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MINIMAL GRAPHS WITH DISJOINT DOMINATING AND TOTAL DOMINATING SETS

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Abstract

A graph G is a DTDP-graph if it has a pair (D, T) of disjoint sets of vertices of G such that D is a dominating set and T is a total dominating set of G. Such graphs were studied in a number of research papers. In this paper we study further properties of DTDP-graphs and, in particular, we characterize minimal DTDP-graphs without loops.

Keywords: domination, total domination.

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1. Introduction

The theory of domination in graphs is well studied in the literature. For recent books on the topic we refer the reader to [5,6]. Let $G = (V_G, E_G)$ be a graph with vertex set V_G and edge set E_G, where multi-edges and multi-loops are allowed. We remark that such a graph is also called a *multigraph* in the literature. A set of vertices $D \subseteq V_G$ in G is a *dominating set* of G if every vertex in $V_G \setminus D$ is adjacent to a vertex in D, while D is a *total dominating set*, abbreviated TD-*set*, of G if every vertex has a neighbor in D. We note that a vertex incident with a loop totally dominates all its neighbors (and therefore also itself).

A DT-*pair* in a graph G is a pair (D, T) of disjoint sets of vertices of G such that $D \cup T = V_G$, where D is a dominating set and T is a TD-set of G. A graph that has a DT-pair is called a DTDP-*graph* (standing for "dominating, total dominating, partitionable graph"). A connected graph G is a *minimal* DTDP-*graph*, if G is a DTDP-graph and no proper spanning subgraph of G is a DTDP-graph.

In this paper we study further properties of DTDP-graphs. We proceed as follows. The necessary graph theory notation and terminology is given in Section 1.1. In Section 2, we present selected known results on DTDP-graphs. In Section 3, elementary properties of DTDP-graphs are presented. We define an important class of DTDP-graphs, which we call 2-subdivision graphs, in Section 4. In Section 5, we define what we have coined a "good subgraph" of a graph, and show that these subgraphs play a role in determining non-minimal DTDP-graphs. Our main results, namely Theorems 6.1 and 6.2, are presented in Section 6. These results provide a structural characterization of minimal DTDP-graphs without loops. We conclude the paper with an open problem section to stimulate further research in the area.

1.1. Notation and terminology

For notation and graph theory terminology we generally follow [5,6,14]. Let $G = (V_G, E_G)$ be a graph with possible multi-edges and multi-loops. The *neighborhood*, denoted by $N_G(v)$, of a vertex v in G is the set of vertices adjacent to v, while its *closed neighborhood*, denoted by $N_G[v]$, is the set $N_G(v) \cup \{v\}$. (Observe that if G has a loop incident with v, then $N_G(v) = N_G[v]$.) In general, for a subset $X \subseteq V_G$ of vertices, the *neighborhood* of X, denoted by $N_G(X)$, is the set $\bigcup_{v \in X} N_G(v)$, and the *closed* neighborhood of X, denoted by $N_G[X]$, is the set $N_G(X) \cup X$. Two vertices are *neighbors* if they are adjacent. A *spanning supergraph* F of the graph G is a graph with the same vertex set as G and whose edge set contains E_G as a subset, that is, $V_G = V_F$ and $E_G \subseteq E_F$.

The *degree* of a vertex v in G, denoted by $d_G(v)$, is the number of edges incident with v plus twice the number of loops incident with v. A vertex of degree one is called a *leaf*, and the only neighbor of a leaf is called its *support vertex* (or simply, its *support*). A *strong support vertex* is a support vertex with at least two leaves as neighbors. A *weak support vertex* is a support vertex with exactly one leaf neighbor. The set of leaves, the set of weak supports, the set of strong supports, and the set of all supports of G is denoted by L_G, S'_G, S''_G, and S_G, respectively. We denote by $E_G(v)$ the set of edges incident with a vertex v in G.

If A and B are disjoint sets of vertices of G, then we denote by $E_G(A, B)$ the set of edges in G joining vertices in A and vertices in B. For one-element sets we write $E_G(v, B)$, $E_G(A, u)$, and $E_G(u, v)$ instead of $E_G(\{v\}, B)$, $E_G(A, \{u\})$, and $E_G(\{u\}, \{v\})$, respectively.

For $n \geq 1$, we denote a complete graph, a path, and a cycle on n vertices by K_n, P_n and C_n, respectively. We emphasize that K_n and P_n are simple graphs, as is the cycle C_n when $n \geq 3$. However, the cycle C_1 is the graph of order 1 with one loop, and the cycle C_2 is the graph of order and size 2 with two (repeated) edges. The *corona* $G \circ K_1$ of a graph G, also denoted $\mathrm{cor}(G)$ in the literature, is the graph obtained from G by adding for each vertex $v \in V_G$ a new vertex v' and the edge vv'. For an integer $k \geq 1$ we let $[k] = \{1, \ldots, k\}$.

2. Known Results

In this section, we present selected known results on DTDP-graphs. Beginning with the classical 1962 result of Ore [16], who was the first to observe that the vertex set of a graph without isolated vertices can be partitioned into two dominating sets, various graph theoretic properties and parameters of graphs having disjoint dominating sets of different types were studied in a large number of papers. Properties of DTDP-graphs (and properties of graphs with disjoint TD-sets) were extensively studied, for example, in [1–4,8,9,12–15,17], to mention just a few. In particular, it was proved in [12] that every connected graph with minimum degree at least two and different from C_5 is a DTDP-graph. A constructive characterization of all DTDP-graphs was given in [13].

3. Elementary Properties of DTDP-Graphs

In this section, we present some elementary properties of DTDP-graphs that we will need when presenting our main results. As an immediate consequence of the definition of a DT-pair, we have the following observations, which we shall use throughout the paper and at times without referencing.

Observation 3.1. *If (D, T) is a* DT*-pair in a graph G, then every leaf of G belongs to D, while every support of G is in T, that is, $L_G \subseteq D$ and $S_G \subseteq T$.*

Observation 3.2. *If every component of a graph G is a* DTDP*-graph, then G is a* DTDP*-graph.*

Observation 3.3. *A* DTDP*-graph is not a minimal* DTDP*-graph if it contains parallel edges.*

We observe that every spanning supergraph of a DTDP-graph is a DTDP-graph, and every DTDP-graph is a spanning supergraph of some minimal DTDP-graph. Hence, minimal DTDP-graphs can be viewed as *skeletons* of DTDP-graphs, in the sense that every DTDP-graph contains as a spanning subgraph a skeleton.

Let G be a graph and let v be a support vertex in G. If G' is obtained from G by adding a new leaf, say v', adjacent to v, then (D, T) is a DT-pair in G if and only if $(D \cup \{v'\}, T)$ is a DT-pair in G'. This yields the following observation, which shows that the addition of new leaves adjacent to an existing support vertex of a graph preserves the property of being a (minimal) DTDP-graph.

Observation 3.4. *Let v be a support vertex in a graph G. If G' is a graph obtained from G by adding a new leaf, say v', adjacent to v, then G is a* (*minimal*) DTDP*-graph if and only if G' is a* (*minimal*) DTDP*-graph.*

4. 2-Subdivision Graphs of a Graph

In this section, we define 2-subdivision graphs of a graph, and show that this class of graphs is important when discussing DTDP-graphs. Informally, given a multigraph H, we consider a 2-subdivision (obtained by adding two vertices on each edge and each loop) of H called H'. Then we contract some vertices in the neighborhoods of the original vertices, that is, vertices in $N_{H'}(v)$ where $v \in V_H$. Finally, we multiply certain leaves where desired.

More formally, let $H = (V_H, E_H)$ be a graph with possible multi-edges and multi-loops. By φ_H we denote a function from E_H to 2^{V_H} that associates with each $e \in E_H$, the set $\varphi_H(e)$ of vertices incident with e. Let X_2 be a set of 2-element subsets of an arbitrary set (disjoint with $V_H \cup E_H$), and let $\xi\colon E_H \to X_2$ be a function such that $\xi(e) \cap \xi(f) = \emptyset$ if e and f are distinct elements of E_H. If $e \in E_H$ and $\varphi_H(e) = \{u, v\}$ ($\varphi_H(e) = \{v\}$, respectively), then we write $\xi(e) = \{u_e, v_e\}$ ($\xi(e) = \{v_e^1, v_e^2\}$, respectively). If $\varphi_H(e) = \{u, v\}$ and $\xi(e) = \{u_e, v_e\}$, then we name the vertex u_e the *co-vertex* of u and the vertex v_e the *far-vertex* of u. Consequently, v_e is the *co-vertex* of v and the vertex u_e the *far-vertex* of v. Let $S_2(H)$ denote the graph obtained from H by inserting two new vertices into each edge and each loop of H. Thus for each vertex $v \in V_H$, the neighbors of v in $S_2(H)$ are the set of co-vertices of v, that is, $N_{S_2(H)}(v) = \{v_e\colon \varphi_H(e) = \{u, v\}$ and $\xi(e) = \{u_e, v_e\}\}$. A graph H and its associated 2-subdivision graph $S_2(H)$ are illustrated in Figure 1.

Formally, the graph $S_2(H)$ has vertex set

$$V_{S_2(H)} = V_H \cup \bigcup_{e \in E_H} \xi(e)$$

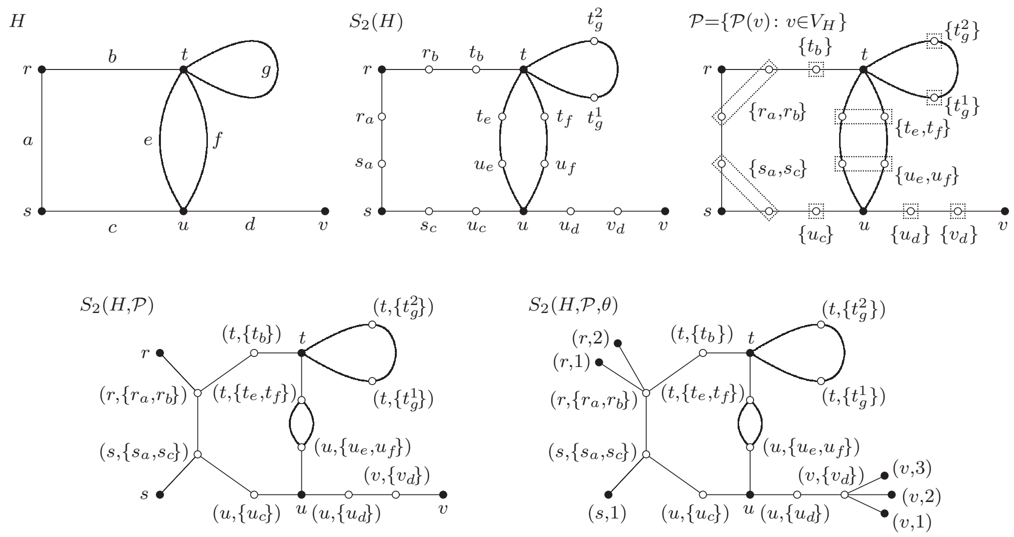

Figure 1. A graph H and 2-subdivision graphs $S_2(H)$, $S_2(H,\mathcal{P})$, and $S_2(H,\mathcal{P},\theta)$.

and edge set $E_{S_2(H)} = E_1 \cup E_2$, where

$$E_1 = \bigcup_{e\in E_H} \{u_e v_e\colon \xi(e) = \{u_e, v_e\}\}, \text{ and}$$

$$E_2 = \bigcup_{v\in V_H} \big(\{vv_e\colon e \in E_H(v)\} \cup \{vv_e^1, vv_e^2\colon e \text{ is a loop incident with } v\}\big).$$

