

AN IMPROVED BOUND ON THE CHROMATIC NUMBER OF THE PANCAKE GRAPHS

LEEN DROOGENDIJK

AND

ELENA V. KONSTANTINOVA

Sobolev Institute of Mathematics, Novosibirsk, 630090, Russia
Novosibirsk State University, Novosibirsk, 630090, Russia

e-mail: drooge001@kpnmail.nl
e_konsta@math.nsc.ru

Abstract

In this paper, an improved bound on the chromatic number of the Pancake graph P_n , $n \geq 9$, is presented. The bound is obtained using a subadditivity property of the chromatic number of the Pancake graph. We also investigate an equitable coloring of P_n . An equitable $(n-1)$ -coloring based on efficient dominating sets is given and optimal equitable 4-colorings are considered for small n . It is conjectured that the chromatic number of P_n coincides with its equitable chromatic number for any $n \geq 2$.

Keywords: Pancake graph, chromatic number, equitable coloring.

2020 Mathematics Subject Classification: 05C15, 05C25, 05C69.

1. INTRODUCTION

The *Pancake graph* P_n , $n \geq 2$, is defined as the Cayley graph over the symmetric group Sym_n with the generating set of all prefix-reversals r_i , $2 \leq i \leq n$, inverting the order of any initial substring of length $1 \cdots i$ of a permutation when multiplied on the right. It is a connected, vertex-transitive, $(n-1)$ -regular graph without loops and multiple edges of order $n!$. It contains all cycles C_l of length l , where $6 \leq l \leq n!$ [15, 25].

A mapping $c : V(\Gamma) \rightarrow \{1, 2, \dots, k\}$ is called a *proper k -coloring* of a graph $\Gamma = (V, E)$ if $c(u) \neq c(v)$ whenever the vertices u and v are adjacent. The *chromatic number* $\chi(\Gamma)$ of a graph Γ is the least number of colors needed to

properly color vertices of Γ . A subset of vertices assigned to the same color forms an independent set, i.e., a proper k -coloring is the same as a partition of the vertex set into k independent sets. The trivial lower and upper bounds on the chromatic number of the Pancake graphs are given as follows

$$(1) \quad 3 \leq \chi(P_n) \leq n - 1 \text{ for any } n \geq 4.$$

Indeed, the graph P_n is $(n-1)$ -regular, hence by Brooks' theorem [4] we have the upper bound. Moreover, $\chi(P_3) = 2$ since $P_3 \cong C_6$, and $\chi(P_4) = 3$ since there are 7-cycles in P_n for any $n \geq 4$ [20] which gives us the lower bound. The Brooks' bound is improved by 1 for graphs with $\omega \leq (\Delta - 1)/2$, where ω and Δ are the size of the maximum clique and the maximum degree of the graph (see [5, 6]). Since $\omega(P_n) = 2$, $\chi(P_n) \leq n - 2$ for any $n \geq 6$. Moreover, there is a proper 3-coloring of P_5 [19]. Thus, we have

$$(2) \quad \chi(P_n) \leq n - 2 \text{ for any } n \geq 5.$$

Catlin's bound for C_4 -free graphs [7], that is $\chi \leq \frac{2}{3}(\Delta + 3)$, gives one more bound for any $n \geq 8$,

$$(3) \quad \chi(P_n) \leq \frac{2}{3}(n + 2).$$

Using structural properties of P_n , the following bounds were obtained in [19].

$$(4) \quad \text{For } 5 \leq n \leq 8, \chi(P_n) \leq \begin{cases} n - k, & \text{if } n \equiv k \pmod{4} \text{ for } k = 1, 3; \\ n - 2, & \text{if } n \text{ is even;} \end{cases}$$

$$(5) \quad \text{for } 9 \leq n \leq 16, \chi(P_n) \leq \begin{cases} n - (k + 2), & \text{if } n \equiv k \pmod{4} \text{ for } k = 1, 3; \\ n - 4, & \text{if } n \text{ is even;} \end{cases}$$

$$(6) \quad \text{for } n \geq 17, \chi(P_n) \leq \begin{cases} n - (k + 4), & \text{if } n \equiv k \pmod{4} \text{ for } k = 1, 2, 3; \\ n - 8, & \text{if } n \equiv 0 \pmod{4}. \end{cases}$$

These bounds improve (2) for $n \geq 7$, however Catlin's bound (3) is still better for all $n > 28$ and some smaller n (for example, $n = 21, 25, 26, 27$). Thus, they are far from good. Meanwhile, the asymptotic bound $\chi(P_n) \leq O\left(\frac{n-1}{\log(n-1)}\right)$ holds for the Pancake graphs which follows from the results for C_3, C_4 -free graphs [10, 16].

