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EXTREMAL GRAPHS AND CLASSIFICATION OF PLANAR GRAPHS BY MC-NUMBERS[1]

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Abstract

A path in an edge-colored graph is called *monochromatic* if all the edges in the path have the same color. An edge-coloring of a connected graph G is called a *monochromatic connection coloring* (MC-*coloring* for short) if any two vertices of G are connected by a monochromatic path in G. For a connected graph G, the *monochromatic connection number* (MC-*number* for short) of G, denoted by $mc(G)$, is the maximum number of colors that ensure G has a monochromatic connection coloring by using this number of colors. This concept was introduced by Caro and Yuster in 2011. They proved that $mc(G) \leq m-n+k$ if $\kappa(G) \leq k-1$. In this paper we characterize all graphs G with $mc(G) = m-n+\kappa(G)+1$ and $mc(G) = m-n+\kappa(G)$, respectively, where $\kappa(G)$ is the connectivity of G. We also prove that $mc(G) \leq m-n+4$ if G is a planar graph, and classify all planar graphs by their monochromatic connection numbers.

Keywords: monochromatic connection coloring (number), connectivity, planar graph, minors.

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1. Introduction

All graphs considered in this paper are simple, finite and undirected. For notation and terminology not defined here we refer to the book [2]. We use $\kappa(G)$ to denote the connectivity of a graph G, and $\chi(G)$ to denote the chromatic number of G. A planar graph is an *outerplanar graph* if it has an embedding with every vertex on the boundary of the unbounded face. If the vertex-set $V(G)$ of a graph G can be partitioned into k independent subsets $U_1, \dots, U_k$ such that every vertex of U_i connects every vertex of U_j in G for any $i \neq j$, then we call G a *complete k-partite graph.* For nonempty and pairwise disjoint k sets $V_1, \dots, V_k$ of vertices, if every vertex of V_i is adjacent to every vertex of V_j for any $i \neq j$, then we say that $V_1, \dots, V_k$ *form a complete k-partite graph.* Note that here each V_i is not necessarily an independent set. If there is no confusion, we always use m and n to denote the number of edges and the number of vertices of a graph, respectively. Sometimes, we also use $e(G)$ and $|V(G)|$ to denote the two numbers, respectively. For a graph G, $d_G(v)$ denotes the degree of a vertex v in G. We use P_n, C_n, S_n, K_n^- to denote a path with n vertices, a cycle with n edges, a star with n edges and a graph obtained from K_n by removing one edge, respectively. Analogically, a *k-path* or a *k-cycle* is a path or a cycle with k edges. For an edge $e = xy$ of G, G/e is called the *contraction* graph that is obtained from G by deleting e and then identifying x and y, which means replacing the two vertices x and y by a *new vertex* such that the new vertex is incident with all the edges which were incident with either x or y in G before. Suppose G and H are vertex-disjoint graphs. Then let $G \vee H$ denote the *join* of G and H, which is obtained from G and H by adding an edge between every vertex of G and every vertex of H, and let $G + H$ denote the graph with vertex-set $V(G) \cup V(H)$ and edge-set $E(G) \cup E(H)$. If $G = H$, we also denote $G + H$ by $2G$.

Generally, the notation $[k]$ refers to the set $\{1, 2, \dots, k\}$ of integers. An *edge-coloring* of G is a mapping from $E(G)$ to a set of positive integers, say $[k]$. A *monochromatic subgraph* is a subgraph whose edges are assigned to the same color. An edge-coloring of a connected graph G is called a *monochromatic connection coloring* (MC-*coloring* for short) if any two vertices of G are connected by a monochromatic path in G, and the edge-colored graph G is called *monochromatic connected.* An *extremal monochromatic connection coloring* (*extremal* MC-*coloring* for short) of G is a monochromatic connection coloring of G that uses the maximum number of colors. For a connected graph G, the *monochromatic connection number* (MC-*number* for short) of G, denoted by $mc(G)$, is the number of colors in an extremal monochromatic connection coloring of G. Huang and Li in [8] recently showed that it is NP-hard to compute the monochromatic connection number for a given graph.

Suppose Γ is an edge-coloring of G and i is a color of $\Gamma(G)$. The *i-induced* *subgraph* is a subgraph of G induced by all the edges with color i. We also call an i-induced subgraph a *color-induced subgraph*. Suppose F is the i-induced subgraph. If F is a single edge, then we call the color i and F *trivial*. Otherwise, they are called *nontrivial*. For a subgraph H of G, we denote $\Gamma|_H$ as the edge-coloring of H by restricting the edge-coloring Γ of G to H.

An edge-coloring of G is *simple* if any two nontrivial color-induced subgraphs intersect in at most one vertex. Caro and Yuster in [5] proved that each color-induced subgraph in a graph is a tree under any extremal MC-colorings of the graph and there exists a simple extremal MC-coloring for every connected graph. If there are t edges in a color-induced subgraph, then we say that the subgraph *wastes* $t-1$ colors. Suppose Γ is an MC-coloring of G and $\mathcal{H}$ is the set of all nontrivial color-induced subgraphs H. Then Γ wastes $w(\Gamma) = \Sigma_{H\in\mathcal{H}}\,(e(H)-1)$ colors. Thus, the number of colors used in G is equal to $m - w(\Gamma)$. If Γ is an extremal MC-coloring of G, then since each color-induced subgraph is a tree, we have that $w(\Gamma) = \Sigma_{H\in\mathcal{H}}\,(e(H)-1) = \Sigma_{H\in\mathcal{H}}\,(|V(H)|-2)$, and thus $mc(G) = m - \Sigma_{H\in\mathcal{H}}\,(|V(H)|-2)$.

For a connected graph G, we can obtain an MC-coloring by coloring a spanning tree monochromatically and coloring every other edge with a trivial color. Therefore, $mc(G) \geq m-n+2$ for every connected graph G. Caro and Yuster in [5] obtained the following results.

Theorem 1.1 [5]**.** *Let G be a connected graph with $n \geq 3$. If G satisfies one of the following properties, then $mc(G) = m-n+2$.*

(1) $\kappa(\overline{G}) \geq 4$, *where $\overline{G}$ is the complement of G;*

(2) *G is triangle-free;*

(3) $\Delta(G) < n - \frac{2m-3(n-1)}{n-3}$;

(4) *the diameter of G is greater than or equal to three;*

(5) *G has a cut-vertex.*

Theorem 1.2 [5]**.** *Let G be a connected graph. Then*

(1) $mc(G) \leq m-n+\chi(G)$;

(2) $mc(G) \leq m-n+k+1$ *if* $\kappa(G) = k$.

A graph G is called *s-perfectly-connected* if $V(G)$ can be partitioned into $s+1$ parts $\{v\}, V_1, \ldots, V_s$, such that each V_i induces a connected subgraph, $V_1, \ldots, V_s$ form a complete s-partite graph, and v has precisely one neighbor in each V_i. We call v a *special vertex.*

Proposition 1.3 [5]**.** *If $\delta(G) = s$, then $mc(G) \leq m-n+s$, unless G is s-perfectly-connected, in which case $mc(G) = m-n+s+1$.*

Jin *et al.* in [10] characterized all graphs with $mc(G) = m-n+\chi(G)$. Li *et al.* in [11, 12] generalized the concept of MC-coloring. For more knowledge about the monochromatic connection of graphs, we refer to [1, 3, 4, 6, 7, 9, 13, 14]. Caro and Yuster in [5] showed that the bound of the second result of Theorem 1.2 is sharp, and they studied wheel graphs, outerplanar graphs and planar graphs with minimum degree three.

The rest of this paper is organized as follows. In Section 2, we characterize all graphs G with $mc(G) = m - n + \kappa(G) + 1$ and $mc(G) = m - n + \kappa(G)$, respectively, where $\kappa(G)$ is the connectivity of G. In Section 3, we classify all planar graphs by their monochromatic connection numbers.

2. EXTREMAL GRAPHS G WITH $\kappa(G) = k$

For a graph G with connectivity $\kappa(G) = k$, we know that $mc(G) \leq m - n + k + 1$. In this section, we characterize all graphs with $mc(G) = m - n + \kappa(G) + 1$ and $mc(G) = m - n + \kappa(G)$, respectively. These results will be used in the next section for the classification of planar graphs.

Let $\mathcal{S}$ be a set of trees. Then we use $V(\mathcal{S})$ to denote $\bigcup_{T\in\mathcal{S}} V(T)$, and $|\mathcal{S}|$ to denote the number of trees in $\mathcal{S}$. Suppose that G is a graph with $\kappa(G) = k$ and Γ is an MC-coloring of G. Let $S = \{w_1, \ldots, w_k\}$ be a vertex-cut of G and $A_1, \ldots, A_t$ be the components of $G - S$. For a vertex $x \in V(A_i)$, we always use $\mathcal{T}_x$ to denote the set of nontrivial trees connecting x and a vertex in $\bigcup_{j\neq i} V(A_j)$. Since x connects every vertex of $\bigcup_{j\neq i} V(A_j)$ by a nontrivial tree, we have $\bigcup_{j\neq i} V(A_j) \subseteq V(\mathcal{T}_x)$.

