bn}}(T)$.*

Proof. Let T be a tree and T' a subtree of T of order $|V(T)| - k$. Consider an i_{bn}-broadcast f on T. First suppose that f covers all edges of T and let P be a diametrical path of T. By Lemma 9(ii),

$$\text{(1)} \qquad \sigma(f) \geq \left\lceil \frac{1}{2}\ell(P) \right\rceil = \mathrm{rad}(T).$$

Certainly, $\mathrm{rad}(T') \leq \mathrm{rad}(T)$ and $i_{\mathrm{bn}}(T') \leq \mathrm{rad}(T')$, therefore, by (1),

$$i_{\mathrm{bn}}(T') \leq \mathrm{rad}(T') \leq \mathrm{rad}(T) \leq \sigma(f) = i_{\mathrm{bn}}(T).$$

Assume therefore that $U_f^E \neq \emptyset$. Let $\mathcal{C}$ be the collection of components of $T - U_f^E$, that is, $\mathcal{C}$ is the collection of subtrees of T induced by the edges covered by f. We successively delete vertices $v_1, \ldots, v_k$ from T until only T' remains. At each step the vertex that we delete is a leaf of the subtree of T obtained in the previous step, hence each resulting graph is a tree. After each step we adjust the broadcast as necessary so that the resulting broadcast is maximal bn-independent and has cost no greater than the cost of f. We do this using a recursive algorithm.

Let $T_0 = T$, $f = f_0$ and $\mathcal{C}_0 = \mathcal{C}$. For $i \geq 0$, assume that we have constructed, in Step i, a tree T_i that contains T' as subtree, obtained by deleting the vertices $v_1, \ldots, v_i$, and a maximal bn-independent broadcast f_i on T_i such that $\sigma(f_i) \leq \sigma(f)$. Denote by $\mathcal{C}_i$ the collection of subtrees of T_i induced by the edges covered by f_i.

Step $i + 1$. Let v_i be a leaf of $T_i - T'$ and u_i the stem adjacent to v_i. Let H_i be the subtree in $\mathcal{C}_i$ that contains v_i. Define $T_{i+1} = T_i - v_i$ and $F_{i+1} = H_i - v_i$. We consider four cases, depending on v_i.

Case 1. If $f_i(v_i) = 0$, let f_{i+1} be the restriction of f_i to T_{i+1}. Then f_{i+1} is a bn-independent and dominating broadcast, and $\sigma(f_{i+1}) = \sigma(f_i)$. The subtree H_i in $\mathcal{C}_i$ containing v_i becomes the subtree F_{i+1} in $\mathcal{C}_{i+1}$, which has the same number of broadcasting vertices as H_i, while all other subtrees in $\mathcal{C}_i$ remain unchanged in $\mathcal{C}_{i+1}$. Since f_i is maximal independent, f_{i+1} is maximal bn-independent, by Proposition 7.

Case 2. Suppose $f_i(v_i) \geq 2$. Since f_i is bn-independent, $f_i(u_i) = 0$. Define the broadcast f_{i+1} on T_{i+1} by $f_{i+1}(u_i) = f_i(v_i) - 1$ and $f_{i+1}(v) = f_i(v)$ for $v \in V(T_{i+1}) \setminus \{u_i\}$. Then $\sigma(f_{i+1}) < \sigma(f_i)$ and the edges of T_{i+1} covered by f_{i+1} are exactly the same as the edges of T_{i+1} covered by f_i. Hence f_{i+1} is bn-independent and dominating. The subtree F_{i+1} in $\mathcal{C}_{i+1}$ has the same number of broadcasting vertices as H_i, while all other subtrees remain unchanged. By Proposition 7, f_{i+1} is maximal bn-independent.

Case 3. Suppose $f_i(v_i) = 1$ and u_i hears f_i from a vertex $w_i \neq v_i$. Then $w_i \in V(H_i)$. Let f'_{i+1} be the restriction of f_i to T_{i+1}. Now $\sigma(f'_{i+1}) = \sigma(f_i) - 1$

and f'_{i+1} covers the same edges of T_{i+1} as f_i does, so f'_{i+1} is bn-independent and dominating. However, there are fewer broadcasting vertices in the component F_{i+1} than in H_i.

If F_{i+1} has at least two broadcasting vertices, we let $f_{i+1} = f'_{i+1}$ and proceed as in Case 2. Assume therefore that w_i is the only broadcasting vertex in F_{i+1}. If $F_{i+1} = T'$, then f_{i+1} is maximal bn-independent and we are also done. We therefore assume further that H_i is not the only component in $\mathcal{C}_i$.

By definition of F_{i+1} and the subtrees in $\mathcal{C}_i$, every vertex in the f'_{i+1}-boundary of w_i also belongs to its private f'_{i+1}-boundary. Therefore, the broadcast f_{i+1} on T_{i+1} defined by $f_{i+1}(w_i) = f_i(w_i) + 1$ and $f_{i+1}(v) = f'_{i+1}(v)$ otherwise, is bn-independent, and $\sigma(f_{i+1}) = \sigma(f'_{i+1}) + 1 = \sigma(f_i)$. Moreover, since f'_{i+1} is dominating and T_{i+1} is connected, there is at least one subtree $G_i \in \mathcal{C}_i \setminus \{H_i\}$ which contains a vertex that hears the broadcast f_{i+1} from w_i. Hence F_{i+1} and G_i belong to the same component in $\mathcal{C}_{i+1}$. Consequently, all subtrees of T_{i+1} in $\mathcal{C}_{i+1}$ contain at least two broadcasting vertices. By Proposition 7, f_{i+1} is maximal bn-independent.

Case 4. Suppose $f_i(v_i) = 1$ and u_i belongs to the private f_i-boundary of v_i. Then f_i does not cover any edges incident with u_i other than $v_i u_i$. Define f_{i+1} by $f_{i+1}(u_i) = 1$ and $f_{i+1}(v) = f_i(v)$ if $v \neq u_i$. Then $\sigma(f_{i+1}) = \sigma(f_i)$. We now proceed as in Case 3 to show that f_{i+1} is a maximal bn-independent broadcast on T_{i+1}.

It follows that the broadcast f_k constructed in Step k is a maximal independent broadcast on $T_k = T'$ such that $\sigma(f_k) \leq \sigma(f) = i_{\mathrm{bn}}(T)$. Consequently, $i_{\mathrm{bn}}(T') \leq \sigma(f_k) \leq i_{\mathrm{bn}}(T)$. ∎

We need another result before we can prove Theorem 6. We first prove this result for a generalized spider (definition below) and deduce the required, more general, result by using Theorem 10. For $k \geq 3$ and $n_i \geq 1$, $i = 1, \ldots, k$, the *generalized spider* $S = S(n_1, n_2, \ldots, n_k)$ is the tree which has exactly one vertex b with $\deg(b) = k$, called the *head* of S, and for which the k components of $S - b$ are paths of length $n_1 - 1, \ldots, n_k - 1$, respectively. The paths from b to the end-vertices of S are the *legs* of S. The following result by Neilson [19] is required in the proof of Lemma 12.