In such a graph $S_2(H)$, we let $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V_H\}$ be a family in which $\mathcal{P}(v)$ is a partition of the set $N_{S_2(H)}(v)$ for each $v \in V_H \subset V_{S_2(H)}$. Further, we let $S_2(H,\mathcal{P})$ denote the graph (possibly with multi-loops and multi-edges) with vertex set

$$V_{S_2(H,\mathcal{P})} = V_H \cup \bigcup_{v\in V_H} (\{v\} \times \mathcal{P}(v))$$

and edge set $E_{S_2(H,\mathcal{P})}$ defined as follows. A vertex $v \in V_H$ is adjacent to (v,A), for every $A \in \mathcal{P}(v)$, by a single edge. Further, if u and v are adjacent vertices in H, then for $A \in \mathcal{P}(v)$ and $B \in \mathcal{P}(u)$, the vertices (v,A) and (u,B) are joined in $S_2(H,\mathcal{P})$ by $|\{e \in E_H\colon \varphi_H(e) = \{u,v\},\ v_e \in A, u_e \in B\}|$ edges. In particular, if u and v are vertices in H that are not adjacent, and $A \in \mathcal{P}(v)$ and $B \in \mathcal{P}(u)$, then the vertices (v,A) and (u,B) are not adjacent in $S_2(H,\mathcal{P})$. Similarly, we can determine the number of edges between vertices (v,A) and (u,B) (and the number of loops incident with (v,A) and (u,B)) if $A, B \in \mathcal{P}(v)$. As an illustration, the graph $S_2(H,\mathcal{P})$ associated with a graph H and a given partition $\mathcal{P}$ is shown in Figure 1. We remark that

$$N_{S_2(H,\mathcal{P})}(v) = \{(v,A)\colon A \in \mathcal{P}(v)\}$$

if $v \in V_H \subseteq V_{S_2(H,\mathcal{P})}$, and

$$N_{S_2(H,\mathcal{P})}((v,A)) = \{v\} \cup \bigcup_{u \in N_H(v)} \{(u,B) \colon B \in \mathcal{P}(u) \text{ and } N_{S_2(H)}(A) \cap B \neq \emptyset\}$$

if $(v,A) \in \bigcup_{v \in V_H} (\{v\} \times \mathcal{P}(v))$. Intuitively, $S_2(H,\mathcal{P})$ is the graph obtained from $S_2(H)$ as follows: for every vertex $v \in V_H$ and every set $A \in \mathcal{P}(v)$, we replace the vertices in the set $A \in \mathcal{P}(v)$ with a new vertex (v,A), which becomes adjacent to all former neighbors of the vertices belonging to A. We remark that in the special case when $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ and $\mathcal{P}(v) = \{\{x\} \colon x \in N_H(v)\}$ for each $v \in V_H$, the graph $S_2(H,\mathcal{P})$ is isomorphic to $S_2(H)$.

For a positive function $\theta \colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$, we let

$$L_\theta \colon L_{S_2(H,\mathcal{P})} \to L_{S_2(H,\mathcal{P})} \times \mathbb{N}$$

be a function such that $L_\theta(x) = \{(x,i) \colon i \in [\theta(x)]\}$ for $x \in L_{S_2(H,\mathcal{P})}$. Finally, let $S_2(H,\mathcal{P},\theta)$ denote the graph obtained from $S_2(H,\mathcal{P})$ by replacing each leaf x of $S_2(H,\mathcal{P})$ by its copies $(x,1),\ldots,(x,\theta(x))$ and adding an edge joining each newly added vertex to the support vertex adjacent to x in $S_2(H,\mathcal{P})$. As an illustration, the graph $S_2(H,\mathcal{P},\theta)$ associated with a graph H, a given partition $\mathcal{P}$ and a given function θ, is shown in Figure 1. We remark that in the special case when $\theta(x) = 1$ for every leaf x of $S_2(H,\mathcal{P})$, the graph $S_2(H,\mathcal{P},\theta)$ is isomorphic to $S_2(H,\mathcal{P})$.

The graphs $S_2(H)$, $S_2(H,\mathcal{P})$, and $S_2(H,\mathcal{P},\theta)$ are said to be 2-*subdivision graphs* of H (for a family $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ of partitions $\mathcal{P}(v)$ of neighborhoods $N_{S_2(H)}(v)$ where $v \in V_H \subset V_{S_2(H)}$, and for a positive function $\theta \colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$). Let $V^o_{S_2(H)}$, $V^o_{S_2(H,\mathcal{P})}$, and $V^o_{S_2(H,\mathcal{P},\theta)}$ be sets of vertices such that

$$\begin{array}{lcll} V^o_{S_2(H)} & = & V_{S_2(H)} \setminus V^n_{S_2(H)} & = V_H, \\ V^o_{S_2(H,\mathcal{P})} & = & V_{S_2(H,\mathcal{P})} \setminus V^n_{S_2(H,\mathcal{P})} & = V_H, \\ V^o_{S_2(H,\mathcal{P},\theta)} & = & V_{S_2(H,\mathcal{P},\theta)} \setminus V^n_{S_2(H,\mathcal{P},\theta)}, & \end{array}$$

where $V^n_{S_2(H)} = \bigcup_{e \in E_H} \xi(e)$, and $V^n_{S_2(H,\mathcal{P})} = V^n_{S_2(H,\mathcal{P},\theta)} = \bigcup_{v \in V_H} (\{v\} \times \mathcal{P}(v))$.

As a consequence of the definition of the 2-subdivision graphs $S_2(H)$ and $S_2(H,\mathcal{P})$, we have the following observation.

Observation 4.1. *If H is a graph with no isolated vertex and $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ is a family in which $\mathcal{P}(v)$ is a partition of the neighborhood $N_{S_2(H)}(v)$ for each $v \in V_H \subset V_{S_2(H)}$, then the following statements hold.*

(1) *If $v \in V_H$, then $N_{S_2(H)}(v)$ and $N_{S_2(H,\mathcal{P})}(v)$ are nonempty subsets of $V_{S_2(H)} \setminus V_H$ and $V_{S_2(H,\mathcal{P})} \setminus V_H$, respectively.*

(2) *If $v \in V_H$, then $d_{S_2(H)}(v) = d_H(v)$ and $d_{S_2(H,\mathcal{P})}(v) = |\mathcal{P}(v)|$.*

(3) *If* $x \in V_{S_2(H)} \setminus V_H$*, then* $|N_{S_2(H)}(x) \cap V_H| = |N_{S_2(H)}(x) \cap (V_{S_2(H)} \setminus V_H)| = 1$.

(4) *If* $v \in V_H$ *and* $A \in \mathcal{P}(v)$*, then* $|N_{S_2(H,\mathcal{P})}((v,A)) \cap (V_{S_2(H,\mathcal{P})} \setminus V_H)| = |A|$ *and* $|N_{S_2(H,\mathcal{P})}((v,A)) \cap V_H| = 1$.

The next result follows trivially from the construction of the 2-subdivision graph $S_2(H,\mathcal{P},\theta)$.

Observation 4.2. *If no vertex of a graph* H *is an isolated vertex, then the* 2*-subdivision graph* $S_2(H,\mathcal{P},\theta)$ *is a* DTDP*-graph for every family* $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ *of partitions* $\mathcal{P}(v)$ *of neighborhoods* $N_{S_2(H)}(v)$ *where* $v \in V_H$*, and for every positive function* $\theta \colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$*. In addition,* $\left(V^o_{S_2(H,\mathcal{P},\theta)}, V^n_{S_2(H,\mathcal{P},\theta)}\right)$ *is a* DT*-pair in* $S_2(H,\mathcal{P},\theta)$.

It follows from Observation 4.2 that $\left(V^o_{S_2(H)}, V^n_{S_2(H)}\right)$, $\left(V^o_{S_2(H,\mathcal{P})}, V^n_{S_2(H,\mathcal{P})}\right)$, and $\left(V^o_{S_2(H,\mathcal{P},\theta)}, V^n_{S_2(H,\mathcal{P},\theta)}\right)$ are DT-pairs in $S_2(H)$, $S_2(H,\mathcal{P})$, and $S_2(H,\mathcal{P},\theta)$, respectively. Consequently, every 2-subdivision graph $S_2(H,\mathcal{P},\theta)$ is a DTDP-graph. Simple examples presented in Figures 2 and 3 illustrate the fact that if $S_2(H,\mathcal{P},\theta)$ is a minimal DTDP-graph depends on the family of partitions $\mathcal{P}$. In Figure 3 we present examples of possible 2-subdivision graphs $S_2(K_1^2,\mathcal{P},\theta)$ of K_1^2, where K_1^s denotes a graph of order 1 and size s. We remark that of all the graphs in Figures 2 and 3, only $S_2(C_2)$, $S_2(P_4)$, and $S_2(K_1^2,\mathcal{P}_4)$ are minimal DTDP-graphs. In the next two propositions we study these relations more precisely.

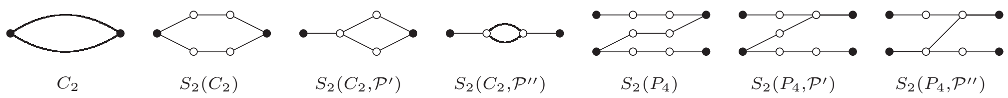

Figure 2. Graphs C_2, $S_2(C_2)$, $S_2(C_2,\mathcal{P}')$, $S_2(C_2,\mathcal{P}'')$, $S_2(P_4)$, $S_2(P_4,\mathcal{P}')$, and $S_2(P_4,\mathcal{P}'')$.

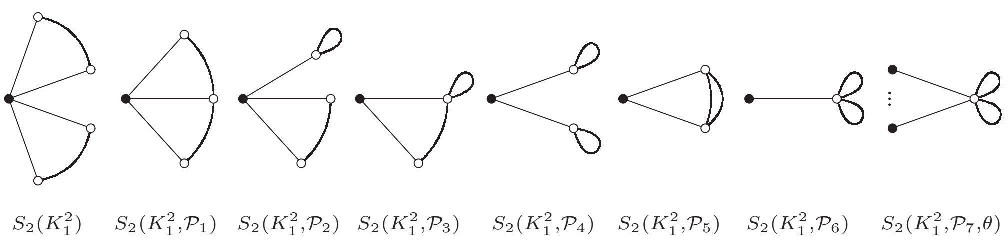

Figure 3. Possible 2-subdivision graphs of K_1^2.

Proposition 4.3. *Let* H *be a graph without isolated vertices and let* $\mathcal{P} = \{\mathcal{P}(v) : v \in V_H\}$ *be a family in which* $\mathcal{P}(v)$ *is a partition of the set* $N_{S_2(H)}(v)$ *for* $v \in$

V_H. Then the 2-subdivision graph $S_2(H,\mathcal{P})$ (and $S_2(H,\mathcal{P},\theta)$ for every positive function $\theta\colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$) is a DTDP*-graph but not a minimal* DTDP*-graph if there exists a vertex $v \in V_H$ and a set $A \in \mathcal{P}(v)$ such that A contains at least two elements belonging to different edges or loops and at least one of them is not a pendant edge of H.*

Proof. By Observation 4.2, $S_2(H,\mathcal{P})$ is a DTDP-graph and the pair $(D,T) = \left(V^o_{S_2(H,\mathcal{P})}, V^n_{S_2(H,\mathcal{P})}\right)$ is a DT-pair in $S_2(H,\mathcal{P})$. It suffices to observe that some proper spanning subgraph of $S_2(H,\mathcal{P})$ is a DTDP-graph. We consider two possible cases.

