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ON DISTANCE MAGIC LABELINGS OF HAMMING GRAPHS AND FOLDED HYPERCUBES

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Abstract

Let $\Gamma = (V, E)$ be a graph of order n. A *distance magic labeling* of Γ is a bijection $\ell\colon V \to \{1, 2, \dots, n\}$ for which there exists a positive integer k such that $\sum_{x \in N(u)} \ell(x) = k$ for all vertices $u \in V$, where $N(u)$ is the neighborhood of u. A graph is said to be *distance magic* if it admits a distance magic labeling.

The *Hamming graph* $\mathrm{H}(D, q)$, where D, q are positive integers, is the graph whose vertex set consists of all words of length D over an alphabet of size q in which two vertices are adjacent whenever the corresponding words differ in precisely one position. The well-known *hypercubes* are precisely the Hamming graphs with $q = 2$. Distance magic hypercubes were classified in two papers from 2013 and 2016. In this paper we consider all Hamming graphs. We provide a sufficient condition for a Hamming graph to be distance magic and as a corollary provide an infinite number of pairs (D, q) for which the corresponding Hamming graph $\mathrm{H}(D, q)$ is distance magic.

A *folded hypercube* is a graph obtained from a hypercube by identifying pairs of vertices at maximal distance. We classify distance magic folded hypercubes by showing that the dimension-D folded hypercube is distance magic if and only if D is divisible by 4.

Keywords: distance magic labeling, distance magic graph, Hamming graph, folded hypercube.

2020 Mathematics Subject Classification: 05C78, 05C50.

1. Introduction

Problems concerning existence of distance magic labelings of graphs (distributions of the positive integers 1 through the order of the graph to its vertices so that the sum of labels "around a vertex" is independent of the vertex) have attracted many researchers in recent years (see for instance [1, 2, 5, 8, 11] and the dynamic survey [10]). Special attention has been given to regular graphs (see for instance [6, 7, 12, 15]) since distance magic labelings of regular graphs correspond to so called equalized incomplete tournaments in which the aim is to schedule the tournament in such a way that the total strength of the teams a given team plays is the same for all teams (see for instance [9]). It is well known and easy to see that for a regular *distance magic* graph (a graph admitting a distance magic labeling) of valency r and order n the sum of labels on the neighbours of each vertex (called the *magic constant*) equals $r(n+1)/2$, thus implying that there are no regular distance magic graphs of odd valency [16].

One of the very well-known families of regular graphs that was considered in this context are the hypercube graphs (see the next paragraph for a definition). Since the valency of the D-dimensional hypercube is D, the above remark implies that only the even dimension hypercubes are potential candidates for being distance magic. Even though it was initially conjectured [1] that the only distance magic hypercube is the 4-cycle, it was later proved [7, 12] that the D-dimensional hypercube is distance magic if and only if $D \equiv 2 \pmod 4$. In other words, of the feasible dimensions D (even, so as to obtain even valency) the ones where D is not divisible by 4 are distance magic while the ones where D is divisible by 4 are not.

Hypercubes are very special examples of Hamming graphs. The *Hamming graph* $\mathrm{H}(D,q)$, where D and q are positive integers, is the graph whose vertex set consists of all D-tuples whose components are elements of a given set of size q, and two vertices are adjacent whenever the corresponding tuples differ in precisely one coordinate. The hypercubes correspond to the graphs $\mathrm{H}(D,2)$. Hamming graphs have been studied quite extensively as they have some very nice properties. For instance, they are *distance-transitive* [4, Section 9.2] (for each possible distance d and for any two ordered pairs of vertices at distance d there is an automorphism mapping the first ordered pair to the second one), implying that they possess a great amount of symmetry. They play an important part in various areas of science such as the theory of error-correcting codes and the theory of communication networks to name just two.

Another very well-known and commonly studied family of distance-transitive graphs, which is in fact closely related to the hypercubes, is the family of *folded hypercubes*. These are the graphs obtained from the hypercubes by identifying all pairs of vertices at maximal distance (one can equivalently take the hypercube of one dimension less and then add the edges between all pairs of vertices at maximal distance — see Section 2 for a precise definition). Note that this implies that the valencies of the D-dimensional hypercube and the dimension-D folded hypercube are the same (namely D). Just as the hypercubes their folded counterparts have some very important and widely studied properties. Apart from being distance-transitive they are, just as all Hamming graphs, Cayley graphs of abelian groups.

In this paper we investigate distance magic Hamming graphs. We provide a necessary condition for a Hamming graph to be distance magic (see Proposition 3.1) and at the same time a sufficient condition for it to be distance magic (see Theorem 3.2). We also prove the following result.

Theorem 1.1. *Let p be a prime and let t, d be positive integers where d is not divisible by p. Then the Hamming graph $\mathrm{H}(dq, q)$, where $q = p^t$, is distance magic.*

We also classify distance magic folded hypercubes. Perhaps somewhat surprisingly it turns out that the distance magic folded hypercubes are precisely the folds of those even-valency hypercubes which are not distance magic. More precisely, we prove the following result (we remark that we were informed by the referees that while this paper was under review the same result was independently obtained by Tian, Hou, Hou and Gao [18]).

Theorem 1.2. *Let $D \geq 3$ be an integer. Then the dimension-D folded hypercube is distance magic if and only if D is divisible by 4.*

The paper is organized as follows. In Section 2 we review a few definitions and gather some preliminary results that will be needed in the remaining sections. In Section 3 we provide both the above mentioned necessary condition (Proposition 3.1) and sufficient condition (Theorem 3.2) for a Hamming graph to be distance magic. We also prove Theorem 1.1. Section 4 is devoted to the proof of Theorem 1.2. In Section 5 we propose some open problems and suggest a few possible directions for future research.

