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# COVERING THE EDGES OF A RANDOM HYPERGRAPH BY CLIQUES

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#### Abstract

We determine the order of magnitude of the minimum clique cover of the edges of a binomial, *r*-uniform, random hypergraph  $G^{(r)}(n,p)$ , *p* fixed. In doing so, we combine the ideas from the proofs of the graph case (r = 2) in Frieze and Reed [*Covering the edges of a random graph by cliques*, Combinatorica 15 (1995) 489–497] and Guo, Patten, Warnke [*Prague dimension of random graphs*, manuscript submitted for publication].

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# 1. INTRODUCTION

For an r-uniform hypergraph (briefly, an r-graph) H = (V, E) and a set S, a representation of H on S is an assignment of subsets  $S_v \subset S$ ,  $v \in V$ , in such a way that for each  $R \in \binom{V}{r}$  we have  $\bigcap_{v \in R} S_v \neq \emptyset$  if and only if  $R \in E$ . To observe that any r-graph admits such a representation, assign to each vertex v the set  $\{e : v \in e \in E\}$  of all edges e containing v. Then,  $\{v_1, \ldots, v_r\} \in E$  if and only if  $\bigcap_{i=1}^r S_{v_i} \neq \emptyset$ .

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**Definition.** The representation number  $\theta_1(H)$  of H is the smallest cardinality of a set S which admits a representation of H. Equivalently,  $\theta_1(H)$  is also the smallest number of cliques needed to cover all edges of H (see Appendix for a proof of the equivalence).

It is perhaps interesting to note (cf. [6]) that the maximum of  $\theta_1(H)$  over all *r*-graphs *H* on *n* vertices equals the Turán number for the *r*-uniform clique  $K_{r+1}^{(r)}$ on r+1 vertices which is unknown even for r=3.

We determine a typical order of magnitude of the parameter  $\theta_1$  for a class of large random r-graphs. Given integers  $n, r \geq 2$ , and a real 0 , let $<math>G^{(r)}(n, p)$  denote the random r-graph obtained by independent inclusion of each r-set with probability p. In particular, the number of edges of  $G^{(r)}(n, p)$  is binomially distributed with expectation  $\binom{n}{r}p$ . We say that a property of r-sets  $\mathcal{P}$  holds asymptotically almost surely, abbreviated to a.a.s., if the probability  $\operatorname{Prob}(G^{(r)}(n, p) \in \mathcal{P}) \to 1$  as  $n \to \infty$ . Throughout the paper p remains independent of n, while all logarithms are natural and denoted by log.

**Theorem 1.** For every integer  $r \ge 2$  and a constant  $0 , there exist positive constants <math>c_1$  and  $c_2$  such that a.a.s.

$$c_1 \frac{n^r}{(\log n)^{r/(r-1)}} \le \theta_1(G^{(r)}(n,p)) \le c_2 \frac{n^r}{(\log n)^{r/(r-1)}}.$$

The case r = 2 was proved by Frieze and Reed [3], and, in a stronger form, by Guo, Patten, and Warnke [4]. Here we follow the ideas from there, some of which have already originated in a paper by Alon, Kim, and Spencer [1]. The lower bound follows immediately from the upper bound on the order of the largest clique in  $G^{(r)}(n, p)$  (see below). Hence, in the remaining sections we focus on the upper bound only.

**Proof of the lower bound in Theorem 1.** Recall that p and r are constants independent of n. The expected number of cliques of order

$$t := \left\lceil \left( \frac{r!}{\log(1/p)} \log n \right)^{1/(r-1)} \right\rceil + r$$

in  $G^{(r)}(n,p)$  is

$$\binom{n}{t}p^{\binom{t}{r}} \le \left(\frac{en}{t}p^{(t-r)^{r-1}/r!}\right)^t \le \left(\frac{en}{t}e^{-\log n}\right)^t = o(1)$$

as  $n \to \infty$ . Hence, a.a.s., there are no cliques of order t (or higher). On the other hand, by Chebyshev's inequality, there are, a.a.s., at least  $\frac{1}{2}p\binom{n}{r}$  edges in

 $G^{(r)}(n,p)$ . Therefore, a.a.s., one needs at least

$$\frac{\frac{1}{2}p\binom{n}{r}}{\binom{t}{r}} \ge \frac{1}{2}p\left(\frac{n}{t}\right)^r = \Omega\left(\frac{n^r}{(\log n)^{r/(r-1)}}\right)$$

cliques to cover all edges of  $G^{(r)}(n, p)$ .

For the proof of the upper bound we now define a crucial notion. Let an r-graph G on a vertex set V, two integers  $0 \le s < j \le |V|$ , and a set  $S \in \binom{V}{s}$  be given. A subset  $J \subset V$  is called an (S, j)-clique in G if

- $J \supset S, |J| = j$ , and
- $E_J := \{f \in {J \choose r} : f \cap (J \setminus S) \neq \emptyset\} \subset E(G)$ , that is, E(G) contains all *r*-element subsets of *J* except those which are subsets of *S*.

Note that  $|E_J| = {j \choose r} - {s \choose r}$ . Moreover, for s = 0, an  $(\emptyset, j)$ -clique is just any copy of the clique  $K_j^{(r)}$  in G, while for j < r every  $J \in {V \choose j}$ ,  $J \supseteq S$ , is a (trivial) (S, j)-clique (with  $E_J = \emptyset$ ).

# 2. An Expanding Property of Random r-Graphs

Throughout the paper V is an *n*-vertex set of the random r-graph  $G^{(r)}(n,p)$ . Given  $s \ge r-1$ , and a set  $S \in \binom{V}{s}$ , let X(S) be the number of (S, s+1)cliques in  $G^{(r)}(n,p)$ . In other words, X(S) counts the common neighbors of all (r-1)-element subsets of S. Clearly,

(1) 
$$\mathbf{E}(X(S)) = (n-s)p^{\binom{s}{r-1}} := \mu(s).$$

The next lemma asserts that, for a wide range of s and for every s-element set S of vertices there are roughly the same number of (S, s + 1)-cliques in  $G^{(r)}(n, p)$ . Let

$$k = \left\lfloor (\alpha \log n)^{1/(r-1)} \right\rfloor$$

for sufficiently small  $\alpha > 0$ .