Theorem 11 [19, Theorem 3.1.4]. *For any path P_n such that $n \neq 3$, $i_{\mathrm{bn}}(P_n) = \left\lceil \frac{2n}{5} \right\rceil$.*

Lemma 12. *For integers $r \geq 2$ and $s \geq 1$, consider the generalized spider $S = S(r - 1, r, s)$ with head b and legs $X = (b, x_1, \ldots, x_{r-1})$, $Y = (b, y_1, \ldots, y_r)$ and $Z = (b, z_1, \ldots, z_s)$. Suppose there exists an i_{bn}-broadcast f on S that covers all edges on Z.*

(i) *Then $\sigma(f) = i_{\mathrm{bn}}(S) \geq \left\lceil \frac{1}{2}(r + s) \right\rceil$.*

(ii) *If there are f-uncovered edges on X, then $\sigma(f) = i_{\mathrm{bn}}(S) \geq \left\lceil \frac{1}{2}(r+s) \right\rceil + 1$.*

Proof. Let P and Q be the paths $X \cup Z$ and $Y \cup Z$, respectively. First suppose that f covers all edges on P and hence also on X. The bound follows immediately from Lemma 9(ii) if f overdominates x_{r-1} or z_s. If f overdominates neither x_{r-1} nor z_s, then no broadcasting vertex on P broadcasts to y_r. Hence $V_f^+ \setminus V(P) \neq \emptyset$, and again the result follows from Lemma 9(ii), in which we now have that $\sigma_1 + \sigma_2 \geq 1$.

Suppose next that f covers all edges on Q. Then $\sigma(f) \geq \left\lceil \frac{1}{2}\ell(Q) \right\rceil = \left\lceil \frac{1}{2}(r+s) \right\rceil$. If there is an uncovered edge, say $x_t x_{t+1}$, on X, then $\{x_{t+2}, \ldots, x_{r-1}\}$ contains a vertex in V_f^+; hence $\sum_{v \in \mathrm{Off}(Q)} f(v) \geq 1$. By Lemma 9(ii), $\sigma(f) \geq \left\lceil \frac{1}{2}\ell(Q) \right\rceil + 1 = \left\lceil \frac{1}{2}(r+s) \right\rceil + 1$, as required.
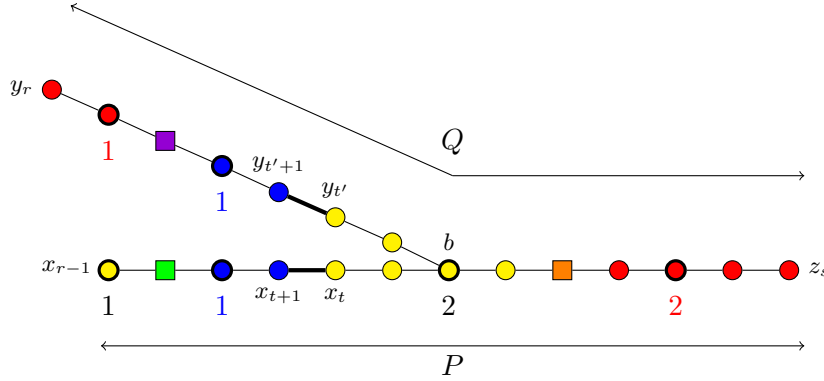

Figure 4. The spider $S(6,6,7)$ with paths P and Q, and uncovered edges $x_t x_{t+1}$ and $y_{t'} y_{t'+1}$.

Assume therefore that both P and Q have f-uncovered edges (see Figure 4). Since f covers all edges on Z, both X and Y have f-uncovered edges. Let t and t', respectively, be the smallest indices such that $x_t x_{t+1}$ and $y_{t'} y_{t'+1}$ are uncovered, and consider the paths

$$X' = (x_{t+1}, \ldots, x_{r-1}),$$
$$Y' = (y_{t'+1}, \ldots, y_r),$$
$$R = (x_t, \ldots, b, z_1, \ldots, z_s), \quad \text{and}$$
$$R' = (y_{t'}, \ldots, b, z_1, \ldots, z_s).$$

Then $|V(X')| = r - t - 1$ and $|V(Y')| = r - t'$, while $\ell(R) = s + t$ and $\ell(R') = s + t'$. Let S' be the subtree of $S - \{x_t x_{t+1}, y_{t'} y_{t'+1}\}$ that contains b and hence also both paths R and R'. For each $A \in \{X', Y', S'\}$, denote the restriction of f to A by f_A. By Corollary 8, f_A is a maximal bn-independent broadcast on A,

hence $\sigma(f_A) \geq i_{\mathrm{bn}}(A)$. By definition, $f_{S'}$ covers all edges on R and R'. Applying Lemma 9(ii) in turn to R and R', we obtain that

$$(2) \qquad \sigma(f_{S'}) \geq \max\left\{\left\lceil\frac{1}{2}\ell(R)\right\rceil, \left\lceil\frac{1}{2}\ell(R')\right\rceil\right\} = \max\left\{\left\lceil\frac{s+t}{2}\right\rceil, \left\lceil\frac{s+t'}{2}\right\rceil\right\}.$$

Theorem 11 gives

$$(3) \quad \sigma(f_{X'}) \geq i_{\mathrm{bn}}(X') = \left\lceil\frac{2(r-t-1)}{5}\right\rceil \text{ and } \sigma(f_{Y'}) \geq i_{\mathrm{bn}}(Y') = \left\lceil\frac{2(r-t')}{5}\right\rceil.$$

By Proposition 7, each component of $T - U_f^E$ contains at least two broadcasting vertices. Hence $r - t - 1 \geq 4$ and $r - t' \geq 4$. This implies that $\left\lceil\frac{4}{5}(r-t)\right\rceil > \left\lceil\frac{1}{2}(r-t)\right\rceil$ and $\left\lceil\frac{4}{5}(r-t')\right\rceil > \left\lceil\frac{1}{2}(r-t')\right\rceil$. Now, if $t \geq t'$, then $r - t' \geq r - t$ and $s + t \geq s + t'$. Substitution in (2) and (3) gives

$$\sigma(f) = \sigma(f_{S'}) + \sigma(f_{X'}) + \sigma(f_{Y'}) \geq \left\lceil\frac{s+t}{2}\right\rceil + \left\lceil\frac{4(r-t)}{5}\right\rceil$$
$$> \left\lceil\frac{s+t}{2}\right\rceil + \left\lceil\frac{r-t}{2}\right\rceil \geq \left\lceil\frac{1}{2}(r+s)\right\rceil.$$