Case 1. Assume there exists a non-pendant edge e in H, say $\varphi_H(e) = \{v,u\}$, and let f be an edge (or a loop) incident with v and such that v_e and v_f (v_e and v_f^1 or v_e and v_f^2, if f is a loop) belong to the same set A which is an element of the partition $\mathcal{P}(v)$. Let B be the only set belonging to $\mathcal{P}(u)$ that contains u_e. Now from the properties of the DT-pair (D,T) in $S_2(H,\mathcal{P})$ it follows that if $\{u_e\} \subsetneq B$, then (D,T) is a DT-pair in the proper spanning subgraph $S_2(H,\mathcal{P}) \setminus (v,A)(u,B)$ of $S_2(H,\mathcal{P})$. Similarly, if $B = \{u_e\}$, then $(D \cup \{(u,\{u_e\})\}, T \setminus \{(u,\{u_e\})\})$ is a DT-pair in the proper spanning subgraph $S_2(H,\mathcal{P}) - u(u,\{u_e\})$ of $S_2(H,\mathcal{P})$, where $u(u,\{u_e\})$ is the edge joining the vertex u and the vertex $(u,\{u_e\})$.

Case 2. Assume now that e is a loop incident with a vertex v in H, and f is a pendant edge or a loop incident with v in H ($f \neq e$) and such that $\{v_e^1, v_e^2\} \cap A \neq \emptyset$ and $v_f \in A$ (or $\{v_e^1, v_e^2\} \cap A \neq \emptyset$ and $\{v_f^1, v_f^2\} \cap A \neq \emptyset$, if f is a loop) for some $A \in \mathcal{P}(v)$. If f is a pendant edge, and, without loss of generality, if $v_e^1, v_f \in A$, then we consider three subcases: (i) $v_e^2 \in A$; (ii) $v_e^2 \notin A$ and $\{v_e^2\} \in \mathcal{P}(v)$; (iii) $v_e^2 \notin A$ and $\{v_e^2\} \subsetneq B \in \mathcal{P}(v)$. In the first case (D,T) is a DT-pair in the spanning subgraph obtained from $S_2(H,\mathcal{P})$ by removing one loop incident with the vertex (v,A). In the second case the pair $(D \cup \{(v,\{v_e^2\})\}, T \setminus \{(v,\{v_e^2\})\})$ is a DT-pair in $S_2(H,\mathcal{P}) - v(v,\{v_e^2\})$. In the third case (D,T) is a DT-pair in $S_2(H,\mathcal{P}) - (v,A)(u,B)$.

Finally assume that every set $A \in \mathcal{P}(v)$ that contains at least two elements is a subset of the set $\bigcup_f \{v_f^1, v_f^2\}$, where the summation is over all loops incident with v in H. Let H_v be the subgraph of H generated by all loops incident with v. Then H_v is isomorphic to K_1^s (where s is the number of loops incident with v in H), and we consider the graphs $S_2(H_v)$ and $S_2(H_v,\mathcal{P}_v)$, where $\mathcal{P}_v = \{B \cap N_{S_2(H_v)}(v)\colon B \in \mathcal{P}(v),\ B \cap N_{S_2(H_v)}(v) \neq \emptyset\}$. It is obvious that $S_2(H_v,\mathcal{P}_v)$ is an induced subgraph of $S_2(H,\mathcal{P})$, and $(\{v\}, V_{S_2(H_v,\mathcal{P}_v)} \setminus \{v\})$ is a DT-pair in $S_2(H_v,\mathcal{P}_v)$. It remains to prove that some proper spanning subgraph of $S_2(H_v,\mathcal{P}_v)$ has a DT-pair (D_v,T_v) such that $v \in D_v$. This is obvious if $S_2(H_v,\mathcal{P}_v)$ contains parallel edges (see Observation 3.3) or a loop adjacent to another loop or to at least 2 edges. Thus, assume that $S_2(H_v,\mathcal{P}_v)$ contains neither parallel edges nor a loop adjacent to another loop or to at least 2 edges. We may also assume that no two mutually adjacent vertices of degree 2 are adjacent to v. Let $u \in N_{S_2(H_v)}(v)$ be a vertex of minimum degree in $S_2(H_v, \mathcal{P}_v)$. Certainly, $d_{S_2(H_v,\mathcal{P}_v)}(u) \geq 2$. Let w be a vertex belonging to $N_{S_2(H_v,\mathcal{P}_v)}(u) \setminus \{v\}$. The choice of u and the assumption that no two mutually adjacent vertices of degree 2 are adjacent to v imply that w is of degree at least 3 in $S_2(H_v, \mathcal{P}_v)$. Consequently, w has a neighbor in $V_{S_2(H_v,\mathcal{P}_v)} \setminus \{v, u\}$. This implies that $(\{v, u\}, V_{S_2(H_v,\mathcal{P}_v)} \setminus \{v, u\})$ is a DT-pair in $S_2(H_v, \mathcal{P}_v) - uv$, and completes the proof. ∎

Proposition 4.4. *Let H be a graph without isolated vertices, and let $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ be a family in which $\mathcal{P}(v)$ is a partition of the set $N_{S_2(H)}(v)$ for every $v \in V_H$. If $\mathcal{P}$ is such that for every $v \in V_H \setminus L_H$ and every non-pendant edge e incident with v, the singleton $\{v_e\}$ is an element of $\mathcal{P}(v)$, then both 2-subdivision graphs $S_2(H)$ and $S_2(H, \mathcal{P})$ are minimal* DTDP*-graphs or neither of them is a minimal* DTDP*-graph.*

Proof. For ease of observation, we assume that H has only one support vertex, say v. Let $u^1, \ldots, u^k$ be the leaves adjacent to v in H. Let H' be the subgraph of H induced by the vertices $v, u^1, \ldots, u^k$. It follows from the properties of $\mathcal{P}$ that the 2-subdivision graph $S_2(H, \mathcal{P})$ results from $S_2(H)$ replacing the tree $S_2(H')$ rooted at v by the tree $S_2(H', \mathcal{P}')$ rooted at v and defined for the family $\mathcal{P}' = \{\mathcal{P}'(x) \colon x \in V_{H'}\}$ in which $\mathcal{P}'(v) = \{A \in \mathcal{P} \colon A \subseteq \{v_{vu^1}, \ldots, v_{vu^k}\}\}$ and $\mathcal{P}'(u^i) = \mathcal{P}(u^i) = \{u^i_{vu^i}\}$ (for $i \in [k]$). Now, the fact that both $S_2(H')$ and $S_2(H', \mathcal{P}')$ are minimal DTDP-graphs implies that both $S_2(H)$ and $S_2(H, \mathcal{P})$ are minimal DTDP-graphs or neither of them is a minimal DTDP-graph. ∎

Definition 1. If e is a pendant edge in H and $\varphi_H(e) = \{v, u\}$, where v is a support vertex of degree at least 2 and u is a leaf, then the edge vv_e in $S_2(H)$ is called a *far part of the pendant edge* e in H. If e is a loop incident with a vertex v in H, then the edges vv_e^1 and vv_e^2 in $S_2(H)$ are said to be *twin parts of the loop* e in H.

It follows from Propositions 4.3 and 4.4 that if $S_2(H, \mathcal{P}, \theta)$ is a minimal DTDP-graph, then $\mathcal{P}$ can only contract far parts of adjacent pendant edges in H or twin parts of a loop in H.

Corollary 4.5. *If H is a connected graph of size at least 2, then the 2-subdivision graphs $S_2(H, \mathcal{P})$ and $S_2(H, \mathcal{P}, \theta)$ are minimal* DTDP*-graphs (for every family $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ in which $\mathcal{P}(v)$ is a partition of the set $N_{S_2(H)}(v)$ for each $v \in V_H$ and for every positive function $\theta \colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$) if and only if H is a star.*

Our aim is to recognize graphs which are present in non-minimal DTDP-graphs. For this purpose, let $\mathcal{F}$ be the family of all graphs defined as follows.

Definition 2 (The family $\mathcal{F}$)**.**
(1) We start with a rooted tree, say T, in which $d_T(x) \leq 2$ for every vertex $x \in V_T \setminus \{r\}$ (where r is a root of T) and form 2-subdivision graphs $S_2(T)$ and $S_2(T, \mathcal{P})$ for the family $\mathcal{P} = \{\mathcal{P}(x) \colon x \in V_T\}$ in which $\mathcal{P}(x)$ is a partition of the set $N_{S_2(T)}(x)$, and where $\mathcal{P}(r) = \{N_{S_2(T)}(r)\}$, while $\mathcal{P}(x) = \{\{y\} \colon y \in N_{S_2(T)}(x)\}$ if $x \in V_T \setminus \{r\}$.
(2) From the choice of $\mathcal{P}$ it follows that r is a leaf in $S_2(T, \mathcal{P})$ (and $(r, N_{S_2(T)}(r))$ is the only neighbor of r in $S_2(T, \mathcal{P})$. Finally, if $\theta \colon L_{S_2(T,\mathcal{P})} \to \mathbb{N}$ is a function such that $\theta(r)$ is a positive integer and $\theta(x) = 1$ for every $x \in L_{S_2(T,\mathcal{P})} \setminus \{r\}$, then we form the 2-subdivision graph $S_2(T, \mathcal{P}, \theta)$ from $S_2(T, \mathcal{P})$ replacing the leaf r by its copies $(r, 1), \ldots, (r, \theta(r))$ (adjacent to $(r, N_{S_2(T)}(r))$ in $S_2(H, \mathcal{P}, \theta)$).

To illustrate this definition, consider the graphs drawn in Figure 4 that can be obtained from the tree T in the leftmost drawing. We remark that the graph $S_2(T, \mathcal{P}, \theta)$ belonging to the family $\mathcal{F}$ can be obtained from the graph $S_2(T)$ by identifying the neighbors of the root r in the graph $S_2(T)$ into one new vertex, namely the vertex $(r, N_{S_2(T)}(r))$, and joining this new vertex to the vertex r and to all far-vertices of r, and thereafter replacing the leaf r with $\theta(r)$ copies of r, each of which is joined to the new vertex $(r, N_{S_2(T)}(r))$. From this remark and from Observation 4.2 we have the following corollary, which we shall use in the proof of Theorem 5.6. For a graph G, we denote the distance between two vertices u and v in G by $d_G(u, v)$.

Corollary 4.6. *Every tree belonging to the family $\mathcal{F}$ is a* DTDP*-tree, that is, if T is a tree rooted at vertex r, and $d_T(x) \leq 2$ for every vertex $x \in V_T \setminus \{r\}$, $1 \leq |N_T(r) \cap L_T| \leq d_T(r) - 1$, and $d_T(x, r) \equiv 2 \pmod 3$ for every $x \in L_T \setminus N_T(r)$, then T is a* DTDP*-tree. In particular, if T is a wounded spider rooted at vertex r, that is, if $1 \leq |N_T(r) \cap L_T| \leq d_T(r) - 1$ and $d_T(x, r) = 2$ for every $x \in L_T \setminus N_T(r)$, then T is a minimal* DTDP*-tree.*

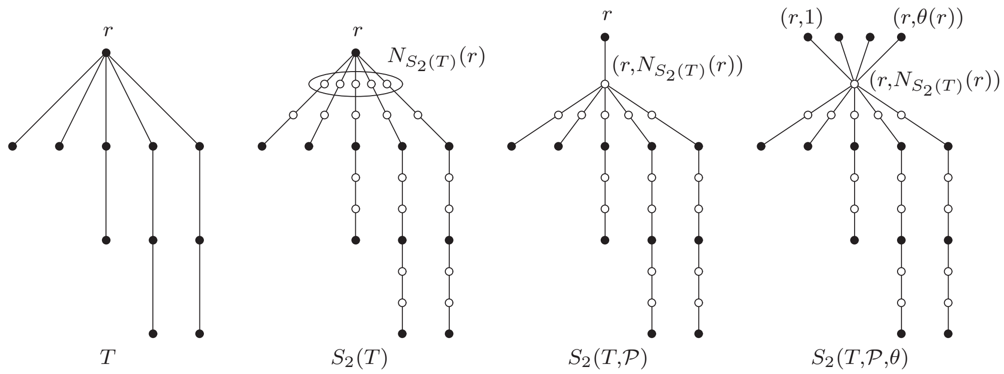

Figure 4. A graph $S_2(T, \mathcal{P}, \theta)$ belongs to the family $\mathcal{F}$.