Let $\mathcal{A}_{n,k}$ be the set of graphs $K_{k-1} \vee H$, where H is a connected graph with $|V(H)| = n - k + 1$ and H has a cut-vertex.

Theorem 2.1. *Suppose $k \geq 2$ and G is a graph with $\kappa(G) = k$. Then $mc(G) = m-n+k+1$ if and only if either $G \in \mathcal{A}_{n,k}$ or G is a k-perfectly-connected graph.*

Proof. If G is a k-perfectly-connected graph, then by Proposition 1.3, $mc(G) = m - n + k + 1$. If $G = K_{k-1} \vee H$ is a graph in $\mathcal{A}_{n,k}$, then let Γ be an edge-coloring of G such that a spanning tree of H is the only nontrivial tree. Then Γ is an MC-coloring of G and Γ wastes $n - k - 1$ colors. By Theorem 1.2, $mc(G) = m - n + k + 1$.

Next, we prove that either $G \in \mathcal{A}_{n,k}$ or G is a k-perfectly-connected graph if $mc(G) = m - n + k + 1$. Suppose that Γ is an extremal MC-coloring of G and $\mathcal{S}$ is the set of all non-trivial trees. Let $S = \{w_1, \ldots, w_k\}$ be a vertex-cut and $A_1, \ldots, A_t$ be the components of $G - S$. We distinguish the following cases.

Case 1. There is a component, say A_1, and a vertex u of A_1, such that $V(A_1) \subseteq V(\mathcal{T}_u)$.

Let $\mathcal{T}_u = \{T_1, \ldots, T_r\}$. Since u connects every vertex of $\bigcup_{i=2}^{t} V(A_i)$ by a nontrivial tree in $\{T_1, \ldots, T_r\}$, we have $\bigcup_{i\in[t]} V(A_i) \subseteq V\big(\bigcup_{i\in[r]} T_i\big)$. Since any two trees of $\{T_1, \ldots, T_r\}$ share a common vertex u and Γ is simple, we have $\bigcup_{i\in[r]} T_i$ is a tree. Moreover, $\left|V\left(\bigcup_{i\in[r]} T_i\right) \cap S\right| \geq r$. Therefore, $\bigcup_{i\in[r]} T_i$ wastes at least $n-(k-r)-1-r = n-k-1$ colors. Since $mc(G) = m-n+k+1$, we have $\mathcal{S} = \{T_1, \ldots, T_r\}$ and $\left|V\left(\bigcup_{i\in[r]} T_i\right) \cap S\right| = r$. Thus, $\left|V(T_i) \cap S\right| = 1$, say $V(T_i) \cap S = \{w_i\}$.

If $A_1 = \{u\}$, then since $\kappa(G) = k$ and $d_G(u) \leq |S| = k$, $\delta(G) = k$. By Proposition 1.3, $mc(G) = m-n+k+1$ implies that G is a k-perfectly-connected graph.

If $|V(A_1)| \geq 2$, then $r = 1$; otherwise, there are at least two nontrivial trees in $\mathcal{S}$. Suppose $v \in V(A_1) \setminus \{u\}$ and $v \in V(T_1)$. Let $w \in \left(\bigcup_{i=2}^{t} V(A_i)\right) \cap V(T_2)$. Then there is a nontrivial tree T_j connecting w and v. Since $v \in V(T_j)$ and $v \notin V(T_2)$, $T_j \neq T_2$. However, $\{u, w\} \subseteq V(T_j) \cap V(T_2)$, a contradiction. Therefore, $\mathcal{S} = \{T_1\}$. Since $mc(G) = m-n+k+1$, we have $\left|V(T_1)\right| = n-k+1$. Recall that $V(T_1) \cap S = \{w_1\}$. Let $S' = S \setminus \{w_1\}$. Then T_1 is a spanning tree of $G - S'$. Thus, $G - S'$ is connected and w_1 is a cut-vertex of $G - S'$. Since T_1 is the unique nontrivial tree of G, we have $G[S'] = K_{k-1}$ and $G = G[S'] \vee (G - S')$. Therefore, $G \in \mathcal{A}_{n,k}$.

Case 2. For each component A_i of $G - S$ and each vertex $u \in V(A_i)$, $V(A_i) \setminus V(\mathcal{T}_u) \neq \emptyset$.

For a vertex u of A_1, denote $A = V(A_1) \setminus V(\mathcal{T}_u)$ and $v \in A$. Let $w \in V(A_2)$, and let $\mathcal{F}$ be the set of nontrivial trees connecting w and a vertex of A. Since Γ is simple, we have $\left|V(\mathcal{T}_u) \cap S\right| \geq \left|\mathcal{T}_u\right|$ and $\left|V(\mathcal{F}) \cap S\right| \geq \left|\mathcal{F}\right|$. So, $\mathcal{T}_u$ wastes at least $n-k-|A|-1$ colors, and $\mathcal{F}$ wastes at least $|A|$ colors. Since $mc(G) = m-n+k+1$, $\mathcal{T}_u$ wastes precisely $n-k-|A|-1$ colors, $\mathcal{F}$ wastes precisely $|A|$ colors and $\mathcal{S} = \mathcal{T}_u \cup \mathcal{F}$. The conclusion that $\mathcal{F}$ wastes precisely $|A|$ colors implies that $V(A_2) \cap V(T) = \{w\}$ for each $T \in \mathcal{F}$. Since $V(A_2) \nsubseteq V(\mathcal{T}_w)$, there is at least one vertex in $V(A_2) \setminus V(\mathcal{T}_w)$, say $w' \in V(A_2) \setminus V(\mathcal{T}_w)$. Then there is no tree of $\mathcal{T}_u \cup \mathcal{F}$ that contains both v and w', which contradicts that $\mathcal{S} = \mathcal{T}_u \cup \mathcal{F}$. ∎

For convenience, we define three sets of graphs G, say $\mathcal{B}_{n,k}^1$, $\mathcal{B}_{n,k}^2$ and $\mathcal{B}_{n,k}^3$, with $\kappa(G) = k$ in the following.

$\mathcal{B}_{n,k}^1$ denotes the set of graphs G that satisfies the following four conditions.

1. $V(G)$ can be partitioned into k nonempty sets $\{u\}, U_1, \ldots, U_{k-1}$ such that the subgraph induced by each $U_i \cup \{u\}$ is connected,
2. $U_1, \ldots, U_{k-1}$ form a complete $(k-1)$-partite graph,
3. u has precisely two neighbors in U_t for $t \in [k-1]$ as well as one neighbor in U_i for $i \neq t$,
4. G is neither a k-perfectly-connected graph nor a graph of $\mathcal{A}_{n,k}$.

$\mathcal{B}^2_{n,k}$ denotes the set of graphs $K_{k-2} \vee H'$, where H' is a graph with connectivity 2 and $|V(H')| = n-k+2$, and $K_{k-2} \vee H'$ is neither a k-perfectly-connected graph nor a graph of $\mathcal{A}_{n,k}$.

$\mathcal{B}^3_{n,k}$ denotes the set of graphs $K^-_{k-1} \vee G'$, where G' is a connected graph of order $n-k+1$ with a cut-vertex.

Lemma 2.2. *For every graph $G \in \mathcal{B}^3_{n,k}$, G is neither a k-perfectly-connected graph nor a graph of $\mathcal{A}_{n,k}$.*

Proof. Suppose $G \in \mathcal{B}^3_{n,k}$ and $G = H \vee H'$, where $H = K^-_{k-1}$ and H' is a connected graph of order $n-k+1$ with a cut-vertex. It is obvious that there are at most $k-2$ vertices of G with degree $n-1$. Since every graph of $\mathcal{A}_{n,k}$ has at least $k-1$ vertices of degree $n-1$, $\mathcal{B}^3_{n,k} \cap \mathcal{A}_{n,k} = \emptyset$. Suppose that G is a k-perfectly-connected graph and v is a special vertex of G. If $v \in V(H')$, then H is a complete graph, a contradiction. If $v \in V(H)$, then $H' = K_{n-k+2}$, a contradiction to that H' has a cut-vertex. Therefore, G is neither a k-perfectly-connected graph nor a graph of $\mathcal{A}_{n,k}$. ■

Combining Lemma 2.2 and the definitions of $\mathcal{B}^1_{n,k}$ and $\mathcal{B}^2_{n,k}$, we have that for every graph $G \in \mathcal{B}^1_{n,k} \cup \mathcal{B}^2_{n,k} \cup \mathcal{B}^3_{n,k}$, G is neither a k-perfectly-connected graph nor a graph of $\mathcal{A}_{n,k}$. Since $\kappa(G) = k$, by Theorem 2.1, $mc(G) \leq m-n+k$.

Lemma 2.3. *If $G \in \mathcal{B}^1_{n,k} \cup \mathcal{B}^2_{n,k} \cup \mathcal{B}^3_{n,k}$, then $mc(G) = m-n+k$.*

Proof. Since $mc(G) \leq m-n+k$, we only need to prove that $mc(G) \geq m-n+k$ below.