In Section 2 of this paper, we present a new upper bound which improves Catlin's bound (3). By using a subadditivity property of the chromatic number of P_n and known chromatic numbers for $n \leq 9$, the new bound is obtained. We have $\chi(P_3) = 2$ since $P_3 \cong C_6$, and $\chi(P_4) = 3$ since there are 7-cycles in P_n , $n \geq 4$. An

n	2	3	4	5	6	7	8	9
$\chi(P_n)$	2	2	3	3	4	4	4	4

Table 1. Chromatic numbers of the Pancake graphs P_n , $2 \leq n \leq 9$.

example of a proper 3-coloring for P_5 was given in [19]. An *optimal coloring* of a graph is a proper coloring with the fewest number of colors. An optimal 4-coloring for P_6 was computed by Pisanski (University of Primorska, Koper, Slovenia) [23] and Azarija (University of Ljubljana, Slovenia), so $\chi(P_6) = 4$. Since P_{n-1} is an induced subgraph of P_n , then $\chi(P_7)$ is at least 4, and from (4) we have $\chi(P_7) = 4$. Optimal 4-colorings for P_8 and P_9 were computed by Ghodrati (Sharif University, Tehran, Iran) [13]. By (5), $4 \leq \chi(P_n) \leq 12$, where $10 \leq n \leq 16$, however, proper 4-colorings in these cases are unknown. The known chromatic numbers are presented in the Table 1.

In Section 3 an equitable coloring is considered. A graph Γ is said to be *equitably k -colorable* if Γ has a proper k -coloring such that the sizes of any two color classes differ by at most one. Such coloring is said to be *strongly equitable* if all color classes have the same size. The *equitable chromatic number* $\chi_{=}(\Gamma)$ is the smallest integer k such that Γ is equitably k -colorable. Equitable coloring was introduced by Meyer in 1973 in the context of scheduling problems [22]. Moreover, it was conjectured that every connected graph with maximum degree Δ has an equitable coloring with Δ or fewer colors, with the exceptions of complete graphs and odd cycles. A strengthened version [8] of the conjecture states that each such graph has an equitable coloring with exactly Δ colors, with one additional exception, a complete bipartite graph in which both sides of the bipartition have the same odd number of vertices. A survey on equitable colorings can be found in [21].

In Section 3.1 an equitable $(n-1)$ -coloring based on efficient dominating sets in the Pancake graphs $P_n, n > 2$, is presented. Moreover, in Section 3.2 optimal equitable 4-colorings for P_5, P_6 and P_7 are described.

Let us note that any equitable coloring of P_n with at most n colors has the property that the sizes of all color classes are equal since every integer at most n divides $n!$. Thus, we have a *strongly equitable coloring* [12].

Since equitable coloring is a proper coloring with an additional condition, the inequality $\chi(P_n) \leq \chi_{=}(P_n)$ holds for any $n \geq 2$. However, since all above optimal colorings are strongly equitable we have conjectured.

Conjecture 1. For any $n \geq 2$,

$$\chi(P_n) = \chi_{=}(P_n).$$

2. IMPROVED UPPER BOUND

Our main result is given by the following theorem.

Theorem 1. *For any $n \geq 9$, the following bound holds for the Pancake graph P_n .*

$$(7) \quad \chi(P_n) \leq 4 \left\lfloor \frac{n}{9} \right\rfloor + \chi(P_m),$$

where m is the remainder when n is divided by 9.