5. Good Subgraphs of a Graph

In this section, we define what we have coined a "good subgraph" of a graph. We show that the presence of such subgraphs play a key role in determining non-minimal DTDP-graphs. Let Q be a subgraph without isolated vertices of a graph H, and let E_Q^- denote the set of edges belonging to $E_H \setminus E_Q$ that are incident with a vertex of Q. Let E be a set such that $E_Q^- \subseteq E \subseteq E_H \setminus E_Q$, and let A_E denote a set of arcs obtained by assigning an orientation for each edge in E. Then by $H^*(A_E)$ we denote the partially oriented graph obtained from H by replacing the edges in E by the arcs belonging to A_E. If $e \in E$, then by e_A we denote the arc in A_E that corresponds to e. By H_0 we denote the subgraph of $H^*(A_E)$ induced by the vertices that are not the initial vertex of an arc belonging to A_E, i.e., by the set $\{v \in V_H \colon d^+_{H^*(A_E)}(v) = 0\}$.

We say that Q is a *good subgraph of* H if there exist a set of edges E (where $E_Q^- \subseteq E \subseteq E_H \setminus E_Q$) and a set of arcs A_E such that in the resulting graph $H^*(A_E)$, which we simply denote by H^* for notational convenience, the arcs in A_E form a family $\mathcal{F} = \{F_v \colon v \in V_Q\}$ of arc disjoint digraphs F_v indexed by the vertices of Q and such that the following hold.

(1) For every v in Q, the digraph F_v is the union of a family, say $\mathcal{P}_v$, of arc disjoint oriented paths that begin at v.

(2) If $u \in V_{H^*} \setminus V_Q$, then $d^+_{H^*}(u) \le 1$.

(3) If $u \in V_{H^*}$, then $d^-_{H^*}(u) < d_{H^*}(u)$.

(4) If $x \in V_{F_v} \cap V_{F_u}$ and $v \neq u$, then $d^+_{F_v}(x) = 0$ or $d^+_{F_u}(x) = 0$.

One example of a good subgraph Q is shown in Figure 5. For clarity, the edges of Q are bold, and the digraphs F_v, F_u, F_w, and F_z are represented by four types arrows.

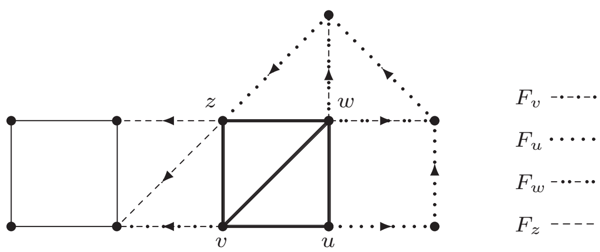

Figure 5. A bold subgraph is a good subgraph in the host graph.

From the definition of a good subgraph we immediately have the following observation.

Observation 5.1. *Neither a leaf nor a support vertex of a graph H belongs to a good subgraph in H.*

Observation 5.1 also implies that not every graph has a good subgraph. In particular, a corona graph (that is, a graph in which each vertex is a leaf or it is adjacent to exactly one leaf) has no good subgraph. On the other hand, if Q is a graph with no isolated vertex, and H is the graph obtained from Q by attaching at least one pendant edge to each vertex of Q and thereafter subdividing these new edges, then Q is a good subgraph in H. This proves that every graph without isolated vertices can be a good subgraph of some graph.

Proposition 5.2. *If e is a loop incident with a vertex v in a connected graph H, then the subgraph H_e of H with vertex set $V_{H_e} = \{v\}$ and edge set $E_{H_e} = \{e\}$ is a good subgraph in H if and only if $H \neq C_1$ and v is not adjacent to a pendant edge in H.*

Proof. If H_e is a good subgraph in H, then, by Observation 5.1, the only vertex v of H_e cannot be a support vertex in H.

Assume now that the vertex v is not a support vertex in H. Then v is neither a support vertex nor a leaf in H (as e is a loop incident with v). If $N_H(v) = \{v\}$, then $H = K_1^s$ $(s \geq 2)$ and, certainly, $H_e = K_1^1$ is a good subgraph in H. Thus assume that $N_H(v) \neq \{v\}$. In this case, the set $N_H(v) \setminus \{v\}$ is nonempty and it consists of two disjoint subsets N_v^1 and N_v^2, where $N_v^1 = \{x \in N_H(v) \setminus \{v\} \colon N_H(x) = \{v\}\}$ and $N_v^2 = \{x \in N_H(v) \setminus \{v\} \colon \{v\} \subsetneq N_H(x)\}$. Consequently, the set $E_{H_e}^-$ of edges or loops belonging to $E_H \setminus E_{H_e} = E_H \setminus \{e\}$ that are incident with v, consists of three disjoint subsets E_v^l, E_v^1, and E_v^2, where E_v^l is the set of loops incident with v which are distinct from e, $E_v^1 = E_H(v, N_v^1)$ (note that every edge in E_v^1 is a multi-edge), and $E_v^2 = E_H(v, N_v^2)$. Now we orient all edges in $E_{H_e}^-$. First, for every $s \in N_v^1$ we choose two edges belonging to $E_H(v,s)$, say f^s and g^s. Let A_E be the set of arcs obtained from $E_{H_e}^-$ by assigning any orientation to every loop in E_v^l, every edge in E_v^2 is oriented toward a vertex in N_v^2, while edges belonging to E_v^1 are oriented in such a way that for every vertex $s \in N_v^1$ one chosen edge joining v and s, say f^s, is oriented from s to v, and all other edges belonging to $E_H(v,s) \setminus \{f^s\}$ are oriented toward s, see Figure 6. Let $\mathcal{P}_v$ be the family of oriented paths that consists of oriented 1-cycles (v, h_A, v) (for every $h \in E_v^l$), oriented 2-cycles (v, g_A^s, s, f_A^s, v) (for every $s \in N_v^1$), oriented 1-paths (v, k_A, x) (for every $x \in N_v^2$ and every $k \in E_H(v,x)$), and (v, l_A, y) (for every $y \in N_v^1$ and every $l \in E_H(v,y) \setminus \{f^y, g^y\}$). Finally, let F_v be the digraph with vertex set $N_H[v]$ and arc set A_e. From the choice of $\mathcal{P}_v$ one can readily observe that F_v and $\mathcal{P}_v$ have the properties (1)–(4) stated in the definition of a good subgraph. Consequently, H_e is a good subgraph in H. ∎

For $s \geq 1$, by K_2^s we denote a graph of order 2 and size s in which the vertices are joined by exactly s edges.

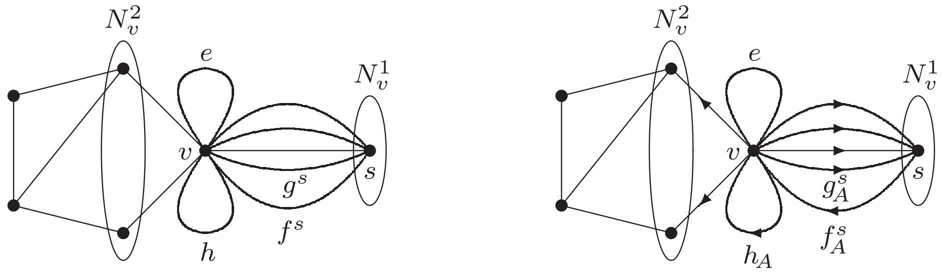

Figure 6. An example to Proposition 5.2.

Proposition 5.3. *If e is an edge joining two vertices, say v and u, in a connected graph H, then the subgraph H_e of H with vertex set $V_{H_e} = \{u, v\}$ and edge set $E_{H_e} = \{e\}$ is a good subgraph in H if and only if $H \notin \{C_2, C_3\}$ and neither v nor u is adjacent to a pendant edge in H.*

Proof. It is easy to observe that if H_e is a good subgraph in H, then $H \neq K_2^2$, $H \neq K_3$, and, by Observation 5.1, neither v nor u is adjacent to a leaf in H.

Thus assume that $H \neq K_2^2$, $H \neq K_3$, e is an edge joining vertices v and u in H, and neither v nor u is adjacent to a leaf in H. We shall prove that H_e is a good subgraph in H. We consider two cases, namely $N_H(\{v, u\}) = \{v, u\}$ and $\{v, u\} \subsetneq N_H(\{v, u\})$.

Case 1. $N_H(\{v, u\}) = \{v, u\}$. In this case, H is a graph of order 2. If e is the only edge joining v and u in H, then H_e is a good subgraph in H if and only if each of the vertices v and u is incident with a loop in H. If v and u are joined by two parallel edges in H, then H_e is a good subgraph in H if and only if at least one of the vertices v and u is incident with a loop in H (or, equivalently, H_e is not a good subgraph in H if $H = K_2^2$). Finally, if v and u are joined by at least three parallel edges in H, then H_e is always a good subgraph in H. In every case it is straightforward to recognize arcs or directed paths forming the families of directed paths $\mathcal{P}_v$ and $\mathcal{P}_u$ and desired digraphs F_v and F_u in graphs $H_1, \ldots, H_6$ shown in Figure 7.

Case 2. $\{v, u\} \subsetneq N_H(\{v, u\})$. In this case, the set $N_{vu}^- = N_H(\{v, u\}) \setminus \{v, u\}$ is nonempty, and, by our assumption, no vertex belonging to N_{vu}^- is a leaf in H. The set N_{vu}^- consists of five subsets: $N_v^1 = \{x \in N_{vu}^- \colon N_H(x) = \{v\}\}$, $N_u^1 = \{x \in N_{vu}^- \colon N_H(x) = \{u\}\}$, $N_{vu}^1 = \{x \in N_{vu}^- \colon N_H(x) = \{v, u\}\}$, $N_v^2 = \{x \in N_{vu}^- \colon \{v\} \subsetneq N_H(x)\}$, and $N_u^2 = \{x \in N_{vu}^- \colon \{u\} \subsetneq N_H(x)\}$. The sets N_v^1, N_u^1, N_{vu}^1, and $N_v^2 \cup N_u^2$ are disjoint, and it is possible that some of them are empty. Let A_E be a set of arcs obtained by assigning an orientation to every edge belonging to the set $E_{H_e}^-$, that is, to every edge incident with v or u and different from e. The set A_E and families $\mathcal{P}_v$ and $\mathcal{P}_u$ of directed paths that begin at v and u, respectively, are defined in the following way.

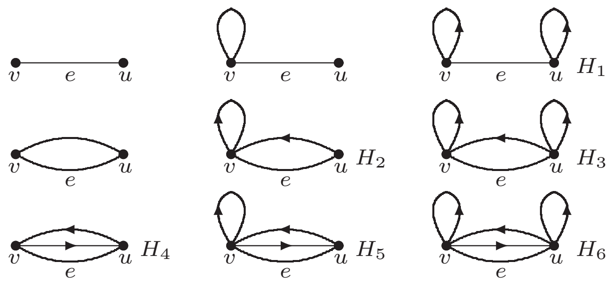

Figure 7. Examples to Case 1.

(1) To every loop f incident with x we assign an arbitrary orientation f_A, and add the 1-cycle (x, f_A, x) to $\mathcal{P}_x$ if $x \in \{v, u\}$.

(2) If an edge f belongs to $E_H(\{v, u\}, N_v^2 \cup N_u^2)$, then by f_A we denote the orientation of f toward $N_v^2 \cup N_u^2$. In addition, if $\varphi_H(f) = \{x, y\}$, where $x \in \{v, u\}$ and $y \in N_x^2$, then we add the 1-path (x, f_A, y) to $\mathcal{P}_x$.

