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EXTREMAL GRAPHS FOR EVEN LINEAR FORESTS IN BIPARTITE GRAPHS

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Abstract

Zarankiewicz proposed the problem of determining the maximum number of edges in an (n, m)-bipartite graph containing no complete bipartite graph $K_{a,b}$. In this paper, we consider a variant of the Zarankiewicz problem and determine the maximum number of edges of an (n, m)-bipartite graph without containing a linear forest consisting of even paths. Moveover, all these extremal graphs are characterized in a recursion way.

Keywords: bipartite graph, linear forest, extremal graph, Turán number.

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1. Introduction

Our notation in this paper is standard. Let $G = (V(G), E(G))$ be a simple graph, where $V(G)$ is the vertex set with size $v(G)$ and $E(G)$ is the edge set

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with size $e(G)$. The *degree* of $v \in V(G)$, the number of edges incident to v, is denoted by $d_G(v)$ and the set of neighbors of v is denoted by $N_G(v)$. For a given set $X \subseteq V(G)$, let $N_X(v) = N_G(v) \cap X$ and $d_X(v) = |N_X(v)|$. Moreover, for $S \subseteq V(G)$, the induced subgraph of G by S is denoted by $G[S]$. Let G and H be two disjoint graphs, denote by $G \cup H$ the disjoint union of G and H, and by kG the disjoint union of k copies of a graph G. Denote by $G + H$ the graph obtained from $G \cup H$ by adding edges between all vertices of G and all vertices of H. Denote by P_k a *path* on k vertices (P_0 denotes the empty graph). We often refer to a path by the natural sequence of its vertices, writing, $P_k = x_1x_2 \cdots x_k$ and calling P_k a path from x_1 to x_k. We call a path *even* if it contains even number of vertices and *odd* otherwise. Similarly, denote by C_k a *cycle* on k vertices, writing, $C_k = x_1x_2 \cdots x_kx_1$. For $1 \leq i \leq j \leq k$, we use $x_iP_kx_j$ to denote the sub-path $x_ix_{i+1} \cdots x_{j-1}x_j$ of P_k and $x_iC_kx_j$ denote the path $x_ix_{i+1} \cdots x_{j-1}x_j$ in C_k. An (n,m)-*bipartite graph* $B_{n,m}$ is a bipartite graph of order $n+m$ whose vertices can be divided into two disjoint sets X and Y with $|X| = n$ and $|Y| = m$ such that every edge joins one vertex in X to the other vertex in Y. Moreover, denote by $K_{n,m}$ the complete (n,m)-bipartite graph (if $n \geq m = 0$, then $K_{n,m}$ denote the independent set on n vertices).

The *Turán number* of a graph H, $\mathrm{ex}(n,H)$, is the maximum number of edges in a graph of order n which does not contain H as a subgraph. Denote by $\mathbf{EX}(n,H)$ the set of graphs on n vertices with $\mathrm{ex}(n,H)$ edges containing no H as a subgraph and call a graph in it an *extremal graph* for H. Often, there are several extremal graphs. Similarly, for a given bipartite graph H, the maximum value $e(B_{n,m})$ under the condition that $B_{n,m}$ does not contain H as a subgraph is denoted by $\mathrm{ex}(n,m;H)$. Furthermore, if a bipartite graph $B_{n,m}$ with $\mathrm{ex}(n,m;H)$ edges does not contain H as a subgraph, then this bipartite graph is called a *bipartite extremal graph* for H. We say that a graph is H-*free* if it does not contain H as a subgraph.

In 1941, Turán [14] proved that the extremal graph without containing K_r as a subgraph is the Turán graph $T_{r-1}(n)$, i.e., the complete $(r-1)$-partite graph on n vertices with partite sizes as equal as possible. Later, Moon [12] (only when $r-1$ divides $n-k+1$) and Simonovits [13] showed that $K_{k-1} + T_{r-1}(n-k+1)$ is the unique extremal graph containing no copy of kK_r for sufficiently large n.

In 1959, Erdős and Gallai [3] proved the following well-known result.

Theorem 1 (Erdős and Gallai, [3]). *If G is a graph on $n \geq k$ vertices, then*

$$\mathrm{ex}\,(n,P_k) \leq \frac{1}{2}(k-2)n \tag{1}$$

with equality if and only if $n = (k-1)t$, where t is an integer.

Furthermore, Erdős and Gallai in [3] also determined $\mathrm{ex}\,(n,M_k)$ for all values of n and k, where M_k is the union of k disjoint edges. Recently, Bushaw and

Kettle in [2] also determined the Turán number of k disjoint copies of P_ℓ with $\ell \geq 3$ and also characterized all extremal graphs for sufficiently large n. Furthermore, Gorgol [6] studied the Turán number of disjoint copies of connected graphs. Let H be a connected graph on ℓ vertices; with the aid of the two graphs $\mathrm{Ex}(n-k\ell+1,H)\cup K_{k\ell-1}$ and $\mathrm{Ex}(n-k+1,H)+K_{k-1}$, she presented a lower bound for $\mathrm{ex}(n,kH)$, where $\mathrm{Ex}(n-k\ell+1,H)\in\mathbf{EX}(n-k\ell+1,H)$ and $\mathrm{Ex}(n-k+1,H)\in\mathbf{EX}(n-k+1,H)$.