To prove this result, we need more notation. Let $[n] = \{1, 2, \dots, n\}$. We consider a permutation $\pi = [\pi_1 \pi_2 \dots \pi_n]$ written as a string in one-line notation, where $\pi_i = \pi(i)$ for any $i \in [n]$. For $K \subset [n]$, let $P_{n,K}$ be the induced subgraph of P_n whose vertex set consists of all permutations π with $\pi_1 \in K$. By the symmetry of P_n , for any k -element subset K of $[n]$, the induced subgraph $P_{n,K}$ is isomorphic to $P_{n,[k]}$, which is abbreviated to $P_{n,k}$.

We define a map $f_{n,k} : P_{n,k} \rightarrow P_k$ by removing the elements that are not in $[k]$. For example, for $n = 5$ and $k = 3$, the vertex $[14352]$ of $P_{5,3}$ is mapped to the vertex $[132]$ of P_3 . It is clear that this mapping is a graph homomorphism. Note that $f_{n,k}$ is surjective, but not necessarily an isomorphism. In fact, $P_{n,k}$ is not even connected unless $k = n$ or $n = 2$.

Since an r -coloring of a graph Γ is equivalent to a graph homomorphism from Γ to the complete graph K_r , this property implies that

$$(8) \quad \chi(P_{n,k}) \leq \chi(P_k).$$

Since $P_{n,k}$ always contains a subgraph isomorphic to P_k (e.g., the subgraph of all vertices that end with $k+1, k+2, \dots, n$) it even follows that $\chi(P_{n,k}) = \chi(P_k)$.

One more useful property says that all fibers $f_{n+1,n} : P_{n+1,n} \rightarrow P_n$ are of size n , which means that $|f_{n+1,n}^{-1}(v)| = n$ for every $v \in V(P_n)$. Indeed, for any permutation of length n , one can insert $n+1$ at n different positions: the first position is forbidden since the vertices of $P_{n+1,n}$ start with an element from $[n]$.

The following theorem immediately gives the general upper bound of (7) with taking into account $\chi(P_9) = 4$.

Theorem 2. *The chromatic number of the Pancake graph is subadditive, i.e.,*

$$(9) \quad \chi(P_{n+m}) \leq \chi(P_n) + \chi(P_m)$$

for all positive integers n and m .

Proof. Let the vertices of P_{n+m} be partitioned into sets U and W such that U contains permutations whose first element is in $[n]$ and W contains permutations

whose first element is in $\{n+1, \dots, n+m\}$. The subgraphs $P_{n+m,U}$ and $P_{n+m,W}$ are isomorphic to $P_{n+m,n}$ and $P_{n+m,m}$, respectively. Hence, by (8) the graph $P_{n+m,U}$ is $\chi(P_n)$ -colorable, and $P_{n+m,W}$ is $\chi(P_m)$ -colorable. Using disjoint color sets on both subgraphs proves the desired inequality. ■

3. EQUITABLE COLORING AND OPTIMAL COLORINGS

It was shown in the Introduction that there are optimal colorings of the Pancake graphs found by computer experiments. Such computations do not provide us any structural insight. In this section, we consider colorings of the Pancake graphs $P_n, n \geq 3$, using their structural properties. More precisely, we present equitable colorings of P_n in $(n-1)$ colors. Let us note that a proper $(n-1)$ -coloring of P_n was first considered in [19]. The new equitable $(n-1)$ -coloring avoids some incorrectness in description with having an easy and visual representation.

An equitable coloring is not the same as a *perfect coloring* for which the multiset of colors of all neighbors of a vertex depends only on its own color [11]. A perfect coloring gives a partition known as an *equitable partition* [14] which is used in algebraic combinatorics, graph theory and coding theory. In coding theory, such kind of partitions are known as perfect codes [1, 17]. Some general information about equitable partitions can be found in [2].

The notion of perfect codes was generalized to the Pancake graphs in a natural way in [9]. An independent set D of vertices in a graph Γ is an *efficient dominating set* (or 1-perfect code) if each vertex not in D is adjacent to exactly one vertex in D . There are n efficient dominating sets in P_n [9, 24] given by

$$(10) \quad D_i = \{[i \pi_2 \cdots \pi_n]\},$$

where $\pi_k \in [n] \setminus \{i\}, k \in [n] \setminus \{1\}, i \in [n]$. It is obvious that $|D_{i_1} \cap D_{i_2}| = \emptyset, i_1, i_2 \in [n], i_1 \neq i_2$, which immediately gives a proper n -coloring. Moreover, this coloring is perfect and equitable.