(3) If $x \in N_v^1 \cup N_u^1$, and y is the only neighbor of x (belonging to $\{v, u\}$), then (as in the proof of Proposition 5.2) we choose two edges belonging to $E_H(x, y)$, say f^x and g^x, one of them, say f^x, obtain an orientation from x to y, and all other edges belonging to $E_H(x, y) \setminus \{f^x\}$ are oriented toward x. In this case, the 2-cycle (y, g_A^x, x, f_A^x, y) and all the 1-paths (y, h_A, x), for every $h \in E_H(x, y) \setminus \{f^x, g^x\}$ (if this set is nonempty), are added to $\mathcal{P}_y$.

(4) If the set N_{vu}^1 is nonempty, then we distinguish two cases.

(a) If at least one of the sets $E_H(v, N_v^1 \cup N_u^1) \cup E_v^l$ and $E_H(u, N_v^1 \cup N_u^1) \cup E_u^l$ is nonempty, say $E_H(v, N_v^1 \cup N_u^1) \cup E_v^l \neq \emptyset$, then for every $z \in N_{vu}^1$, we choose two edges belonging to $E_H(z, \{v, u\})$, say $f^z \in E_H(z, v)$ and $g^z \in E_H(z, u)$, orient f^z toward v, all other edges belonging to $E_H(z, \{v, u\}) \setminus \{f^z\}$ are oriented toward z, and to every edge in $E_H(v, u) \setminus \{e\}$ (if this set is nonempty) we choose an arbitrary orientation, say from u to v. Now the 2-path (u, g_A^z, z, f_A^z, v), the 1-paths (u, h_A, z) (for every $h \in E_H(u, z) \setminus \{g^z\}$) and (u, k_A, v) (for every $k \in E_H(u, v) \setminus \{e\}$) are added to $\mathcal{P}_u$, while the 1-paths (v, l_A, z) (for every $l \in E_H(v, z) \setminus \{f^z\}$) to $\mathcal{P}_v$.

(b) If both the sets $E_H(v, N_v^1 \cup N_u^1) \cup E_v^l$ and $E_H(u, N_v^1 \cup N_u^1) \cup E_u^l$ are empty, then we consider two subcases.

(b1) If $|N_{vu}^1| \geq 2$, and if C is a smallest subset of $E_H(\{v, u\}, N_{vu}^1)$ that covers the vertices in $\{v, u\} \cup N_{vu}^1$, then we orient the edges in C toward $\{v, u\}$, the edges in $E_H(\{v, u\}, N_{vu}^1) \setminus C$ toward N_{vu}^1, and the edges belonging to $E_H(v, u) \setminus \{e\}$ (if this set is nonempty) in an arbitrary way, again say from u to v. We may assume that $N_{vu}^1 = \{z_1, \ldots, z_k\}$, $C = \{f^{z_1}, \ldots, f^{z_k}\}$, and $\varphi_H(f^{z_i}) = \{z_i, x_i\}$, where $x_i \in \{v, u\}$ for $i \in [k]$. Let $D = \{g^{z_1}, \ldots, g^{z_k}\}$, where $\varphi_H(g^{z_i}) = \{z_i, y_i\}$

and y_i is the only element of $\{v, u\} \setminus \{x_i\}$ for $i \in [k]$. Now, we add all 1-paths (u, l_A, v) (for every $l \in E_H(v, u) \setminus \{e\}$) and 1-paths (u, l_A, z_i) (if $i \in [k]$ and $l \in E_H(u, z_i) \setminus (C \cup D)$) to $\mathcal{P}_u$, while we add 1-paths (v, p_A, z_i) (if $i \in [k]$ and $p \in E_H(v, z_i) \setminus (C \cup D)$) to $\mathcal{P}_v$. Finally, we add the 2-path $(y_i, g_A^{z_i}, z_i, f_A^{z_i}, x_i)$, $i \in [k]$, to $\mathcal{P}_v$ ($\mathcal{P}_u$, respectively) if and only if $y_i = v$ ($y_i = u$, respectively).

(b2) If $|N_{vu}^1| = 1$, say $N_{vu}^1 = \{z\}$, then, since $V_H = \{v, u, z\}$ and $H \neq K_3$, H is a proper spanning supergraph of K_3 and, therefore, it has parallel edges (as $E_v^l = E_u^l = \emptyset$). Without loss of generality, we assume that v is incident with parallel edges. There are five cases to consider, and they are sketched in Figure 8. In each of these cases, let f^z and g^z be an edge belonging to $E_H(v, z)$ and $E_H(u, z)$, respectively. We orient f^z toward v, all other edges belonging to $E_H(\{v, u\}, z) \setminus \{f^z\}$ we orient toward z, and the edges belonging to $E_H(v, u) \setminus \{e\}$ (if $E_H(v, u) \setminus \{e\} \neq \emptyset$) are directed toward u. Now, the 2-path (u, g_A^z, z, f_A^z, v) and 1-paths (u, h_A, z) ($h \in E_H(u, z) \setminus \{g^z\}$) form the family $\mathcal{P}_u$, while 1-paths (v, l_A, z) ($l \in E_H(v, z) \setminus \{f^z\}$) and (v, p_A, u) ($p \in E_H(v, u) \setminus \{e\}$) form the family $\mathcal{P}_u$.

Let F_v and F_u be digraphs generated by arcs belonging to families $\mathcal{P}_v$ and $\mathcal{P}_u$, respectively. Since families $\mathcal{P}_v$ and $\mathcal{P}_u$ consist of 1- and 2-paths, we observe that the digraphs F_v and F_u, and families $\mathcal{P}_v$ and $\mathcal{P}_u$ have the properties (1)–(4) stated in the definition of a good subgraph. Consequently, H_e is a good subgraph in H. ∎

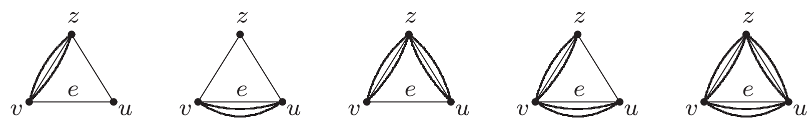

Figure 8. An example to Observation 5.3.

Remark 1. Let H be a graph without isolated vertices and let I be a proper subset of V_H. Then the induced subgraph $H[I]$ is a good subgraph in H if $1 \leq d_{H[I]}(v) < d_H(v)$ for every $v \in I$, and $N_H(x) \setminus I \neq \emptyset$ for every $x \in N_H(I) \setminus I$.

A proof of this statement is similar to the proofs of Propositions 5.2 and 5.3 and is omitted.

As a consequence of Observation 5.1 and Propositions 5.2 and 5.3, we have the following two corollaries.

Corollary 5.4. *A connected graph has a good subgraph if and only if it has a good subgraph generated by a loop or by an edge, that is, if and only if* $H \notin \{C_1, C_2, C_3\}$ *and it has an edge* (*or a loop*) *which is neither a pendant edge nor adjacent to a pendant edge in* H.

Corollary 5.5. *A tree* T *of order at least* 2 *has a good subgraph if and only if it has an edge which is neither a pendant edge nor adjacent to a pendant edge in* T.

It might be thought that a 2-subdivision graph of a graph having a good subgraph is not a minimal DTDP-graph, but it is not true in general since, for example, K_1^2 has a good subgraph and its 2-subdivision graph $S_2(K_1^2, \mathcal{P}_4)$ shown in Figure 3 is a minimal DTDP-graph. For this reason in the next theorem (which is important in our characterization of the minimal DTDP-graphs) we only consider 2-subdivision graphs without loops, that is, 2-subdivision graphs in which no twin parts corresponding to a loop are contracted into a single edge and, in consequence, forming a loop in the 2-subdivision graph.

Theorem 5.6. *Let H be a connected graph without isolated vertices, and let $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ be a family in which $\mathcal{P}(v)$ is a partition of the neighborhood $N_{S_2(H)}(v)$ for every $v \in V_H$. If H has a good subgraph and $\mathcal{P}$ does not contract in $S_2(H, \mathcal{P})$ any twin parts corresponding to a loop in H, then the 2-subdivision graph $S_2(H, \mathcal{P}, \theta)$ is a non-minimal* DTDP*-graph (for every positive function $\theta \colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$).*

Proof. Let Q be a good subgraph in H. By Corollary 5.4 we may assume that Q is a good subgraph generated by a loop or by an edge. By Observation 4.2 the 2-subdivision graph $S_2(H, \mathcal{P}, \theta)$ is a DTDP-graph. We shall prove that $S_2(H, \mathcal{P}, \theta)$ is not a minimal DTDP-graph. By Observation 3.4 it suffices to show that $S_2(H, \mathcal{P})$ is a non-minimal DTDP-graph. By virtue of Proposition 4.3 it suffices to show this non-minimality only in the case when $\mathcal{P}$ contracts in $S_2(H, \mathcal{P})$ far parts of adjacent pendant edges in H (since we have assumed that $\mathcal{P}$ does not contract in $S_2(H, \mathcal{P})$ any twin parts corresponding to any loop in H). In such case it is possible to observe that $S_2(H, \mathcal{P})$ is a non-minimal DTDP-graph if and only if $S_2(H)$ is a non-minimal DTDP-graph. Thus it remains to prove that $S_2(H)$ is a non-minimal DTDP-graph.

Assume first that the good subgraph Q in H is generated by a loop, say by a loop e incident with a vertex v. It is obvious that $H = K_1^s$ has a good subgraph and $S_2(K_1^s)$ is a non-minimal DTDP-graph if and only if $s \geq 2$. Thus assume that H is a connected graph of order at least 2. For simplicity, as far as possible, we adopt the notation from the proof of Proposition 5.2. For ease of presentation, we assume that $N_v^1 = \{v^1, \ldots, v^k\}$ and $E_H(v, v^i) = \{e_i^1, \ldots, e_i^{j_i}\}$ (where $j_i \geq 2$ as every edge in $E_H(v, N_v^1)$ is a multi-edge) for every $v^i \in N_v^1$. We may assume that A_E is an orientation of E_Q^- (of the set of edges or loops belonging to $E_H \setminus E_Q = E_H \setminus \{e\}$ that are incident with v) such that every loop belonging to E_v^l obtain an arbitrary direction, every edge belonging to E_v^2 is directed toward N_v^2, and edges belonging to E_v^1 are oriented in such a way that for every vertex $v^i \in N_v^1$ the edge e_i^1 is directed from v^i to v, and all other edges belonging to $E_H(v, v^i)$ are directed toward v^i. Let F_v be the digraph generated by the arcs belonging to A_E. Let $\mathcal{P}_v$ be the family consisting of all directed 2-cycles $(v, e_i^2, v^i, e_i^1, v)$ (for $i \in [k]$) and of all directed 1-paths (and 1-cycles) generated by all other arcs of F_v, see the left part of Figure 9. The digraph F_v and the family $\mathcal{P}_v$, as in the proof of Proposition 5.2, have the properties (1)–(4) stated in the definition of a good subgraph, implying that Q is a desired good subgraph in H.