The related results on the Turán number of paths, forests may be referred to [1, 4, 9, 10, 15] and the references therein.

On the other hand, Zarankiewicz in [16] proposed a variant of Turán's problem: determine the maximum number of edges of an (n,m)-bipartite graph containing no copy of $K_{a,b}$. This problem has attracted wide attention (see [5] and the references therein). Later, Gyárfás, Rousseau and Schelp [7] considered a variant of the Zarankiewicz problem for paths. Their result can be stated as follows:

Theorem 2 (Gyárfás, Rousseau and Schelp, [7]). *Let $n \geq m$. Then*

$$\mathrm{ex}\,(n,m;P_{2\ell}) = \begin{cases} nm, & \text{for } m\leq \ell-1;\\ (\ell-1)n, & \text{for } \ell-1<m<2(\ell-1);\\ (\ell-1)(n+m-2\ell+2), & \text{for } m\geq 2(\ell-1). \end{cases} \tag{2}$$

Furthermore,

(a) *If $m \leq \ell-1$, then the unique extremal graph is $K_{n,m}$.*

(b) *If $\ell-1<m<2(\ell-1)$, then the unique extremal graph is $K_{\ell-1,n}\cup K_{m-\ell+1,0}$.*

(c) *If $m\geq 2(\ell-1)$, then the extremal graphs are $K_{\ell-1,m-\ell+1}\cup K_{\ell-1,n-\ell+1}$; or $K_{\ell-1,i}\cup K_{\ell-1,n-i}$ for $i=0,1,\ldots,\lfloor n/2\rfloor$, when $m=2(\ell-1)$.*

Remark 3. In [7], Gyárfás, Rousseau and Schelp also determined the bipartite Turán numbers for odd paths. Moreover, Li, Tu and Jin [11] determined $\mathrm{ex}\,(n,m;M_k)$ for all values of n,m and k, where M_k is the union of k disjoint edges.

Theorem 4 (Li, Tu, and Jin, [11]). *Let $n\geq m\geq k$. Then*

$$\mathrm{ex}\,(n,m;M_k)=(k-1)n.$$

Moreover, the unique extremal graph is $K_{k-1,n}\cup K_{m-k+1,0}$.

A *linear forest* is a graph consisting of paths. Motivated by the above results, we will study the bipartite Turán numbers of linear forests.

Let $r_j=\sum_{i=1}^{j}k_i$ with $k_1\geq k_2\geq\cdots\geq k_j$. Define $f\,(n,m;k_1,\ldots,k_j)=$

$$\begin{cases} nm, & \text{for } m\leq r_j-1;\\ (r_j-1)n, & \text{for } r_j\leq m\leq 2(r_j-1);\\ (r_j-1)(n-k_j+1)+(k_j-1)(m-r_j+1), & \text{for } m\geq 2r_j-1. \end{cases}$$

The main result of this paper can be stated as follows.

Theorem 5. *Let $n \geq m$ and $k_1 \geq \cdots \geq k_\ell \geq 1$ with $\ell \geq 1$. Then*

$$\mathrm{ex}(n,m;P_{2k_1}\cup\cdots\cup P_{2k_\ell})=\max\{f(n,m;k_1),f(n,m;k_1,k_2),\ldots,f(n,m;k_1,\ldots,k_\ell)\}.$$

Furthermore, we have $\mathbf{EX}\,(n,m;P_{2k_1}\cup\cdots\cup P_{2k_\ell})$

$$=\mathbf{EX}\,(n,m;P_{2k_1})\cup\cdots\cup\mathbf{EX}\left(n,m;P_{2k_1}\cup\cdots\cup P_{2k_{\ell-1}}\right)\cup\mathcal{B},$$

where

$$\mathcal{B}=\begin{cases} K_{n,m}, & \text{when } m\leq r_\ell-1;\\ K_{r_\ell-1,n}\cup\overline{K}_{m-r_\ell+1}, & \text{when } r_\ell\leq m\leq 2r_\ell-3;\\ K_{r_\ell-1,i}\cup K_{r_\ell-1,n-i}: i\in\{0,1,\ldots,k_\ell-1\}, & \text{when } m=2r_\ell-2;\\ K_{r_\ell-1,n-k_\ell+1}\cup K_{k_\ell-1,m-r_\ell+1}, & \text{when } m\geq 2r_\ell-1.\end{cases}$$

In other words, all extremal graphs are characterized in a recursive way.

The rest of this paper is organized as follows. In Section 2, several technical lemmas are presented. In Section 3, we will prove Theorem 5. In Section 4, we give an open problem for conclusion.

2. Several Technical Lemmas

In 1981, Jackson [8] proved the following theorem which is useful in the proof of our main theorem.