On the other hand, if $t < t'$, then $s + t' > s + t$ and $r - t - 1 \geq r - t'$, and, from (2) and (3),

$$\sigma(f) > \left\lceil\frac{s+t'}{2}\right\rceil + \left\lceil\frac{4(r-t')}{5}\right\rceil \geq \left\lceil\frac{1}{2}(r+s)\right\rceil.$$

In both cases we see that $i_{\mathrm{bn}}(S) = \sigma(f) \geq \left\lceil\frac{1}{2}(r+s)\right\rceil + 1$, as required. ∎

Corollary 13. *Let $Q = (u_0, \ldots, u_s, \ldots, u_d)$ and $P = (u_0, \ldots, u_s, w_1, \ldots, w_k)$ be paths of a tree T such that u_0 and w_k are leaves of T, $\{u_{s+1}, \ldots, u_d\} \cap \{w_1, \ldots, w_k\} = \emptyset$ and $\ell(P) < \ell(Q)$. Let P' and P'' be, respectively, the $u_0 - u_s$ and $u_s - w_k$ subpaths of P. Suppose there exists an i_{bn}-broadcast f on T that covers all edges on P'.*

(i) *Then $\sigma(f) = i_{\mathrm{bn}}(T) \geq \left\lceil\frac{k+s+1}{2}\right\rceil > \frac{1}{2}\ell(P)$.*

(ii) *If some edges on P'' are f-uncovered, then $\sigma(f) = i_{\mathrm{bn}}(T) \geq \left\lceil\frac{s+k+1}{2}\right\rceil + 1 > \frac{1}{2}\ell(P) + 1$.*

Proof. Note that T contains the spider $S = S(k, k+1, s)$ as subtree. Hence, by Theorem 10, $i_{\mathrm{bn}}(T) \geq i_{\mathrm{bn}}(S)$. Since all conditions of Lemma 12 are satisfied for S with respect to its (maximal) path of length $k + s$, the result follows from Lemma 12. ∎

4.3.   Outline of proof of Theorem 6

Let T be a non-radial tree and P a diametrical path of T. We may think of T as drawn with P on a horizontal line so that we can refer to a vertex or edge of P as lying to the left (or right) of another vertex or edge. Assume that the maximum split-set(s) of T has (have) cardinality m, where either $m = 1$ or $m = 2$. Note that when $m = 1$, Definition 1 implies that T has odd diameter, whereas T has even diameter when $m = 2$; that is, $\mathrm{diam}(T) \equiv m \pmod 2$.

In the proof of Theorem 6 we consider three types of vertex partitions of T into subtrees. First, for a maximum split-set M of T, there is the associated partition of T into $m + 1$ uniquely radial subtrees. Then, in order to obtain a contradiction, we assume that $i_{\mathrm{bn}}(T) = \gamma_b(T)$ instead of $\gamma_b(T) + 1$, and consider a maximal bn-independent broadcast f of cost $\sigma(f) = i_{\mathrm{bn}}(T) = \gamma_b(T)$. We show that f leaves some edges $e_1, \ldots, e_k$ of P uncovered, and consider the subtrees $T_1, \ldots, T_{k+1}$ of T obtained by deleting $e_1, \ldots, e_k$, as well as the restrictions f_i of f to T_i. Then we consider a succession of partitions of T into two subtrees — the trees L_i to the left of e_i, and R_i to the right, for $i = 1, \ldots, k$, together with the restrictions g_i and h_i of f to L_i and R_i, respectively. As we proceed from left to right, we iteratively determine $\sigma(g_i)$. Our aim is to construct a strictly increasing sequence of indices $i_1, i_2, \ldots$ of infinite length such that $\sigma(g_{i_j}) \geq \mathrm{diam}(L_{i_j})$, thus obtaining a contradiction. To this end we also consider a variety of subtrees of L_i and R_i.

We state and prove a number of claims within the proof; the end of the proof of each claim is indicated by an open square ($\square$). When we consider different cases, the end of the proof of each case is indicated by a solid diamond ($\blacklozenge$).

4.4.   Proof of Theorem 6

We restate the theorem for convenience.

Theorem 6. *Let T be a tree whose maximum split-set(s) have cardinality $m \in \{1, 2\}$. Suppose that for each maximum split-set M, every component of $T - M$ is uniquely radial. Then $i_{\mathrm{bn}}(T) = \gamma_b(T) + 1$.*

Proof. Suppose, contrary to the statement, that $i_{\mathrm{bn}}(T) \leq \gamma_b(T)$; by Proposition 3 we thus assume that $i_{\mathrm{bn}}(T) = \gamma_b(T)$. Let $P = (v_0, v_1, \ldots, v_d)$ be a diametrical path of T and consider an i_{bn}-broadcast f on T. By Theorem 1,

$$(4) \qquad \sigma(f) = \gamma_b(T) = \frac{1}{2}(\mathrm{diam}(T) - m) < \frac{1}{2}\mathrm{diam}(T).$$

By Lemma 9(ii), f does not cover all edges of P. Let $e_1, \ldots, e_k$, $k \geq 1$, be the uncovered edges on P; say $e_i = x_i y_i$ (where x_i is to the left of y_i), and also let $y_0 = v_0$ and $x_{k+1} = v_d$. Note that it is possible that $x_{i+1} = y_i$ for some i; this

happens precisely when some vertex in $\mathrm{Touch}(P) \setminus V(P)$ broadcasts to exactly one vertex on P. Let $T_1, \ldots, T_{k+1}$ be the components of $T - \{e_1, \ldots, e_k\}$, where T_i is the component whose leftmost and rightmost vertices on P are y_{i-1} and x_i, respectively. For each i, let P_i be the subpath of P on T_i, and f_i the restriction of f to T_i. Since $k \geq 1$, $|V_f^+| \geq 2$. By Corollary 8, each f_i is a maximal bn-independent broadcast on T_i, hence $\sigma(f_i) \geq i_{\mathrm{bn}}(f_i)$ for each i. By Proposition 7, $|V_{f_i}^+| \geq 2$ for each i.

Suppose f_i is not an i_{bn}-broadcast and let g_i be an i_{bn}-broadcast on T_i instead. Then $\sigma(g_i) < \sigma(f_i)$. By maximality, g_i is a dominating broadcast on T_i (Proposition 3). However, this implies that $g = (f - f_i) \cup g_i$ is a dominating broadcast on T with cost $\sigma(g) < \sigma(f) = \gamma_b(T)$, a contradiction. Hence f_i is an i_{bn}-broadcast on T_i for each i.

