

## BIREGULAR (AND REGULAR) PLANAR CAGES

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### Abstract

We study the *Cage Problem* for biregular planar graphs. This problem has being widely studied for biregular graphs (without the planarity hypothesis). An  $(\{r, m\}; g)$ -graph is a biregular graph whose vertices have degrees  $r$  and  $m$ , for  $2 \leq r < m$ , and girth  $g$ . An  $(\{r, m\}; g)$ -cage is an  $(\{r, m\}; g)$ -graph of minimum order. In this paper, we determine the triplets of values  $(\{r, m\}; g)$  for which there exist planar  $(\{r, m\}; g)$ -graphs and for all values we construct examples. Furthermore, we bound the order of the  $(\{r, m\}; g)$ -cages and in many instances we build examples that reach the bounds.

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### 1. INTRODUCTION

We only consider finite simple graphs. The girth of a graph is the length of a smallest cycle. An  $(r, g)$ -graph is a  $r$ -regular graph with girth  $g$ . An  $(r, g)$ -cage

is an  $(r, g)$ -graph of minimum order,  $n(r, g)$ . These graphs were introduced by Tutte in 1947 (see [21]). The *Cage Problem* consists of finding the  $(r, g)$ -cages for any pair integers  $r \geq 2$  and  $g \geq 3$ . However, this challenge has proven to be very difficult even though the existence of  $(r, g)$ -graphs was proved by Erdős and Sachs in 1963 (see [12]).

There is a known natural lower bound for the order of a cage, called *Moore's lower bound* and denoted by  $n_0(r, g)$ . It is obtained by counting the vertices of a rooted tree,  $T_{(g-1)/2}$  with radius  $(g-1)/2$ , if  $g$  is odd; or the vertices of a “double-tree” rooted at an edge (that is, two different rooted trees  $T_{(g-3)/2}$  with the root vertices incident to an edge), if  $g$  is even (see [11, 13]). Consequently, the challenge is to find  $(r, g)$ -graphs with minimum order. In each case, the smallest known example is called a *record graph*. For a complete review about known cages, record graphs, and different techniques and constructions see [13] and for some specific constructions related with finite geometries see [5]. To consult other new approaches to the Cage Problem see [6, 8, 15, 17].

As a generalization of this problem, in 1981, Chartrand, Gould and Kapoor introduced in the concept of *biregular cage* [10]. A biregular  $(\{r, m\}; g)$ -cage, for  $2 \leq r < m$ , is a graph of girth  $g \geq 3$  whose vertices have degrees  $r$  and  $m$  and are of the smallest order among all such graphs. See [2, 3, 4, 14] for *record biregular graphs* and the bounds given by those.

Let  $n_0(\{r, m\}; g)$  be the order of an  $(\{r, m\}; g)$ -cage. There is also a Moore lower bound tree construction for biregular cages (see [10]), which gives the following bounds.

**Theorem 1.** *For  $r < m$  the following bounds hold.*

$$\begin{aligned} n_0(\{r, m\}; g) &\geq 1 + \sum_{i=1}^{t-1} m(r-1)^i \text{ for } g = 2t + 1, \\ n_0(\{r, m\}; g) &\geq 1 + \sum_{i=1}^{t-2} m(r-1)^i + (r-1)^{t-1} \text{ for } g = 2t, \end{aligned}$$

Also, in the same paper Chartrand, *et al.* proved the following result.

**Theorem 2.** *For  $2 < m$  the following hold.*

$$\begin{aligned} n(\{2, m\}; g) &= \frac{m(g-2) + 4}{2} \text{ for } g = 2t, \\ n(\{2, m\}; g) &= \frac{m(g-1) + 2}{2} \text{ for } g = 2t + 1. \end{aligned}$$

**Theorem 3.** *For  $r < m$ ,  $n(\{r, m\}; 4) = r + m$ .*

Moreover, there are different generalizations of this problem. For example, those introduced by Filipovski, Ramos Rivera and Jajcay in 2019 [16], that consist

in the construction and study of biregular bipartite cages: in this case they also have two vertex degrees but all the vertices in each part have the same degree (see also [7]).