Theorem 6 (Jackson, [8])**.** *Let $k\geq 2$ and $B_{n,m}$ be a bipartite graph with partite sets X and Y such that $|X|=n\geq 2$ and $|Y|=m\geq k$. Assume that each vertex of X has degree at least k. If $m\leq n$, then $B_{n,m}$ contains a cycle of length at least $2k$.*

The following simple lemmas are also needed.

Lemma 7. *Let $k_1>k_2\geq 1$ and $B_{n,m}$ be a bipartite graph with partite sets X and Y. Let $C=x_0y_0\cdots x_{k_1+k_2-1}y_{k_1+k_2-1}x_0$ be a cycle of length $2k_1+2k_2$ in $B_{n,m}$ with $\{x_0,\ldots,x_{k_1+k_2-1}\}\in X$, $\{y_0,\ldots,y_{k_1+k_2-1}\}\in Y$. If there is a vertex $z\in V(B_{n,m})\setminus V(C)$ with $d_{V(C)}(z)\geq k_1+1$, then $B_{n,m}[V(C)\cup\{z\}]$ contains a cycle $C_{2\ell}$ with $k_1\leq\ell<k_1+k_2$.*

Proof. Without loss of generality, we may assume $z\in X$. Since $d_{V(C)}(z)\geq k_1+1\geq 3$, it is easy to see $y_i,y_{i+j}\in N_{V(C)}(z)$ with $2\leq j\leq k_2-1$ for some $i\in\{0,\ldots,k_1+k_2-1\}$, where the indices are take under the additive group $\mathbb{Z}_{k_1+k_2}$. Hence, $zy_ix_iy_{i-1}\cdots y_{i+j+1}x_{i+j+1}y_{i+j}z$ is a cycle in $B_{n,m}[V(C_{2k_1+2k_2})\cup\{z\}]$ with length 2ℓ, where $k_1\leq\ell<k_1+k_2$. The assertion holds. ∎

Lemma 8. *Let $B_{n,m}$ be a bipartite graph. Let $P^1 = u_1u_2\cdots u_{2\ell}$ be an even path in $B_{n,m}$. If $d_{V(P^1)}(u_1) + d_{V(P^1)}(u_{2\ell}) \geq \ell + 1$, then $B_{n,m}[V(P^1)]$ contains a copy of $C_{2\ell}$.*

Proof. Since $d_{V(P^1)}(u_1)+d_{V(P^1)}(u_{2\ell}) \geq \ell+1$, there exists an integer j such that u_j is adjacent to $u_{2\ell}$ and u_{j+1} is adjacent to u_1. Thus $u_1u_{j+1}P^1u_{2\ell}u_jP^1u_1$ is a cycle of length 2ℓ, and we are done. ■

The following two lemmas follows easily in a similar way like in the proof of Lemma 8. We omit their proofs.

Lemma 9. *Let $B_{n,m}$ be a bipartite graph with partite sets X and Y. Let $P^1 = u_1u_2\cdots u_{k_1}$ be an odd path in $B_{n,m}$ with both end vertices in X and $y \in Y\setminus V(P^1)$. If $d_{V(P^1)}(u_1) + d_{V(P^1)}(y) \geq \lfloor k_1/2\rfloor + 1$, then $B_{n,m}[V(P^1) \cup \{y\}]$ contains a path on $k_1 + 1$ vertices.*

Lemma 10. *Let $B_{n,m}$ be a bipartite graph with partite sets X and Y. Let $C^1 = u_1u_2\cdots u_{2k_1}$ be a cycle in $B_{n,m}$, $x \in Y\setminus V(C^1)$ and $y \in Y\setminus V(C^1)$. If $d_{V(C^1)}(x)+ d_{V(C^1)}(y) \geq k_1+1$, then $B_{n,m}[V(C^1)\cup\{x, y\}]$ contains a path on $2k_1+2$ vertices.*

Lemma 11. *Let $B_{n,m}$ be a bipartite graph with partite sets X and Y. Let $P^1 = u_1u_2\cdots u_{k_1}$ and $P^2 = v_1v_2\cdots v_{k_2}$ be two vertex-disjoint paths in $B_{n,m}$. If $d_{V(P^2)}(u_1) + d_{V(P^2)}(u_{k_1}) \geq \ell$ with $\ell \leq \lfloor k_2/2\rfloor$, then $B_{n,m}[V(P^1) \cup V(P^2)]$ contains a path on $2\lfloor k_1/2\rfloor + 2\ell$ vertices.*