Let G' be the proper spanning subgraph obtained from $S_2(H)$ by removing the "middle" edge $v_e^1v_e^2$ from the 3-cycle corresponding to the loop e of Q, and the third edge from the 4-path corresponding to the last arc in every directed path in $\mathcal{P}_v$, as illustrated in the right part of Figure 9. Formally, G' is the proper spanning subgraph of $S_2(H)$ with edge set $E_{G'} = E_{S_2(H)} \setminus (\{v_e^1v_e^2\} \cup \{vv_f^2 \colon f \in E_v^l\} \cup R_v^1 \cup R_v^2)$, where

$$R_v^1 = \bigcup_{i=1}^{k} \left\{ vv_{e_i^1}, v^iv^i_{e_i^3}, \ldots, v^iv^i_{e_i^{j_i}} \right\} \text{ and } R_v^2 = \bigcup_{u \in N_v^2} \{uu_g \colon g \in E_H(v,u)\}.$$

All that remains to prove is that G' is a DTDP-graph. It suffices to observe that every component of G' is a 2-subdivision graph. Let G'_v be the component of G' containing the vertex v. We note that G'_v belongs to the family $\mathcal{F}$ and, therefore, it is a DTDP-graph by Corollary 4.6. If the set N_v^2 is empty, then $G' = G'_v$ and the desired result follows. Thus assume that the set N_v^2 is non-empty. Then, since every edge belonging to the set

$$\bigcup_{u \in N_v^2} \{uu_g \colon g \in E_H(v,u)\}$$

joins a vertex in N_v^2 to a vertex in $V_{G'_v}$, while every edge belonging to the set

$$\{v_e^1v_e^2\} \cup \left\{vv_f^2 \colon f \in E_v^l\right\} \cup \bigcup_{i=1}^{k} \left\{ vv_{e_i^1}, v^iv^i_{e_i^3}, \ldots, v^iv^i_{e_i^{j_i}} \right\}$$

joins two vertices belonging to $V_{G'_v}$, the subgraph $G'' = G' - V_{G'_v}$ is an induced subgraph of G' and, in addition, G'' is a 2-subdivision graph, $G'' = S_2(H - V_{G'_v})$. Thus, by Observation 4.2, G'' is DTDP-graph. Consequently, since G'_v and G'' are DTDP-graphs, the proper spanning subgraph G' of $S_2(H)$ is a DTDP-graph and, therefore, $S_2(H)$ is a non-minimal DTDP-graph. (For example, in Figure 9 it is $G' = S_2(C_4) \cup S_2(K_{1,10}^2, \mathcal{P}, \theta)$, where $K_{1,10}^2$ is a star $K_{1,10}$ with two subdivided edges, $\mathcal{P}$ in $S_2(K_{1,10}^2)$ contracts all ten neighbors of the vertex corresponding to the central vertex of $K_{1,10}$ (or $K_{1,10}^2$), and finally the "new" pendant edge in $S_2(H, \mathcal{P})$ is replaced by twin pendant edges (using the function θ).)

Assume now that the good subgraph Q in H is generated by an edge, say by an edge e which joins two vertices v and u in H. We know that $S_2(H)$ is a DTDP-graph and we shall prove that $S_2(H)$ is a non-minimal DTDP-graph.

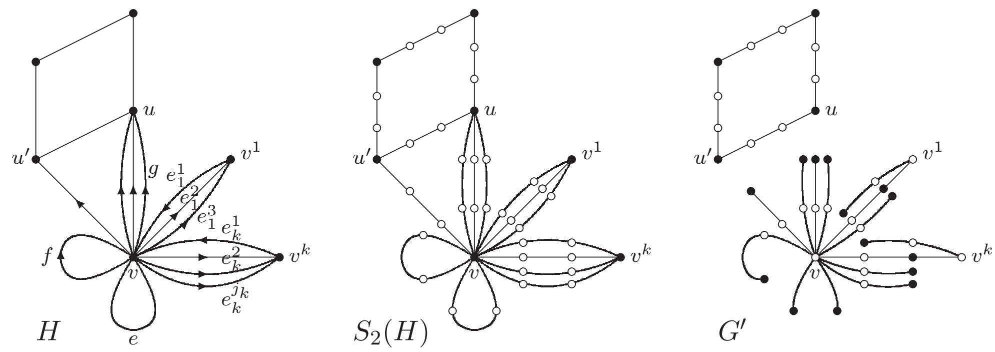

Figure 9. Graphs H, $S_2(H)$, and a spanning subgraph G' of $S_2(H)$.

We already know that $S_2(H)$ is a non-minimal DTDP-graph if H has a good subgraph generated by a loop. Thus assume that no subgraph of H generated by a loop is a good subgraph in H. Consequently, since H is a connected graph of order at least 2, every loop in H is incident with a support vertex, and, in particular, neither v nor u is incident with a loop. Certainly, neither v nor u is a support vertex in H. As in the proof of Proposition 5.3 we consider two cases, namely $N_H(\{v,u\}) = \{v,u\}$ and $\{v,u\} \subsetneq N_H(\{v,u\})$.

Case 1. $N_H(\{v,u\}) = \{v,u\}$. In this case, since neither v nor u is incident with a loop, $H = K_2^s$ and $s \geq 1$. From the fact that K_2^s has a good subgraph it follows that $s \geq 3$. Certainly, $S_2(K_2^s)$ is a non-minimal DTDP-graph if $s \geq 3$.

Case 2. $\{v,u\} \subsetneq N_H(\{v,u\})$. For simplicity we use the same notation as in the second part of the proof of Proposition 5.3. Consider the orientation A_E of E_Q^-, the families $\mathcal{P}_v$ and $\mathcal{P}_u$, and the digraphs F_v and F_u, introduced in Case 2 of the proof of Proposition 5.3. Let G' be the spanning subgraph of $S_2(H)$ obtained from $S_2(H)$ by removing the middle edge $v_e u_e$ from the 4-path (v, v_e, u_e, u) corresponding to the edge e, and the third edge from each 4-path corresponding to the last arc in every directed path in $\mathcal{P}_v$ or $\mathcal{P}_u$, see the lower part of Figure 10. As in the first part of the proof, G' is a DTDP-graph. Consequently, the proper spanning subgraph G' of $S_2(H)$ is a DTDP-graph and therefore $S_2(H)$ is a non-minimal DTDP-graph. ■

It follows from Corollary 5.4 that every path P_n (with $n \geq 6$) and every cycle C_m (with $m \geq 4$) has a good subgraph, and, therefore, Observation 4.2, Theorem 5.6, and a simple verification justify the following remark about minimal DTDP-paths and minimal DTDP-cycles.

Remark 2. If C_m is a cycle of size m, then $S_2(C_m)$ is a DTDP-graph for every positive integer m, but $S_2(C_m)$ is a minimal DTDP-graph if and only if $m \in$

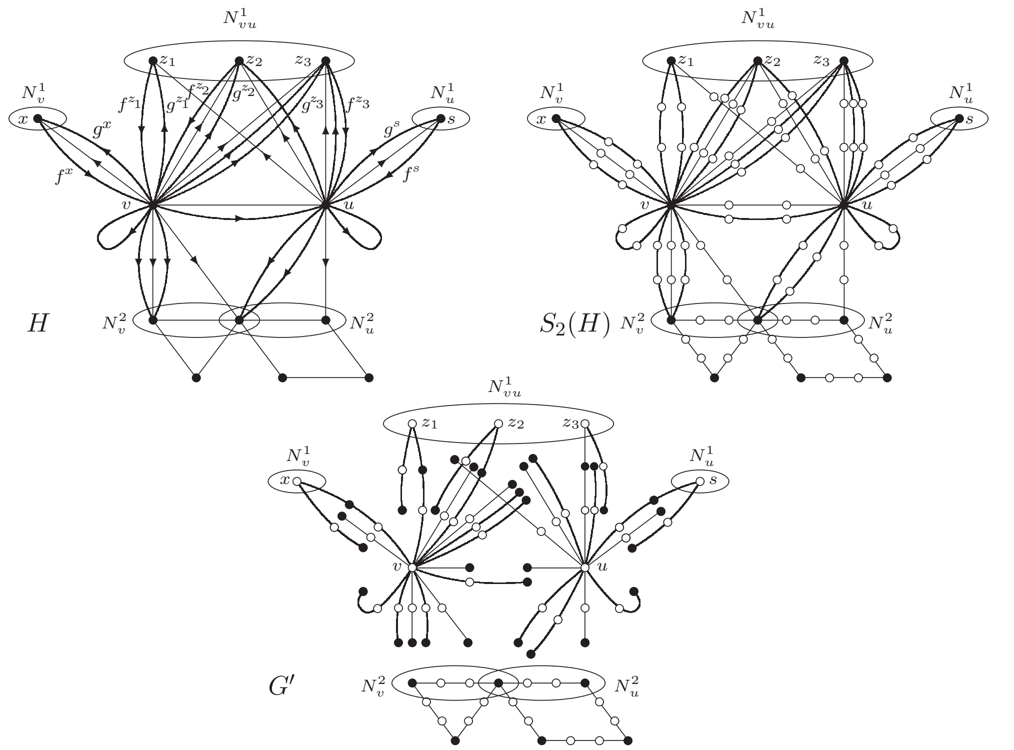

Figure 10. Graphs H, $S_2(H)$, and a spanning subgraph G' of $S_2(H)$.

$\{1, 2, 3\}$. If P_n is a path of order n, then $S_2(P_n)$ is a DTDP-graph for every integer $n \geq 2$, while $S_2(P_n)$ is a minimal DTDP-graph if and only if $n \in \{2, 3, 4, 5\}$.

6. Structural Characterization of the DTDP-Graphs

The next theorem presents general properties of DT-pairs in minimal DTDP-graphs.

Theorem 6.1. *A connected minimal* DTDP*-graph G is a* 2*-subdivision graph $S_2(H, \mathcal{P}, \theta)$ of some connected graph H, where $\mathcal{P} = \{\mathcal{P}(v) : v \in V_H\}$ is a family in which $\mathcal{P}(v)$ is a partition of the neighborhood $N_{S_2(H)}(v)$ for $v \in V_H$, and $\theta : L_{S_2(H,\mathcal{P})} \to \mathbb{N}$ is a positive function.*

Proof. Let G be a connected minimal DTDP-graph, and let (D, T) be a DT-pair in G. We proceed with the following series of claims, that yield structural properties of the graph G.

Claim 1. *The set D is a maximal independent set in G.*

Proof. If the set D were not independent, then two vertices belonging to D, say x and y, would be adjacent, and then (D,T) would be a DT-pair in $G-xy$, contradicting the minimality of G. Now, since D is both an independent set and a dominating set of G, the set D is a maximal independent set in G. $\square$

Claim 2. *Every component of* $G[T]$ *is a star or it is a graph of order* 1 *and size* 1.

Proof. Since T is a TD-set of G, by definition, $G[T]$ has no isolated vertex. Consequently, every component of $G[T]$ is either of order 1 and size at least 1 or has order at least 2. From the minimality of G, a component of order 1 in $G[T]$ has exactly one loop incident with its only vertex. Now let F be a component of order at least 2 in $G[T]$. To prove that F is a star, it suffices to show that if distinct vertices are adjacent in $G[T]$, then at least one of them is a leaf in $G[T]$. If x and y are adjacent in $G[T]$ and neither of them is a leaf in $G[T]$, then (D,T) would be a DT-pair in $G-xy$, violating the minimality of G. $\square$

Claim 3. *If* $x \in T$, *then* $|N_G(x) \setminus T| = 1$ *or* $N_G(x) \setminus T$ *is a nonempty subset of* L_G. *In addition, if* x *is a leaf in a star of order at least* 3 *in* $G[T]$, *then* $N_G(x) \setminus T$ *is a nonempty subset of* L_G.