Finally, we would like to mention the closely related degree diameter problem; determine the largest graphs or digraphs of given maximum degree and given diameter. This problem has also been studied in the context of embeddability, namely.

Let  $S$  be an arbitrary connected, closed surface (orientable or not) and let  $n_{\Delta,D}(S)$  be the largest order of a graph of maximum degree at most  $\Delta$  and diameter at most  $D$ , embeddable in  $S$ .

For a good survey on both the degree diameter problem and its embedded version see [19].

### 1.1. Our contributions

In this paper we study a variation of the cage problem, namely we want to find the minimum order of regular and biregular planar cages.

The  $g$  parameter of a planar cage is a lower bound for the minimum length of a face in the graph's embedding. It is not difficult to prove that, besides to the cycles, the unique regular planar cages are the *Platonic Solids* (Section 2).

Given a triad  $\{r, m, g\}$ , with  $2 \leq r < m$  and  $3 \leq g$ , for which a planar  $(\{r, m\}; g)$ -graph exists, we denote as  $n_p(\{r, m\}; g)$  the order of an  $(\{r, m\}; g)$ -planar cage. In Section 3, we discover that the set of triads  $(\{r, m\}; g)$  for which planar  $(\{r, m\}; g)$ -graphs may exist is

$$\begin{aligned} &\{(\{r, m\}, 3) \mid 2 \leq r \leq 5, r < m\}, \quad \{(\{r, m\}, 4) \mid 2 \leq r \leq 3, r < m\}, \\ &\{(\{r, m\}, 5) \mid 2 \leq r \leq 3, r < m\}, \quad \{(\{2, m\}, g) \mid 2 < m, 6 \leq g\}. \end{aligned}$$

We provide upper and lower bounds for all  $n_p(\{r, m\}; g)$ . We construct planar  $(\{r, m\}; g)$ -graphs for all possible triads and construct planar  $(\{r, m\}; g)$ -cages for

$$\begin{aligned} &(\{2, m\}, 3), (\{3, m\}, 3), (\{4, m\}, 3), (\{5, 6\}, 3), (\{5, 7\}, 3), \\ &(\{2, m\}, 4), (\{3, 4 \leq m \leq 13\}, 4), (\{3, m = 5k - 1\}, 4) \text{ with } k \geq 3, \\ &(\{2, m\}, 5) \text{ and } (\{2, m\}, 6). \end{aligned}$$

We also remark that, for the triplets of the form  $(\{5, m\}; 3)$  and  $(\{3, m\}; 4)$  for which we cannot assert that we have found a planar biregular cage, we provide constructions with a small excess (i.e., the upper bounds provided by the constructions and the lower bounds for  $n_p(\{5, m\}; 3)$  and  $n_p(\{3, m\}; 4)$ , respectively, differ only by a small constant).

For completeness and comparison, at the end of each section or subsection we include the order of the known  $(r; g)$ -cages and  $(\{r, m\}; g)$ -cages in the classical problem (that is, when the cages are not necessarily planar).

In Section 4 we provide the proofs for some technical lemmas that we will use throughout the paper. Finally, in Section 5 we state some conclusions and further research directions.

## 2. REGULAR PLANAR CAGES

The  $(2, g)$ -graphs are cycles on  $g$  vertices. Since these graphs are both planar and known to be cages, it follows that these graphs are the  $(2, g)$ -planar cages. In this section we will investigate the existence of  $(r, g)$ -planar graphs for  $r \geq 3$ , and find the smallest ones.

It is well known that any planar graph has at least one vertex of degree at most 5, hence  $3 \leq r \leq 5$ . Recall that if  $G$  is a planar  $(r, g)$ -graph, then its planar dual  $G^*$  is also planar, all its faces are of size  $r$ , and its girth is bounded above by  $r$ ; while the degree of its vertices is bounded below from  $g$ . Using these, when  $r \geq 3$ , we can deduce that  $g \leq 5$ . Thus,  $3 \leq g \leq 5$ , and  $(r, g)$ -planar graphs may only exist for  $r, g \in \{3, 4, 5\}$ .