If $G \in \mathcal{B}^1_{n,k}$, then let T_i be a spanning tree of $G\big[U_i \cup \{u\}\big]$ for $i \in [k-1]$. We color the edges of T_i with i and color any other edges with trivial colors. Then the edge-coloring is an MC-coloring of G, which uses $m-n+k$ colors. Thus, $mc(G) \geq m-n+k$.

If $G \in \mathcal{B}^2_{n,k}$, then $G = K_{k-2} \vee H'$. We color the edges of G such that a spanning tree of H' is the unique nontrivial color-induced subgraph. The edge-coloring is obviously an MC-coloring of G, which uses $m-n+k$ colors. Thus, $mc(G) \geq m-n+k$.

If $G \in \mathcal{B}^3_{n,k}$, then $G = K^-_{k-1} \vee G'$. Let T be a spanning tree of G' and let F be a 2-path obtained by connecting one vertex of G' and two nonadjacent vertices of K^-_{k-1}. We color the edges of G such that $\{T, F\}$ is the set of nontrivial color-induced subgraphs. The edge-coloring is obviously an MC-coloring of G, which uses $m-n+k$ colors. Thus, $mc(G) \geq m-n+k$. ■

Theorem 2.4. *Suppose $k \geq 3$, and G is a graph with $\kappa(G) = k$. Then $mc(G) = m-n+k$ if and only if $G \in \mathcal{B}^1_{n,k} \cup \mathcal{B}^2_{n,k} \cup \mathcal{B}^3_{n,k}$.*

Proof. If $G \in \mathcal{B}^1_{n,k} \cup \mathcal{B}^2_{n,k} \cup \mathcal{B}^3_{n,k}$, then by Lemma 2.3, $mc(G) = m - n + k$.

Suppose $mc(G) = m - n + k$. We will prove that $G \in \mathcal{B}^1_{n,k} \cup \mathcal{B}^2_{n,k} \cup \mathcal{B}^3_{n,k}$. Suppose that $S = \{v_1, \dots, v_k\}$ is a vertex-cut of G and $G - S$ has r components $A_1, \dots, A_r$. Let Γ be an extremal MC-coloring of G and $u \in V(A_i)$. Then Γ wastes $n - k$ colors. Since Γ is simple, any two trees of $\mathcal{T}_u$ intersect only at u. Thus, $\mathcal{T}_u$ wastes

$$\begin{aligned} &\left|\bigcup_{l\neq i} V(A_l)\right| + \left|V(\mathcal{T}_u) \cap V(A_i)\right| + \left|V(\mathcal{T}_u) \cap S\right| - 1 - \left|\mathcal{T}_u\right| \\ &= n - k - \left|V(A_i) \setminus V(\mathcal{T}_u)\right| + \left(\left|V(\mathcal{T}_u) \cap S\right| - \left|\mathcal{T}_u\right|\right) - 1 \end{aligned} \tag{1}$$

colors.

Claim 2.5. *Suppose $U \subseteq V(A_1)$. Then $\bigcup_{w\in U} \mathcal{T}_w$ wastes at least*

$$|U| + \left|\bigcup_{l=2}^{r} V(A_l)\right| - 1$$

colors.

Proof. Let $U = \{a_1, \dots, a_q\}$ and let $\mathcal{F}_i = \mathcal{T}_{a_i} \setminus \bigcup_{l=1}^{i-1} \mathcal{T}_{a_l}$. Suppose $\mathcal{F}_i$ contains c_i vertices of U. Then $\Sigma_{i\in[q]} c_i \geq q = |U|$. Since each tree of $\mathcal{F}_i$ connects one vertex of S and one vertex of $\bigcup_{l=2}^{r} V(A_l)$, $\mathcal{F}_i$ wastes at least c_i colors if $c_i \neq 0$. Since $\mathcal{F}_i = \mathcal{T}_{a_1}$ wastes at least $\left|\bigcup_{l=2}^{r} V(A_l)\right| + c_1 - 1$ colors by equality (1), $\bigcup_{w\in U} \mathcal{T}_w$ wastes at least

$$\begin{aligned} \sum_{i\in[q]} w_i &\geq \left|\bigcup_{l=2}^{r} V(A_l)\right| + c_1 - 1 + \sum_{i=2}^{q} c_i = \left|\bigcup_{l=2}^{r} V(A_l)\right| - 1 + \sum_{i\in[q]} c_i \\ &\geq \left|\bigcup_{l=2}^{r} V(A_l)\right| + |U| - 1 \end{aligned}$$

colors. □

Claim 2.6. *If T is a 2-path of G, then the two leaves of T are nonadjacent.*

Proof. Suppose the two leaves of T are adjacent. Then recolor every edge of T by a trivial color. It is easy to verify that the new coloring is an MC-coloring of G. However, the new coloring wastes less colors, a contradiction to the assumption that Γ is extremal. □

The proof of Theorem 2.4 continues by distinguishing the following cases.

Case 1. There is a component, say A_1, and a vertex u of A_1 such that $A_1 \subseteq V(\mathcal{T}_u)$.

Let $\mathcal{T}_u = \{T_1, \ldots, T_t\}$ and $B = \bigcup_{l=2}^{r} V(A_l)$. Here T_i is a tree colored with i. Each T_i contains at least one vertex of S.

Case 1.1. $V(A_1) = \{u\}$.

Since S is a vertex-cut of order k and $\kappa(G) = k$, u connects every vertex of S, that is, $S = N(u)$.

If there is a tree of $\mathcal{T}_u$, say T_t, which contains at least two vertices of S, then by equality (1), $\mathcal{T}_u$ wastes at least $n-k$ colors. Since $mc(G) = m-n+k$, $\mathcal{T}_u$ wastes precisely $n-k$ colors. Thus, T_t contains precisely two vertices of S (say v_t, v_{t+1}), and T_l contains precisely one vertex of S for $l \in [t-1]$ (say v_l). Therefore, $\mathcal{T}_u$ is the set of all nontrivial trees of G. Since Γ is simple, any two trees of $\mathcal{T}_u$ share a common vertex u. Let $U_i = V(T_i) \setminus \{u\}$ for $i \in [t]$ and $U_i = \{v_{i+1}\}$ for $t+1 \le i \le k-1$. Then $u, U_1, \ldots, U_{k-1}$ form a partition of $V(G)$ and each $G[U_i \cup \{u\}]$ is connected. Moreover, $|U_i \cap N(u)| = 1$ for $i \neq t$ and $|U_t \cap N(u)| = 2$. Since there is no nontrivial tree connecting a vertex of U_i and a vertex of U_j if $i \neq j$, $U_1, \ldots, U_{k-1}$ form a complete $(k-1)$-partite graph. Since $mc(G) \neq m-n+k+1$, by Theorem 2.1, G is neither a k-perfectly-connected graph nor a graph of $\mathcal{A}_{n,k}$. Thus, $G \in \mathcal{B}^1_{n,k}$.

If every tree of $\mathcal{T}_u$ contains precisely one vertex of S, say $V(T_i) \cap S = \{v_i\}$ for $i \in [t]$, then $\mathcal{T}_u$ wastes $n-k-1$ colors. Thus, there is a nontrivial tree T that wastes one color, in other words, T is a 2-path. So, $\mathcal{T}_u \cup \{T\}$ is the set of all nontrivial trees of G. Since T is a 2-path, by Claim 2.6, the two leaves of T are nonadjacent. Let $U_i = V(T_i) \setminus \{u\}$ for $i \in [t]$ and $U_i = \{v_i\}$ for $t+1 \le i \le k$. Since Γ is simple, the two leaves of T cannot appear in the same set U_i. Thus, there are two different integers i, j of $[k]$ such that one leaf of T is in U_i and the other leaf is in U_j. Then $U_1, \ldots, U_i \cup U_j, \ldots, U_k$ form a complete $(k-1)$-partite graph. Since $mc(G) \neq m-n+k+1$, by Theorem 2.1, G is neither a k-perfectly-connected graph nor a graph of $\mathcal{A}_{n,k}$. Recalling the definition of $\mathcal{B}^1_{n,k}$, we get $G \in \mathcal{B}^1_{n,k}$.

Case 1.2. $t = 1$.

From the assumption, $\bigcup_{i\in[r]} V(A_i) \subseteq V(T_1)$. Then T_1 wastes $n-k+\left|V(T_1) \cap S\right| - 2$ colors. Since Γ wastes $n-k$ colors, either T_1 is the only nontrivial tree and $\left|V(T_1) \cap S\right| = 2$, or $\left|V(T_1) \cap S\right| = 1$ and there is a 2-path F such that $\{F, T_1\}$ is the set of all nontrivial trees. Let $V = V(T_1)$ and $U = V(G) \setminus V$.

If $\left|V(T_1) \cap S\right| = 2$, then since T_1 is the unique nontrivial tree of Γ, we have that $G[U] = K_{k-2}$ and $G = G[U] \vee G[V]$. Since S is a vertex-cut with $|S| = k$, $V(T_1) \cap S$ is a vertex-cut of $G - U$, then $G[V]$ is a graph with connectivity 2. Since G is neither a k-perfectly-connected graph nor a graph of $\mathcal{A}_{n,k}$, we have $G \in \mathcal{B}^2_{n,k}$.