2. Preliminaries

Throughout this paper all graphs are simple, undirected, finite and connected. For a graph $\Gamma = (V, E)$ of order $n = |V|$ a *distance magic labeling* of Γ is a bijection $\ell \colon V \to \{1, 2, \ldots, n\}$ for which there exists a positive integer k such that the *weight* $w(u) = \sum_{x \in N(u)} \ell(x)$ is equal to k for all $u \in V$ (here $N(u)$ is the set of all neighbors of u). If such a labeling exists then Γ is said to be *distance magic* and the constant k is called the *magic constant* of Γ.

As already mentioned, a regular distance magic graph is necessarily of even valency. We first point out a useful observation for regular graphs that can be extracted from a result from [15] that links the property of being distance magic to existence of certain eigenvectors for the (potential) eigenvalue 0 of the corresponding adjacency matrix. For a graph Γ of order n let A denote its adjacency matrix (an $n \times n$ matrix with rows and columns indexed by the elements of V such that for each $x, y \in V$ the (x, y)-entry of A is 1 if $y \in N(x)$ and is 0 otherwise). Then $\lambda \in \mathbb{C}$ is an *eigenvalue* of Γ if λ is an eigenvalue of A. The following result can be extracted from [15, Lemma 2.1].

Lemma 2.1. *Let $\Gamma = (V, E)$ be a regular graph of order n and even valency. If Γ is distance magic, then 0 is an eigenvalue of Γ.*

For the purposes of our arguments in Sections 3 and 4 we now provide a somewhat different approach to defining the Hamming graphs and the folded hypercubes. Let D and q be positive integers and let R be a commutative ring with unity of order q. Let $R^{\#}$ be the set of all nonzero elements of R. The vertex set of $\mathrm{H}(D, q)$ can then be thought of as the free R-module R^D (the elements of which are simply all D-tuples of elements of R) with basis $\{\mathbf{e}_i \colon 1 \le i \le D\}$, where $\mathbf{e}_i$ is the D-tuple with zeros on all components but the i-th component which is the unity 1 of R (see for instance [14]). Letting $\oplus$ be the componentwise addition in R^D the neighborhood of a vertex $u \in R^D$ in $\mathrm{H}(D, q)$ is then

$$\big\{u \oplus (x \cdot \mathbf{e}_i) \colon 1 \le i \le D,\ x \in R^{\#}\big\}, \tag{1}$$

where $x \cdot \mathbf{e}_i$ is to be understood as the usual scalar multiplication in the module R^D (i.e., $x \cdot \mathbf{e}_i$ is the tuple with zeros on all components but the i-th component which equals x). Observe that $\mathrm{H}(D, q)$ is a regular graph of valency $D(q-1)$ and order q^D.

Let now $D \ge 3$ and consider the hypercube $\mathrm{Q}_D = \mathrm{H}(D, 2)$. Observe first that in these graphs each vertex u has precisely one vertex at maximal distance D, namely $u \oplus \mathbf{j}$, where $\mathbf{j} = \mathbf{e}_1 \oplus \mathbf{e}_2 \oplus \cdots \oplus \mathbf{e}_D$ is the all-ones vector. The vertex $u \oplus \mathbf{j}$ is called the *antipodal vertex* of u. This enables us to make the corresponding quotient graph where we identify each pair of antipodal vertices (and retain the edges in a natural way). More formally, for a vertex u of Q_D we denote the set $\{u, u \oplus \mathbf{j}\}$ by $\overline{u} = \overline{u \oplus \mathbf{j}}$. Pick distinct vertices u, v of Q_D. Note that since $D \ge 3$, either there are no edges between $\overline{u}$ and $\overline{v}$ in Q_D, or there is a perfect matching between these two sets. The *antipodal quotient* of Q_D (or the *folded* Q_D) is the graph with vertex set $\{\overline{u} : u \in V(\mathrm{Q}_D)\}$, where $\overline{u}$ and $\overline{v}$ are adjacent if and only if there is a perfect matching between $\overline{u}$ and $\overline{v}$ in Q_D. The folded Q_D will be denoted by $\overline{\mathrm{Q}_D}$ and will also be called the *dimension-D folded hypercube*. Note that $\overline{\mathrm{Q}_D}$ is of order 2^{D-1} and is regular of valency D. We mention that an alternative description of $\overline{\mathrm{Q}_D}$ is obtained by first taking Q_{D-1} and then adding edges between each pair of antipodal vertices in Q_{D-1}. The hypercube Q_4 and its fold, the complete bipartite graph $K_{4,4}$, are depicted in Figure 1, where three pairs of antipodal vertices in Q_4 and their counterparts in $\overline{\mathrm{Q}_4}$ are pointed out.

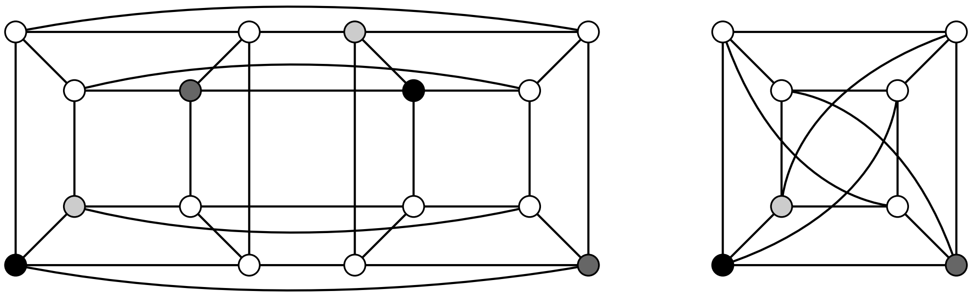

Figure 1. The hypercube Q_4, three pairs of its antipodal vertices and the corresponding fold $\overline{Q_4}$.