To present a proper $(n-1)$ -coloring of the Pancake graphs based on efficient dominating sets we need to define such sets for induced subgraphs P_{n-1} of P_n .

Due to the hierarchical structure, for any $n \geq 3$ the graph P_n has n copies of $P_{n-1}(i)$ with the vertex set $V_i = \{[\pi_1 \cdots \pi_{n-1} i]\}$, where $\pi_k \in [n] \setminus \{i\}, k \in [n-1], |V_i| = (n-1)!$. Any two copies $P_{n-1}(i), P_{n-1}(j), i \neq j$, are connected by $(n-2)!$ edges $\{[i \pi_2 \cdots \pi_{n-1} j], [j \pi_{n-1} \cdots \pi_2 i]\}$, where $[i \pi_2 \cdots \pi_{n-1} j]r_n = [j \pi_{n-1} \cdots \pi_2 i]$. Prefix-reversals $r_j, 2 \leq j \leq n-1$, define internal edges in all n copies $P_{n-1}(i)$, and the prefix-reversal r_n defines external edges between copies.

Efficient dominating sets of $P_{n-1}(j), j \in [n]$, contain all permutations with the last element fixed to j and the first element fixed [18], namely

$$(11) \quad D_i^j = \{[i \pi_2 \cdots \pi_{n-1} j]\},$$

where $i, j \in [n], i \neq j, \pi_k \in [n] \setminus \{i, j\}, k \in [n] \setminus \{1, n\}$. For any $i \in [n]$, the sets (10) and (11) are given by the following obvious relationship

$$(12) \quad D_i = \bigcup_{j=1, j \neq i}^n D_i^j.$$

3.1. Equitable $(n - 1)$ -coloring

We now present an equitable $(n - 1)$ -coloring based on efficient dominating sets. Let

$$(13) \quad D = \{D_i^j : i, j \in [n], i \neq j\}, \quad |D| = n(n - 1),$$

and

$$(14) \quad D^j = \{D_i^j : i \in [n], i \neq j\}, \quad j \in [n], \quad |D^j| = n - 1,$$

where

$$(15) \quad |D_i^j| = (n - 2)!.$$

Note that D partitions the vertices of P_n , and D^j partitions the vertices of P_n that end with j . We now define a graph Q_n whose vertices are the elements of D , and $X, Y \in D$ are adjacent in Q_n if and only if a vertex of X is adjacent to a vertex of Y in P_n . According to the properties of the Pancake graphs, we immediately see that vertices D_i^j and $D_{i'}^{j'}$ are adjacent in Q_n if and only if one of the following statements is true.

(A1) $j = j'$ and $i \neq i'$.

(A2) $i = j'$ and $j = i'$.

It is obvious that a proper coloring c of Q_n trivially extends to a proper coloring of P_n in such a way that any vertex of P_n belongs to exactly one efficient dominating set $X \in D$ and we give it the color $c(X)$.

We now have reduced the problem to find a proper $(n - 1)$ -coloring for Q_n . First, let us show an idea of such colorings for the graphs Q_4 and Q_6 . The graphs are presented on Figures 1 and 2 such that the vertices corresponding to the set $D_i^j, i, j \in [n], i \neq j$, are denoted by labels ij . The vertices are arranged in a Hamiltonian cycle such that all vertices with the same last element are grouped together and form 3- and 5-cliques, respectively. Within each clique, the first element of labeling is cyclically incremented. Obviously, the elements of each clique must all have different colors, but the pictures suggest that we can ‘almost’ cyclically repeat a color pattern chosen on the first clique. The only collisions occur with the ‘long’ chords that connect antipodal vertices. We see that if we

exchange the color of one end of each long chord with the color of the vertex counterclockwise next to it on the cycle, we obtain a proper coloring. It is clear that the proper colorings of Q_4 and Q_6 are equitable. Indeed, by (13)–(15) and from the construction, Q_4 has 3 color classes of cardinality 4 each, and Q_6 has 5 color classes of cardinality 6 each. However, these colorings are not perfect since the multisets of colors of all neighbors are different for different vertices having the same color. For example, in Q_4 the red vertex labeled 14 has one blue and two green neighbors, while the red vertex labeled 32 has one green and two blue neighbors. In Q_6 the green vertex labeled 31 has two blue neighbors, but the green vertex labeled 64 has only one blue neighbor.