Proof. Let k_1 be even. If v_i is adjacent to u_1, then v_{i-1} and v_{i+1} are not adjacent to u_{k_1}. Otherwise, there is a path $v_1P^2v_{i-1}u_{k_1}P^1u_1v_iP^2v_{k_2}$ (or $v_{k_2}P^2v_{i+1}u_{k_1}P^1 u_1v_iP^2v_1$) on k_1+k_2 vertices, and we are done. Let v_j be the vertex of P^2 which is adjacent to u_1 or u_{k_1} with j maximum. Note that v_1 and v_{k_2} are not adjacent to u_1, as otherwise we are done. Since $d_{V(P^2)}(u_1)+d_{V(P^2)}(u_{k_1}) \geq \ell$, we have $j \geq 2\ell$. Thus, $u_1P^1u_{k_1}v_jP^2v_1$ (or $u_{k_1}P^1u_1v_jP^2v_1$) is a path on at least $k_1 + 2\ell$ vertices and hence we are done when k_1 is even. Now assume that k_1 is odd. Without loss of generality, let $u_1, u_{k_1} \in X$. If v_i is adjacent to u_1, then v_{i-2} and v_{i+2} is not adjacent to u_{k_1}. Otherwise, there is a path on $k_1 + k_2 - 1 \geq 2\lfloor k_1/2\rfloor + 2\ell$ vertices, and we are done. Thus, if v_p is adjacent to u_1 and v_q is adjacent to u_{k_1} with $p \neq q$, then we have $|p - q| \geq 4$ and $|p - q|$ is even. (1) k_2 is even. Without loss of generality, let $v_1 \in X$. Then v_2 is not adjacent to u_1 and u_{k_1}, as otherwise we are done. As the previous case, we choose v_j of P^2 which is adjacent to u_1 or u_{k_1} with j maximum. Hence, by $d_{V(P^2)}(u_1) + d_{V(P^2)}(u_{k_1}) \geq \ell$, we have $j \geq 4\lceil \ell/2\rceil \geq 2\ell$. Thus, $v_1P^2v_ju_1P^1u_{k_1}$ (or $v_1P^2v_ju_{k_1}P^1u_1$) is a path on $k_1 + 2\ell$ vertices, and we are done. (2) k_2 is odd. If both end-vertices of P^2 belong to X, then the result follows similarly as the previous proof when k_2 is even. Let $v_1, v_{k_2} \in Y$. Then v_3 is adjacent to neither u_1 nor u_{k_1}. Otherwise, it is easy to see that $u_1P^1u_{k_1}v_3P^2v_{k_2}$ (or $u_{k_1}P^1u_1v_3P^2v_{k_2}$) is a path on $2\lfloor k_1/2\rfloor + 2\ell$ vertices, and we are done. The rest proof of this case is similar as before (taking v_j as previous cases). The proof of the lemma is complete. ∎

Lemma 12. *Let $B_{n,m}$ be a bipartite graph with partite sets X and Y. Let $P^1 = u_1u_2\cdots u_{k_1}$ and $P^2 = v_1v_2\cdots v_{k_2}$ be two vertex-disjoint paths in $B_{n,m}$ such that $u_1, u_{k_1} \in X$. Assume that there is a vertex $y \in Y \setminus \big(V(P^1)\cup V(P^2)\big)$ which is not adjacent to end-vertices of P^2. If $d_{V(P^2)}(y) + d_{V(P^2)}(u_1) \geq \ell + 1$ with $2\ell \leq k_2$, then $B_{n,m}[V(P^1)\cup V(P^2)]$ contains either $P_{k_1+2\ell-1}$ or $P_{k_1-1}\cup P_{k_2+2}$ or $P_{k_1}\cup C_{2\ell'}$ as a subgraph, where $\ell' \geq \ell$.*

Proof. Let U be the set of neighbors of u_1 in P_2 and V be the set of neighbors of y in P_2. Then we have $U \subseteq \{v_{k_2-2\ell+3}, v_{k_2-2\ell+4}, \ldots, v_{2\ell-3}, v_{2\ell-2}\}$. Otherwise, $B_{n,m}$ contains a copy of $P_{k_1+2\ell-1}$, and we are done. Moreover, we have $V \subseteq \{v_i, v_{i+2}, \ldots, v_{i+2\ell-6}, v_{i+2\ell-4}\}$ for some $i \in \{2, 3, \ldots, k_2-2\ell+3\}$. Otherwise, $B_{n,m}$ contains a copy of $P_{k_1}\cup C_{2\ell'}$ with $\ell' \geq \ell$, and we are done. Note that $U\cap V = \emptyset$. We have $U \subseteq \{v_{i+1}, v_{i+3}, \ldots, v_{i+2\ell-7}, v_{i+2\ell-5}\}$. Since $d_{V(P^2)}(y) + d_{V(P^2)}(u_1) \geq \ell + 1$, there exist j_1 and j_2 such that $v_{j_1}, v_{j_2} \in U$ and $v_{j_1+1}, v_{j_2+1} \in V$. Hence $u_2P^1u_{k_1}$ and $v_1P^2v_{j_1}u_1v_{j_2}P^2v_{j_1+1}yv_{j_2+1}P^2v_{k_2}$ form a copy of $P_{k_1-1}\cup P_{k_2+2}$. We finish the proof of the lemma. ∎

Given a bipartite graph $B_{n,m}$ with partite sets X and Y, denote by $L(x, y)$ the set of edges incident with vertices $x \in X$ and $y \in Y$ and $e(x, y)$ the size of $L(x, y)$. We will prove the main lemma of this section.