**Claim 2.** Let  $\mathcal{A}$  be the event that for all  $r-1 \leq s \leq k-1$  and all  $S \in {\binom{V}{s}}$ 

$$|X(S) - \mathbf{E}(X(S))| \le n^{-1/3} \mathbf{E}(X(S)).$$

Then,  $\operatorname{Prob}(\mathcal{A}) = 1 - o(1)$ .

**Proof.** Note that  $\mu(s)$  is a decreasing function of s and, by the definition of k above,  $\binom{k-1}{r-1} \leq (k-1)^{r-1} \leq \alpha \log n$ . Thus, for large n,

(2) 
$$\mu(s) \ge \mu(k-1) \ge (n/2)p^{\binom{k-1}{r-1}} \ge (n/2)p^{\alpha \log n} \ge \frac{1}{2}n^{0.99},$$

if only  $\alpha \log(1/p) \le 0.01$ .

Recall that, for every  $S \in {\binom{V}{s}}$ , the random variable X(S) counts the number of (S, s + 1)-cliques and so, it is binomially distributed with expectation given by (1). Thus, by Chernoff's bound (see, e.g., [5], Corollary 2.3, Ineq. (2.9)), assuming n is sufficiently large,

$$\operatorname{Prob}(|X(S) - \mu(s)| > n^{-1/3}\mu(s)) \le 2\exp\{-n^{-2/3}\mu(s)/3\} \le \exp\{-n^{1/4}\},$$

where the last inequality follows from (2). Finally, by the union bound, summing over all choices of s and S,

$$\operatorname{Prob}(\neg \mathcal{A}) \le kn^k \exp\{-n^{1/4}\} = o(1).$$

### 3. Proof of Theorem 1

The upper bound in Theorem 1 will be a consequence of Claim 2 given in Section 2 and Lemma 4 to be stated below.

#### 3.1. Notation

Before stating Lemma 4, we introduce a few parameters used therein. Very roughly, the lemma will claim the existence of a sequence of r-graphs  $G_1, \ldots, G_{i_0}$ ,

$$i_0 = \left\lceil \frac{r+1}{r-1} \log \log n \right\rceil,$$

which begins with the random r-graph  $G_1 := G^{(r)}(n, p)$  and maintains throughout certain properties. As the proof of Lemma 4 will reveal, each next graph  $G_{i+1}$ will be derived from  $G_i$  by a random deletion of cliques of order  $k_i$ , where, for sufficiently small  $\alpha > 0$ ,

$$k_i = \left\lfloor \left(\frac{\alpha \log n}{i}\right)^{1/(r-1)} \right\rfloor$$

(Note that  $k_1 = k$  defined earlier.)

In addition, some random edges of  $G_i$  will be deleted as well. The random procedure will be designed in such a way that the graphs  $G_i$  will shrink at the rate of 1/e, thus resembling random r-graphs  $G^{(r)}(n, p_i)$ , where

$$p_i = p e^{1-i}.$$

The resemblance will be manifested by the behavior of the number of (S, j)-cliques in  $G_i$ , which, for all  $0 \le s < j \le k_i$ , will be close to the quantity

(3) 
$$\mu_i(s,j) = \binom{n-s}{j-s} p_i^{\binom{j}{r} - \binom{s}{r}}.$$

Note that  $\mu_i(s, j)$  is the expected number of (S, j)-cliques in a random r-graph  $G^{(r)}(n, p_i)$ .

In particular, for  $s \ge r-1$ , the quantity  $\mu_1(s, s+1) = \mu(s)$  has been defined in (1). Note also that

(4) 
$$\frac{\mu_{i+1}(s,j)}{\mu_i(s,j)} = (1/e)^{\binom{j}{r} - \binom{s}{r}}.$$

Let us now prove some bounds on  $\mu_i(s, j)$ .

**Claim 3.** For all  $1 \leq i \leq i_0$ , all  $0 \leq s < j \leq k_i$ , sufficiently small  $\alpha$  and sufficiently large n,

(a)  $\frac{\mu_i(s+1,j)}{\mu_i(s,j)} \le n^{-0.99};$ (b)  $\mu_i(s,j) \ge n^{0.99}.$ 

**Proof.** (a) Since  $s < k_i \leq \left(\frac{\alpha \log n}{i}\right)^{1/(r-1)}$ , we have  $is^{r-1} < \alpha \log n$ . Hence, for sufficiently small  $\alpha$  and large n,

$$\frac{\mu_i(s+1,j)}{\mu_i(s,j)} = \frac{j-s}{n-s} p_i^{-\binom{s}{r-1}} < \frac{j}{n} (e^i/p)^{s^{r-1}} \le \frac{j}{n} (e/p)^{is^{r-1}} < \frac{j}{n} (e/p)^{\alpha \log n} < n^{-0.99}.$$

(b) By (a),  $\mu_i(s, j)$  decreases with growing s. Thus, similarly to (a), since

$$ij^{r-1} \le ik_i^{r-1} \le \alpha \log n,$$

we have

$$\mu_i(s,j) \ge \mu_i(j-1,j) = (n-j+1)p_i^{\binom{j-1}{r-1}} \ge \frac{n}{2} \left(p/e^{i-1}\right)^{j^{r-1}} \ge \frac{n}{2} (p/e)^{\alpha \log n} \ge n^{0.99},$$
for sufficiently small  $\alpha$ .