If $\left|V(T_1) \cap S\right| = 1$, then suppose $F = x_1e_1ye_2x_2$ and $V(T_1) \cap S = \{w\}$. If, by symmetry, $x_1 \in V(T_1)$, then $V(F) \cap V(T_1) = \{x_1\}$. Let $w' \in V(T_1) \setminus \{x_1\}$.

Then $w'x_2$ is a trivial edge of G. Let $T = T_1 \cup w'x_2$ and let Γ' be an edge-coloring of G such that T is the only nontrivial tree of G. Then Γ' is an extremal MC-coloring of G with $|V(T) \cap S| = 2$, this case has been discussed above. If $\{x_1, x_2\} \cap V(T_1) = \emptyset$, then $G[U] = K_{k-1}^-$ and $G = G[U] \vee G[V]$. Moreover, $G[V]$ is a connected graph with a cut-vertex w. Thus, $G \in \mathcal{B}_{n,k}^3$.

Case 1.3. $\left|V(A_1)\right| \geq 2$ and $t \geq 2$.

If $\left|V(A_1)\right| \geq 3$, then there are two trees of $\mathcal{T}_u$, say T_1, T_2, such that either $\left|V(T_1) \cap V(A_1)\right| \geq 3$ or $\left|V(T_1) \cap V(A_1)\right| = \left|V(T_2) \cap V(A_1)\right| = 2$. Let $w_i \in V(T_i) \cap B$ for $i \in [2]$. If $\left|V(T_1) \cap V(A_1)\right| \geq 3$, then there are trees of $\mathcal{T}_{w_2} \setminus \mathcal{T}_u$ connecting w_2 and $V(T_1) \cap V(A_1) \setminus \{u\}$. It is obvious that $\mathcal{T}_{w_2} \setminus \mathcal{T}_u$ wastes at least two colors. Since $\mathcal{T}_u$ wastes at least $n-k-1$ colors, $\mathcal{T}_{w_2} \cup \mathcal{T}_u$ wastes at least $n-k-1+2 = n-k+1$ colors, which contradicts that Γ is an extremal MC-coloring of G. If $\left|V(T_1) \cap V(A_1)\right| = \left|V(T_2) \cap V(A_1)\right| = 2$, say $\{z_i\} = V(T_i) \cap V(A_1) \setminus \{u\}$ for $i \in [2]$. Then there is a nontrivial tree F_1 connecting w_1, z_2, and a nontrivial tree F_2 connecting w_2, z_1. Since Γ is simple, we have $F_1 \neq F_2$. Since $\{F_1, F_2\} \cap \mathcal{T}_u = \emptyset$, $\{F_1, F_2\} \cup \mathcal{T}_u$ wastes at least $n-k+1$ colors, a contradiction. Therefore, $\left|V(A_1)\right| = 2$. Let $V(A_1) = \{z, u\}$ and let T_1 contain z, u. Then $V(T_i) \cap V(A_1) = \{u\}$ for $i \geq 2$.

Since $t \geq 2$, we have $B \setminus V(T_1) \neq \emptyset$. Then z connects every vertex of $B \setminus V(T_1)$ by a nontrivial tree, $\mathcal{T}_z \setminus \mathcal{T}_u$ is not an empty set. It is obvious that $\mathcal{T}_u$ wastes at least $n-k-1$ colors and $\mathcal{T}_z \setminus \mathcal{T}_u$ wastes at least one color. Since $mc(G) = m-n+k$, $\mathcal{T}_u$ wastes precisely $n-k-1$ colors and $\mathcal{T}_z \setminus \mathcal{T}_u$ wastes precisely one color. Therefore, $\mathcal{T}_z \setminus \mathcal{T}_u$ has only one member, and the member is a 2-path (denoting the 2-path by F, then $\mathcal{T}_z \setminus \mathcal{T}_u = \{F\}$). So, $|B \setminus V(T_1)| = 1$ and $t = 2$. Then $\mathcal{T}_u = \{T_1, T_2\}$ and $\mathcal{S} = \{F, T_1, T_2\}$ is the set of all nontrivial trees. We can also get that each tree of $\mathcal{S}$ intersects with S at only one vertex. So, F and T_2 are 2-paths.

Let Γ' be an edge-coloring of G obtained from Γ by recoloring $T' = T_1 \cup F$ with 1 and recoloring any other edges with trivial colors. Then the new coloring is also an MC-coloring of G. Since Γ' wastes $n-k$ colors, Γ' is an extremal MC-coloring of G. Then T' is the unique nontrivial tree of Γ' and $\left|V(T') \cap S\right| = 2$, this case has been discussed in Case 1.2.

Case 2. For each $i \in [r]$ and each $u \in V(A_i)$, $V(A_i) \setminus V(\mathcal{T}_u) \neq \emptyset$ (then each A_l has an order at least two).

If there is an integer $i \in [r]$ such that $\left|\bigcup_{l \neq i} V(A_l)\right| \geq 3$, then let $u \in V(A_i)$ and let $A' = V(A_i) \setminus V(\mathcal{T}_u)$. Then $\mathcal{T}_u$ wastes at least $n - |A'| - k - 1$ colors. By Claim 2.5, $\bigcup_{w \in A'} \mathcal{T}_w$ wastes at least $|A'| + |\bigcup_{l \neq i} V(A_l)| - 1$ colors. Since $\left(\bigcup_{w \in A'} \mathcal{T}_w\right) \cap \mathcal{T}_u = \emptyset$, $\mathcal{T}_u \cup (\bigcup_{w \in A'} \mathcal{T}_w)$ wastes at least $n-k+1$ colors, a contradiction. Therefore, $|\bigcup_{l \neq i} V(A_l)| \leq 2$ for each $i \in [r]$, and $\left|V(A_i)\right| = 2$ for $i \in [r]$ and $r = 2$. Let $V(A_1) = \{x_1, x_2\}$ and $V(A_2) = \{y_1, y_2\}$. Then each nontrivial tree contains at most two of $\{x_1, x_2, y_1, y_2\}$. Therefore, there is a nontrivial tree $T_{i,j}$ connecting x_i, y_j for $i, j \in [2]$, and the four nontrivial trees are pairwise different. Since $n = k + 4$ in this case and Γ wastes $n - k = 4$ colors, each $T_{i,j}$ is a 2-path and there is no other nontrivial tree. By Claim 2.6, the two leaves of each $T_{i,j}$ are nonadjacent. Thus, $\overline{G} = \{x_1y_1, x_1y_2, x_2y_1, x_2y_2\}$ is a 4-cycle. Choose a vertex of S, say v_1. Let $T = \bigcup_{i\in[2]}(v_1x_i \cup v_1y_i)$. Then T is a tree of G. Let Γ' be an edge-coloring of G such that T is the only nontrivial tree. Then Γ' is an MC-coloring of G and it wastes three colors, which contradicts that Γ is an extremal MC-coloring of G. ■

3. Classification of Planar Graphs

In this section, we consider the monochromatic connection numbers of all planar graphs. Since the connectivity of a planar graph is at most five, the monochromatic connection number of a planar graph is less than or equal to $m - n + 6$. In fact, we get that $m - n + 2 \le mc(G) \le m - n + 4$ if G is a planar graph. We characterize all planar graphs G of $\kappa(G) = k$ with $mc(G) = m - n + r$, for $1 \le k \le 5$ and $2 \le r \le 4$.

It is well-known that a graph is *outerplanar* if and only if it does not contain a K_4-minor or a $K_{2,3}$-minor, and an outerplanar graph with connectivity 2 contains a vertex of degree 2. Moreover, if $\kappa(G) = 2$, then the exterior face of an outerplanar graph G is a Hamiltonian cycle, called the *boundary* of G. A forest is called a *linear forest* if every component of the forest is a path (possibly a single vertex).

Lemma 3.1. *Let H be a graph. Then the following is satisfied.*

(1) *$K_1 \vee H$ is a planar graph if and only if H is an outerplanar graph.*

(2) *$2K_1 \vee H$ is a planar graph if and only if H is either a cycle or linear forest.*

(3) *$K_2 \vee H$ is a planar graph if and only if H is a linear forest.*

(4) *If H is an outerplanar graph with $\kappa(H) = 2$ and $|V(H)| \ge 4$, then H contains two nonadjacent vertices of degree 2.*

Proof. Notice that $K_1 \vee H$ is a planar graph if H is an outerplanar graph. On the other hand, if $K_1 \vee H$ is a planar graph but H is not an outerplanar graph, then H contains either a K_4-minor or a $K_{2,3}$-minor. Therefore, $K_1 \vee H$ contains either a K_5-minor or a $K_{3,3}$-minor, a contradiction.