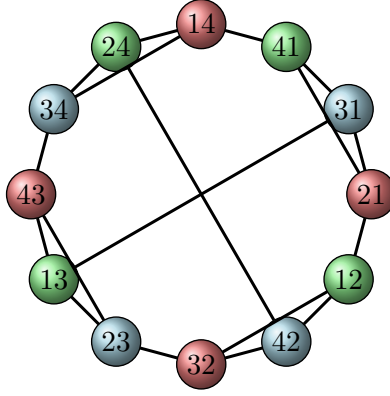


Figure 1. The equitable 3-coloring of Q_4 .

Note that this coloring is exactly the greedy coloring for the vertex sequence that starts with the vertex labeled $1n$ and then counterclockwise follows the cycle.

Now we formalize and prove this observation. First we define a map

$$(16) \quad f : D \rightarrow [n - 1]$$

such that

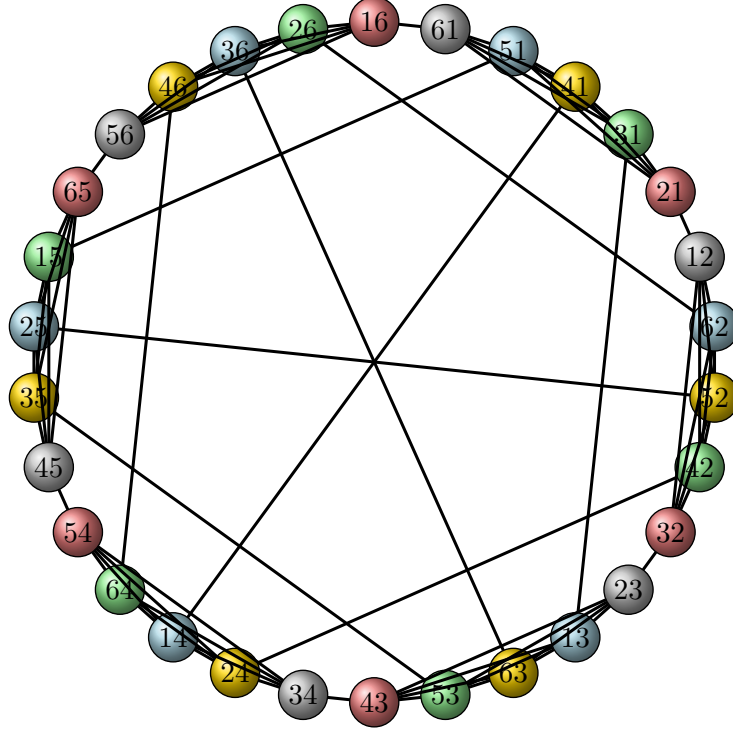
$$(17) \quad f(D_i^j) = \begin{cases} i - j, & \text{if } i > j; \\ n + i - j, & \text{otherwise.} \end{cases}$$

Note that f indeed has all its values in $[n - 1]$, and that the restriction of f to D^j is injective, i.e.,

$$f(D_i^j) = f(D_{i'}^j) \text{ if and only if } i = i'.$$

Next we let $k = \frac{n}{2}$ and define a coloring

$$(18) \quad c : D \rightarrow [n - 1]$$

Figure 2. The equitable 5-coloring of Q_6 .

by

$$(19) \quad c(D_i^j) = f(D_i^j) + \varepsilon,$$

where

$$(20) \quad \varepsilon = \begin{cases} +1, & \text{if } j > k \text{ and } f(D_i^j) = k; \\ -1, & \text{if } j > k \text{ and } f(D_i^j) = k + 1; \\ 0, & \text{otherwise.} \end{cases}$$

If n is odd, the first two cases cannot occur, since $f(D_i^j)$ is an integer, but k is not. The restriction of c to D^j is still injective, since it is either equal to f for $j \leq k$ or it is equal to f with at most two adjacent function values exchanged.