Finally, as an important part of the forthcoming lemma is a sequence of rgraphs  $G_1, \ldots, G_{i_0}$ , for given  $0 \leq s < j \leq k_i$  and  $S \in \binom{V}{s}$ , we denote by  $N_i(S, j)$ the number of (S, j)-cliques in  $G_i$ . Note that  $N_1(S, s + 1)$  is the deterministic counterpart of the random variable X(S) appearing in Claim 2. In particular, if  $G_1 \in \mathcal{A}$ , then

(5) 
$$|N_1(S,s+1) - \mu_1(s,s+1)| \le n^{-1/3} \mu_1(s,s+1).$$

Note also that, for s = 0,  $N_i(\emptyset, j)$  is just the number of cliques of order j in  $G_i$ , in particular,  $N_i(\emptyset, r) = |G_i|$ , the number of edges of  $G_i$ . (From now on, we will denote the number of edges of an r-graph G by |G|.) Finally, notice that by a comment at the end of Section 1 and by (3), for j < r,

(6) 
$$N_i(S,j) = \binom{n-s}{j-s} = \mu_i(s,j).$$

#### 3.2. Statement of Lemma 4 and proof of Theorem 1

Here we state a crucial, technical lemma from which Theorem 1 will follow. Out of the three properties listed therein, the second one,  $Q_i$ , is there just to facilitate the proof. All parameters appearing in the statement have been defined in the previous subsection.

**Lemma 4.** For every n-vertex r-graph  $G_1 \in A$ , where the event A is defined in Claim 2, there exist a descending sequence of r-graphs

$$G_1 \supset G_2 \supset \cdots \supset G_{i_0}$$

and an ascending sequence of families of cliques

$$\emptyset = \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_{i_0}$$

such that the three properties below hold.

 $(\mathcal{P}_i)$  For all  $2 \leq i \leq i_0$ ,  $\mathcal{C}_i$  is a clique cover of  $G_1 - G_i$  and

$$|\mathcal{C}_i \setminus \mathcal{C}_{i-1}| \le \frac{2p_{i-1}n^r}{k_{i-1}^r}$$

 $(\mathcal{Q}_i)$  For all  $1 \leq i \leq i_0$ , all  $0 \leq s \leq k_i - 1$ , and all  $S \in {\binom{V}{s}}$ ,

(7) 
$$|N_i(S,s+1) - \mu_i(s,s+1)| \le in^{-1/3}\mu_i(s,s+1).$$

 $(\mathcal{R}_i)$  For all  $1 \leq i \leq i_0$ , all  $0 \leq s < j \leq k_i$ , and all  $S \in {\binom{V}{s}}$ ,

(8) 
$$|N_i(S,j) - \mu_i(s,j)| \le n^{-1/4} \mu_i(s,j).$$

Note that for j = s + 1, property  $\mathcal{R}_i$  is overwritten by  $\mathcal{Q}_i$ . This is because,  $i \leq i_0 \leq \frac{r+1}{r-1} \log \log n$  and thus, for all n, the right-hand side of (7) is smaller than the right-hand side of (8). Also, by (6), for j < r the left-hand side of (8) equals 0. For the same reason, whenever  $s \leq r - 2$ , the left-hand side of (7) equals 0. This means that in these cases properties  $\mathcal{Q}_i$  and  $\mathcal{R}_i$  hold trivially.

We defer the proof of Lemma 4 for later. Now, we give a short proof of Theorem 1 based on Claim 2 and Lemma 4.

**Proof of Theorem 1.** By Claim 2, the random r-graph  $G_1 = G^{(r)}(n, p)$  a.a.s. satisfies event  $\mathcal{A}$  and so, we are in position to fix  $G_1 \in \mathcal{A}$  and apply Lemma 4. Obviously, the union  $\mathcal{C}$  of the clique cover  $\mathcal{C}_{i_0}$  and the edge set of the graph  $G_{i_0}$  form a clique cover of  $G_1$ . Further, recalling that  $|G_{i_0}| = N_{i_0}(\emptyset, r)$ , we have, by

 $\mathcal{P}_i, i = 2, \ldots, i_0$ , and by  $\mathcal{R}_{i_0}$  applied only in one special case of  $S = \emptyset, j = r$ , and  $i = i_0$ ,

$$\begin{split} |\mathcal{C}| &= |\mathcal{C}_{i_0}| + |G_{i_0}| = \sum_{i=1}^{i_0-1} |\mathcal{C}_{i+1} \setminus \mathcal{C}_i| + N_{i_0}(\emptyset, r) \leq \sum_{i=1}^{i_0-1} \frac{2p_i n^r}{k_i^r} + (1+n^{-1/4})\mu_{i_0}(0,r) \\ &\leq \sum_{i=1}^{i_0-1} \frac{2pn^r i^{\frac{r}{r-1}}}{e^{i-1}(\alpha \log n)^{r/(r-1)}} + 2\binom{n}{r} p_{i_0} \leq \frac{2epn^r}{(\alpha \log n)^{r/(r-1)}} \sum_{i=1}^{\infty} \frac{i^{\frac{r}{r-1}}}{e^i} + \frac{2epn^r}{e^{i_0}} \\ &\leq \frac{2epCn^r}{(\alpha \log n)^{r/(r-1)}} + \frac{2epn^r}{(\alpha \log n)^{(r+1)/(r-1)}} = \frac{2epC(1+o(1))n^r}{(\alpha \log n)^{r/(r-1)}}, \end{split}$$
where  $C = \sum_{i=1}^{\infty} \frac{i^{\frac{r}{r-1}}}{e^i}.$ 

4. Preparations for the Proof of Lemma 4

First, we are going to show that property  $\mathcal{R}_i$  is, in some sense, redundant. Nevertheless, we found it convenient to state it explicitly in Lemma 4.

# 4.1. $Q_i$ implies $\mathcal{R}_i$

This short subsection is devoted to proving the following implication.