It is obvious that $2K_1 \vee S_3$ contains a $K_{3,3}$ as a subgraph, and $2K_1 \vee (K_3 + K_1)$ contains a K_5-minor. Therefore, H does not have vertices of degrees at least three when $2K_1 \vee H$ is a planar graph. Then each component of H is either a cycle or a path. If H has two components H_1, H_2 such that H_1 is a cycle, then H has a $(K_3 + K_1)$-minor. Thus, $2K_1 \vee H$ has a K_5-minor, a contradiction. Therefore,

H is either a cycle or a linear forest if $2K_1 \vee H$ is a planar graph. On the other hand, if H is either a cycle or a linear forest, then $2K_1 \vee H$ is clearly a planar graph.

If H is a linear forest, then $K_2 \vee H$ is obviously a planar graph. If $K_2 \vee H$ is a planar graph, then since $2K_1 \vee H$ is a subgraph of $K_2 \vee H$, H is either a cycle or a linear forest. Since $K_2 \vee H$ contains a K_5-minor if one component of H is a cycle, H is a linear forest.

If H is an outerplanar graph with connectivity 2 and $|V(H)| = 4$, then H has two nonadjacent vertices of degree 2. If $|V(H)| \geq 5$ and H does not have any chord, then H has two nonadjacent vertices of degree 2. If $|V(H)| \geq 5$ and H has a chord $e = xy$, then the two $\{x, y\}$-components, say H_1 and H_2, are outerplanar graphs with connectivity 2. For $i \in [2]$, if $\left|V(H_i)\right| \geq 4$, then by induction, H_i has a vertex $z_i \notin \{x, y\}$ such that $d_{H_i}(z_i) = 2$; if $H_i = K_3$, let $\{z_i\} = V(H_i) \setminus \{x, y\}$. Then z_1, z_2 are two nonadjacent vertices of degree 2 in H. ∎

Let $\mathcal{P}_1$ denote the set of graphs $G = v \vee H$, where H is a connected outerplanar graph with a cut-vertex.

Lemma 3.2. *Let G be a planar graph with $\kappa(G) = 2$. Then $mc(G) = m - n + 3$ if and only if $G \in \mathcal{P}_1$.*

Proof. By Lemma 3.1 (1) and Theorem 2.1, G is a planar graph and $mc(G) = m-n+3$ if $G \in \mathcal{P}_1$. Suppose $mc(G) = m-n+3$. Then by Theorem 2.1, G is either a 2-perfectly-connected graph or a graph in $\mathcal{A}_{n,2}$. If $G \in \mathcal{A}_{n,2}$, then $G = v \vee H$ and H is a connected graph with a cut-vertex. Then by Lemma 3.1 (1), H is a connected outerplanar graph with a cut-vertex. If G is a 2-perfectly-connected graph, then $V(G)$ can be partitioned into three nonempty sets $\{v\}, A, B$ such that A, B form a complete bipartite graph. Let $|A| \leq |B|$. Then $1 \leq |A| \leq 2$; otherwise, G contains a $K_{3,3}$ as a subgraph. If $|A| = 1$, say $A = \{x\}$, then by Lemma 3.1 (1), $G[B]$ is a connected outerplanar graph. Let $H = G[B \cup v]$. Then H is a connected outerplanar graph with a cut-vertex and $G = x \vee H$, and so $G \in \mathcal{P}_1$. If $|A| = 2$, that is, $G[A] = K_2$, then $G[B]$ is a path by Lemma 3.1 (3). Let $A = \{x, y\}$ and $N(v) = \{x, z\}$, Then $G - x = (y \vee G[B]) \cup vz$. Since $G[B]$ is a path, $G - x$ is an outerplanar graph with a cut-vertex z. Since $G = x \vee (G - x)$, we get $G \in \mathcal{P}_1$. ∎

Let $\mathcal{P}_2 = \{v \vee H : H$ is an outerplanar graph with $\kappa(H) = 2$ and $H \neq u \vee P_{n-2}\}$.

Lemma 3.3. *Let G be a planar graph with $\kappa(G) = 3$. Then*
(1) *$mc(G) = m - n + 3$ if and only if $G \in \{2K_1 \vee P_{n-2}\} \cup \mathcal{P}_2$;*
(2) *$mc(G) = m - n + 4$ if and only if $G = K_2 \vee P_{n-2}$.*

Proof. By Lemma 3.1 (3) and Theorem 2.1, $K_2 \vee P_{n-2}$ is a planar graph with $mc(K_2 \vee P_{n-2}) = m - n + 4$. Next, we prove that $G = K_2 \vee P_{n-2}$ if $mc(G) =$

$m-n+4$. Suppose $mc(G)=m-n+4$. Then either $G \in \mathcal{A}_{n,3}$ or G is a 3-perfectly-connected graph. If G is the latter, then $V(G)$ can be partitioned into four parts v, V_1, V_2, V_3, such that each V_i induces a connected subgraph, V_1, V_2, V_3 form a complete 3-partite graph, and v has precisely one neighbor in each V_i. Let $|V_1| \leq |V_2| \leq |V_3|$. If $|V_1|=|V_2|=1$, then $G[V_1 \cup V_2]$ is an edge, say e. Thus, $G = e \vee G[V_3 \cup v]$. By Lemma 3.1 (3), since G is a graph with $\kappa(G)=3$, $G[V_3 \cup v]$ is a path of order $n-2$. Therefore, $G = K_2 \vee P_{n-2}$. If $|V_2| \geq 2$, then $G[V_1 \cup V_2 \cup V_3]$ contains a K_5-minor, a contradiction. If $G \in \mathcal{A}_{n,3}$, then $G = K_2 \vee H$. By Lemma 3.1 (3), since G is a graph with $\kappa(G)=3$, $G = K_2 \vee P_{n-2}$. Therefore, $mc(G)=m-n+4$ if and only if $G = K_2 \vee P_{n-2}$.

If $mc(G)=m-n+3$, then $G \in \mathcal{B}^1_{n,3} \cup \mathcal{B}^2_{n,3} \cup \mathcal{B}^3_{n,3}$. If $G \in \mathcal{B}^3_{n,3}$, then $V(G)$ can be partitioned into two parts U, V such that $G[U] = K_2^- = 2K_1$, $G[V]$ is a connected graph with a cut-vertex and $G = G[U] \vee G[V]$. Note that $\kappa(G)=3$. By Lemma 3.1 (2), we get that $G[V]$ is a path. If $G \in \mathcal{B}^2_{n,3}$, then $G = K_1 \vee H$, where H is a graph with connectivity 2. Since G is planar, by Lemma 3.1 (1), H is an outerplanar graph with connectivity 2 (recall that connectivity of H is possibly 1 or 2). Therefore, $G \in \mathcal{P}_2$. If $G \in \mathcal{B}^1_{n,3}$, then $V(G)$ can be partitioned into three parts v, A, B, such that v has two neighbors in A and one neighbor in B, and A, B form a complete bipartite graph.

If $G[A] = K_2$, then by Lemma 3.1 (3), $G[B]$ is a path P_{n-3}. Thus, $G = K_2 \vee P_{n-2}$, a contradiction to the assumption that $mc(G)=m-n+3$. If $G[A]=2K_1$, then $G = G[A] \vee G[B \cup v]$. By Lemma 3.1 (2), $G[B \cup v]$ is either a path P_{n-3} or a cycle C_{n-3}. Since v has precisely one neighbor in B, $G[B \cup v]$ is a path. Thus, $G = 2K_1 \vee P_{n-2}$.

If $|A| \geq 3$, then $|B| \leq 2$. Let x be the neighbor of v in B. Since $mc(G) = m-n+3$, we have $G \neq K_2 \vee P_{n-2}$. If $|B|=2$, that is, $G[B]=K_2$, then $G = x \vee (G-x)$, where $x = N_G(v) \cap B$. Thus, $G-x$ is an outerplanar graph with connectivity 2. If $|B|=1$, then $V(B)=\{x\}$ and $G = x \vee (G-x)$, and thus $G-x$ is an outerplanar graph with connectivity 2. Therefore, $G \in \mathcal{P}_2$. ∎

Lemma 3.4. *Suppose G is a planar graph with $\kappa(G)=k$ and S is a vertex-cut with $|S|=k$. Then $G[S]$ is either a cycle or a linear forest.*

Proof. Let u, v be two vertices in different components of $G-S$. Since G is a graph with $\kappa(G)=k$, there are k internally disjoint uv-paths $L_1, \ldots, L_k$. Let H be a graph obtained from $\bigcup_{i \in [k]} L_i$ by contracting all edges but those incident with u and v. Then $H = K_{2,k}$ is a minor of G with one part S. Thus, by Lemma 3.1 (2), $G[S]$ is either a cycle or a linear forest. ∎

Lemma 3.5. *Let G be a planar graph with $\kappa(G)=k$ and S be a vertex-cut with $|S|=k$. Suppose Γ is an extremal* MC*-coloring of G such that $G[S]$ does not contain nontrivial edges. Then we have the following.*

(1) *If* $k = 4$ *and* $G[S]$ *is not a* 4*-cycle, then* $mc(G) = m - n + 2$*;*

(2) *If* $k = 5$*, then* $mc(G) = m - n + 2$*.*

In addition, if $k = 4$ *and* $G[S]$ *does not contain nontrivial edges under any extremal* MC*-colorings, then* $mc(G) = m - n + 2$*.*

Proof. By Lemma 3.4, G has a $K_{2,k}$-minor with one part S. Since G is a planar graph, by Lemma 3.1 (2), $G[S]$ is either a cycle or a linear forest. Let $A_1, \dots, A_r$ be the components of $G - S$.