3. The Hamming Graphs

We start by an easy consequence of Lemma 2.1 and the fact that the eigenvalues of Hamming graphs are known. In particular, the eigenvalues of the Hamming graph $\mathrm{H}(D,q)$, where D and q are positive integers with $q \geq 2$, are of the form (see for instance [4, Theorem 9.2.1])

$$\lambda_j = q(D-j) - D, \quad 0 \leq j \leq D. \tag{2}$$

It thus follows that 0 is an eigenvalue of $\mathrm{H}(D,q)$ if and only if there exists a positive integer j with $0 \leq j \leq D$ such that $j = (q-1)D/q$. Since $q-1$ and q are relatively prime such an integral j thus exists if and only if q divides D. This proves the following necessary condition for $\mathrm{H}(D,q)$ to be distance magic.

Proposition 3.1. *Let D and q be positive integers with $q \geq 2$. If the Hamming graph $\mathrm{H}(D,q)$ is distance magic, then D is a multiple of q.*

Our next goal is to provide a sufficient condition for a Hamming graph $\mathrm{H}(D,q)$ to be distance magic. The idea of how to obtain a suitable distance magic labeling is somewhat similar to the one used in [12], where it was proved that all hypercubes Q_D with $D \equiv 2 \pmod 4$ are distance magic.

Theorem 3.2. *Let d and q be positive integers with $q \geq 2$. Let R be a commutative ring of order q with unity 1 and let $D = dq$. Let $M \in R^{D\times D}$ be a matrix such that each row of M has d zero entries while the remaining entries are all equal to 1. If the natural action of M on the free R-module R^D is a bijection, then the Hamming graph $\mathrm{H}(D,q)$ is distance magic.*

Proof. Suppose $M \in R^{D\times D}$ satisfies the conditions of the theorem and let $f\colon R \to \{0,1,\ldots,q-1\}$ be any bijection. We define a labeling of vertices of $\Gamma = \mathrm{H}(D,q)$ as follows (recall that we can view the vertex set of Γ as the underlying set of the free R-module R^D). For each $u \in R^D$ we set

$$\ell(u) = 1 + \sum_{i=1}^{D} f((Mu)_i) q^{D-i}, \tag{3}$$

where $(Mu)_i$ represents the i-th component of the D-tuple Mu from R^D. Observe that the above sum is performed in the ring of integers even though $(Mu)_i$ is an element of the ring R.

Now, since $1+(q-1)(1+q+q^2+\cdots+q^{D-1}) = q^D$, we have that $1 \le \ell(u) \le q^D$ holds for all $u \in R^D$. Moreover, since M induces a permutation of R^D and f is also a bijection, it is clear that $\ell(u) = \ell(v)$ if and only if $u = v$. To complete the proof we thus only need to show that all vertices have the same weight (which by the remarks from the Introduction must be $D(q-1)(q^D+1)/2$).

Since the set of neighbors of a vertex $u \in R^D$ can be described as in (1), it follows that the weight of u is

$$w(u) = \sum_{j=1}^{D} \sum_{x \in R^\#} \ell(u \oplus (x \cdot \mathbf{e}_j)).$$

Note that each of the two sums is again performed within the ring of integers. By (3) we get

$$\begin{aligned} w(u) &= \sum_{j=1}^{D} \sum_{x \in R^\#} \left(1 + \sum_{i=1}^{D} f((M(u \oplus (x \cdot \mathbf{e}_j)))_i) q^{D-i} \right) \\ &= D(q-1) + \sum_{j=1}^{D} \sum_{x \in R^\#} \sum_{i=1}^{D} f((M(u \oplus (x \cdot \mathbf{e}_j)))_i) q^{D-i} \\ &= D(q-1) + \sum_{i=1}^{D} \left(q^{D-i} \sum_{x \in R^\#} \sum_{j=1}^{D} f((Mu \oplus x \cdot M_j)_i) \right), \end{aligned} \tag{4}$$

where M_j denotes the j-th column of the matrix M. Recall that by assumption each row of M consists of d zero entries and $D-d$ entries which are all equal to 1. This implies that for a given $x \in R^\#$ precisely d of the numbers $f((Mu \oplus x \cdot M_j)_i)$ in the above sum over j are equal to $f((Mu)_i)$, while the remaining $D-d = (q-1)d$ are equal to $f((Mu)_i + x)$, where the last sum is taken within the ring R. Since $x \neq 0$ and f is bijective, this is never equal to $f((Mu)_i)$. Moreover, for $x, y \in R^\#$, where $x \neq y$, the numbers $f((Mu)_i + x)$ and $f((Mu)_i + y)$ are again different. This shows that in the double sum

$$\sum_{x \in R^\#} \sum_{j=1}^{D} f((Mu \oplus x \cdot M_j)_i)$$

each of the numbers from $\{0, 1, 2, \dots, q-1\}$ appears precisely $(q-1)d$ times. It thus follows that

$$
\begin{aligned}
w(u) &= D(q-1) + \sum_{i=1}^{D} \big(q^{D-i}(q-1)d(0+1+2+\cdots+q-1)\big) \\
&= D(q-1) + (q-1)d(q-1)q/2\big(1+q+q^2+\cdots+q^{D-1}\big) \qquad (5)\\
&= D(q-1) + D(q-1)\big(q^D-1\big)/2 \\
&= D(q-1)\frac{q^D+1}{2},
\end{aligned}
$$

thus finally proving that ℓ is indeed a distance magic labeling of Γ. ∎

We now show that at least in certain specific situations one can always find a matrix M satisfying the assumptions from Theorem 3.2, thereby proving that the corresponding Hamming graphs are distance magic. For the sake of completeness we briefly review some well-known connections between circulant matrices and polynomials.