For $i = 1, \ldots, k$, let L_i and R_i (L for "left", R for "right") be the subtrees of $T - e_i$ that contain x_i and y_i, respectively. Let $P(L_i)$ and $P(R_i)$ be the subpaths of P on L_i and R_i, respectively. Also denote the restriction of f to L_i and R_i by g_i and h_i, respectively. As in the case of the f_i, the g_i and h_i are i_{bn}-broadcasts on L_i and R_i, respectively.

Claim 14. *None of the uncovered edges $e_1, \ldots, e_k$ is a split-edge of T.*

Proof. Suppose that for some j, $j = 1, \ldots, k$, e_j is a split-edge of T on P. Consider L_j and R_j, and note that $g_j = \bigcup_{i=1}^{j} f_i$ and $h_j = \bigcup_{i=j+1}^{k+1} f_i$. Since $|V_{f_i}^+| \geq 2$ for each i, $|V_{g_j}^+|, |V_{h_j}^+| \geq 2$. Now, if $m = 1$, then for $M = \{e_j\}$, both L_j and R_j are radial subtrees of T. On the other hand, if $m = 2$, then $M = \{e_j, e'\}$ for some $e' \in E(P)$. We may assume without loss of generality that e_j lies to the left of e' on P. Then L_j is a radial subtree of T. In either case, L_j is uniquely radial, which contradicts $|V_{g_j}^+| \geq 2$. We conclude that no e_i is a split-edge. $\square$

We continue with the proof of Theorem 6 and consider two cases, depending on whether $k = 1$ or $k \geq 2$.

Case 1. $k = 1$. Then $\mathrm{diam}(T) = \ell(P_1) + \ell(P_2) + 1$. As stated in (4), $\gamma_b(T) = \frac{1}{2}(\mathrm{diam}(T) - m)$. On the other hand, $\gamma_b(T) = \sigma(f_1) + \sigma(f_2)$. Since f_i covers each edge of P_i, Lemma 9(ii) implies that $\sigma(f_i) \geq \ell(P_i)/2$ for $i = 1, 2$. Therefore

$$(5) \qquad \mathrm{diam}(T) - m = 2(\sigma(f_1) + \sigma(f_2)) \geq \ell(P_1) + \ell(P_2) = \mathrm{diam}(T) - 1.$$

We conclude that $m = 1$ and equality holds throughout (5). In particular,

$$(6) \qquad\qquad\qquad \gamma_b(T_i) = \frac{1}{2}\ell(P_i) \text{ for } i = 1, 2,$$

hence $\ell(P_i)$ is even. Since e_1 is not a split-edge, Definition 1 implies that P_i is not a diametrical path of T_i for at least one $i \in \{1, 2\}$. Assume without loss of

generality that P_1 is not a diametrical path of T_1. Since f_1 is an i_{bn}-broadcast on T_1 that covers all edges on P_1, Corollary 13 implies that $i_{\text{bn}}(T_1) > \frac{1}{2}\ell(P_1)$, which contradicts (6). This completes the proof of Case 1. ♦

Case 2. $k \geq 2$. We first state and prove another claim.

Claim 15. *Each P_i, $i = 2, \ldots, k$, has even length.*

Proof. Since the edges e_i are uncovered, no x_i or y_i is overdominated, for $i = 1, \ldots, k$. Since f is bn-independent, each edge of P_i is covered exactly once. Since each vertex in $V_{f_i}^+$ covers an even number of edges on P_i, the result follows. □

The same argument shows that if P_1 or P_{k+1} has odd length, then y_0 or x_{k+1}, respectively, is overdominated. Since every edge of P_i is covered, Lemma 9(ii) implies that, for each i,

$$(7) \qquad\qquad \sigma(f_i) \geq \left\lceil \frac{1}{2}\ell(P_i) \right\rceil .$$

Since

$$\sum_{i=1}^{k+1} \sigma(f_i) = \gamma_b(T) = \frac{1}{2}(\operatorname{diam}(T) - m) = \frac{1}{2}\left[\sum_{i=1}^{k+1} \ell(P_i) + k - m \right],$$

we see that

$$2\sum_{i=1}^{k+1} \sigma(f_i) - \sum_{i=1}^{k+1} \ell(P_i) = k - m \geq 0.$$

Hence either $k = m = 2$ and $\sigma(f_i) = \frac{1}{2}\ell(P_i)$ for each i, or $k - m > 0$ and there is at least one value of i for which $\sigma(f_i) > \frac{1}{2}\ell(P_i)$. In the former case we proceed as in Case 1 (but with $m = 2$) to obtain a contradiction. Hence we may assume that the latter case holds. For such an i, since each edge of P_i is covered exactly once, either (a) $V_{f_i}^+ \setminus V(P_i) \neq \emptyset$, or (b) $i \in \{1, k+1\}$, $\ell(P_i)$ is odd and an end-vertex of P_i is overdominated (or both (a) and (b) hold).

We consider two subcases, depending on whether $\ell(P_1)$ is odd or even, beginning with the case where $\ell(P_1)$ is odd. The analysis for this case can be repeated with only a trivial modification when $\ell(P_1)$ is even.

Case 2(a). Suppose $\ell(P_1)$ is odd. By (7), $\sigma(f_1) \geq \frac{1}{2}(\ell(P_1) + 1)$. Since $\ell(P_1)$ is odd while $\ell(P_2)$ is even, and $P(L_2)$ consists of P_1 followed by e_1 followed by P_2, $P(L_2)$ has even length $\ell(P_1) + \ell(P_2) + 1$. Moreover,

$$(8) \qquad \sigma(g_2) = \sigma(f_1) + \sigma(f_2) \geq \frac{1}{2}[\ell(P_1) + \ell(P_2) + 1] = \frac{1}{2}\ell(P(L_2)).$$

Our next goal is to show that there exists an even integer, say $2t$, such that $P(L_{2t})$ is a diametrical path of L_{2t} (see Figure 5).
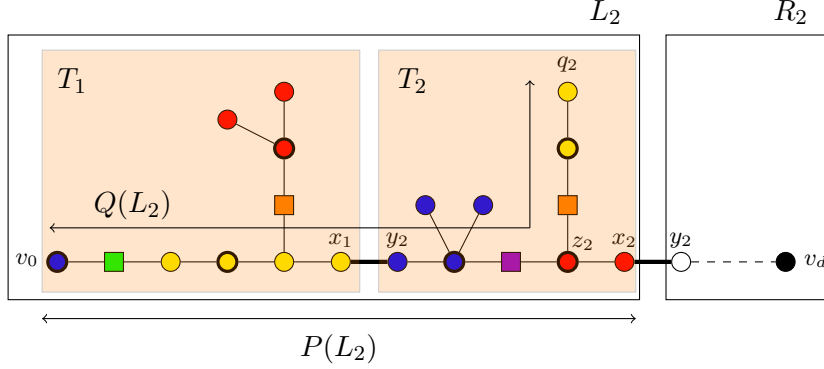
Figure 5. A step in the proof of Theorem 2.6.