In terms of the above intuitive approach, f is the coloring that cyclically repeats along the cycle, and c is the coloring that exchanges the colors near the ends of long chords.

Theorem 3. *The coloring c is proper and equitable for $P_n, n > 2$.*

Proof. Indeed, suppose that $X = D_i^j$ and $Y = D_{i'}^{j'}$ are adjacent.

If $j = j'$, then $i \neq i'$ by (A1). By the injectivity of c for fixed j , X and Y must have different colors.

By (A2) the only other possibility is that $Y = D_j^i$. Without loss of generality we assume that $j < i$.

First we handle the case that n is odd. Then $c(X) = c(D_i^j) = f(D_i^j) = i - j$, and $c(Y) = c(D_j^i) = f(D_j^i) = n + j - i$. The equality $c(X) = c(Y)$ implies $n = 2i - 2j$, so n is even. There is a contradiction.

In the last part of the proof, we assume that n is even, so k is an integer. We do a case analysis based on the position of k .

1. If $j < i \leq k$, then $c(X) = i - j$, and $c(Y) = n + j - i$. The equality $c(X) = c(Y)$ implies $2i = n + 2j = 2k + 2j$, so $i = k + j > k$, and this gives a contradiction.

2. If $j \leq k < i$, then $c(X) = i - j$, and $c(Y) = n + j - i + \varepsilon$. The equality $c(X) = c(Y)$ implies $i - j = n + j - i + \varepsilon$, so $2k + 2j = 2i - \varepsilon$. Then ε must be even, hence $\varepsilon = 0$, and $i = k + j$ or $f(Y) = f(D_j^i) = n + j - i = k$. Since $i > k$ we have $\varepsilon = 1$ by (20) which gives a contradiction.

3. If $k < j < i$, then $c(X) = i - j + \varepsilon_1$ and $c(Y) = n + j - i + \varepsilon_2$. The equality $c(X) = c(Y)$ implies $i - j + \varepsilon_1 = n + j - i + \varepsilon_2$, so $2i + \varepsilon_1 = 2k + 2j + \varepsilon_2$, and hence ε_1 and ε_2 have the same parity. There are the following possibilities.

- (i) If $\varepsilon_1 = \varepsilon_2$, then $i - j = k$. Since $k < j < i \leq n = 2k$, we have $k = i - j < 2k - k = k$ which gives a contradiction.
- (ii) If $\varepsilon_1 = -1$, $\varepsilon_2 = 1$, then as in case (i), we have $k + 1 = i - j < 2k - k = k$ which leads to a contradiction.
- (iii) If $\varepsilon_1 = 1$, $\varepsilon_2 = -1$, then $i - j = k - 1$. Again, since $k < j < i \leq 2k$, this is only possible if $j = k + 1$ and $i = n = 2k$. Then $f(D_i^j) = k - 1$, so $\varepsilon_1 = 0$ by (20), and we have a contradiction.

In all cases, the assumption of color equality leads to a contradiction, which finishes the proof that c is a proper coloring. Moreover, by (18) we have $(n - 1)$ color classes each of which has a cardinality $|D^j| \cdot |D_i^j| = n(n - 2)!$ (see (14), (15)) which finishes the proof of the statement. ■

3.2. Optimal colorings

It is obvious that above equitable $(n - 1)$ -coloring produces an optimal coloring for P_3 and P_4 (see Table 1). However, for $n > 4$ a proper coloring of Q_n can never produce an optimal coloring for P_n . Indeed, by (2) for $n > 4$ we have $\chi(P_n) < n - 1$ and $\chi(Q_n) \geq n - 1$ since Q_n contains an $(n - 1)$ -clique.

Now we give a simple optimal 4-coloring of P_5 , P_6 or P_7 . We define an *even* (*odd*) prefix-reversal r_i , $2 \leq i \leq n$, if it corresponds to an even (odd) permutation. By [19, Lemma 4], if $i \equiv 0, 1 \pmod{4}$, then r_i is an even prefix-reversal. Similar,

r_i is an odd prefix-reversal if $i \equiv 2, 3 \pmod{4}$. By [19, Lemma 6], the Pancake graph $P_n, n \geq 3$, has $n!/\ell$ independent even ℓ -cycles where $6 \leq \ell \leq 2n$.