Suppose $q = p^t$ is a prime power, let d be a positive integer coprime to p, let $D = dq$ and let $R = \mathrm{GF}(q)$ be the finite field of order q. In this case R^D is of course a vector space over R, and so a matrix $M \in R^{D\times D}$ induces a bijection on R^D if and only if its rows are linearly independent. The subspace spanned by all the rows of M can be considered as a linear code over R. Therefore, in the case that M is a *circulant matrix* (a matrix obtained by taking a first row and then obtaining each of the next rows by making a right cyclic shift of the row above it) one obtains what is called a *cyclic code* over R (see for instance [13]). It is well known that in this case each codeword (i.e., a vector in the subspace) can be represented by a polynomial of degree at most $D-1$ in a natural way (or equivalently, by an element of the quotient ring $R[x]/(x^D-1)$), where a vector $[a_1, a_2, \dots, a_D]^T$ corresponds to the polynomial $a_1 + a_2x + a_3x^2 + \cdots + a_{D-1}x^{D-2} + a_Dx^{D-1}$. Note that, as $x^D = 1$ in $R[x]/(x^D-1)$, multiplication by x corresponds to a one step cyclic shift of the coefficients. The cyclic code corresponding to the circulant matrix M is thus equivalent to the principal ideal of $R[x]/(x^D-1)$ generated by the polynomial corresponding to the first (or any other) row of M (called the *generating polynomial* of the code). One thus obtains the well-known fact that the subspace corresponding to such a circulant matrix M is the full space R^D (or equivalently, M is invertible) if and only if the corresponding generating polynomial is coprime to x^D-1 in $R[x]$.

Proposition 3.3. *Let $q = p^t$ be a prime power, let d be a positive integer coprime to p, let $D = dq$ and let $R = \mathrm{GF}(q)$ be the field of order q. Then the polynomial*

$$g(x) = x^{D-1} + x^{D-2} + \cdots + x^{d+1} + x \in R[x]$$

is coprime to the polynomial $x^D - 1$.

Proof. Note first that since R is a field of characteristic p we have that

$$x^D - 1 = \left(x^d - 1\right)^{p^t} = (x-1)^{p^t}\left(1 + x + x^2 + \cdots + x^{d-1}\right)^{p^t}.$$

Since $g(1) = d(q-1) = -d \neq 0$ (recall that d is coprime to p), we thus need to verify that $\gcd(h(x), g(x)) = 1$, where $h(x) = 1 + x + x^2 + \cdots + x^{d-1}$. Now,

$$g(x) = x - x^d + h(x)\left(x^d + x^{2d} + \cdots + x^{(q-1)d}\right),$$

and so $\gcd(h(x), g(x))$ divides $x(1 - x^{d-1})$ and consequently also $x^{d-1} - 1$. But as $\gcd(h(x), g(x))$ clearly also divides $(x-1)h(x) = x^d - 1$, this implies that $\gcd(h(x), g(x))$ divides $x^{d-1}(x-1)$, which finally implies $\gcd(h(x), g(x)) = 1$, as claimed. ■

Observe that the circulant matrix over R corresponding to the polynomial $g(x)$ from Proposition 3.3 has precisely d zeros in each row while all the remaining coefficients are equal to 1. Combining together Theorem 3.2 and Proposition 3.3 we thus get the proof of Theorem 1.1. To illustrate the corresponding assignment of labels to vertices of the Hamming graphs $\mathrm{H}(dp^t, p^t)$, where the prime p is coprime to d, we consider two examples.

Example 3.4. Let us have a closer look at the Hamming graph $\Gamma = \mathrm{H}(3,3)$. In this case the corresponding field R is the field $\mathbb{Z}_3$ of order 3 and the polynomial $g(x)$ from Proposition 3.3 is $g(x) = x^2 + x \in \mathbb{Z}_3[x]$. The corresponding matrix M is therefore

$$M = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Letting $f\colon \mathbb{Z}_3 \to \{0,1,2\}$ be the "identity" bijection we thus see that (3) assigns to each vertex $u = [j_1, j_2, j_3]^T$ of Γ the label

$$1 + (j_1 + j_2) + (j_1 + j_3)3 + (j_2 + j_3)9,$$

where addition within each of the three parentheses is performed modulo 3. In particular, the labels of the vertices are as given in the following table (where for each of the nine possibilities for the pair j_1 and j_2 the labels of the three vertices $[j_1, j_2, j]^T$, $j \in \mathbb{Z}_3$, are given in the column denoted by $[j_1, j_2, j]$) and as they are given on Figure 2, where the Hamming graph $\mathrm{H}(3,3)$, together with the assigned labels is depicted.