Claim 16. *For $p \in \left\{1, \ldots, \left\lfloor \frac{k}{2} \right\rfloor\right\}$, suppose $P(L_{2i})$ is not a diametrical path of L_{2i} for each $i = 1, \ldots, p-1$. Then $\sigma(g_{2p}) \geq \frac{1}{2}\ell(P(L_{2p}))$. If, in addition, $P(L_{2p})$ is not a diametrical path of L_{2p}, then $\sigma(g_{2p}) \geq \frac{1}{2}\ell(P(L_{2p})) + 1$.*

Proof. Let $Q(L_{2i})$ be a diametrical path of L_{2i}, $i = 1, \ldots, p$. Since $v_0 = y_0$ is a peripheral vertex of T, we may assume without loss of generality that v_0 is also a peripheral vertex of $Q(L_{2i})$. Let z_{2i} be the vertex nearest to x_{2i} common to $Q(L_{2i})$ and $P(L_{2i})$, and let q_{2i} be the end-vertex of $Q(L_{2i})$ antipodal to v_0. By assumption, $d(z_{2i}, q_{2i}) > d(z_{2i}, x_{2i})$. If $z_{2i} \in V(T_1)$, then g_{2i} covers every edge on the $y_0 - z_{2i}$ subpath of $P(L_{2i})$. By Corollary 13, $i_{\mathrm{bn}}(L_{2i}) = \sigma(g_{2i}) > \frac{1}{2}\ell(P(L_{2i}))$, and since $\frac{1}{2}\ell(P(L_{2i}))$ is even, $\sigma(g_{2i}) \geq \frac{1}{2}\ell(P(L_{2i})) + 1$. We therefore assume henceforth that $z_{2i} \notin V(T_1)$.

Suppose $p = 1$. We know from (8) that $\sigma(g_2) \geq \frac{1}{2}\ell(P(L_2))$, hence the first part of the claim holds. Assume $\ell(Q(L_2)) > \ell(P(L_2))$. Since $z_2 \notin V(T_1)$, we deduce that $z_2 \in V(T_2)$. Since f_2 covers every edge on the $y_1 - z_2$ subpath of P_2, Corollary 13 implies that $i_{\mathrm{bn}}(T_2) = \sigma(f_2) > \frac{1}{2}\ell(P_2)$. Since $\ell(P_2)$ is even, $\sigma(f_2) \geq \frac{1}{2}\ell(P_2) + 1$. Therefore $\sigma(g_2) = \sigma(f_1) + \sigma(f_2) \geq \frac{1}{2}(\ell(P_1) + \ell(P_2) + 1) + 1$, that is,

$$(9) \qquad\qquad \sigma(g_2) \geq \frac{1}{2}\ell(P(L_2)) + 1,$$

and the second part of the claim holds.

Suppose $p \geq 2$. Assume that we have shown the result for $1, \ldots, p-1$, and consider L_{2p}. Since $\ell(P(L_{2p})) = \ell(P(L_{2p-2})) + \ell(P_{2p-1}) + \ell(P_{2p}) + 2$, our assumption and (7) prove the first part of the claim, namely

$$
\begin{aligned}
\sigma(g_{2p}) &= \sigma(g_{2p-2}) + \sigma(f_{2p-1}) + \sigma(f_{2p}) \\
(10) \qquad &\geq \frac{1}{2}\big[\ell(P(L_{2p-2}) + 2 + \ell(P_{2p-1}) + \ell(P_{2p})\big] = \frac{1}{2}\ell(P(L_{2p})).
\end{aligned}
$$

Now assume $\ell(Q(L_{2p}) > \ell(P(L_{2p}))$. Then $z_{2p} \in V(T_\alpha)$ for some $\alpha \in \{2, 3, \ldots, 2p\}$. Let G be the subtree of $L_{2p} - x_{\alpha-1}y_{\alpha-1}$ that contains $y_{\alpha-1}$ and hence also z_{2p}. Let $P(G)$ be the subpath of P in G, and g the restriction of f to G. As in the case of the f_i and g_i, the broadcast g is maximal bn-independent. We may apply Corollary 13 to G, with the $y_{\alpha-1} - z_{2p}$ subpath of $P(L_{2p})$ as P', to obtain that $\sigma(g) > \frac{1}{2}\ell(P(G))$.

- Suppose $\alpha = 2$. Then $\ell(P(L_{2p})) = \ell(P_1) + \ell(P(G)) + 1$ and $\ell(P(G))$ is even (since $\ell(P(L_{2p}))$ is even and $\ell(P_1)$ is odd), hence $\sigma(g) \geq \frac{1}{2}\ell(P(G)) + 1$. Therefore

$$\sigma(g_{2p}) = \sigma(f_1) + \sigma(g) \geq \frac{1}{2}\left[\ell(P_1) + 1\right] + \frac{1}{2}[\ell(P(G)) + 2] = \frac{1}{2}\ell(P(L_{2p})) + 1.$$

- Suppose $\alpha \geq 3$ is even. Then

$$\ell(P(L_{2p})) = \ell(P(L_{\alpha-2})) + \ell(P_{\alpha-1}) + \ell(P(G)) + 2.$$

It follows that $\ell(P(G))$ is even, hence $\sigma(g) \geq \frac{1}{2}\ell(P(G)) + 1$. By assumption, $\sigma(g_{\alpha-2}) \geq \frac{1}{2}\ell(P(L_{\alpha-2})) + 1$, hence

$$\sigma(g_{2p}) \geq \frac{1}{2}\left[\ell(P(L_{\alpha-2})) + \ell(P_{\alpha-1}) + \ell(P(G))\right] + 2 = \frac{1}{2}\ell(P(L_{2p})) + 1.$$

- Suppose $\alpha \geq 3$ is odd. Then

$$\ell(P(L_{2p})) = \ell(P(L_{\alpha-1}) + \ell(P(G)) + 1$$

and $\ell(P(G))$ is odd, so $\sigma(g) \geq \frac{1}{2}[\ell(P(G)) + 1]$. By assumption, $\sigma(g_{\alpha-1}) \geq \frac{1}{2}\ell(P(L_{\alpha-1})) + 1$. Therefore

$$\sigma(g_{2p}) = \sigma(g_{\alpha-1}) + \sigma(g) \geq \frac{1}{2}[\ell(P(L_{\alpha-1})) + \ell(P(G)) + 1] + 1$$
$$\geq \frac{1}{2}\ell(P(L_{2p})) + 1. \qquad \square$$

Applying Claim 16 to $p = \lfloor\frac{k}{2}\rfloor$, we obtain that $\sigma(g_{2p}) \geq \frac{1}{2}\ell(P(L_{2p})) + 1$. We show next that this result produces a contradiction.