Let Γ be one of P_5, P_6, P_7 . Then the subgraph H generated by the even prefix-reversals r_4 and r_5 is a spanning subgraph of Γ consisting of disjoint 10-cycles $C_{10} = (r_5 r_4)^5$. Since even prefix-reversals preserve the parity of the permutation, all vertices of H that are on the same cycle have the same parity. Since all other edges of Γ correspond to odd permutations, they have one endpoint on an ‘even’ 10-cycle and the other endpoint on an ‘odd’ 10-cycle. Thus, we can 2-color the even cycles and 2-color the odd cycles using two other colors. This results in an equitable 4-coloring of Γ where each of the color classes has $n!/4$ vertices. Since $\chi(P_6) = \chi(P_7) = 4$ this coloring is optimal for P_6 and P_7 .

4. DISCUSSION

Conjecture 1 holds for some simple examples of graphs, such as even cycles, bipartite graphs with equal parts. Any Cayley graph over the symmetric group generated by a set of transpositions gives a bipartite graph with two equal color classes. The statement holds for the Hamming graphs $H(d, q)$ whose (equitable) chromatic number is q . A regular graph with a Hoffman coloring always gives a strongly equitable coloring [3]. Hoffman’s lower bound is known as $\chi(\Gamma) \geq 1 - \lambda_1/\lambda_v$, where λ_1 and λ_v are the largest and the smallest eigenvalues of Γ . If equality holds, an optimal coloring of Γ is called a Hoffman coloring.

The bound given by Theorem 1 depends on exact values of the chromatic number for the Pancake graph P_n . Its current version uses the known value $\chi(P_9) = 4$. As soon as we would know that $\chi(P_{10}) = 4$, the subadditivity property given by Theorem 2 implies a stronger version of Theorem 1.

Clearly the subadditivity property only makes sense for graphs that are hierarchical in some sense. The property trivially holds for hypercube graphs, since these are 2-colorable. It also holds trivially for a Mycielskian hierarchy. A less trivial example is given by the Burnt Pancake graphs (the proof is similar to the proof given above for the Pancake graphs).

Acknowledgements

The research work of the second author is supported by the Mathematical Center in Akademgorodok under agreement No. 075-15-2019-1613 with the Ministry of Science and Higher Education of the Russian Federation. The authors thank Alexander Kostochka and Sergey Avgustinovich for useful discussions on equitable colorings. The authors thank the reviewers for their valuable comments which improved the manuscript.