j	$[0,0,j]$	$[0,1,j]$	$[0,2,j]$	$[1,0,j]$	$[1,1,j]$	$[1,2,j]$	$[2,0,j]$	$[2,1,j]$	$[2,2,j]$
0	1	11	21	5	15	22	9	16	26
1	13	23	6	17	27	7	12	19	2
2	25	8	18	20	3	10	24	4	14

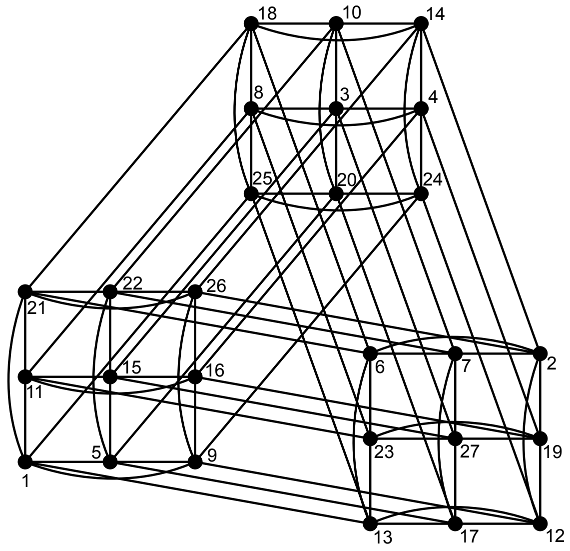

Figure 2. The Hamming graph H$(3, 3)$ with a distance magic labeling.

Example 3.5. Let us now consider the Hamming graph $\Gamma = \mathrm{H}(12, 4)$. In this case the order $4^{12} = 16\,777\,216$ of Γ is way too large to display the labels of all vertices of Γ, and so we only demonstrate how to compute the label of one particular vertex of Γ. Observe that the corresponding field R is the field $\mathrm{GF}(4) = \{0, 1, a, 1+a\}$ of order 4 with $1+1 = 0$ and $a^2 = 1+a$. This time the $g(x)$ from Proposition 3.3 is $g(x) = x^{11} + x^{10} + x^9 + \cdots + x^4 + x \in \mathrm{GF}(4)[x]$, and so the corresponding matrix M is the circulant matrix with the first row equal to

$$(0\ 1\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 1\ 1).$$

Let $f\colon \mathrm{GF}(4) \to \{0, 1, 2, 3\}$ be the map defined by $f(0) = 0$, $f(1) = 1$, $f(a) = 2$ and $f(1+a) = 3$. Consider the vertex

$$u = [a,\ 1+a,\ 1+a,\ 0,\ 0,\ 1,\ a,\ a,\ 1+a,\ a,\ a,\ 1+a]^T$$

of Γ. It is easy to see that

$$Mu = [a,\ 0,\ 1,\ 0,\ a+1,\ a+1,\ 0,\ 1,\ 1,\ 0,\ 0,\ 0]^T$$

(recall that when computing Mu addition (and multiplication) has to be performed in GF(4)). By (3) we now get that the label $\ell(u)$ of u is equal to

$$1+2\cdot4^{11}+0\cdot4^{10}+1\cdot4^{9}+0\cdot4^{8}+3\cdot4^{7}+3\cdot4^{6}+0\cdot4^{5}+1\cdot4^{4}+1\cdot4^{3}+0\cdot4^{2}+0\cdot4^{1}+0\cdot4^{0} = 8\,712\,513.$$

We now provide appropriate matrices M for another infinite family of Hamming graphs (having a nonempty intersection with the one from Theorem 1.1).

In fact, we prove that at least when $d = 1$ we can simply take R to be the residue class ring $\mathbb{Z}_q$ in Theorem 3.2 and use a natural generalization of the matrix M corresponding to the polynomial from Proposition 3.3 to apply Theorem 3.2.

Proposition 3.6. *Let $q \geq 2$ be a positive integer. Then the Hamming graph $\mathrm{H}(q, q)$ is distance magic.*

Proof. We let $R = \mathbb{Z}_q$ be the residue class ring of order q and let $M \in R^{q\times q}$ be the circulant matrix corresponding to the row $[0, 1, 1, \ldots, 1]$. To be able to apply Theorem 3.2 we only need to show that the matrix M induces a permutation of the free R-module R^q. This is of course equivalent to showing that the corresponding map is injective.

Observe that $M = \mathbf{J} - \mathbf{I}$, where $\mathbf{J}, \mathbf{I} \in R^{q\times q}$ are the all-ones matrix and the identity matrix, respectively. Suppose $u, v \in R^q$ are such that $Mu = Mv$. Then $\mathbf{J}(u - v) = u - v$, and since the q-tuple $\mathbf{J}(u - v)$ has all of its components equal, say to $x \in R$, this implies that $u - v$ has all of its components equal to x. But then the above equality implies that in R one has $x = qx = 0$, so that $u = v$, as claimed.