Proof. Assume that $x \in T$. Then $N_G(x) \setminus T$ is a nonempty subset of D (since $D = V_G \setminus T$ is a dominating set in G). Therefore, since $L_G \subseteq D$ (by Observation 3.1), either $N_G(x) \setminus T$ is a nonempty subset of L_G or $N_G(x) \setminus (L_G \cup T)$ is nonempty. It remains to prove that if $N_G(x) \setminus (L_G \cup T)$ is nonempty, then $|N_G(x) \setminus T| = 1$. Assume that $y \in N_G(x) \setminus (L_G \cup T)$. Then, since D is independent and $y \in D \setminus L_G$, the set $N_G(y) \setminus \{x\}$ is a nonempty subset of T, say $x' \in N_G(y) \setminus \{x\}$. Now suppose that $|N_G(x) \setminus T| \geq 2$, and let y' be any vertex in $(N_G(x) \setminus T) \setminus \{y\}$. Then, since x is dominated by y' and y is totally dominated by x', the pair (D,T) is a DT-pair in $G-xy$, contradicting the minimality of G. Assume now that x is a leaf in a star of order at least 3 in $G[T]$. Let x' be the only neighbor of x in $G[T]$, and let $y \in N_G(x) \setminus \{x'\}$. It remains to show that y is a leaf in G. Suppose that y is not a leaf in G. Then $N_G(y) \setminus \{x\} \neq \emptyset$, and, if $x'' \in N_G(y) \setminus \{x\}$, then the pair $(D \cup \{x\}, T \setminus \{x\})$ is a DT-pair in $G-xy$, a contradiction. $\square$

We now return to the proof of the theorem. Let $G = (V_G, E_G, \varphi_G)$ be a graph (where, as usually, $\varphi_G(e)$ denotes the set of vertices incident with $e \in E_G$). Assume that G is a minimal DTDP-graph, and let (D,T) be a DT-pair in G. The minimality of G implies that G has neither multi-edges nor multi-loops. With respect to Observation 3.4 we may assume that G has no strong supports. This assumption, together with Claim 3, imply that every vertex v belonging to T has exactly one neighbor in D, and, in addition, this unique neighbor of v is a leaf in G if v is a leaf of a star of order at least 3 in $G[T]$. Consequently, if e

is an edge in $G[T]$, then the subset $N_G(\varphi_G(e)) \setminus T$ of D is of order 1 or 2. This implies that the triple $H = (V_H, E_H, \varphi_H)$ in which $V_H = D$, $E_H = E_{G[T]}$, and $\varphi_H \colon E_H \to 2^{V_H}$ is a function such that $\varphi_H(e) = N_G(\varphi_G(e)) \setminus T$ for each edge $e \in E_H$, is a well-defined graph with possible multi-edges or multi-loops. (If e is an edge in $G[T]$ and $\varphi_H(e) = \{a, b\}$, then e is an edge which joins the vertices a and b in H. Similarly, if e is an edge or a loop in $G[T]$ and $\varphi_H(e) = \{a\}$, then e is a loop which joins a to itself in H.) Now, to restore the graph G from the multi-graph H, we first form $S_2(H)$ inserting two new vertices into each edge and each loop of H. More precisely, if an edge e joins vertices a and b in the multi-graph H (that is, if $\varphi_H(e) = \{a, b\}$), then by a_e and b_e we denote the two mutually adjacent vertices inserted into the edge e, where a_e and b_e are adjacent to a and b, respectively. (If e is a loop incident with a vertex a (that is, if $\varphi_H(e) = \{a\}$), then by a_e^1 and a_e^2 we denote two mutually adjacent vertices inserted into the loop e and both adjacent to a.) Let $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ be a family in which the partition $\mathcal{P}(v)$ of the set $N_{S_2(H)}(v)$ (for $v \in V_H$ ($\subseteq V_{S_2(H)}$)) is defined as follows.

- If e is a loop in $G[T]$ incident with a vertex $N_G(v)$, then we let the 2-element set $\{v_e^1, v_e^2\}$ be an element of $\mathcal{P}(v)$.
- If e is an edge (not a loop) in $G[T]$ and $\varphi_G(e) \subseteq N_G(v)$, then we choose both one-element sets $\{v_e^1\}$ and $\{v_e^2\}$ to belong to $\mathcal{P}(v)$.
- If $\{e_1, \ldots, e_k\}$ is the edge set of a star in $G[T]$ and the central vertex of this star is in $N_G(v)$ (or exactly one vertex of this star is in $N_G(v)$, if $k = 1$), then we select the set $\{v_{e_1}, \ldots, v_{e_k}\}$ as an element of $\mathcal{P}(v)$.

From the above definition of the family $\mathcal{P}$, the 2-subdivision graph $S_2(H, \mathcal{P})$ ($= S_2(H, \mathcal{P}, \theta)$ if $\theta(x) = 1$ for every $x \in L_{S_2(H,\mathcal{P})}$) obtained from $S_2(H)$ is isomorphic to the graph G, see Figure 11 for an illustration. ■

From Theorem 6.1 every minimal DTDP-graph is a 2-subdivision graph of some graph. The converse, however, is not true in general. It is easy to check that neither of the 2-subdivision graphs $S_2(H)$, $S_2(H, \mathcal{P})$, and $S_2(H, \mathcal{P}, \theta)$ presented in Figure 1 is a minimal DTDP-graph. In our last theorem we present the main structural characterization of minimal DTDP-graphs without loops.

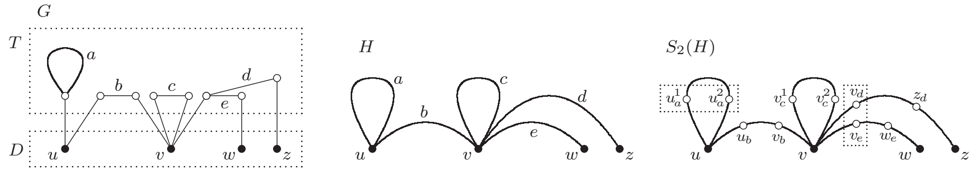

Figure 11.

Theorem 6.2. *Let G be a connected graph of order at least* 3 *that has no loops. Then the following statements are equivalent.*

(1) *The graph G is a minimal* DTDP*-graph.*

(2) *Either* (2a) $G \in \{C_3, C_6, C_9\}$ *or* (2b) *G is a* 2*-subdivision graph, say* $G = S_2(H, \mathcal{P}, \theta)$ (*where H is a connected graph of order at least* 2*,* $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V_H\}$ *is a family in which $\mathcal{P}(v)$ is a partition of the set $N_{S_2(H)}(v)$ for $v \in V_H$, and* $\theta\colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$ *is a positive function*) *in which* (2b1) *the pair* $(V^o_G, V^n_G) = (V^o_{S_2(H,\mathcal{P},\theta)}, V^n_{S_2(H,\mathcal{P},\theta)})$ *is the only* DT*-pair and, in addition, in which* (2b2) *every component of $G[V^n_G]$ is a star.*

(3) *The graph G is a* 2*-subdivision graph, say $G = S_2(H, \mathcal{P}, \theta)$, where either* (3a) $H \in \{C_1, C_2, C_3\}$ *or* (3b) *H is a connected graph of order at least* 2 *in which* (3b1) *every non-pendant edge* (*and every loop*) *is adjacent to a pendant edge,* (3b2) $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V_H\}$ *is a family in which every $\mathcal{P}(v)$ is a partition of the set $N_{S_2(H)}(v)$ for every $v \in V_H$, and $\mathcal{P}$ contracts in $S_2(H, \mathcal{P})$ only far parts of adjacent pendant edges of H* (*if any*)*, and* (3b3) $\theta\colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$ *is a positive function.*

Proof. (1) ⇒ (2) Assume first that G is a connected minimal DTDP-graph. Then G has no multi-edges and Theorem 6.1 implies that G is a 2-subdivision graph, i.e., $G = S_2(H, \mathcal{P}, \theta)$ (for some connected graph H without a good subgraph (by Theorem 5.6), some family $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V_H\}$ in which $\mathcal{P}(v)$ is a partition of the neighborhood $N_{S_2(H)}(v)$ (for $v \in V_H$) which contracts at most far parts of pendant edges (by Proposition 4.3), and a positive function $\theta\colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$). By Observation 3.4 we may assume that G has no strong supports, and therefore we may assume that $G = S_2(H, \mathcal{P})$ (as $S_2(H, \mathcal{P})$ and $S_2(H, \mathcal{P}, \theta)$ are isomorphic if $\theta(x) = 1$ for every $x \in L_{S_2(H,\mathcal{P})}$). By Observation 4.2, the pair $(D, T) = (V^o_G, V^n_G)$ is a DT-pair in G. In addition, the minimality of G, Theorem 6.1 and our assumption that G has no loop imply that every component of $G[V^n_G]$ is a star. Thus, it remains to prove that either G is a cycle of length 3, 6 or 9, or the pair (V^o_G, V^n_G) is the only DT-pair in G. We consider three cases depending on $\Delta(H)$.

Case 1. $\Delta(H) = 1$. In this case, $H = P_2$ and $G = S_2(H, \mathcal{P}) = P_4$ (as by our assumption G has no strong supports). Moreover, $(V^o_G, V^n_G) = (L_G, S_G)$ is the only DT-pair in G.

Case 2. $\Delta(H) = 2$. In this case, either $H = C_m$ ($m \ge 1$) or $H = P_n$ ($n \ge 3$). But, since $S_2(H, \mathcal{P})$ is a minimal DTDP-graph, Remark 2 implies that either $H = C_m$ and $m \in \{1, 2, 3\}$, or $H = P_n$ and $n \in \{3, 4, 5\}$. Now, depending on $\mathcal{P}$, $S_2(C_1, \mathcal{P}) = C_3$ or $S_2(C_1, \mathcal{P}) = C_1 \circ K_1$ and only C_3 has the desired properties (as $C_1 \circ K_1$ has a loop). It is also a simple matter to observe that $S_2(P_3, \mathcal{P}) = S_2(P_3) = P_7$ or $S_2(P_3, \mathcal{P}) = P_3 \circ K_1$ and each of these graphs has the desired properties. Simple verifications and Proposition 4.3 show that each of the graphs $S_2(C_2,\mathcal{P})$ (see Figure 2), $S_2(C_3,\mathcal{P})$, $S_2(P_4,\mathcal{P})$, and $S_2(P_5,\mathcal{P})$ is a minimal DTDP-graph if and only if $S_2(C_2,\mathcal{P}) = S_2(C_2) = C_6$, $S_2(C_3,\mathcal{P}) = S_2(C_3) = C_9$, $S_2(P_4,\mathcal{P}) = S_2(P_4) = P_{10}$, and $S_2(P_5,\mathcal{P}) = S_2(P_5) = P_{13}$, respectively. Certainly, each of these four graphs has the desired properties.

Case 3. $\Delta(H) \geq 3$. In this case we claim that $(D,T) = (V_G^o, V_G^n)$ is the only DT-pair in G. Suppose, to the contrary, that (D',T') is another DT-pair in G. Then, since D and D' are maximal independent sets in G (by Theorem 6.1) and $D \neq D'$, each of the sets $D \setminus D'$ and $D' \setminus D$ is a nonempty subset of T' and T, respectively. Let v be a vertex of maximum degree among all vertices in $D \setminus D' \subseteq T'$. Since $v \in T'$, it follows from Theorem 6.1 that $d_H(v) \geq 2$. If $H = K_1^s$ and $s \geq 2$ (as $d_H(v) \geq 3$), then K_1^1 would be a good subgraph in H, which is impossible. Hence, H has order at least 2. We consider two possible cases.

Case 3.1. *There is a loop, say e, at v.* In this case, a pendant edge, say f, is incident with v, as otherwise, by Proposition 5.2, the subgraph generated by e would be a good subgraph in H, which again is impossible. Assume that f joins the vertex v with a leaf u in H. We claim that v belongs to the set D'. Let A be the only set in $\mathcal{P}(v)$ which contains the vertex v_f. By Observation 3.1, $u \in D'$ and $(u_f, \{u_f\}) \in T'$. Thus, $(v,A) \in T'$ as T' is a TD-set of G and (v,A) is the only neighbor of $(u_f,\{u_f\})$ in T'. Finally, this implies that $v \in D'$ as D' is a dominating set of G and v is the only neighbor of $(v,A) \in T' = V_G \setminus D'$. Consequently, $v \in D'$ and $v \in D \setminus D'$ (by the choice of v), a contradiction.