Lemma 13. *Let $k_1 \geq \cdots \geq k_\ell \geq 1$, $\ell \geq 2$ and $n \geq 2\sum_{i=1}^{\ell} k_i - 1$. Let $B_{n,n}$ be a bipartite graph with partite sets X and Y. Assume that $B_{n,n}$ contains $P_{2k_1}\cup \cdots \cup P_{2k_{\ell-1}} \cup P_{2k_\ell-2}$ as a subgraph. If*

$$(3)\qquad e(x, y) \geq \sum_{i=1}^{\ell} k_i + k_\ell - 1 \text{ for every } x \in X \text{ and for every } y \in Y,$$

then $B_{n,n}$ contains $P_{2k_1}\cup\cdots\cup P_{2k_\ell}$ as a subgraph.

Proof. Let $F_\ell = P_{2k_1}\cup\cdots\cup P_{2k_\ell}$ and $F'_\ell = P_{2k_1}\cup\cdots\cup P_{2k_{\ell-1}}\cup P_{2k_\ell-2}$ (for $k_\ell = 1$, let $F'_\ell = P_{2k_1}\cup\cdots\cup P_{2k_{\ell-1}}\cup P_{2k_{\ell-1}}$). Let $\mathcal{F}$ be the set of subgraphs of $B_{n,n}$ consisting of paths and cycles and containing F'_ℓ as a subgraph. By the condition of the lemma, $\mathcal{F}$ is not empty. Choose $F \in \mathcal{F}$ with minimum number of components and maximum number of cycles. Let $F = C^1\cup\cdots\cup C^s\cup P^1\cup\cdots\cup P^t$. Moreover, we choose F with $v\left(P^1\cup\cdots\cup P^t\right)$ as large as possible and $v\left(C^1\cup\cdots\cup C^s\right)$ as small as possible. Let C^i contains a copy of P_{2x_i} for $i \in \{1, \ldots, s\}$ and P^j contains a copy of $P_{2x_{s+j}}$ for $j \in \{1, \ldots, t\}$, where $x_i = \sum_{x\in X_i} x$ and $\{X_1, \ldots, X_{s+t}\}$ is a partition of $\{k_1, \ldots, k_{\ell-1}, k_\ell - 1\}$. Let $X' = X \setminus V(F)$, $Y' = Y \setminus V(F)$ and $B' = B_{n,n} \setminus V(F)$. Suppose that $B_{n,n}$ does not contain F_ℓ as a subgraph. We will prove the lemma by contradictions. Note that if $k_\ell - 1 \notin X_i$, then $v(C^i) \leq 2x_i + 2k_\ell - 2$ or $v(P^{i-s}) \leq 2x_i + 2k_\ell - 1$. If $k_\ell - 1 \in X_i$, then $v(C^i) = 2x_i$ or $v(P^{i-s}) \leq 2x_i + 1$. Thus, we have

$$\min\{|V(F) \cap X|, |V(F) \cap Y|\} \geq \sum_{i=1}^{\ell} k_i + (\ell - 1)k_\ell, \tag{4}$$

with equality holds if and only if $k_1 = \cdots = k_\ell$, $s = 0$, $v(P^1) = 2k_\ell - 1$, $v(P^i) = 4k_\ell - 1$ for $i = 2, \ldots, \ell$ and all end-vertices of P^i lie in X (suppose that $k_\ell - 1 \in X_1$). Since $|X| = |Y| \geq 2\sum_{i=1}^{\ell} k_i - 1$, we have $\min\{|X'|, |Y'|\} \geq k_\ell - 1$. Without loss of generality, let $|X'| \leq |Y'|$. If $|X'| \geq k_\ell$, then there is a vertex in Y' with degree less than k_ℓ. Otherwise, it follows from Theorem 6 that there is a cycle of length at least $2k_\ell$, and hence $B_{n,n}$ contains a copy of F_ℓ. If $|X'| = k_\ell - 1$, then the equality of (4) holds. Hence we have $|Y'| \geq 1$ and each vertex in Y' has degree less than $|X'| \leq k_\ell - 1$. Thus, in both cases, there is a vertex, say y', in Y' with degree less than k_ℓ. We divide the proof into the following two cases.

Case 1. $t \geq 1$.