It follows that the matrix M satisfies the requirements from Theorem 3.2, and so $\mathrm{H}(q, q)$ is indeed distance magic. ■

We conclude this section by a few remarks on a possibility of how to use Theorem 3.2 for a circulant matrix even when $d \neq 1$ and q is not a prime power. Namely, suppose q is any positive integer (not necessarily a prime power) and let R simply be the residue class ring $\mathbb{Z}_q$. Suppose d is coprime to q and simply consider the $D \times D$ circulant matrix over $\mathbb{Z}_q$ corresponding to the polynomial $g(x)$ from Proposition 3.3 (this time viewed over $\mathbb{Z}_q$). If this matrix is invertible over $\mathbb{Z}_q$ one can of course still apply Theorem 3.2. Nevertheless, it turns out that this matrix is not always invertible. To determine when this is indeed the case one can use [3, Theorem 2.2] which states that this matrix is invertible over $\mathbb{Z}_q$ if and only if for each prime p dividing q the polynomial from $\mathbb{Z}_p[x]$ corresponding to $g(x)$ is coprime to $x^D - 1$ (in $\mathbb{Z}_p[x]$). For instance, if $q = 15$ and $d = 2$ (and so $D = 30$) one can check that $x^{29} + x^{28} + x^{27} + \cdots + x^3 + x$ when viewed either as an element of $\mathbb{Z}_3[x]$ or $\mathbb{Z}_5[x]$ is coprime to $x^{30} - 1$, implying that the corresponding circulant matrix is invertible over $\mathbb{Z}_{15}$. This thus shows that $\mathrm{H}(30, 15)$ is distance magic. On the other hand, setting $q = 24$ and $d = 5$ (so that $D = 120$) one can verify that the polynomial $x^{119} + x^{118} + x^{117} + \cdots + x^6 + x \in \mathbb{Z}_3[x]$ is not coprime to $x^{120} - 1$ (their common factor in $\mathbb{Z}_3[x]$ is $x^2 + x - 1$), and so the corresponding circulant matrix is not invertible over $\mathbb{Z}_{24}$. Of course, this does not prove that $\mathrm{H}(120, 24)$ is not distance magic. It only shows that this method fails for this pair of q and d.

4. THE FOLDED HYPERCUBES

In this section we prove Theorem 1.2. As we already mentioned, while this paper was under review it came to our attention that the same result was independently proved in [18]. We would like to point out that, as in our case, the proof in [18] heavily relies on the method introduced by Gregor and Kovár in [12] where one uses a certain matrix over a suitable $\mathbb{Z}_2$-vector space to obtain a distance magic labeling. Nevertheless, while [18] works directly with the vertices of the folded hypercubes we construct a distance magic labeling of a folded hypercube from a suitable labeling of the corresponding hypercube and also use a somewhat different matrix, which implies that the obtained labelings are slightly different. Nevertheless, the key idea originates in [12] which indicates that this method has potential for further use in other types of regular graphs.

We now proceed towards the proof of Theorem 1.2 without further reference to [18]. Recall that we denote the dimension-D folded hypercube by $\overline{\mathrm{Q}_D}$ and that it is obtained by identifying pairs of antipodal vertices (that is pairs of the form $\{u, u \oplus \mathbf{j}\}$, where $\mathbf{j}$ is the all-ones vector) of Q_D in a natural way. We start our analysis of the folded hypercubes by an easy observation that will prove useful in our arguments.

Lemma 4.1. *Let $D \geq 3$ be an integer, let $V = \mathbb{Z}_2^D$ be the D-dimensional vector space over $\mathbb{Z}_2$, let Q_D be the D-dimensional hypercube and let $\overline{\mathrm{Q}_D}$ be the folded Q_D. Let $\overline{u}$ be a vertex of $\overline{\mathrm{Q}_D}$. Then the set of neighbours of $\overline{u}$ in $\overline{\mathrm{Q}_D}$ consists of precisely those vertices $\overline{v}$ of $\overline{\mathrm{Q}_D}$, for which one of v and $v \oplus \mathbf{j}$ is adjacent with u in Q_D.*

As with the Hamming graphs we first record a necessary condition for a folded hypercube to be distance magic which is based on Lemma 2.1 and the fact that the eigenvalues of the folded hypercubes are known. According to [4, p. 264] the eigenvalues of the dimension-D folded hypercube $\overline{\mathrm{Q}_D}$ are of the form $D - 4j$, $j \leq D/2$, and so this graph has 0 as one of its eigenvalues if and only if D is divisible by 4. This thus proves the following result.

Proposition 4.2. *Let $D \geq 3$ be an integer. If $\overline{\mathrm{Q}_D}$ is distance magic, then D is divisible by 4.*

To prove Theorem 1.2 we thus need to prove that all folded hypercubes $\overline{\mathrm{Q}_D}$ with D divisible by 4 indeed admit a distance magic labeling. For the rest of this section we therefore assume that $D = 4d$ for some positive integer d. The following result provides the main tool for finding a suitable labeling of $\overline{\mathrm{Q}_D}$.

Proposition 4.3. *Let $D \geq 3$ be an integer and let $V = \mathbb{Z}_2^D$ be the D-dimensional vector space over $\mathbb{Z}_2$. Then the dimension-D folded hypercube $\overline{\mathrm{Q}_D}$ is distance* *magic if and only if there exists a labeling ℓ of the vertices of Q_D with the following properties.*

(i) $1 \leq \ell(u) \leq 2^{D-1}$ *for each* $u \in V$*;*

(ii) $\ell(u) = \ell(v)$ *if and only if either* $u = v$ *or* $u = v \oplus \mathbf{j}$*;*

(iii) *For each* $u \in V$ *we have* $\sum_{v \in N(u)} \ell(v) = D(2^{D-1}+1)/2$.

Proof. Recall that since $\overline{\mathrm{Q}_D}$ is of order 2^{D-1} and valency D its magic constant (in the case that $\overline{\mathrm{Q}_D}$ is distance magic) is $k = D(2^{D-1}+1)/2$.