Suppose Γ is an extremal MC-coloring of G such that $G[S]$ does not contain nontrivial edges. We use $\mathcal{S}$ to denote the set of all nontrivial trees of G. For each $T \in \mathcal{S}$, let $x_T = |V(T) \cap S|$ when $|V(T) \cap S| \geq 2$ and let $x_T = 1$ when $|V(T) \cap S| \leq 1$. Suppose T is a tree of $\mathcal{S}$ such that x_T is maximum. Since $G[S]$ is not a complete graph, we have $x_T \geq 2$.

Without loss of generality, suppose A_1 is a minimum component of $G - S$. Choose two vertices u, v from A_1, A_2, respectively. Let $U = V(A_1) \setminus V(\mathcal{T}_u)$. Denote $\mathcal{F}$ as the set of nontrivial trees connecting v and a vertex of U (if $U = \emptyset$, then $\mathcal{F} = \emptyset$). Then $\mathcal{T}_u$ wastes $n - k - |U| - 1 + \Sigma_{T' \in \mathcal{T}_u} (x_{T'} - 1)$ colors and $\mathcal{F}$ wastes at least $|U| + \Sigma_{T' \in \mathcal{F}} (x_{T'} - 1)$ colors. Assume $\mathcal{T} = \mathcal{T}_u \cup \mathcal{F}$. Then $\mathcal{T}$ wastes

$$w_{\mathcal{T}} \geq n - k - 1 + \Sigma_{T' \in \mathcal{T}} (x_{T'} - 1) \tag{2}$$

colors. Moreover, the equality will mean that each tree of $\mathcal{F}$ intersects with $\bigcup_{i \neq 1} A_i$ only at v if $\mathcal{F} \neq \emptyset$. Since $G[S]$ does not contain nontrivial edges, if $T' \in \mathcal{S} \setminus \mathcal{T}$, then T' wastes at least $x_{T'} - 1$ colors. Then Γ wastes

$$w_{\Gamma} \geq n - k - 1 + \Sigma_{T' \in \mathcal{S}} (x_{T'} - 1) \tag{3}$$

colors. If the equality of (3) holds, then the equality of (2) will hold. Therefore, the equality of (3) will mean that each tree of $\mathcal{F}$ intersects with $\bigcup_{i \neq 1} A_i$ only at v if $\mathcal{F} \neq \emptyset$.

Claim 3.6. *If it does not simultaneously happen that* $G[S]$ *is a* 4*-cycle and* $x_T = 2$*, then* $mc(G) = m - n + 2$*.*

Proof. Note that $G[S]$ is either a cycle or a linear forest. Therefore, $\overline{G[S]}$ contains a 5-cycle if $|S| = 5$, and $\overline{G[S]}$ contains a $2K_2$ if $|S| = 4$.

Suppose $x_T \geq 4$. If $k = 4$, then $w_\Gamma \geq n - 2$. If $k = 5$ and $x_T \geq 5$, then $w_\Gamma \geq n - 2$. If $k = 5$ and $x_T = 4$, then let $S \setminus V(T) = \{u'\}$. Since $\overline{G[S]}$ contains a 5-cycle, u' does not connect a vertex of $S \setminus \{u'\}$ in $G[S]$. Therefore, u' connects this vertex by a nontrivial tree different from T. Thus, $w_\Gamma \geq n - 2$.

Suppose $x_T = 3$. If $k = 4$, then let $S \setminus V(T) = \{u\}$. Since $\overline{G[S]}$ contains a $2K_2$, u does not connect a vertex of $S \setminus \{u\}$ in $G[S]$. Therefore, u connects this vertex by a nontrivial tree different from T. Thus, $w_\Gamma \geq n - 2$. If $k = 5$, then let

$\{u, v\} = S \setminus V(T)$. Since $\overline{G[S]}$ contains a 5-cycle, u connects a vertex of $S \setminus \{u\}$ by a nontrivial tree T_1, and v connects a vertex of $S \setminus \{v\}$ by a nontrivial tree T_2. No matter $T_1 = T_2$ or not, Γ wastes at least $n - 2$ colors.

Suppose $x_T = 2$. Since T is a tree of $\mathcal{S}$ such that x_T is maximum, for any two different pairs of nonadjacent vertices of S, there are two different nontrivial trees connecting them, respectively. Therefore, $\Sigma_{T' \in \mathcal{S}} (x_{T'} - 1) \geq e(\overline{G[S]})$. Since $\overline{G[S]}$ contains a 5-cycle for $k = 5$ and $\overline{G[S]}$ contains a $2K_2$ for $k = 4$, if Γ wastes at most $n - 3$ colors, then $k = 4$ and $\overline{G[S]} = 2K_2$. Note that it does not simultaneously happen that $G[S]$ is a 4-cycle and $x_T = 2$. Thus, Γ wastes at least $n - 2$ colors, and then $mc(G) = m - n + 2$. □

By Claim 3.6, the former two results hold. Now we prove that if $k = 4$ and $G[S]$ does not contain nontrivial edges under any extremal MC-colorings, then $mc(G) = m - n + 2$. If it does not simultaneously happen that $G[S]$ is a 4-cycle and $x_T = 2$, then by Claim 3.6, $mc(G) = m - n + 2$. Thus, we only need to prove that subject to the conditions that $G[S]$ is a 4-cycle and $x_T = 2$, we can get a contradiction if $mc(G) \geq m - n + 3$.

Assume that $G[S]$ is a 4-cycle and $x_T = 2$. Then let $E(\overline{G[S]}) = \{v_1v_2, v_3v_4\}$. Suppose, to the contrary, that $mc(G) \geq m - n + 3$. Since $x_T = 2$, there is a nontrivial tree T_1 connecting v_1, v_2, and a nontrivial tree T_2 connecting v_3, v_4. Then Γ wastes at least

$$n - k - 1 + \Sigma_{T' \in \mathcal{S}} (x_{T'} - 1) \geq n - k - 1 + (x_{T_1} - 1) + (x_{T_2} - 1) = n - 3 \tag{4}$$

colors. Since $mc(G) \geq m - n + 3$, Γ wastes exactly $n - 3$ colors, and so the equality of (4) holds. Since the equality of (4) will mean that the equality of (3) holds, each tree of $\mathcal{F}$ intersects with A_2 only at v if $\mathcal{F} \neq \emptyset$. In addition, T_1 and T_2 are the only two trees each of which intersects with S at more than one vertex.

If $\mathcal{S} \neq \mathcal{T}$, then $\mathcal{S}' = \mathcal{S} \setminus \mathcal{T} \neq \emptyset$. Since T_1 and T_2 are the only two trees each of which intersects with S at more than one vertex, $\mathcal{T}$ wastes at least

$$n - k - 1 + \Sigma_{T' \in \mathcal{T} \cap \{T_1, T_2\}} (x_{T'} - 1)$$

colors, and Γ wastes at least

$$n - k - 1 + \Sigma_{T' \in \mathcal{T} \cap \{T_1, T_2\}} (x_{T'} - 1) + \Sigma_{T' \in \mathcal{S}' \cap \{T_1, T_2\}} \big(e(T') - 1\big)$$

colors. Let $T' \in \mathcal{S}'$. Since $k = 4$ and Γ wastes exactly $n - 3$ colors, T' is a 2-path and $T' \in \mathcal{S}' \cap \{T_1, T_2\}$, say $T' = T_1$. Let $T^* = v_1v_3 \cup v_2v_3$ and let Γ' be an edge-coloring of G obtained from Γ by recoloring T^* with a new nontrivial colors and recoloring all edges of T_1 with new trivial colors. Then Γ' is an extremal MC-coloring of G and $G[S]$ contains nontrivial edges, a contradiction.

If $\mathcal{S} = \mathcal{T}$ and $U \neq \emptyset$, then each tree of $\mathcal{F}$ intersects with $V(A_2)$ only at v. Suppose $\left|\bigcup_{l\neq 1} V(A_l)\right| \geq 2$ and $v' \in \bigcup_{l\neq 1} V(A_l) \setminus \{v\}$. Since $U \neq \emptyset$, there is a nontrivial tree T'' connecting v' and a vertex of U. However, T'' is not a member of $\mathcal{T}$, a contradiction to that $\mathcal{S} = \mathcal{T}$. Thus, $\left|\bigcup_{l\neq 1} V(A_l)\right| = 1$, in other words, $G - S$ has two components A_1, A_2 and $\left|V(A_2)\right| = 1$. Note that A_1 is a minimum component of $G - S$, $\left|V(A_1)\right| = 1$. Therefore, $G = 2K_1 \vee C_4$ and $G[S] = C_4$. Let F' be a 2-path connecting the two components of $G - S$ in G, and let F'' be a 3-path of $G[S]$. Suppose Γ' is an edge-coloring of G such that F', F'' are all nontrivial trees. Then Γ' is an extremal MC-coloring of G and $G[S]$ contains nontrivial edges, a contradiction.