Claim 5. For all  $1 \leq i \leq i_0$ , if  $G_i \in Q_i$ , then  $G_i \in \mathcal{R}_i$ 

**Proof.** The key idea is to view (S, j)-cliques as a result of an iterative process of vertex by vertex "extensions" of the set S (with all required edges present). Fix  $0 \leq s < j \leq k, S = \{v_1, \ldots, v_s\} \in \binom{V}{s}$  and let  $\mathcal{N}_i(S, j)$  be the set of all (S, j)-cliques in  $G_i$ . Recall that  $\mathcal{N}_i(S, j) = |\mathcal{N}_i(S, j)|$ , that is,  $\mathcal{N}_i(S, j)$  counts the number of sets  $\{v_{s+1}, \ldots, v_j\} \subset V \setminus S$  such that  $J = S \cup \{v_{s+1}, \ldots, v_j\}$  is an (S, j)-clique in  $G_i$ . Similarly, we define  $\mathcal{N}'_i(S, j)$  as the number of sequences  $(v_{s+1}, \ldots, v_j)$  of j - s distinct vertices in  $V \setminus S$  such that, again,  $J = S \cup \{v_{s+1}, \ldots, v_j\}$  is an (S, j)-clique in  $G_i$ . Equivalently,

$$N'_{i}(S,j) = |\{(v_{s+1},\ldots,v_{j}): S \cup \{v_{s+1},\ldots,v_{j}\} \in \mathcal{N}_{i}(S,j)\}|.$$

We have, obviously,

(9) 
$$N'_i(S,j) := (j-s)!N(S,j).$$

For all  $s + 1 \leq \ell \leq j$  and  $v_{s+1}, \ldots, v_{\ell}$ , by property  $\mathcal{Q}_i$  applied to the set  $S_{\ell} := S \cup \{v_{s+1}, \ldots, v_{\ell}\}$ , setting  $V_{\ell} = \{v_{\ell+1} : S_{\ell} \cup \{v_{\ell+1}\} \in \mathcal{N}_i(S_{\ell}, \ell+1)\}$ ,

$$\left| |V_{\ell}| - \mu_i(s+\ell,s+\ell+1) \right| \le i n^{-1/3} \mu_i(s+\ell,s+\ell+1).$$

Note that, by definition (3) and the standard combinatorial identity  $\sum_{h=0}^{t-1} {h \choose r-1} = {t \choose r}$ ,

$$\prod_{h=s}^{j-1} \mu_i(h,h+1) = (n-s)_{(j-s)} p_i^{\sum_{h=s}^{j-1} \binom{h}{r-1}} = (j-s)! \mu_i(s,j).$$

Thus, observing that  $N_i'(S,j) = \prod_{\ell=s}^{j-1} |V_\ell|$ , we arrive at

$$\left(1 - in^{-1/3}\right)^{j-s} (j-s)! \mu_i(s,j) \le N'_i(S,j) \le \left(1 + in^{-1/3}\right)^{j-s} (j-s)! \mu_i(s,j).$$

Comparing with (9) and canceling (j - s)! sidewise, this yields

$$\left(1 - in^{-1/3}\right)^{j-s} \mu_i(s,j) \le N_i(S,j) \le \left(1 + in^{-1/3}\right)^{j-s} \mu_i(s,j).$$

Finally, as  $(j-s)in^{-1/3} \le k_i(i_0+1)n^{-1/3} = o(n^{-1/4})$ , we conclude that

$$|N_i(S,j) - \mu_i(s,j)| \le n^{-1/4} \mu_i(s,j)$$

which means that  $G_i$ , indeed, satisfies property  $\mathcal{R}_i$ .

### 4.2. A random procedure

We intend to prove Lemma 4 by induction on *i*. Suppose that for some  $1 \leq i \leq i_0 - 1$ , a graph  $G_i$  and a clique cover  $C_i$  of  $G_1 - G_i$  satisfy properties  $\mathcal{P}_i$ ,  $\mathcal{Q}_i$ , and  $\mathcal{R}_i$ . To obtain  $(G_{i+1}, \mathcal{C}_{i+1})$ , we apply a random procedure during which we simultaneously select

•  $\mathcal{K}_i$  — a random collection of cliques in  $G_i$  of order  $k_i$ , each chosen independently with probability

(10) 
$$q_i := \frac{1}{(1+n^{-1/4})\mu_i(r,k_i)}$$

and

•  $\mathcal{E}_i$  — a random collection of edges  $f \in G_i$ , viewed as r-vertex cliques, each f chosen independently with probability

(11) 
$$q_{i,f} = 1 - (1 - q_i)^{(1 + n^{-1/4})\mu_i(r,k_i) - N_i(f,k_i)}.$$

Then, we set

- $C_{i+1} := C_i \cup K_i \cup E_i$ , and
- $G_{i+1} = G_i (\bigcup \mathcal{K}_i \cup \mathcal{E}_i)$ , where  $\bigcup \mathcal{K}_i$  is the set of edges covered by the union of cliques in  $\mathcal{K}_i$ .

(The idea of using such a random procedure has appeared in a similar context already in [1].)

The selections of  $\mathcal{K}_i$  and  $\mathcal{E}_i$  are performed simultaneously, that is, independently of each other. Note also that the exponent in (11) is, due to property  $\mathcal{R}_i$ , nonnegative. Finally, observe that for an edge  $f \in G_i$ , the probability that  $f \in G_{i+1}$  equals

$$(1-q_i)^{N_i(f,k_i)}(1-q_{i,f}) = (1-q_i)^{(1+n^{-1/4})\mu_i(r,k_i)} \sim \frac{1}{e},$$

which explains the definition of  $p_i$  given earlier.

# 4.3. $\mathcal{R}_i$ implies $\mathcal{P}_{i+1}$

The following result is the first ingredient of the forthcoming probabilistic proof of Lemma 4.

**Claim 6.** For all  $1 \le i \le i_0 - 1$ , if  $G_i \in \mathcal{R}_i$ , then, with probability at least 0.49, the pair  $(G_{i+1}, \mathcal{C}_{i+1})$  satisfies property  $\mathcal{P}_{i+1}$ .

**Proof.** Recall that, by the random procedure described in Subsection 4.2,

(12) 
$$\mathcal{C}_{i+1} \setminus \mathcal{C}_i = \mathcal{K}_i \cup \mathcal{E}_i,$$

where  $\mathcal{K}_i$  is a collection of  $k_i$ -cliques and  $\mathcal{E}_i$  is a collection of edges selected randomly and independently from  $G_i$ .