Suppose first that there exists a labeling ℓ of Q_D with the above three properties and note that properties (i) and (ii) imply that each of the values from $\{1, 2, \ldots, 2^{D-1}\}$ is assigned to precisely two (antipodal) vertices of Q_D. An appropriate distance magic labeling $\overline{\ell}$ of $\overline{\mathrm{Q}_D}$ can then simply be obtained by setting

$$\overline{\ell}(\{u, u \oplus \mathbf{j}\}) = \ell(u) = \ell(u \oplus \mathbf{j}).$$

The above remark implies that $\overline{\ell}$ is a bijection, and so property (iii) of ℓ and Lemma 4.1 imply that it is a distance magic labeling of $\overline{\mathrm{Q}_D}$ with magic constant k.

For the converse suppose that $\overline{\mathrm{Q}_D}$ is distance magic and let $\overline{\ell}$ be a distance magic labeling of $\overline{\mathrm{Q}_D}$. Define a labeling ℓ of the vertices of Q_D by setting $\ell(u) = \overline{\ell}(\overline{u})$. By definition, ℓ has properties (i) and (ii). That it also has property (iii) follows from Lemma 4.1. ∎

An appropriate labeling ℓ of the corresponding hypercube Q_D will be defined in a similar way as was done in Section 3 for the Hamming graphs and uses an idea similar to the one used in [12]. For a positive integer n let $\mathbf{I}_n$ denote the identity matrix over $\mathbb{Z}_2$ of dimension $n \times n$. Similarly, for positive integers n, m let $\mathbf{J}_{n,m}$ and $\mathbf{0}_{n,m}$ denote the all-ones matrix and the all-zeros matrix over $\mathbb{Z}_2$ of dimension $n \times m$, respectively. Let $M' \in \mathbb{Z}_2^{D \times D}$ be the matrix which has the following form in its $(2d-1, 2, 2d-1) \times (2d-1, 2, 2d-1)$ block representation

$$\begin{pmatrix} \mathbf{I}_{2d-1} & \mathbf{0}_{2d-1,2} & \mathbf{J}_{2d-1,2d-1} \\ \mathbf{0}_{2,2d-1} & \mathbf{I}_2 & \mathbf{J}_{2,2d-1} \\ \mathbf{J}_{2d-1,2d-1} & \mathbf{0}_{2d-1,2} & \mathbf{I}_{2d-1} \end{pmatrix}.$$

Observe that in each row of M', precisely $1 + 2d - 1 = D/2$ entries are equal to 0 and the remaining $D/2$ entries are equal to 1. Let M be the $D \times D$ matrix obtained from M' by replacing the first row of M' with the all-zeros row, that is

$$M_{ij} = \begin{cases} 0 & \text{if } i = 1, \\ M'_{ij} & \text{otherwise.} \end{cases}$$

Recall that $M \in \mathbb{Z}_2^{D \times D}$ so that all computations involving the matrix M are performed modulo 2. We now use M to define a labeling ℓ of Q_D in a very similar way as we have done for the Hamming graphs in Section 3. For a vertex u of Q_D let $\ell(u)$ be the integer $1+m(u)$, where $m(u)$ is the integer corresponding to the vector Mu in the binary number system, that is

$$\ell(u) = 1 + \sum_{i=1}^{D} (Mu)_i 2^{D-i}. \tag{6}$$

We point out that while the product Mu is computed modulo 2 (and so $(Mu)_i$ is an element of $\mathbb{Z}_2$), the sum $\sum_{i=1}^{D}(Mu)_i 2^{D-i}$ is to be computed within the ring of integers. For instance, for $D = 4$ the corresponding matrix M is

$$M = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{pmatrix},$$

and so the vertex $u = [1,0,1,0]^T$ gets the label $1+(0\cdot 2^3+0\cdot 2^2+1\cdot 2^1+1\cdot 2^0) = 4$. We now show that the above defined labeling ℓ of Q_D satisfies the three properties from Proposition 4.3.

Proposition 4.4. *Let $D = 4d$ be a positive integer divisible by* 4 *and let the labeling ℓ of the vertices of the hypercube Q_D be defined as in the above paragraph. Then the labeling ℓ satisfies all three properties from Proposition* 4.3.

Proof. Observe first that the fact that the first row of M is an all-zeros row implies that for each vertex u the label $\ell(u)$ is at most $1+2^{D-2}+2^{D-3}+\cdots+1 = 2^{D-1}$, showing that ℓ satisfies property (i) from Proposition 4.3.

To prove that it has property (ii) observe first that $M\mathbf{j} = \mathbf{0}_{D,1}$ since each row of M (other than the first one which is an all-zeros row) consists of $D/2$ ones and $D/2$ zeros (recall that $D/2$ is even and the computations are performed modulo 2). Therefore,

$$M(u \oplus \mathbf{j}) = Mu \oplus M\mathbf{j} = Mu, \text{ and thus } \ell(u) = \ell(u \oplus \mathbf{j})$$

holds for each vertex u. Observe that removing the first row and first column of M and then adding (over $\mathbb{Z}_2$) each of the first $2d-2$ (an even number) rows of this matrix to each of the last $2d-1$ rows of this matrix results in an upper triangular matrix with ones on the diagonal. This shows that M is of rank $D-1$, and so the kernel of this matrix has dimension 1. The kernel thus consists of the zero vector and the vector $\mathbf{j}$, thereby proving that $Mu = Mv$ implies that either $v = u$ or $v = u \oplus \mathbf{j}$. This confirms property (ii) of ℓ.