Case 1(a). There is an even path in $P^1, \ldots, P^t$. Without loss of generality, let P^1 be an even path and $x \in X$, $y \in Y$ be two end-vertices of P^1. Then, by the maximality of $v\left(P^1 \cup \cdots \cup P^t\right)$, x and y are not adjacent to each vertex of B'. Moreover, by the minimality of $s + t$, x and y are not adjacent to each vertex of $C^1 \cup \cdots \cup C^s$. If $k_\ell - 1 \notin X_{s+1}$ and $v(P^1) = 2x_{s+1} + 2k_\ell - 2$, then we can repartition $\{k_1, \ldots, k_\ell\}$ into $X_1^*, \ldots, X_{s+t}^*$ such that $k_\ell - 1 \in X_{s+1}^*$. Thus, we may assume that $v(P^1) \leq 2x_{s+1} + 2k_\ell - 4$ for $k_\ell - 1 \notin X_{s+1}$ (i.e., $k_\ell - 1$ belongs to $X_1 \cup \cdots \cup X_s$) and $v(P^1) = 2x_{s+1}$ for $k_\ell - 1 \in X_{s+1}$. Hence, we have $d_{V(P^1)}(x) + d_{V(P^1)}(y) \leq x_{s+1} + k_\ell - 2$ for $k_\ell - 1 \notin X_{s+1}$ and $d_{V(P^1)}(x) + d_{V(P^1)}(y) \leq x_{s+1}$ for $k_\ell - 1 \in X_{s+1}$. Otherwise, by Lemma 8, there is a cycle on $v(P^1)$ in $B_{n,n}[V(P_1)]$, contradicting our choice of F. For $i = 2, \ldots, t$, we have $d_{V(P^i)}(x) + d_{V(P^i)}(y) \leq x_{s+i} - 1$. Otherwise, by Lemma 11, there is an $F \in \mathcal{F}$ with $s + t - 1$ components, contradicting the minimality of $s + t$. Combining the above arguments, since either $k_\ell - 1 \in X_{s+1}$ or $k_\ell - 1 \in X_1$ with $v(P_1) \leq 2x_1 + 2k_\ell - 2$, we have

$$e(x, y) \leq \max\left\{\sum_{i=s+1}^{s+t} x_i, \sum_{i=s+1}^{s+t} x_i + k_\ell - 3\right\} \leq \sum_{i=1}^{s+t} x_i + k_\ell - 1 = \sum_{i=1}^{\ell} k_i + k_\ell - 2,$$

contradicting (3). Hence, we finish the proof of Case 1 when $P^1, \ldots, P^t$ contains an even path.

Case 1(b). There is no even path in $P^1, \ldots, P^t$. Since $|X'| \geq |Y'|$, without loss of generality, assume that P^1 is an odd path with both end-vertices in X. As in Case 1(a), we may assume that $v(P^1) \leq 2x_{s+1} + 2k_\ell - 3$ for $k_\ell - 1 \notin X_{s+1}$

and $v(P^1) = 2x_{s+1} + 1$ for $k_\ell - 1 \in X_{s+1}$. Let x be an end-vertex of P^1 and y' be a vertex in Y' with $d_{B'}(y') \le k_\ell - 1$.

Clearly, x is not adjacent to any vertex of C^i for $i = 1, \dots, s$. Otherwise, we will get a contradiction to the minimality of $s + t$. By Lemma 7, we have $d_{V(C_i)}(y') \le 2x_i$. Otherwise there is an $F' \in \mathcal{F}$ with smaller $v(C^1 \cup \cdots \cup C^s)$, contradicting our choice of F. For the path P^1, we have $d_{V(P^1)}(x) + d_{V(P^1)}(y') \le \lfloor v(P^1)/2 \rfloor$. Otherwise, by Lemma 9, the subgraph of $B_{n,n}$ induced by $V(P^1) \cup \{y'\}$ contains path on $v(P^1) + 1$ vertices, contradicting the choice of F. Moreover, we have $d_{V(P^i)}(x) + d_{V(P^i)}(y') \le x_{s+i}$ for $i = 2, \dots, t$. Otherwise, it follows from Lemma 12 that the subgraph of $B_{n,m}$ induced by $V(P^1) \cup V(P^i) \cup \{y'\}$ contains $P_{k_1+2x_i-1}$ or $P_{k_1-1} \cup P_{k_2+2}$ or $P_{k_1} \cup C_{2x'_{s+i}}$ as a subgraph ($x'_{s+i} \ge x_{s+i}$), contradicting our choice of F. Combining the above arguments, since either $k_\ell - 1 \in X_{s+1}$ or $k_\ell - 1 \notin X_{s+1}$ with $v(P_1) \le 2x_1 + 2k_\ell - 3$, we have

$$e(x, y') \le \sum_{i=1}^{s} x_i + \max\left\{\sum_{i=s+1}^{s+t} x_i, \sum_{i=s+1}^{s+t} x_i + k_\ell - 2\right\} \le \sum_{i=1}^{s+t} x_i + k_\ell - 1 = \sum_{i=1}^{\ell} k_i + k_\ell - 2,$$

contradicting (3). Thus, the proof of Case 1 is complete.

Case 2. $t = 0$. Clearly, $P_{2k_\ell-2}$ of F'_ℓ is contained in a cycle of F. Without loss of generality, let $k_\ell - 1 \in X_1$. Then the length of C_1 is $2x_1$, the length of C_i is at most $2x_i + 2k_\ell - 2$ for $i = 2, \dots, p$. Otherwise, G contains F_ℓ as a subgraph, and hence we are done. First, we have that C_i is not connected to any $C_{\ell \ne i}$. Otherwise, it is easy to see that G contains a copy $F' \in \mathcal{F}$ with smaller number of components than F, contradicting our choice of F. Note that $|X'| = |Y'| \ge k_\ell$. There is a vertex $x' \in X'$ and a vertex $y' \in Y'$ such that $d_{B'}(x') \le k_\ell - 1$ and $d_{B'}(y') \le k_\ell - 1$. If there is a vertex $x \in X'$ which is adjacent to C_1, then $d_{B'}(x) = 0$. Otherwise, it is easy to see that $B_{n,n}$ contains a copy of F_ℓ. Moreover, by the minimality of $s + t$, x is not adjacent to each vertex of C^i for $i = 2, \dots, p$. Furthermore, we have $d_{V(C_1)}(x) + d_{V(C_1)}(y') \le x_1$. Otherwise, it follows from Lemma 10 that $B_{n,n}$ contains a copy of F_ℓ. Combining the above arguments, we have