As each  $k_i$ -clique is drawn with the same probability  $q_i$ , the quantity  $|\mathcal{K}_i|$  is binomially distributed with expectation  $\mathbf{E}|\mathcal{K}_i| = N_i(0, k_i) \times q_i$ . This, for large ncan be estimated, using property  $\mathcal{R}_i$ , the definition (3) of  $\mu_i(s, j)$ , and the divergence  $k_i \to \infty$  as  $n \to \infty$  (cf. definitions of  $k_i$  and  $i_0$  in Subsection 3.1), as follows:

$$\mathbf{E}|\mathcal{K}_i| = N_i(0, k_i) \times q_i = \frac{N_i(0, k_i)}{(1 + n^{-1/4})\mu_i(r, k_i)} \le \frac{\mu_i(0, k_i)}{\mu_i(r, k_i)} = \frac{(n)_r}{(k_i)_r} p_i$$
$$\le \left(\frac{n}{k_i - r + 1}\right)^r p_i = (1 + o(1))(n/k_i)^r p_i \le 1.01(n/k_i)^r p_i.$$

Similarly, quantity  $|\mathcal{E}_i|$  has a general binomial distribution with

(13) 
$$\mathbf{E}|\mathcal{E}_i| = \sum_{f \in G_i} q_{i,f}.$$

For  $f \in G_i$ , by property  $\mathcal{R}_i$  and Bernoulli's inequality, we have

(14) 
$$q_{i,f} \le 1 - (1 - q_i)^{2n^{-1/4}\mu_i(r,k_i)} \le 2n^{-1/4}\mu_i(r,k_i)q_i \le 2n^{-1/4}.$$

Consequently, bounding crudely  $|G_i| \leq n^r$ , by (13) and (14),

 $\mathbf{E}|\mathcal{E}_i| \le n^r \times 2n^{-1/4} = O(n^{r-1/4}).$ 

Further, observe that by the definitions of  $p_i, k_i$ , and  $i_0$ ,

$$\frac{k_i^r}{p_i} \le \frac{e^{i}k_i^r}{pe} \le \frac{e^{i_0}k_i^r}{pe} = O\left((\log n)^{\frac{r+1}{r-1} + \frac{r}{r-1}}\right) = O\left((\log n)^{\frac{2r+1}{r-1}}\right).$$

Thus,  $\mathbf{E}|\mathcal{E}_i| = O(n^{r-1/4}) = o((n/k_i)^r p_i)$ , and, by (12), we have

$$\mathbf{E}|\mathcal{C}_{i+1} \setminus \mathcal{C}_i| = \mathbf{E}|\mathcal{K}_i| + \mathbf{E}|\mathcal{E}_i| \le (1.01 + o(1))(n/k_i)^r p_i.$$

Finally, by Markov's inequality,

$$\operatorname{Prob}(|\mathcal{C}_{i+1} \setminus \mathcal{C}_i| > 2(n/k_i)^r p_i) \le 0.51.$$

It means that property  $\mathcal{P}_{i+1}$  holds for  $(G_{i+1}, \mathcal{C}_{i+1})$  with probability at least 0.49.

#### 5. Proof of Lemma 4

We are going to show the existence of sequences  $G_1 \supset G_2 \supset \cdots \supset G_{i_0}$  and  $\emptyset = \mathcal{C}_1 \subset \mathcal{C}_2 \subset \cdots \subset \mathcal{C}_{i_0}$  satisfying properties  $\mathcal{P}_i$ ,  $\mathcal{Q}_i$ , and  $\mathcal{R}_i$ , by induction on  $i = 1, \ldots, i_0$ .

Let us begin with the base case i = 1. For a fixed  $G_1 \in \mathcal{A}$ , property  $\mathcal{Q}_1$  follows from Claim 2 (cf. (5)), while property  $\mathcal{R}_1$  is implied by  $\mathcal{Q}_1$ , as shown in Claim 5.

Assuming now that for some  $i \geq 1$  a pair  $(G_i, C_i)$  satisfies properties  $\mathcal{P}_i$  (only for  $i \geq 2$ ),  $\mathcal{Q}_i$ , and  $\mathcal{R}_i$ . We are going to show that with positive probability the pair  $(G_{i+1}, C_{i+1})$ , chosen randomly according to the procedure described in Subsection 4.2, satisfies  $\mathcal{P}_{i+1}$ ,  $\mathcal{Q}_{i+1}$ , and  $\mathcal{R}_{i+1}$ , and thus, such a pair exists.

By Claim 5, property  $\mathcal{Q}_{i+1}$  implies property  $\mathcal{R}_{i+1}$ . Moreover, by Claim 6, property  $\mathcal{P}_{i+1}$  holds for  $(G_{i+1}, \mathcal{C}_{i+1})$  with probability at least 0.49. Thus, it suffices to prove that  $G_{i+1}$  satisfies property  $\mathcal{Q}_{i+1}$  with probability strictly greater than 0.51. In fact, the latter probability will turn out to be 1 - o(1).

We begin with estimating the expectation of

$$X := N_{i+1}(S, s+1)$$

the (random) number of (S, s+1)-cliques in  $G_{i+1}$ .

**Claim 7.** For all  $1 \le i \le i_0 - 1$ , if  $G_i \in Q_i$ , then, for all  $r - 1 \le s < k_i$ , and all  $S \in {V \choose s}$ ,

$$|\mathbf{E}X - \mu_{i+1}(s, s+1)| \le (i+0.5)n^{-1/3}\mu_{i+1}(s, s+1).$$

**Proof.** Fix  $r-1 \leq s < k_i$  and  $S \in {V \choose s}$  and recall our notation  $E_J$  for the set of all edges of an (S, j)-clique J and  $\mathcal{N}_i(S, j)$  for the family of all (S, j)-cliques in  $G_i$ . By linearity of expectation,

(15) 
$$\mathbf{E}X = \sum_{J \in \mathcal{N}_i(S,s+1)} \operatorname{Prob}(E_J \subset G_{i+1}).$$