Claim 17. *Suppose that $P(L_{2i})$ is not a diametrical path of L_{2i} for each $i = 1, \ldots, \lfloor\frac{k}{2}\rfloor$. Then $\gamma_b(T) \geq \frac{1}{2}\mathrm{diam}(T)$.*

Proof. Suppose k is even. Then $2p = k$ and L_{2p} is followed by $T_{k+1} = T_{2p+1}$, hence

$$\mathrm{diam}(T) = \ell(P) = \ell(P(L_{2p}) + \ell(P_{2p+1}) + 1.$$

By (7) and Claim 16,

$$\gamma_b(T) = \sigma(g_{2p}) + \sigma(f_{2p+1}) \geq \frac{1}{2}[\ell(P(L_{2p})) + \ell(P_{2p+1})] + 1 > \frac{1}{2}\text{diam}(T).$$

Suppose k is odd. Then $2p = k - 1$ and L_{2p} is followed by $T_k = T_{2p+1}$ and $T_{k+1} = T_{2p+2}$, hence

$$\text{diam}(T) = \ell(P) = \ell(P(L_{2p}) + \ell(P_{2p+1}) + \ell(P_{2p+2}) + 2,$$

and

$$\gamma_b(T) = \sigma(g_{2p}) + \sigma(f_{2p+1}) + \sigma(f_{2p+2})$$
$$\geq \frac{1}{2}[\ell(P(L_{2p})) + \ell(P_{2p+1}) + \ell(P_{2p+2}) + 2] = \frac{1}{2}\text{diam}(T). \qquad \square$$

However, Claim 17 contradicts (4). We conclude that there exists a smallest even integer $2t$, $t \geq 1$, such that $P(L_{2t})$ is a diametrical path of L_{2t}. By Claim 16 and the choice of t, $\sigma(g_{2t}) \geq \frac{1}{2}\ell(P(L_{2t})) = \frac{1}{2}\text{diam}(L_{2t})$. Since $\sigma(f) < \frac{1}{2}\text{diam}(T)$, the edge $e_{2t} = x_{2t}y_{2t}$ and the tree R_{2t} exist.

By Claim 14, e_{2t} is not a split-edge of T; that is, there is no split-set M such that $M = \{e_{2t}\}$ (if $m = 1$) or $M = \{e_{2t}, e'\}$ for some $e' \in E(P)$ (if $m = 2$). Hence Definition 1 and the fact that $\ell(P(L_{2t}))$ is even imply that either

(a) $P(R_{2t}) = y_{2t}, \ldots, v_d$ is not a diametrical path of R_{2t}, or

(b) $m = 2$ and $P(R_{2t})$ is a diametrical path of R_{2t}, but there is no edge e' on $P(R_{2t})$ such that $\{e_{2t}, e'\}$ is a split-set of T.

Suppose (b) holds. Then $\text{diam}(T)$ is even, hence $\text{diam}(R_{2t})$ is odd. Since any split-edge e'' of R_{2t} on its diametrical path $P(R_{2t})$ would result in $\{e_{2t}, e''\}$ being a split-set of T, which is not the case, we conclude that R_{2t} is a bicentral radial tree. But then $\gamma_b(R_{2t}) = \frac{1}{2}(\ell(P(R_{2t})) + 1)$, thus

$$\gamma_b(T) = \gamma_b(L_{2t}) + \gamma_b(R_{2t}) \geq \frac{1}{2}[\ell(P(L_{2t})) + (\ell(P(R_{2t})) + 1)] > \frac{1}{2}(\text{diam}(T) - m),$$

a contradiction. Therefore (a) holds. We proceed to show that there is a smallest integer $t_1 > t$ such that $P(L_{2t_1})$ is a diametrical path of L_{2t_1} and $\sigma(g_{2t_1}) \geq \frac{1}{2}\ell(P(L_{2t_1}))$. This proof first considers the subtrees R_i instead of L_i and shows some similarities with the proof of Claim 16.

The next part of the proof is illustrated in Figure 6. Let $Q(R_{2t})$ be a diametrical path of R_{2t}. Since $v_d = x_{k+1}$ is a peripheral vertex of T, we may assume without loss of generality that v_d is a peripheral vertex of $Q(R_{2t})$ as well. Let u_{2t} be the end-vertex of $Q(R_{2t})$ antipodal to v_d and let w_{2t} be the vertex nearest to y_{2t} that is common to $Q(R_{2t})$ and $P(R_{2t})$. Say $w_{2t} \in V(T_{2t+\alpha})$ for some integer

$\alpha \geq 1$. If $2t + \alpha = k + 1$, let $H = R_{2t}$. Otherwise, let H be the subtree of $R_{2t} - e_{2t+\alpha}$ that contains $x_{2t+\alpha}$. Denote the subpath of P in H by $P(H)$, and the restriction of f to H by h. By Corollary 8, h is a maximal bn-independent broadcast on H. Moreover, h covers every edge on the $w_{2t} - x_{2t+\alpha}$ subpath of $P(H)$. By Corollary 13, $\sigma(h) > \frac{1}{2}\ell(P(H))$.
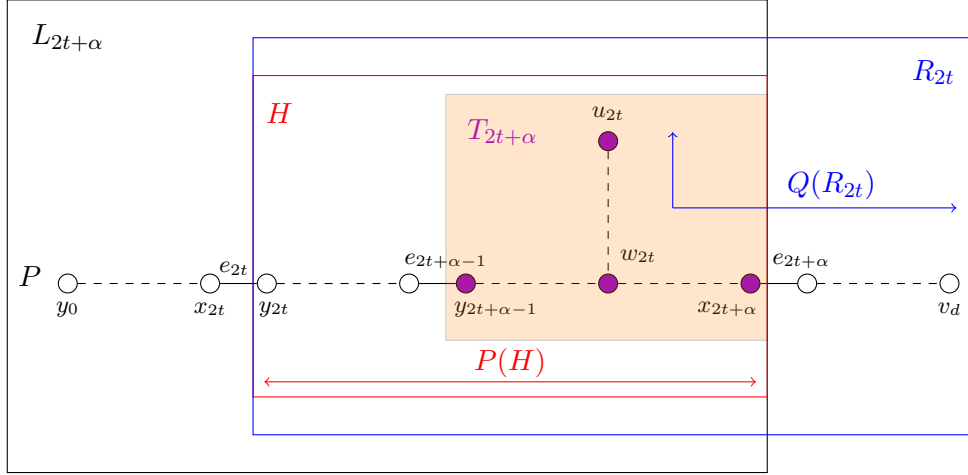

Figure 6. The subtree R_{2t}, its diametrical path $Q(R_{2t})$, and the subtree H of R_{2t}.