If $\mathcal{S} = \mathcal{T}$ and $U = \emptyset$, then $\mathcal{S} = \mathcal{T}_u$. Since each pair of different trees in $\mathcal{T}_u$ intersect only at u, we have $\mathcal{T}_u = \{T_1, T_2\}$. Therefore, $\mathcal{S} = \{T_1, T_2\}$. Let $B_i = V(T_i) \cap \left(S \cup \bigcup_{l\neq 1} V(A_l)\right)$ for $i = [2]$. Then $\left|V(B_1)\right|, \left|V(B_2)\right| \geq 3$. Since T_1 and T_2 intersect only at u, every vertex of B_1 connects every vertex of B_2 by a trivial edge, then $G[B_1 \cup B_2]$ contains a $K_{3,3}$, a contradiction. ■

Lemma 3.7. *Let Γ be a simple extremal* MC*-coloring of G and $e = xy$ be a nontrivial edge in G. Suppose that $mc(G) = e(G) - |V(G)| + x$ and H is the underlying graph of G/e. Then $mc(H) \geq e(H) - |V(H)| + x$.*

Proof. Since Γ is a simple extremal MC-coloring of G and $mc(G) = e(G) - |V(G)| + x$, Γ wastes $|V(G)| - x$ colors. Suppose z is the new vertex of $V(G/e)$. Then any parallel edges are incident with z, and between any two vertices there are at most two parallel edges. Since e is a nontrivial edge, Γ is simple and every color-induced subgraph in G is a tree, we have that any color-induced subgraph of G/e is a tree. It is obvious that any two vertices of G/e are connected by a monochromatic path under $\Gamma|_{G/e}$. Moreover, $\Gamma|_{G/e}$ wastes $|V(G)| - 1 - x = |V(G/e)| - x$ colors.

Suppose there are parallel edges e_1, e_2 between u and z. If there is a trivial and parallel edge between u and z, say e_1, then we delete e_1. Then the resulting graph is also monochromatic connected, and the edge-coloring wastes $|V(G/e)| - x$ colors. If the two parallel edges are nontrivial, then suppose e_1, e_2 are edges of two nontrivial trees T_1, T_2, respectively. Let T be a spanning tree of $T_1 \cup T_2$ containing e_1. Let Γ' be an edge-coloring of $G/e - e_2$ obtained from Γ by recoloring T with a new nontrivial color, and then recoloring any other edges of $E(T_1 \cup T_2) \setminus E(T) \setminus \{e_2\}$ with trivial colors. Then Γ' is an MC-coloring of $G/e - e_2$ and Γ' wastes at most $\left|V(G/e - e_2)\right| - x = \left|V(G/e)\right| - x$ colors. By the above operation, we obtain an underlying graph H of G/e, and a simple MC-coloring Γ'' of H, which wastes at most $|V(H)| - x$ colors. Thus, $mc(H) \geq e(H) - |V(H)| + x$. ■

Lemma 3.8. *Let G be a planar graph and $e = ab$ be an edge of G. If the underlying graph of G/e contains $\{u, v\} \vee P_t$ as a subgraph, u is the new vertex* *and a (and also b) connects two leaves of P_t, then either $N_G(a) \cap I = \emptyset$ and $I \subseteq N_G(b)$, or $N_G(b) \cap I = \emptyset$ and $I \subseteq N_G(a)$, where I is the set of internal vertices of P_t.*

Proof. If $N_G(a) \cap I \neq \emptyset$ and $N_G(b) \cap I \neq \emptyset$, then let G' be a graph obtained from G by contracting all but two pendent edges of P_t. Then G' has a subgraph $K_{3,3}$ with one part $\{a, b, v\}$, and so G also has a $K_{3,3}$-minor, a contradiction. ■

Lemma 3.9. *If G is a planar graph with $\kappa(G) = 4$, then $mc(G) \leq m - n + 3$, and $mc(G) = m - n + 3$ if and only if $G = 2K_1 \vee C_{n-2}$.*

Proof. Suppose $G = \{u, v\} \vee H$, where H is an $(n-2)$-cycle and uv is not an edge of G. Then there is a 2-path P connecting u and v. Let L be a spanning tree of H. Suppose Γ is an edge-coloring such that P and L are all nontrivial trees of G. Then Γ is an MC-coloring of G , which wastes $n - 3$ colors. Thus, $mc(G) \geq m - n + 3$. It is easy to verify that G is neither a graph of $\mathcal{A}_{n,4} \cup \mathcal{B}^1_{n,4} \cup \mathcal{B}^2_{n,4} \cup \mathcal{B}^3_{n,4}$, nor a 4-perfectly-connected graph. Therefore, $mc(G) = m - n + 3$.

Suppose $mc(G) \geq m-n+3$. We prove that $G = 2K_1 \vee C_{n-2}$ below. Suppose $S = \{x_1, x_2, x_3, x_4\}$ is a vertex-cut of G. If $G[S]$ does not contain nontrivial edges under any extremal MC-colorings of G, then by Lemma 3.5, $mc(G) = m - n + 2$. If there is an extremal MC-coloring Γ of G such that $G[S]$ has a nontrivial edge, say $e = x_1x_2$, then by Lemma 3.7 the underlying graph H of G/e satisfies that $mc(H) \geq e(H) - |V(H)| + 3$. Since H is a graph with $\kappa(H) = 3$, H is either $2K_1 \vee P_{n-3}$ or $K_2 \vee P_{n-3}$, or a graph of $\mathcal{P}_2$. Since $\kappa(G) = 4$, if there is a vertex x of H with $d_H(x) = 3$, then either x is the new vertex or x is incident with the new vertex.

Case 1. Either $H = 2K_1 \vee P_{n-3}$ or $H = K_2 \vee P_{n-3}$.

From the assumption, $V(H)$ can be partitioned into two parts $A = \{u, v\}$ and B, such that $H[B] = P_{n-3}$ and $H = H[A] \vee H[B]$. Here, uv is an edge of H if $H = K_2 \vee P_{n-3}$, and uv is not an edge of H if $H = 2K_1 \vee P_{n-3}$. Let $H[B] = v_1e_1v_2e_2 \cdots e_{n-4}v_{n-3}$. If $|B| = 3$, then H contains a spanning subgraph $K_1 \vee C_4$. Since each vertex of $V(H) \setminus \{v_2\}$ has a degree three in H, v_2 is the new vertex and G has a subgraph $K_2 \vee C_4$, a contradiction to the choice that G is a planar graph. Thus, $|V(B)| \geq 4$ and v_1, v_{n-3} are the only two vertices with degree 3 in H. Therefore, the new vertex is either u or v, say u by symmetry. Since $\kappa(G) = 4$, v_1 (and also v_{n-3}) connects x_1, x_2 in G. Then by Lemma 3.8, suppose that x_1 does not connect any vertices of $\{v_2, \ldots, v_{n-4}\}$ and x_2 connects every vertex of $\{v_2, \ldots, v_{n-4}\}$. Since $\kappa(G) = 4$, x_1 connects v. Then $G[B \cup x_1]$ is an $(n - 2)$-cycle and thus $G = 2K_1 \vee C_{n-2}$.

Case 2. $H \in \mathcal{P}_2$.

From the definition of $\mathcal{P}_2$, $H = v \vee R$, where R is an outerplanar graph with connectivity 2. If $R = K_3$, then $|V(G)| = 5$. Since $\kappa(G) = 4$, $G =$

K_5, a contradiction. Thus, $|V(R)| \geq 4$. Since R is an outerplanar graph with connectivity 2, by Lemma 3.1 (4), R has two nonadjacent vertices of degree 2. Moreover, the boundary C of R is a Hamiltonian cycle.

Case 2.1. R has at least three vertices of degree two, say u_1, u_2, u_3.

Note that every vertex of degree 2 in R is either a new vertex or incident with the new vertex in H. Thus, v is the new vertex and each u_i connects both x_1 and x_2 in G. Note that u_1, u_2 and u_3 divide C into three paths. Let H' be a graph obtained from H by contracting all but one edge of each such path. Then the underlying graph of H' is a K_5, and so G also has a K_5-minor, a contradiction.

Case 2.2. R has exactly two vertices of degree two and v is not the new vertex.

Suppose w_1, w_2 are nonadjacent vertices of degree 2 in R. Since v is not the new vertex, w_1, w_2 have a common neighbor z in R, and z is the new vertex.