To estimate  $\operatorname{Prob}(E_J \subset G_{i+1})$ , observe that an (S, s+1)-clique J of  $G_i$  "survives" into  $G_{i+1}$  if none of its edges was selected to  $\mathcal{E}_i$  or belonged to some  $k_i$ -clique selected to  $\mathcal{K}_i$ . The probability of the former event is  $\prod_{f \in E_J} (1 - q_{i,f})$ , while the probability of the latter event is  $(1 - q_i)^{|\mathcal{U}|}$ , where  $\mathcal{U} := \bigcup_{f \in E_J} \mathcal{N}_i(f, k_i)$ . Set

$$m_1 = |\mathcal{U}| - \sum_{f \in E_J} N_i(f, k_i)$$
 and  $m_2 = (1 + n^{-1/4})\mu_i(r, k_i)|E_J|.$ 

Then, using (11), we infer that

(16) 
$$\operatorname{Prob}(E_J \subset G_{i+1}) = (1-q_i)^{|\mathcal{U}|} \prod_{f \in E_J} (1-q_{i,f}) = (1-q_i)^{m_1+m_2}.$$

Next, we separately find lower and upper bounds on  $(1-q_i)^{m_1}$  and  $(1-q_i)^{m_2}$ . By Bonferroni's inequality, property  $\mathcal{R}_i$  (which follows from  $\mathcal{Q}_i$ , see Claim 5), and the monotonicity of  $\mu_i(t, k_i)$  as a function of t (see Claim 3(a)), the quantity  $-m_1$ can be bounded as follows:

$$0 \le -m_1 \le \sum_{g,h \in E_J, g \ne h} N_i(g \cup h, k_i) \le \sum_{g,h \in E_J, g \ne h} (1 + n^{-1/4}) \mu_i(|g \cup h|, k_i)$$
(17) 
$$\le \sum_{g,h \in E_J, g \ne h} (1 + n^{-1/4}) \mu_i(r + 1, k_i) \le |E_J|^2 (1 + n^{-1/4}) \mu_i(r + 1, k_i).$$

(Above, we maximized  $\mu_i(|g \cup h|, k_i)$  by minimizing  $|g \cup h|$  which achieves minimum at r + 1.) Note that

(18) 
$$|E_J| = \binom{s}{r-1} \le k^{r-1} = \Theta(\log n).$$

Consequently, by Claim 3(a) and the definition (10) of  $q_i$ ,

(19)  
$$1 \le (1 - q_i)^{m_1} \le \exp\left\{q_i |E_J|^2 \left(1 + n^{-1/4}\right) \mu_i(r+1, k_i)\right\}$$
$$= \exp\left\{\frac{|E_J|^2 \mu_i(r+1, k_i)}{\mu_i(r, k_i)}\right\} \stackrel{Cl.3(a)}{\le} \exp\left\{|E_J|^2 n^{-0.99}\right\} = 1 + o\left(n^{-0.98}\right).$$

Further, by Claim 3(b), (10), and (18),

$$\frac{q_i|E_J|}{1-q_i} \le 2q_i|E_J| = \frac{O(\log n)}{\left(1+n^{-1/4}\right)\mu_i(r,k_i)} \le 2n^{-0.99}\Theta(\log n) = o\left(n^{-0.98}\right).$$

This implies that

(20) 
$$e^{-|E_J|} \ge (1-q_i)^{m_2} \ge \exp\left\{-\frac{|E_J|}{1-q_i}\right\} \ge e^{-|E_J|} \left(1-\frac{q_i|E_J|}{1-q_i}\right) \ge e^{-|E_J|} \left(1-o(n^{-0.98})\right).$$

Thus, by (16), (19), and (20),

(21) 
$$(1 - o(n^{0.98})) e^{-|E_J|} \le \operatorname{Prob}(E_J \subset G_{i+1}) \le (1 + o(n^{-0.98})) e^{-|E_J|}.$$

Recall that, by property  $Q_i$ ,  $N_i(S, s+1) \leq (1+in^{-1/3})\mu_i(s, s+1)$ , while, by (4),  $\mu_i(s, s+1)e^{-\binom{s}{r-1}} = \mu_{i+1}(s, s+1)$ . Thus, using also (15) and (21), and recalling that  $|E_J| = \binom{s}{r-1}$ , we finally have

$$\mathbf{E}X \le N_i(S, s+1) \left(1 + o(n^{-0.98})\right) e^{-\binom{s}{r-1}}$$
  
$$\le \left(1 + o(n^{-0.98})\right) (1 + in^{-1/3}) \mu_i(s, s+1) e^{-\binom{s}{r-1}}$$
  
$$\stackrel{(4)}{=} \left(1 + o(n^{-0.98})\right) \left(1 + in^{-1/3}\right) \mu_{i+1}(s, s+1)$$
  
$$\le \left(1 + (i+0.5)n^{-1/3}\right) \mu_{i+1}(s, s+1)$$

and, similarly,  $\mathbf{E}X \ge ((1 - (i + 0.5)n^{-1/3}) \mu_{i+1}(s, s+1).$ 

In view of the above claim, to establish property  $Q_{i+1}$  of  $G_{i+1}$ , it remains to show that X is concentrated around its expectation with probability very close to 1. In doing so, similarly to [4], we will utilize the following Azuma-type concentration inequality which can be deduced from [7], Theorem 3.8 (see also [8], Corollary 1.4).

**Lemma 8.** Let  $X_1, \ldots, X_M$  be 0–1 independent random variables and let  $f : \{0,1\}^{[M]} \to \mathbb{R}$  satisfy Lipschitz condition (L) with constants  $c_1, \ldots, c_M$ :

(L) for all  $(z_1, \ldots, z_M) \in \{0, 1\}^{[M]}$  and  $(z'_1, \ldots, z'_M) \in \{0, 1\}^{[M]}$ , and all  $1 \le m \le M$ ,

$$|f(z_1,\ldots,z_M) - f(z'_1,\ldots,z'_M)| \le c_m$$
, whenever  $z_h = z'_h$  for all  $h \ne m$ .

Set

$$X = f(X_1, \dots, X_M), \quad W = \sum_{m=1}^{M} c_m^2 \operatorname{Prob}(X_m = 1), \quad and \quad C = \max_{1 \le m \le M} c_m.$$

Then, for every  $t \geq 0$ ,

$$\operatorname{Prob}(|X - \mathbf{E}X| \ge t) \le 2 \exp\left\{-\frac{t^2}{2(W + Ct)}\right\}.$$

Now, we are ready to provide the last ingredient of the proof of Lemma 4.