To prove that ℓ also has property (iii) from Proposition 4.3 pick a vertex u and recall that the neighbours of u in Q_D are precisely the vectors $u \oplus \mathbf{e}_i$ for

$i \in \{1, 2, 3, \ldots, D\}$. By (6) we have

$$\begin{aligned} \sum_{v \in N(u)} \ell(v) &= \sum_{j=1}^{D} \ell(u \oplus \mathbf{e}_j) = D + \sum_{j=1}^{D} \sum_{i=1}^{D} (M(u \oplus \mathbf{e}_j))_i 2^{D-i} \\ &= D + \sum_{j=1}^{D} \sum_{i=1}^{D} (Mu \oplus M_j)_i 2^{D-i} = D + \sum_{i=1}^{D} \left(2^{D-i} \sum_{j=1}^{D} (Mu \oplus M_j)_i \right). \end{aligned} \tag{7}$$

The first row of M is the all-zeros row, and so $(Mu \oplus M_j)_1 = 0$ for each j, $1 \le j \le D$. Furthermore, as each of the other $D-1$ rows of M contains exactly $D/2$ zeros and exactly $D/2$ ones, $D/2$ of the entries $(Mu \oplus M_j)_i$, $1 \le j \le D$, are equal to 0 and the remaining $D/2$ are equal to 1. It thus follows that

$$\sum_{j=1}^{D} (Mu \oplus M_j)_i = D/2$$

for each $i > 1$. Thus (7) implies that

$$\sum_{v \in N(u)} \ell(v) = D + \frac{D}{2} \sum_{i=2}^{D} 2^{D-i} = D + D\frac{2^{D-1}-1}{2} = D\frac{2^{D-1}+1}{2}. \tag{8}$$

■

A proof of Theorem 1.2 is now at hand. The forward implication was established in Proposition 4.2. For the converse, suppose $D \ge 3$ is divisible by 4. By Proposition 4.4 the labeling ℓ defined in the paragraph preceding that proposition satisfies all three properties from Proposition 4.3, and so the later proposition implies that $\overline{\mathrm{Q}_D}$ is distance magic.

5. Open Problems and Concluding Remarks

We conclude the paper with a few open problems and concluding remarks. Recall that by Proposition 3.1, if a Hamming graph $\mathrm{H}(D, q)$ is distance magic, then D is a multiple of q. In view of Theorem 1.1 the following problem arises quite naturally.

Problem 5.1. Let p be a prime and let t, d be positive integers where d is divisible by p. Determine whether the Hamming graph $\mathrm{H}(dq, q)$, where $q = p^t$, is distance magic or not.

Recall that the D-dimensional hypercube $\mathrm{Q}_D = \mathrm{H}(D, 2)$ is distance magic if and only if D is an even number, which is not divisible by 4. This suggests that the answer to Problem 5.1 might be negative. Observe also that in the case that this turns out not to be true (and thus some Hamming graphs from Problem 5.1 are distance magic) one will not be able to find a distance magic labeling simply using Theorem 3.2 with a circulant matrix having d zero entries and all other entries equal to 1 since there will always be a nonzero vector with all entries equal that will be in its kernel.

One can of course also try to generalize Proposition 3.6 in the sense that one considers the following problem.

Problem 5.2. Let q be a positive integer, which is not a prime power, and let $d \geq 2$ be a positive integer. Determine the necessary and sufficient condition on the pair q and d for the Hamming graph $\mathrm{H}(dq, q)$ to be distance magic.

Let us point out another interesting question in connection with Theorem 1.1 and Proposition 3.6. In both of these two results the distance magic labeling of the corresponding Hamming graph was obtained using Theorem 3.2. The following open problem therefore arises quite naturally.

Problem 5.3. Let $\mathrm{H}(D, q)$ be a Hamming graph with $D = dq$ for some positive integer d, where either $d = 1$ or $q = p^t$ for some prime p and positive integer t, such that d is not divisible by p. Construct a distance magic labeling of $\mathrm{H}(D, q)$ that it is not obtained by means of Theorem 3.2 or prove that such a labeling does not exist.

We remark that at least for the 6-dimensional hypercube $\mathrm{H}(6, 2)$ such labelings do exist. They were obtained quite recently using computers (see [17]).

An important step in the proof of Theorem 1.2 is Proposition 4.3. This result can of course be generalized in the following way. Suppose Γ is a graph of order $2n$ admitting a partition $\mathcal{P}$ of its vertex set into n pairs of independent vertices such that for any two distinct pairs from $\mathcal{P}$ the induced subgraph on the set of the corresponding four vertices either has no edges or is $2K_2$. The *quotient graph* $\Gamma_{\mathcal{P}}$ is then defined to be the graph with vertex set $\mathcal{P}$ in which two vertices are adjacent whenever the corresponding induced subgraph of Γ is $2K_2$. In the language of graph covers one could equivalently say that Γ is a 2-cover over the graph $\Gamma_{\mathcal{P}}$. Using essentially the same arguments as in the proof of Proposition 4.3 one can now show that $\Gamma_{\mathcal{P}}$ (which is of order n) is distance magic if and only if there exists a labeling of the vertices of Γ with labels from $\{1, 2, \ldots, n\}$ such that two different vertices get the same label if and only if they belong to the same pair from $\mathcal{P}$ and the sum of the labels of any vertex of Γ is independent of the given vertex. What is more, completely analogous arguments could be used to generalize the corresponding result to s-covers where $s > 2$. These facts could thus be exploited to analyze other families of graphs which have well-understood 2-covers or perhaps s-covers with $s > 2$.

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