Let $P = R - z$. We prove that $H = vz \vee P$ and P is a path. We first prove that $R = z \vee P$, which implies that each chord of R is incident with z. Suppose, to the contrary, that there is a chord $f = z_1z_2$ of R such that $z \notin \{z_1, z_2\}$. Then z_1, z_2 divide C into two paths L_1 and L_2, say z is an internal vertex of L_1. Since R is an outerplanar graph, z does not connect any internal vertices of L_2 in H. Furthermore, since z is the new vertex, neither x_1 nor x_2 connects internal vertices of L_2 in G. Thus, $\{v, z_1, z_2\}$ is a vertex-cut of G, a contradiction to the assumption that $\kappa(G) = 4$. So, $R = z \vee P$ and P is a path. Since v connects every vertex of R, we have $H = vz \vee P$.

Consider the graph G below. Since w_1, w_2 are vertices of degree 3 and z is the new vertex of H, w_1 (and also w_2) connects x_1 and x_2 in G. Let $I = V(P) \setminus \{w_1, w_2\}$. Since $H = vz \vee P$, by Lemma 3.8, suppose that x_1 does not connect any vertices of I and x_2 connects every vertex of I. Then $D = G[V(P) \cup x_1]$ is a C_{n-2} and $G - v = x_2 \vee D$. Since $\{v, x_2\} \vee D$ is a spanning subgraph of G, v does not connect x_2 by Lemma 3.1 (3). This implies that $G = \{x_2, v\} \vee D$, and so $G = 2K_1 \vee C_{n-2}$.

Case 2.3. R has exactly two vertices of degree two and v is the new vertex.

Suppose a, b are nonadjacent vertices of degree 2 in R. Then a, b divide C into two paths, say L_1 and L_2. Let $L_1 = ae_1z_1e_2 \cdots z_se_{s+1}b$ and $L_2 = af_1w_1f_2 \cdots w_tf_{t+1}b$. Since a, b are vertices of degree 3 in H, a (and also b) connects x_1 and x_2 in G.

If $N_G(x_1) \cap (V(L_1) \setminus \{a, b\}) \neq \emptyset$ and $N_G(x_2) \cap (V(L_1) \setminus \{a, b\}) \neq \emptyset$, then let J be a graph obtained from H by contracting all edges of C but e_1, e_{s+1} and f_1. Then the underlying graph of J is a K_5, and so G has a K_5-minor, a contradiction. Thus, by symmetry, suppose $V(L_1) \subseteq N_G(x_1)$ and $N_G(x_2) \cap V(L_1) = \{a, b\}$. By the same reason, it will happen that $N_G(x_1) \cap (V(L_2) \setminus \{a, b\}) \neq \emptyset$ and $N_G(x_2) \cap (V(L_2) \setminus \{a, b\}) \neq \emptyset$. Thus, $V(L_2) \subseteq N_G(x_2)$ and $N_G(x_1) \cap V(L_2) =$

$\{a, b\}$. Therefore, $N_G(x_1) \cap V(R) = V(L_1)$ and $N_G(x_2) \cap V(R) = V(L_2)$.

If $R = K_1 \vee P_{n-3}$, then $G = 2K_1 \vee C_{n-2}$. We will prove that $R = K_1 \vee P_{n-3}$ below.

Claim 3.10. *Suppose $l = n_1 n_2$ is a chord of R. Then one end of l is contained in $V(L_1) \setminus \{a, b\}$ and the other end of l is contained in $V(L_2) \setminus \{a, b\}$.*

Proof. Suppose, to the contrary, that $\{n_1, n_2\} \subseteq V(L_1)$. Then $S' = \{x_1, x_2, n_1, n_2\}$ is a vertex-cut of G with $|S'| = 4$. However, $d_{G[S']}(x_1) = 3$, a contradiction to Lemma 3.4. □

If, by symmetry, $|V(L_1)| = 3$, then $L_1 = ae_1z_1e_2b$, and so by Claim 3.10, z_1 connects every vertex of L_2. Thus, $R = K_1 \vee P_{n-3}$.

If $|V(L_1)|, |V(L_2)| \geq 4$, then recall that $e = x_1x_2$ is a nontrivial edge under Γ. Suppose e is an edge of a nontrivial tree T. Then there is a nontrivial edge f of T between $\{x_1, x_2\}$ and R. By symmetry, suppose $f = x_1w$, where $w \in V(L_1)$. Let H' be the underlying graph of G/f. Then by Lemma 3.7, $mc(H') \geq e(H') - |V(H')| + 3$. Since H' is a planar graph with $\kappa(H') = 3$, H' is either $2K_1 \vee P_{n-3}$ or $K_2 \vee P_{n-3}$, or a graph of $\mathcal{P}_2$.

Suppose H' is either $2K_1 \vee P_{n-3}$ or $K_2 \vee P_{n-3}$. Let $H' = A \vee P_{n-3}$, where $V(A) = \{y_1, y_2\}$. If $x_2 \in V(A)$ (say $x_2 = y_2$), then since $|L_1| \geq 4$, y_1 is an internal vertex of L_1 and $y_1 \neq w$. This implies that either y_1a or y_1b is an edge of G, a contradiction. If $x_2 \notin \{y_1, y_2\}$, then the degree of x_2 in H' is at most 4. Since $V(L_2) \subseteq N_{H'}(x_2)$ and $|L_2| \geq 4$, we have $|L_2| = 4$ and $A \subseteq V(L_1)$. So, $L_2 = af_1w_1f_2w_2f_3b$. Since $|L_1| \geq 4$, by Claim 3.10, $A = \{w_1, w_2\}$. Let J be a graph obtained from H' by contracting all edges of L_1 but e_2. Then the underlying graph of J is a K_5, and so G has a K_5-minor, a contradiction.

Suppose H' is a graph of $\mathcal{P}_2$. Then $H' = y \vee H''$, where H'' is an outerplanar graph with connectivity 2. If $y = x_2$, then x_2 connects every vertex of R. However, since $N_G(x_2) \cap V(L_1) = \{a, b\}$ and $|V(L_1)| \geq 4$, we get a contradiction. If $y \neq x_2$, then $y \in V(R)$ and thus $R = K_1 \vee P_{n-3}$, a contradiction to the assumption that $|V(L_1)|, |V(L_2)| \geq 4$. ■

Lemma 3.11. *If G is a planar graph with $\kappa(G) = 5$, then $mc(G) = m - n + 2$.*

Proof. Suppose $mc(G) \geq m - n + 3$. Let $S = \{v_1, \ldots, v_5\}$ be a vertex-cut of G and Γ be an extremal MC-coloring of G. If $G[S]$ does not contain nontrivial edges, then by Lemma 3.5, $mc(G) = m - n + 2$, a contradiction. Otherwise, there is a nontrivial edge in $G[S]$, say $e = v_1v_2$. Let H be the underlying graph of G/e. Then by Lemma 3.7, $mc(H) \geq e(H) - |V(H)| + 3$. Since $\kappa(H) = 4$, we have $mc(H) = e(H) - |V(H)| + 3$. Thus, $H = 2K_1 \vee C_{n-2}$, say $H = \{u, v\} \vee C$, where $C = C_{n-2}$. Since each vertex of C has a degree 4 in H, either u or v is the new vertex. By symmetry, let u be the new vertex. Thus, v_1, v_2 connect every vertex of C, in other words, $e \vee C$ is a subgraph of G, a contradiction to the choice that G is planar. ■

Combining Lemmas 3.2, 3.3, 3.9 and 3.11, we get the following conclusions.

Theorem 3.12. *Suppose G is a connected planar graph. Then $mc(G) \leq m-n+4$ and the following results hold.*

(1) *If G is a graph with $\kappa(G) = 1$, then $mc(G) = m - n + 2$;*

(2) *If G is a graph with $\kappa(G) = 2$, then $m - n + 2 \leq mc(G) \leq m - n + 3$ and $mc(G) = m - n + 3$ if and only if $G \in \mathcal{P}_1$;*

(3) *If G is a graph with $\kappa(G) = 3$, then $m-n+2 \leq mc(G) \leq m-n+4$. Moreover, $mc(G) = m - n + 4$ if and only if $G = K_2 \vee P_{n-2}$, and $mc(G) = m - n + 3$ if and only if either $G \in \mathcal{P}_2$, or $G = 2K_1 \vee P_{n-2}$;*

(4) *If G is a graph with $\kappa(G) = 4$, then $m - n + 2 \leq mc(G) \leq m - n + 3$, and $mc(G) = m - n + 3$ if and only if $G = 2K_1 \vee C_{n-2}$;*

(5) *If G is a graph with $\kappa(G) = 5$, then $mc(G) = m - n + 2$.*

For ease of reading, the classification of planar graphs are summarized in the following table (remember that the connectivity $\kappa(G)$ of a planar graph G is at most 5).

$mc(G)$ \ $\kappa(G)$	1	2	3	4	5
$m-n+4$	$\emptyset$	$\emptyset$	$G = K_2 \vee P_{n-2}$	$\emptyset$	$\emptyset$
$m-n+3$	$\emptyset$	$G \in \mathcal{P}_1$	either $G \in \mathcal{P}_2$, or $G = 2K_1 \vee P_{n-2}$	$G = 2K_1 \vee C_{n-2}$	$\emptyset$
$m-n+2$	all	all but the above	all but the above	all but the above	all

Table 1. The classification of planar graphs.

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