Claim 9. For all  $1 \leq i \leq i_0 - 1$ , if  $G_i \in \mathcal{Q}_i$ , then, a.a.s.,  $G_{i+1} \in \mathcal{Q}_{i+1}$ .

**Proof.** Fix  $r-1 \leq s < k_i$  and  $S \in {V \choose s}$ , and notice that if J is an (S, s+1)-clique in  $G_{i+1}$ , then it must have also been an (S, s+1)-clique in  $G_i$ , whose all edges "survived" the random procedure described in Subsection 4.2.

Recall that  $\mathcal{N}_i(\emptyset, k_i)$  denotes the set of all  $k_i$ -cliques in  $G_i$  and that  $\mathcal{N}_i(\emptyset, k_i) = |\mathcal{N}_i(\emptyset, k_i)|$ . We set  $M_1 = \mathcal{N}_i(\emptyset, k_i)$  and  $\mathcal{N}_i(\emptyset, k_i) = \{K_1, \ldots, K_{M_1}\}$ . Let  $X_m$ ,  $m = 1, \ldots, M_1$ , be the indicator random variable which equals 1 if  $K_m \in \mathcal{K}_i$  and 0 otherwise. Similarly, set  $M_2 = |G_i|$  and  $G_i = \{f_1, \ldots, f_{M_2}\}$ , and denote by  $Y_m, m = 1, \ldots, M_2$ , the indicator random variable equal to 1 if  $f_m \in \mathcal{E}_i$  and 0 otherwise.

As the events  $K_m \in \mathcal{K}_i$ ,  $m = 1, \ldots, M_1$ , and  $f_m \in \mathcal{E}_i$ ,  $m = 1, \ldots, M_2$ , fully determine the number of (S, s)-cliques left in  $G_{i+1}$ , there exists a function  $f : \{0, 1\}^{[M_1+M_2]} \to \mathbb{R}$ , such that

$$X = N_{i+1}(S, s+1) = f(X_1, \dots, X_{M_1}, Y_1, \dots, Y_{M_2}).$$

The explicit form of function f is not important for us.

As we are aiming at applying Lemma 8 to X, we need to find constants for which the Lipschitz condition (L) holds and then estimate W. Set

$$c_m = \max |f(x_1, \dots, x_{M_1}, y_1, \dots, y_{M_2}) - f(x'_1, \dots, x'_{M_1}, y_1, \dots, y_{M_2})|,$$

where the maximum is taken over all  $(x_1, ..., x_{M_1})$ ,  $(x'_1, ..., x'_{M_1}) \in \{0, 1\}^{[M_1]}$ and  $(y_1, ..., y_{M_2}) \in \{0, 1\}^{[M_2]}$  such that  $x_h = x'_h$  for all  $h \neq m$ . Similarly, we set

$$d_m = \max |f(x_1, \dots, x_{M_1}, y_1, \dots, y_{M_2}) - f(x_1, \dots, x_{M_1}, y'_1, \dots, y'_{M_2})|,$$

where the maximum is taken over all  $(x_1, \ldots, x_{M_1}) \in \{0, 1\}^{[M_1]}$  and  $(y_1, \ldots, y_{M_2})$ ,  $(y'_1, \ldots, y'_{M_2}) \in \{0, 1\}^{[M_2]}$  such that  $y_h = y'_h$  for all  $h \neq m$ . In other words,  $c_m$  and  $d_m$  are, respectively, upper bounds on the change of X due to flipping the outcome of the event  $K_m \in \mathcal{K}_i$ , respectively,  $f_m \in \mathcal{E}_i$ . Now we estimate the Lipschitz parameters  $c_m$  and  $d_m$  taking into account the position of  $K_m$  and  $f_m$  with respect to the given set S. We begin with  $d_m$  as this case is easier. As an edge of  $G_i$  may belong to at most one (S, s + 1)-clique J, the values of  $X = N_{i+1}(S, s + 1)$  for  $G_{i+1}$  with or without the edge  $f_m$  may differ by at most one. Thus,

$$(22) d_m \le 1$$

for all  $m = 1, ..., M_2$ .

On the other hand, since |J| = s + 1, any such J contains exactly  $\binom{s}{r-1}$  edges of  $G_i$ , so there are altogether

(23) 
$$N_i(S,s+1) \times \binom{s}{r-1} \le N_i(S,s+1)k^{r-1}$$

edges  $f_m \in G_i$  whose removal could affect  $X = N_{i+1}(S, s+1)$ . Thus, for that many edges we put  $d_m = 1$ , while for all other edges  $d_m = 0$ .

Turning to  $c_m$ , by the same token, very crudely, for all  $m = 1, \ldots, M_1$ ,

(24) 
$$c_m \le |K_m| = \binom{k_i}{r} \le k^r$$

as every edge of  $K_m$  may belong to at most one (S, s + 1)-clique J.

Moreover,  $c_m > 0$  only if  $K_m$  contains at least one edge of some (S, s + 1)clique of  $G_i$ . There are  $N_i(S, s + 1)$  such (S, s + 1)-cliques and each contains  $\binom{s}{r-1}$  edges. In turn, by property  $\mathcal{R}_i$ , each edge f is contained in  $N_i(f, k_i) \leq (1 + n^{-1/4})\mu_i(r, k_i) k_i$ -cliques of  $G_i$ . Hence, there are at most

(25) 
$$N_i(S,s+1) \times {\binom{s}{r-1}} \times \max_{f \in G_i} N_i(f,k_i) \le N_i(S,s+1)k^{r-1}(1+n^{-1/4})\mu_i(r,k_i)$$

cliques  $K_m$  in  $G_i$  which share an edge with some (S, s + 1)-clique. This implies that for at most that many indices  $m \in [M_1]$  we have  $c_m > 0$ .

Putting (22)–(25) together, one can bound the parameter W appearing in Lemma 8, using again property  $\mathcal{R}_i$ , the definition (10) of  $q_i$ , and the estimate (14) of  $q_{i,f}$ , as follows.