$$e(x, y') \le \sum_{i=1}^{s} x_i + k_\ell - 1 = \sum_{i=1}^{\ell} k_i + k_\ell - 2,$$

contradicting (3).

Now, assume that each vertex of B' is not adjacent to C_1. Take a vertex $y \in Y \cup V(C_1)$. Hence, we have

$$e(x', y) \le \sum_{i=1}^{s} x_i + k_\ell - 1 = \sum_{i=1}^{\ell} k_i + k_\ell - 2,$$

contradicting (3). Thus, the proof of the lemma is complete. ∎

3. Proof of Theorem 5

In this section, we are ready to present the proof of the main theorem in this paper.

***Proof of Theorem* 5.** Let $k_1 \geq k_2 \geq \cdots \geq k_\ell \geq 1$, $n \geq m \geq 1$ and $\ell \geq 1$. Let $r_j = \sum_{i=1}^{j} k_i$. Let $F_\ell = P_{2k_1} \cup \cdots \cup P_{2k_\ell}$ and $F'_\ell = P_{2k_1} \cup \cdots \cup P_{2k_\ell - 2}$ (for $k_\ell = 1$, let $F'_\ell = P_{2k_1} \cup \cdots \cup P_{2k_{\ell-1}}$). Let $\mathcal{B}$ be the set of bipartite graphs defined in Theorem **5**. We will show that $\mathbf{EX}\,(n, m; P_{2k_1} \cup \cdots \cup P_{2k_\ell})$

$$= \mathbf{EX}\,(n, m; P_{2k_1}) \cup \cdots \cup \mathbf{EX}\left(n, m; P_{2k_1} \cup \cdots \cup P_{2k_{\ell-1}}\right) \cup \mathcal{B}.$$

Hence, the bipartite extremal graphs for F_ℓ are characterized by induction on ℓ.

The theorem holds trivially for $m \leq r_\ell - 1$. For $r_\ell \leq m \leq 2r_\ell - 2$, since $B_{n,m}$ contains P_{2r_ℓ} as a subgraph implies that $B_{n,m}$ contains F_ℓ as a subgraph, the theorem follows from Theorem 2 by an easy observation (consider the extremal graphs in Theorem 2).

Now, let $m \geq 2r_\ell - 1$. We will prove the theorem by induction on ℓ, r_ℓ and $n + m$. Roughly speaking, we apply induction on ℓ to show that $B_{n,m}$ may belong to $\mathbf{EX}\left(n, m; P_{2k_1} \cup \cdots \cup P_{2k_{\ell-1}}\right)$ and on r_ℓ to deduce that $B_{n,m}$ contains a copy of F'_ℓ. Finally, we apply induction on $n+m$ to characterize the remaining extremal graphs. The theorem holds for $\ell = 1$ by Theorem 2. Let $\ell \geq 2$ and assume that the theorem is true for $\ell' < \ell$. For $\ell \geq 2$, the theorem holds for $r_\ell = \ell$ by Theorem 4. Assume that the theorem is true for $r'_\ell < r_\ell$. Let $B_{n,m}$ be a bipartite graph with a partition $X \cup Y$ such that $|X| = n$ and $|Y| = m$. Assume that

$$e\,(B_{n,m}) \geq \max\{f(n, m; k_1), f(n, m; k_1, k_2), \ldots, f(n, m; k_1, \ldots, k_\ell)\}. \tag{5}$$

A basic calculation shows that $f(n, m; k_1, \ldots, k_\ell) > f(n, m; k_1, \ldots, k_{\ell-1}, k_\ell - 1)$ when $k_\ell \geq 2$ (for $k_\ell = 1$, we set $f(n, m; k_1, \ldots, k_{\ell-1}, k_\ell - 1) = f(n, m; k_1, \ldots, k_{\ell-1})$). Then, by (5), we have

$$e\,(B_{n,m}) \geq \max\left\{f(n, m; k_1), f(n, m; k_1, k_2), \ldots, f(n, m; k_1, \ldots, k_\ell - 1)\right\}. \tag{6}$$

Thus, by induction hypothesis, either $B_{n,m}$ contains F'_ℓ as a subgraph or we have $B_{n,m} \in \mathbf{EX}\,(n, m; P_{2k_1}) \cup \cdots \cup \mathbf{EX}\left(n, m; P_{2k_1} \cup \cdots \cup P_{2k_{\ell-1}}\right)$. In fact, we apply induction on ℓ for $k_\ell = 1$ and we apply induction on r_ℓ for $k_\ell \geq 2$. Assume that $B_{n,m}$ contains F'_ℓ as a subgraph. We prove the theorem in the following three cases. The following case is the basis of induction.