REFERENCES

- [1] R. Ahlswede, H. Aydinian and L. Khachatrian, *On perfect codes and related concepts*, Des. Codes Cryptogr. **22** (2001) 221–237.
<https://doi.org/10.1023/A:1008394205999>
- [2] R.A. Bailey, P.J. Cameron, A.L. Gavriluk and S.V. Goryainov, *Equitable partitions of Latin-square graphs*, J. Combin. Des. **27** (2019) 142–160.
<https://doi.org/10.1002/jcd.21634>
- [3] A. Blokhuis, A.E. Brouwer and W.H. Haemers, *On 3-chromatic distance-regular graphs*, Des. Codes Cryptogr. **44** (2007) 293–305.
<https://doi.org/10.1007/s10623-007-9100-7>
- [4] R.L. Brooks, *On colouring the nodes of a network*, Math. Proc. Cambridge Philos. Soc. **37** (1941) 194–197.
<https://doi.org/10.1017/S030500410002168X>
- [5] O.V. Borodin and A.V. Kostochka, *On an upper bound of a graph's chromatic number, depending on the graph's degree and density*, J. Combin. Theory Ser. B **23** (1977) 247–250.
[https://doi.org/10.1016/0095-8956\(77\)90037-5](https://doi.org/10.1016/0095-8956(77)90037-5)
- [6] P.A. Catlin, *A bound on the chromatic number of a graph*, Discrete Math. **22** (1978) 81–83.
[https://doi.org/10.1016/0012-365X\(78\)90049-3](https://doi.org/10.1016/0012-365X(78)90049-3)
- [7] P.A. Catlin, *Another bound on the chromatic number of a graph*, Discrete Math. **24** (1978) 1–6.
[https://doi.org/10.1016/0012-365X\(78\)90167-X](https://doi.org/10.1016/0012-365X(78)90167-X)
- [8] B.-L. Chen, K.-W. Lih and P.-L. Wu, *Equitable coloring and the maximum degree*, European J. Combin. **15** (1994) 443–447.
<https://doi.org/10.1006/eujc.1994.1047>
- [9] I.J. Dejter and O. Serra, *Efficient dominating sets in Cayley graphs*, Discrete Appl. Math. **129** (2003) 319–328.
[https://doi.org/10.1016/S0166-218X\(02\)00573-5](https://doi.org/10.1016/S0166-218X(02)00573-5)
- [10] A. Johansson, *Asymptotic choice number for triangle free graphs*, DIMACS Technical Report **91–4** (1996) 1196.
- [11] D.G. Fon-Der-Flaass, *Perfect 2-colorings of a hypercube*, Sib. Math. J. **48** (2007) 740–745.
<https://doi.org/10.1007/s11202-007-0075-4>
- [12] H. Furmańczyk, M. Kubale and V.V. Mkrtchyan, *Equitable colorings of corona multiproducts of graphs*, Discuss. Math. Graph Theory **37** (2017) 1079–1094.
<https://doi.org/10.7151/dmgt.1992>
- [13] H. Ghodrati, (private communications), 04.04.2019.
- [14] C.D. Godsil, *Compact graphs and equitable partitions*, Linear Algebra Appl. **255** (1997) 259–266.
[https://doi.org/10.1016/S0024-3795\(97\)83595-1](https://doi.org/10.1016/S0024-3795(97)83595-1)

- [15] A. Kanevsky and C. Feng, *On the embedding of cycles in pancake graphs*, Parallel Comput. **21** (1995) 923–936.
[https://doi.org/10.1016/0167-8191\(94\)00096-S](https://doi.org/10.1016/0167-8191(94)00096-S)
- [16] J.H. Kim, *On Brooks’ theorem for sparse graphs*, Combin. Probab. Comput. **4** (1995) 97–132.
<https://doi.org/10.1017/S0963548300001528>
- [17] S. Klavžar, U. Milutinović and C. Petr, *1-perfect codes in Sierpiński graphs*, Bull. Aust. Math. Soc. **66** (2002) 369–384.
<https://doi.org/10.1017/S0004972700040235>
- [18] E.V. Konstantinova, *On some structural properties of star and Pancake graphs*, in: Information Theory, Combinatorics, and Search Theory, Lecture Notes in Comput. Sci. **7777** (2013) 472–487.
https://doi.org/10.1007/978-3-642-36899-8_23
- [19] E.V. Konstantinova, *Chromatic properties of the Pancake graphs*, Discuss. Math. Graph Theory **37** (2017) 777–787.
<https://doi.org/10.7151/dmgt.1978>
- [20] E.V. Konstantinova and A.N. Medvedev, *Cycles of length seven in the Pancake graph*, Diskretn. Anal. Issled. Oper. **17** (2010) 46–55, in Russian.
- [21] K.W. Lih, *Equitable coloring of graphs*, in: Handbook of Combinatorial Optimization, P. Pardalos, D.Z. Du and R. Graham (Ed(s)), (Springer, New York, 2013) 1199–1248.
https://doi.org/10.1007/978-1-4419-7997-1_25
- [22] W. Meyer, *Equitable coloring*, Amer. Math. Monthly **80** (1973) 920–922.
<https://doi.org/10.1080/00029890.1973.11993408>
- [23] T. Pisanski, (private communications), 01.09.2015.
- [24] K. Qiu, *Optimal broadcasting algorithms for multiple messages on the star and pancake graphs using minimum dominating sets*, Congr. Numer. **181** (2006) 33–39.
- [25] J.-J. Sheu, J.J.M. Tan and K.-T. Chu, *Cycle embedding in pancake interconnection networks*, in: Proc. 23rd Workshop on Combinatorial Mathematics and Computation Theory (Taiwan, 2006) 85–92.

Received 15 March 2021

Revised 8 September 2021

Accepted 10 September 2021

Available online 22 September 2021