(26)  

$$W = \sum_{m=1}^{M_1} c_m^2 q_i + \sum_{m=1}^{M_2} d_m^2 q_{i,f} \stackrel{(14)}{\leq} N_i(S,s+1)k^{r-1}(1+n^{-1/4})\mu_i(r,k_i) \times k^{2r} \times q_i$$

$$+ N_i(S,s+1)k^{r-1} \times 1^2 \times 2n^{-1/4} \stackrel{(10)}{\leq} (1+o(1))N_i(S,s+1)k^{3r-1} \stackrel{\mathcal{R}_i}{\leq} \mu_i(s,s+1)k^{3r}.$$

Recall that, by definition (3),  $\mu_i(s, s+1) = (n-s)p_i^{\binom{s}{r-1}} \leq n$ , while, by equality (4),  $\mu_{i+1}(s, s+1) = e^{-\binom{s}{r-1}}\mu_i(s, s+1) \leq \mu_i(s, s+1)$ . Moreover, by Claim 3(b) applied with i+1,  $\mu_{i+1}(s, s+1) \geq n^{0.99}$ . Putting all these facts together, we have

(27) 
$$n^{0.99} \le \mu_{i+1}(s, s+1) \le \mu_i(s, s+1) \le n.$$

Thus, in view of (26), by Lemma 8 with

$$t = \frac{1}{2}n^{-1/3}\mu_{i+1}(s, s+1) \quad \text{and} \quad C = \max\left\{\max_{1 \le m \le M_1} c_m, \ \max_{1 \le m \le M_2} d_m\right\} \le k^r,$$

noting that  $Ct = o\left(\mu_i(s,s+1)k^{3r}\right)$  and taking *n* sufficiently large,

$$\begin{aligned} \operatorname{Prob}(X - \mathbf{E}X \ge t) &\leq 2 \exp\left\{-\frac{\frac{1}{4}n^{-2/3}\mu_{i+1}^2(s, s+1)}{2(\mu_i(s, s+1)k^{3r} + Ct)}\right\} \\ &\leq 2 \exp\left\{-\frac{\mu_{i+1}^2(s, s+1)}{9\mu_i(s, s+1)n^{2/3}k^{3r}}\right\} \stackrel{(27)}{\leq} 2 \exp\left\{-\frac{n^{1.98}}{n^{5/3}k^{3r}}\right\} \leq \exp\left\{-n^{0.3}\right\}. \end{aligned}$$

In view of the above and using Claim 7 and the union bound, a.a.s., for all s and  $S \in {V \choose s}$ ,

$$X = N_{i+1}(S, s+1) \le \mathbf{E}X + t \le \left(1 + (i+0.5)n^{-1/3} + 0.5n^{-1/3}\right)\mu_{i+1}(s, s+1)$$
$$\le \left(1 + (i+1)n^{-1/3}\right)\mu_{i+1}(s, s+1)$$

and, similarly,  $X \ge (1 - (i+1)n^{-1/3}) \mu_{i+1}(s, s+1)$ , which completes the proof of Claim 9.

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# Appendix

The following folklore result was observed by many authors for graphs (r = 2) but there seems to be no published proof of the general case. Here we fill that gap.

**Fact 1.** Let *H* be an *r*-graph and let  $\theta_1(H)$  and  $\tilde{\theta}_1(H)$  stand, respectively, for its representation number and minimum (edge) clique cover number. Then  $\theta_1(H) = \tilde{\theta}_1(H)$ .

For the proof we need a simple observation.

**Observation.** Let H = (V, E) be an r-graph and let  $S = \{S_v \subset S : v \in V\}$  be a representation of H with the smallest set S. Then every element  $s \in S$  belongs to at least r sets in S.

**Proof.** Suppose there is an  $s \in S$  belonging to fewer than r sets in S. Then  $S_s = \{S_v \setminus \{s\} : v \in V\}$  would also be a representation of H which contradicts the minimality of S. Indeed, for such an s and any R with |R| = r,

$$\bigcap_{v \in R} S_v \neq \emptyset \quad \text{if and only if} \quad \bigcap_{v \in R} (S_v \setminus \{s\}) \neq \emptyset.$$

**Proof of Fact 1.** Let  $S = \{S_v \subset S : v \in V\}$  be a minimum representation of H, that is, a representation of size  $|S| = \theta_1(H)$ . By the above Observation, for each  $s \in S$  the set  $C(s) = \{v : s \in S_v\}$  has size  $|C(s)| \ge r$ . What is more important, C(s) is a clique in H. Indeed, if  $\{v_1, \ldots, v_r\} \subset C(s)$ , then  $S_{v_1} \cap \cdots \cap S_{v_r} \ni s$ , thus  $\{v_1, \ldots, v_r\} \in H$ . Moreover, each edge  $\{v_1, \ldots, v_r\} \in H$  is covered by a clique C(s), where  $s \in S_{v_1} \cap \cdots \cap S_{v_r}$ . Hence,  $\tilde{\theta}_1(H) \le \theta_1(H)$ .

Conversely, let  $\{C(s) : s \in S\}$  be a clique cover of H indexed by some (abstract) set S. For every vertex  $v \in V$  consider the set

$$S_v = \{s \in S : v \in C(s)\}.$$

Next, observe that  $\{v_1, \ldots, v_r\} \in E$  if and only if there is some  $s \in S$  with  $\{v_1, \ldots, v_r\} \subset C(s)$ . We will draw two consequences of this equivalence. First, if  $\{v_1, \ldots, v_r\} \in E$ , then there exists  $s \in S_{v_1} \cap \cdots \cap S_{v_r}$ , implying that

$$S_{v_1} \cap \cdots \cap S_{v_r} \neq \emptyset.$$

However, if  $\{v_1, \ldots, v_r\} \notin E$  then  $\{v_1, \ldots, v_r\} \notin C(s)$  for all  $s \in S$ , which means that  $S_{v_1} \cap \cdots \cap S_{v_r} = \emptyset$ . Consequently,  $\{S_v : v \in V\}$  is a representation of H, yielding  $\theta_1(H) \leq \tilde{\theta}_1(H)$ .