It follows that $R_{2t} \neq H$, otherwise we have $\mathrm{diam}(T) = \ell(P) = \ell(P(L_{2t})) + \ell(P(H)) + 1$ and $\sigma(f) = \sigma(g_{2t}) + \sigma(h) > \frac{1}{2}[\ell(P(L_{2t})) + \ell(P(H))] \geq \frac{1}{2}(\mathrm{diam}(T) - m)$, which is not the case. Hence we can consider the trees $L_{2t+\alpha}$ and $R_{2t+\alpha}$. Note that $\ell(P(L_{2t+\alpha})) = \ell(P(L_{2t})) + \ell(P(H)) + 1$.

• Suppose $\alpha > 1$. Then the edge $x_{2t+\alpha-1}y_{2t+\alpha-1}$ on the $w_{2t} - y_{2t}$ subpath of $P(H)$ is uncovered. By Corollary 13(ii), $\sigma(h) > \frac{1}{2}\ell(P(H)) + 1$. If $\ell(P(L_{2t+\alpha}))$ is even, i.e., if α is even, then $\ell(P(H))$ is odd and $\sigma(h) \geq \frac{1}{2}(\ell(P(H)) + 3)$. Hence

$$\sigma(g_{2t+\alpha}) = \sigma(g_{2t}) + \sigma(h) \geq \frac{1}{2}[\ell(P(L_{2t})) + \ell(P(H)) + 3] = \frac{1}{2}\ell(P(L_{2t+\alpha})) + 1.$$

Thus, whether or not $P(L_{2t+\alpha})$ is a diametrical path of $L_{2t+\alpha}$, we have that $\sigma(g_{2t+\alpha}) \geq \frac{1}{2}\ell(P(L_{2t+\alpha})) + 1$.

If $\ell(P(L_{2t+\alpha}))$ is odd, i.e., if α is odd, then $\ell(P(H))$ is even; hence $\sigma(h) \geq \frac{1}{2}\ell(P(H)) + 2$. As in previous cases, it follows that $L_{2t+\alpha+1}$ and $R_{2t+\alpha+1}$ exist. Moreover, $\ell(P(L_{2t+\alpha+1})) = \ell(P(L_{2t})) + \ell(P(H)) + \ell(P_{2t+\alpha+1}) + 2$ and

$$\sigma(g_{2t+\alpha+1}) = \sigma(g_{2t}) + \sigma(h) + \sigma(f_{2t+\alpha+1})$$

$$(11) \qquad \geq \frac{1}{2}[\ell(P(L_{2t})) + \ell(P(H)) + \ell(P_{2t+\alpha+1})] + 2 = \frac{1}{2}\ell(P(L_{2t+\alpha+1})) + 1.$$

Thus, whether or not $P(L_{2t+\alpha+1})$ is a diametrical path of $L_{2t+\alpha+1}$, we have that $\sigma(g_{2t+\alpha+1}) \geq \frac{1}{2}\ell(P(L_{2t+\alpha+1})) + 1$.

• Suppose $\alpha = 1$. Then $H = T_{2t+1}$. Similar to (11), but only knowing that $\sigma(h) > \frac{1}{2}\ell(P(H))$, we obtain that

$$\sigma(g_{2t+2}) \geq \frac{1}{2}\ell(P(L_{2t+2})).$$

Suppose $P(L_{2t+2})$ is not a diametrical path of L_{2t+2}, and let $Q(L_{2t+2})$ be such a path instead. Define z_{2t+2} and q_{2t+2} as in the proof of Claim 16. Since $P(L_{2t})$ is a diametrical path of L_{2t}, $z_{2t+2} \in V(T_{2t+1}) \cup V(T_{2t+2})$.

– Suppose $z_{2t+2} \in V(T_{2t+1})$. Let F be the subtree of T induced by $V(T_{2t+1}) \cup V(T_{2t+2})$, with $P(F)$ the subpath of P in F, and f^* the restriction of f to F. As before, f^* is a maximal bn-independent broadcast on F, hence $\sigma(f^*) \geq i_{\mathrm{bn}}(F)$. Note that $\ell(P(F)) = \ell(P_{2t+1}) + \ell(P_{2t+2}) + 1$, which is odd. Then all edges on the $y_{2t} - z_{2t+2}$ subpath of $P(F)$ are covered, while the edge $x_{2t+1}y_{2t+1}$ on the $z_{2t+2} - x_{2t+2}$ subpath of $P(F)$ is uncovered. By Corollary 13, $\sigma(f^*) > \frac{1}{2}\ell(P(F)) + 1$, that is, $\sigma(f^*) \geq \frac{1}{2}[\ell(P(F)) + 1] + 1$. Now we have $\ell(P(L_{2t+2})) = \ell(P(L_{2t})) + \ell(P(F)) + 1$ and

$$\sigma(g_{2t+2}) = \sigma(g_{2t}) + \sigma(f^*) \geq \frac{1}{2}[\ell(P(L_{2t})) + \ell(P(F)) + 1] + 1 = \frac{1}{2}\ell(P(L_{2t+2})) + 1.$$

– Suppose $z_{2t+2} \in V(T_{2t+2})$. By Corollary 13(i), $\sigma(f_{2t+2}) > \frac{1}{2}\ell(P_{2t+2})$. Since $H = T_{2t+1}$, we also have $\sigma(f_{2t+1}) > \frac{1}{2}\ell(P_{2t+1})$. Since $\ell(P_i)$ is even for each i, we have $\sigma(f_{2t+1}) \geq \frac{1}{2}\ell(P_{2t+1}) + 1$ and $\sigma(f_{2t+2}) \geq \frac{1}{2}\ell(P_{2t+2}) + 1$. Now $\ell(P(L_{2t+2})) = \ell(P(L_{2t}) + \ell(P_{2t+1}) + \ell(P_{2t+2}) + 2$ and

$$\sigma(g_{2t+2}) = \sigma(g_{2t}) + \sigma(f_{2t+1}) + \sigma(f_{2t+2})$$
$$\geq \frac{1}{2}[\ell(P(L_{2t})) + \ell(P_{2t+1}) + \ell(P_{2t+2})] + 2 = \frac{1}{2}\ell(P(L_{2t+2})) + 1.$$

In all cases we have an even integer $2t_1 > 2t$ such that $\sigma(g_{2t_1}) \geq \frac{1}{2}\ell(P(L_{2t_1}))$, and if $P(L_{2t_1})$ is a not a diametrical path of L_{2t_1}, then $\sigma(g_{2t_1}) \geq \frac{1}{2}\ell(P(L_{2t_1})) + 1$. Thus we can repeat the entire process indefinitely, constructing a strictly increasing infinite sequence $t, t_1, \ldots$ which corresponds to the sequence of subtrees $L_{2t} \subsetneq L_{2t_1} \subsetneq \cdots$ of T, which, of course, is impossible. This completes the proof of Case 2(a). ♦