Case 1. $n = m = 2r_\ell - 1$. By Lemma 13, there exist a vertex $x \in X$ and a vertex $y \in Y$ such that $e(x, y) \leq r_\ell + k_\ell - 2$. Otherwise, we will get a contradiction to the fact that $B_{n,n}$ is F_ℓ-free. Let $B_{n-1,n-1} = B_{n,n} - \{x, y\}$. By (5), we have

$$\begin{aligned} e\left(B_{n-1,n-1}\right) &\geq r_\ell(k_\ell - 1) + (2r_\ell - k_\ell)(r_\ell - 1) - (r_\ell + k_\ell - 2) \\ &= (2r_\ell - 2)\left(r_\ell - 1\right). \end{aligned}$$

Since $B_{n-1,n-1}$ does not contain F_ℓ as a subgraph and $n - 1 = 2r_\ell - 2$, we have $B_{n-1,n-1} = K_{r_\ell-1,n-1-i} \cup K_{r_\ell-1,i}$ for $i \in \{0, \dots, k_\ell - 1\}$. So we have $e(x, y) = r_\ell + k_\ell - 2$. Note that x, y can not be adjacent to the larger partite vertex set of $K_{r_\ell-1,n-1-i}$ in $B_{n-1,n-1}$. Otherwise, $B_{n,n}$ contains F_ℓ as a subgraph, a contradiction. Thus, we must have $B_{n-1,n-1} = K_{r_\ell-1,n-k_\ell} \cup K_{r_\ell-1,k_\ell-1}$. Therefore, we have $B_{n,n} = K_{r_\ell-1,n-k_\ell+1} \cup K_{r_\ell,k_\ell-1}$. We finish the proof of our main theorem for $n = m = 2r_\ell - 1$.

From now on, we may suppose that the theorem holds for smaller $n + m$. Clearly, by Case 1 and $n \geq m$, the theorem holds for $n + m \leq 4r_\ell - 2$.

Case 2. $2r_\ell - 1 \leq m < n$. There exists a vertex $x \in X$ with $d(x) \leq r_\ell - 1$. Otherwise, it follows from Theorem 6 that $B_{n,m}$ contains C_{2r_ℓ} and so F_ℓ as a subgraph. Let $B_{n-1,m} = B_{n,m} \setminus \{x\}$. By (5), we have

$$\begin{aligned} e\left(B_{n-1,m}\right) &= e\left(B_{n,m}\right) - d(x) \\ &\geq (m - r_\ell + 1)\left(k_\ell - 1\right) + \left(n - k_\ell\right)\left(r_\ell - 1\right). \end{aligned}$$

By induction hypothesis, we have $B_{n-1,m} = K_{m-r_\ell+1,k_\ell-1} \cup K_{n-k_\ell,r_\ell-1}$ and $d_{B_{n,m}}(y) = r_\ell - 1$. Hence, we have $B_{n,m} = K_{n-r_\ell+1,k_\ell-1} \cup K_{m-k_\ell+1,r_\ell-1}$. Otherwise, $B_{n,m}$ contains F_ℓ as a subgraph, a contradiction.

Case 3. $n = m \geq 2r_\ell$. By Lemma 13, there exist $x \in X$ and $y \in Y$ such that $e(x, y) \leq r_\ell + k_\ell - 2$. Otherwise $B_{n,m}$ contains F_ℓ as a subgraph, contradicting that $B_{n,m}$ is F_ℓ-free. Let $B_{n-1,n-1} = B_{n,n} \setminus \{x, y\}$. By (5), we have

$$\begin{aligned} e\left(B_{n-1,n-1}\right) &\geq e\left(B_{n,n}\right) - (r_\ell + k_\ell - 2) \\ &\geq (n - r_\ell + 1)\left(k_\ell - 1\right) + (n - k_\ell + 1)\left(r_\ell - 1\right) - (r_\ell + k_\ell - 2) \\ &= (n - r_\ell + 1)\left(k_\ell - 1\right) + (n - k_\ell + 1)\left(r_\ell - 1\right). \end{aligned}$$

Since $B_{n-1,n-1}$ does not contain F_ℓ as a subgraph, by induction hypothesis, we have $B_{n-1,n-1} = K_{n-r_\ell,k_\ell-1} \cup K_{n-k_\ell,r_\ell-1}$ and $e(x, y) = r_\ell + k_\ell - 2$. The result follows by an easy observation. ■

4. Conclusion

In [7], Gyárfás, Rousseau and Schelp also characterized the bipartite extremal graphs for $P_{2\ell+1}$. The bipartite extremal graphs for $P_{2\ell+1}$ is quit complicate than the bipartite extremal graphs for $P_{2\ell}$. Thus, it seems hard to characterize the bipartite extremal graphs for linear forest consisting of at least one odd path. We leave this as an open problem for future research.

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