Case 3.2. *No loop is incident with v.* In this case, let $f_1,\ldots,f_k$ be the edges incident with v in H, say $\varphi_H(f_i) = \{v, v^i\}$ for $i \in [k]$ ($k \geq 2$). If at least one of the edges $f_1,\ldots,f_k$ is a pendant edge in H, then $v \in D'$ (similarly as in Case 3.1) and this again contradicts the choice of v. Thus assume that none of the edges $f_1,\ldots,f_k$ is a pendant edge in H. Then, since H has no good subgraph, it follows from Corollary 5.4 that every vertex $v^1,\ldots,v^k$ is incident with a pendant edge in H. Analogously as in Case 3.1, each of the vertices $v^1,\ldots,v^k$ belongs to D' in G. Now, the minimality of G implies in turn that the vertices $(v^i,\{v^i_{f_i}\})$ belong to T' for $i \in [k]$. Consequently, the vertices $(v,\{v_{f_i}\})$ also belong to T', since T' is a TD-set in G and $(v,\{v_{f_i}\})$ is the only neighbor of $(v^i,\{v^i_{f_i}\})$ which is not in D' (for $i \in [k]$). Finally, since all the neighbors $(v,\{v_{f_i}\})$ ($i \in [k]$) of the vertex v are in T', the vertex v has to be in D', a final contradiction proving the implication (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (1) Assume that $G = S_2(H,\mathcal{P},\theta)$ (for some connected graph H, some family $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V_H\}$ in which $\mathcal{P}(v)$ is a partition of the neighborhood $N_{S_2(H)}(v)$ (for $v \in V_H$), and some positive function $\theta\colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$). If G is a cycle of length 3, 6 or 9, then G is a minimal DTDP-graph. Thus assume that (V_G^o, V_G^n) is the only DT-pair in G and every component of $G[V_G^n]$ is a star.

Certainly, G is a DTDP-graph (by Observation 4.2), and we shall prove that G is a minimal DTDP-graph. Suppose, to the contrary, that G is not a minimal DTDP-graph. Thus some proper spanning subgraph G' of G is a DTDP-graph. Let e be any edge belonging to G but not to G', say $\varphi_G(e) = \{v, u\}$. Then, since V_G^o is an independent set in G, either $|\{v, u\} \cap V_G^o| = 1$ or $\{v, u\} \subseteq V_G^n$.

Let (D', T') be a DT-pair in G' and, consequently, in $G - vu$ and G. Therefore $(D', T') = (V_G^o, V_G^n)$ (since (V_G^o, V_G^n) is the only DT-pair in G) and (V_G^o, V_G^n) is a DT-pair in $G - vu$. But this is impossible, as we will see below. Assume first that $|\{v, u\} \cap V_G^o| = 1$, say $v \in V_G^o$ and $u \in V_G^n$. Then $N_G(u) \cap V_G^o = \{v\}$ (by Observation 4.1 (4)) and, therefore, $N_{G-vu}(u) \cap V_G^o = \emptyset$, which contradicts the observation that (V_G^o, V_G^n) is a DT-pair in $G - vu$. Thus assume that $\{v, u\} \subseteq V_G^n$. Because v and u are adjacent in $G[V_G^n]$ and every component of $G[V_G^n]$ is a star, at least one of the vertices v and u is a leaf in $G[V_G^n]$, say v is a leaf in $G[V_G^n]$. Hence, u is the only neighbor of v in $G[V_G^n]$ and, therefore, no neighbor of v belongs to V_G^n in $G - vu$. Thus, V_G^n is not a TD-set of $G - vu$, which contradicts the observation that (V_G^o, V_G^n) is a DT-pair in $G - vu$. We conclude that G is a minimal DTDP-graph.

$(1) \Rightarrow (3)$ Assume again that G is a connected minimal DTDP-graph. Then the equivalence of (1) and (2) implies that either $G \in \{C_3, C_6, C_9\}$ or G is a 2-subdivision graph, say $G = S_2(H, \mathcal{P}, \theta)$ (where H is a connected graph of order at least 2, $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ is a family in which $\mathcal{P}(v)$ is a partition of the set $N_{S_2(H)}(v)$ for $v \in V_H$, and $\theta \colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$ is a positive function). Thus, either $G = S_2(H)$ and $H \in \{C_1, C_2, C_3\}$ or $G = S_2(H, \mathcal{P}, \theta)$, where H is a connected graph of order at least 2 in which every non-pendant edge and every loop is adjacent to a pendant edge (as otherwise H has a good subgraph (by Corollary 5.4) and then $G = S_2(H, \mathcal{P}, \theta)$ would be a non-minimal DTDP-graph (by Theorem 5.6)), $\mathcal{P} = \{\mathcal{P}(v) \colon v \in V_H\}$ is a family in which every $\mathcal{P}(v)$ is a partition of the set $N_{S_2(H)}(v)$ for every $v \in V_H$, and $\mathcal{P}$ contracts in $S_2(H, \mathcal{P})$ only far parts of adjacent pendant edges of H (as otherwise $G = S_2(H, \mathcal{P}, \theta)$ would be a non-minimal DTDP-graph (by Proposition 4.3)), and $\theta \colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$ is a positive function (as we already have observed).

$(3) \Rightarrow (1)$ Assume finally that H, $\mathcal{P}$, and θ have the properties stated in (3). Then $G = S_2(H, \mathcal{P}, \theta)$ is a DTDP-graph (by Observation 4.2). We shall prove that G is a minimal DTDP-graph. Since the minimality of $S_2(H, \mathcal{P}, \theta)$ does not depend on positive values of θ (by Observation 3.4), we may assume that $\theta(x) = 1$ for every $x \in L_{S_2(H,\mathcal{P})}$, and therefore we may assume that $G = S_2(H, \mathcal{P})$. By our assumption $\mathcal{P}$ contracts at most far parts of adjacent pendant edges in H, and since the minimality of $S_2(H, \mathcal{P})$ does not depend on such contractions (by Proposition 4.4), we may assume that $G = S_2(H)$. We shall prove that $G = S_2(H)$ is a minimal DTDP-graph. In order to prove this it suffices to show that G has the properties stated in (2). If $H \in \{C_1, C_2, C_3\}$, then we note that

$G \in \{C_3, C_6, C_9\}$. Thus, since every component of $G[V_G^n]$ is a star (in fact, a star of order 2), it remains to prove that the pair (V_G^o, V_G^n) is the only DT-pair in G if every non-pendant edge and every loop is adjacent to a pendant edge in H. Let (D,T) be a DT-pair in G. We shall prove that $(D,T) = (V_G^o, V_G^n)$. Since the pairs (D,T) and (V_G^o, V_G^n) form partitions of the set V_G, it suffices to show that $V_G^o \subseteq D$ and $V_G^n \subseteq T$. We prove these two containments showing that if e is an edge (a loop, respectively) of H and $\varphi_H(e) = \{v, u\}$ ($\varphi_H(e) = \{v\}$, respectively), then $\varphi_H(e) \subseteq D$ and $\{v_e, u_e\} \subseteq T$ ($\{v_e^1, v_e^2\} \subseteq T$, respectively). We distinguish three possible cases: (1) e is a pendant edge in H; (2) e joins two support vertices in H (or e is a loop incident with a support vertex in H); (3) exactly one of the two end vertices of e is a support vertex in H.

Case 1. *The edge e is a pendant edge in H, say $\varphi_H(e) = \{v, u\}$, where $u \in L_H$.* In this case, (u, u_e, v_e, v) is a 4-path in $G = S_2(H)$ and $d_G(u_e) = d_G(v_e) = 2$. Now $u \in D$ and $u_e \in T$ (by Observation 3.1), and this implies that $v_e \in T$ (since u_e belonging to a TD-set T in G has a neighbor in T and v_e is the only neighbor of u_e in $V_G \setminus D$) and $v \in D$ (since D is a dominating set in G and v is the only neighbor of v_e which is not in D). Consequently, $L_H \cup S_H \subseteq D$ and, in addition, if a pendant edge e joins vertices v and u in H, then $\{v_e, u_e\} \subseteq T$.

Case 2. $\varphi_H(e) = \{v, u\} \subseteq S_H$. In this case, $\varphi_H(e) \subseteq D$ (since $S_H \subseteq D$) and both v_e and u_e must be in T (because $\{v, u\} \subseteq D$, $N_G(v_e) = \{v, u_e\}$, $N_G(u_e) = \{u, v_e\}$, and (D,T) is a DT-pair in G). Similarly, if e is a loop incident with a support vertex v in H, then $\varphi_H(e) = \{v\} \subseteq S_H \subseteq D$ and, certainly, both v_e^1 and v_e^2 are in T.

Case 3. *A non-pendant edge e (which is not a loop) is incident with exactly one support vertex, say $\varphi_H(e) = \{v, u\}$ and $\varphi_H(e) \cap S_H = \{u\}$.* Let $E_H(v)$ denote the set of edges incident with v in H. Since $v \notin S_H$ and e is a non-pendant edge, $|E_H(v)| \geq 2$ and every element of $E_H(v)$ is a non-pendant edge (and it is not a loop). Therefore, since every non-pendant edge is adjacent to a pendant edge in H, each neighbor of v is a support vertex in H and, consequently, $N_H(v) \subseteq S_H \subseteq D$ in G. We claim that v also belongs to D in G. Suppose, to the contrary, that v is in T. Then, since (D,T) is a DT-pair in G, there is an edge f in $E_H(v)$ such that $v_f \in D$. Suppose, without loss of generality, that $\varphi_H(f) = \{v, w\}$. Then $N_G(w_f) = \{w, v_f\} \subseteq D$, and, therefore, $N_G(w_f) \cap T = \emptyset$, which is impossible as T is a TD-set in G. This proves that $v \in D$ and implies that both v and u are in D. Finally, as in Case 2, we observe that both v_e and u_e are in T. This completes the proof. ■

As an immediate consequence of Theorem 6.2, we have the following corollaries.

Corollary 6.3. *If H is a graph in which every vertex is a leaf or it is adjacent to* *at least one leaf, then $S_2(H, \mathcal{P}, \theta)$ is a minimal* DTDP*-graph if and only if $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V_H\}$ is a family of partitions $\mathcal{P}(v)$ of sets $N_{S_2(H)}(v)$ $(v \in V_H)$ which contracts only far parts of adjacent pendant edges (if any), and $\theta\colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$ is a positive function.*

Corollary 6.4. *A tree T of order at least 4 is a minimal* DTDP*-graph if and only if T is a 2-subdivision graph, say $T = S_2(R, \mathcal{P}, \theta)$, where R is a tree in which every non-pendant edge is adjacent to a pendant edge, $\mathcal{P} = \{\mathcal{P}(v)\colon v \in V_H\}$ is a family of partitions $\mathcal{P}(v)$ of sets $N_{S_2(R)}(v)$ $(v \in V_R)$, and $\mathcal{P}$ contracts only far parts of adjacent pendant edges (if any), and $\theta\colon L_{S_2(H,\mathcal{P})} \to \mathbb{N}$ is a positive function.*

7. Open Problems

We close this paper with the following list of open problems that we have yet to settle.

(1) Characterize the graphs with loops which are minimal DTDP-graphs.

(2) The *domatic-total domatic number* of a graph G, denoted $\mathrm{dom}_{\gamma\gamma_t}(G)$, is the maximum number of sets into which the vertex set of G can be partitioned in such a way that the subgraph induced by the set is a DTDP-graph. It is clear that $\mathrm{dom}_{\gamma\gamma_t}(G)$ is a positive integer only for DTDP-graphs. We write $\mathrm{dom}_{\gamma\gamma_t}(G) = 0$ if a graph G is not a DTDP-graph. Give bounds on the domatic-total domatic number of a graph in terms of order. It is quite easy to observe that $\mathrm{dom}_{\gamma\gamma_t}(G) \le |V_G|/3$. For which graphs G is $\mathrm{dom}_{\gamma\gamma_t}(G) = |V_G|/3$? If G is a tree, then $\mathrm{dom}_{\gamma\gamma_t}(G) \le |V_G|/4$. For which trees G is $\mathrm{dom}_{\gamma\gamma_t}(G) = |V_G|/4$?

(3) Study relations between the set of minimal DTDP-graphs and the set of graphs G for which $\gamma\gamma_t(G) = |V_G|$. The reader interested in knowing more about the parameter $\gamma\gamma_t(G)$ is recommended to refer to the book [14].

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