Case 2(b). Suppose $\ell(P_1)$ is even. By Lemma 9(ii), $\sigma(f_1) \geq \frac{1}{2}\ell(P)$. Equivalently, considering L_1, $\sigma(g_1) \geq \frac{1}{2}\ell(P(L_1))$. If $P(L_1)$ is not a diametrical path of L_1, then $\sigma(g_1) \geq \frac{1}{2}\ell(P(L_1)) + 1$. The proof now proceeds exactly as in Case 2(a), except that we consider the subtrees L_i and R_i for odd indices i instead of even ones, because $\ell(P(L_i))$ is even. The resulting contradiction concludes the proof of the theorem. ♦   ■

4.5. Comments on Theorems 5 and 6

Paths P_{3k}, where $k \geq 4$, have split-sets of cardinality $k - 1$, uniquely radial subtrees P_3, and $\gamma_b(P_{3k}) = k$, but by Theorem 11, $i_{\mathrm{bn}}(P_{3k}) = \lceil 6k/5 \rceil < \lceil 4k/3 \rceil$. Therefore Theorem 6 cannot be extended to trees with larger split-sets.

The condition that the radial subtrees of T be uniquely radial is used only in the proof of Claim 14. It is not hard to see that the proof remains valid when we relax this condition to only require that when $m = 1$, at least one of the radial subtrees of T is uniquely radial, while when $m = 2$, both radial subtrees that contain peripheral vertices of T are uniquely radial, but the subtree that lies between the two split-edges need not be uniquely radial. For example, for each of the split-sets of the tree in Figure 1, only one of the radial subtrees is uniquely radial; as shown in Figure 2, $i_{\mathrm{bn}}(T) = \gamma_b(T) + \lceil 2/3 \rceil = 5$.

5. OPEN PROBLEMS

We close with a list of open problems.

Problem 18. Find more trees (other than paths) whose radial subtrees are uniquely radial but for which $i_{\mathrm{bn}}(T) < \gamma_b(T) + \left\lceil \frac{|M|+1}{3} \right\rceil$ when $|M| \geq 3$.

Problem 19. Let T be a tree whose radial subtrees are uniquely radial with radii at least 2 for any maximum split-set M. Is it true that $i_{\mathrm{bn}}(T) = \gamma_b(T) + \left\lceil \frac{|M|+1}{3} \right\rceil$?

Herke [15] showed that if G is a connected graph, then $\gamma_b(G) = \min\{\gamma_b(T) : T$ is a spanning tree of $G\}$.

Problem 20. Is it true that if G is connected, then $i_{\mathrm{bn}}(G) = \min\{i_{\mathrm{bn}}(T) : T$ is a spanning tree of $G\}$?

The proof of the bound $i_{\mathrm{bn}}(G) \leq \lceil 4\gamma_b(G)/3 \rceil$ in [19] shows that equality holds for a graph G only if G has a γ_b-broadcast f such that $f(v) = 1$ for each $v \in V_f^+$, $|V_f^+| \geq 2$, and $N_f(v) \cap N_f(u) = \emptyset$ for any distinct $u, v \in V_f^+$. In this case, f is the characteristic function of an *efficient dominating set* of G. The paths P_6 and P_9 are examples of graphs with efficient dominating sets for which equality holds in the bound.

Problem 21. Determine classes of graphs such that $i_{\mathrm{bn}}(G) = \left\lceil \frac{4\gamma_b(G)}{3} \right\rceil$. Improve the bound for classes of graphs other than trees.

As mentioned in the introduction, $i(G)$ and $i_{\mathrm{bn}}(G)$ are not comparable. Since $i(K_{n,n}) = n$ and $i_{\mathrm{bn}}(K_{n,n}) = 2$ when $n \geq 2$, the ratio $i(G)/i_{\mathrm{bn}}(G)$ is, in general, unbounded. Since $\gamma_b(G) \leq \gamma(G) \leq i(G)$ for all graphs G, we also have that

$i_{bn}(G) \leq \lceil 4i(G)/3 \rceil$ for all graphs G. Paths show that the difference $i_{bn}(G) - i(G)$ can be arbitrarily large, but $i_{bn}(P_n)/i(P_n) \approx 6/5$. In general, the actual value of $i_{bn}(G)$ is likely to be smaller than $4i(G)/3$.

Problem 22. Improve the bound $i_{bn}(G) \leq \left\lceil \frac{4i(G)}{3} \right\rceil$. Bound the ratio $i(G)/i_{bn}(G)$ for specific graph classes.

As mentioned in the introduction, Erwin denoted the minimum weight of a maximal hearing independent broadcast f on G by $i_b(G)$. We prefer the notation $i_h(G)$ (h for "hearing") to avoid confusion with other notation and to be consistent with [18, 19]. A tree T with $i_{bn}(T) = 8$ and $i_h(T) = \text{rad}(T) = 9$ is shown in Figure 7; hence these parameters are distinct. Similar to the result in Theorem 11, Bouchouika, Bouchemakh and Sopena [5] showed that $i_h(P_n) = i_h(C_n) = \lceil 2n/5 \rceil$ for all $n \neq 3$.
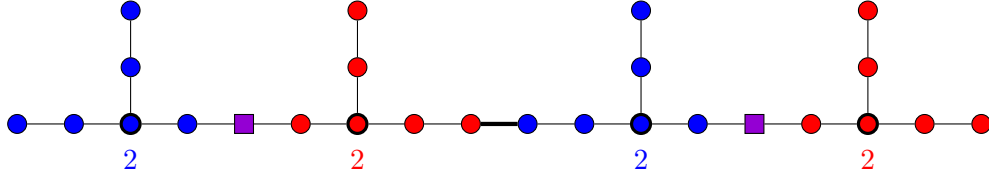

Figure 7. A tree T with $i_{bn}(T) = 8$ and $i_h(T) = \text{rad}(T) = 9$.

Problem 23. (i) For which graphs G is $i_h(G) = i_{bn}(G)$?
 (ii) Can the difference $i_h(G) - i_{bn}(G)$ be arbitrary?
(iii) Is the ratio $i_h(G)/i_{bn}(G)$ bounded?
(iv) Do there exist graphs G such that $i_{bn}(G) > i_h(G)$?

Heggernes and Lokshtanov [13] showed that minimum broadcast domination is solvable in polynomial time for any graph. Their algorithm runs in $O(n^6)$ time for a graph of order n.

Problem 24. Study the complexity of determining $i_{bn}(G)$ for G belonging to various graph classes.

Brewster, Mynhardt and Teshima [6] considered broadcast domination as an integer programming (IP) problem. Its fractional relaxation linear program (LP) has a dual linear program whose IP formulation provides a lower bound for the broadcast number.

Problem 25. Consider the bn-independent broadcast problem as an IP problem and obtain its dual IP formulation via relaxation to an LP problem and its dual.

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