We would like to highlight that if  $G$  is an  $(r, g)$ -graph, then  $G^*$  is not necessarily a  $(g, r)$ -graph.

Let  $G$  be an embedded planar graph. Then we will denote as  $v = v(G)$  its order, as  $e = e(G)$  its size and its number of faces as  $f = f(G)$ . We will now argue the following.

**Theorem 4.** *The planar  $(r, g)$ -cages are the five platonic solids: the tetrahedron, the cube, the octahedron, the dodecahedron and the icosahedron.*

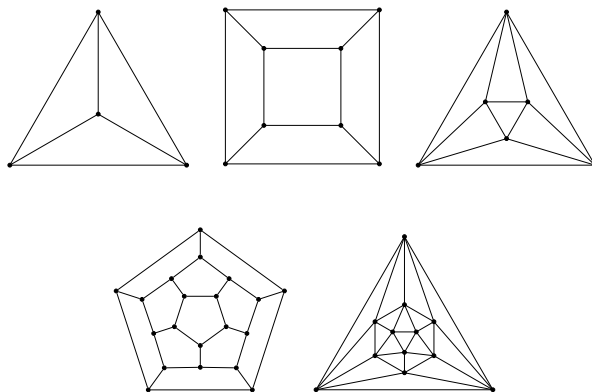


Figure 1. The five platonic solids.

**Proof.** Let  $G$  be a planar  $(r, g)$ -cage. From our discussion above  $r, g \in \{3, 4, 5\}$ . By the Handshaking lemma  $e = \frac{vr}{2}$ . Using the Euler's characteristic equation, we

find that the number of faces of an embedding is such that  $f = \frac{v(r-2)}{2} + 2$ . Also note that the face lengths (i.e., the number of edges of each face) are bounded below by  $g$  and the sum of the lengths of all faces of the embedding equals  $2e$ , this is  $2e \geq g(\frac{v(r-2)}{2} + 2)$ .

Hence  $v, r$  and  $g$  have to satisfy the inequality

$$(1) \quad v(2r - g(r - 2)) - 4g \geq 0.$$

Thus we must have

$$(2) \quad 2r - g(r - 2) > 0.$$

It is easy to check that if  $r, g \in \{3, 4, 5\}$ , then only the pairs  $(r, g)$  satisfying (2) are  $(3, 3), (3, 4), (3, 5), (4, 3)$  and  $(5, 3)$ .

Note that (1) can be rewritten as

$$v \geq \frac{4g}{2r - g(r - 2)}.$$

Also, note that each of the feasible pairs from above provides a lower bound for  $v$ . We argue that these lower bounds are in fact tight and the resulting  $(r, g, v = n_p(r, g))$  triplet possibilities for  $(r, g)$ -cages are

$$(a) (3, 3, 4) \quad (b) (3, 4, 8) \quad (c) (3, 5, 20) \quad (d) (4, 3, 6) \quad (e) (5, 3, 12).$$

This follows as,  $v(2r - g(r - 2)) - 4g = 0$  if and only if the size of all faces is precisely  $g$ . Thus, case (a) corresponds to a map on the plane with vertex degree 3 and face size 3, this is an embedding of the tetrahedron. Similarly, we can see that cases (b), (c), (d) and (e) are the cube, the dodecahedron the octahedron and the icosahedron, respectively. ■

For comparison, the  $(r; 3)$ -cages are the complete graphs  $K_r$  for  $r = \{3, 4, 5\}$  out of which only  $K_4$  is planar (the tetrahedron). Obviously, neither the  $(3; 4)$ -cage nor the  $(3; 5)$ -cage are planar, for the first is  $K_{3,3}$  and the second is the Petersen Graph.

It is interesting to note that intuitively, a cage is a graph with few vertices but with many edges, as many as the fixed girth allows. This observation is confirmed in [1], where the authors proved that the cages with order of the Moore's lower bound are extremal graphs; they have the maximum number of possible edges for an  $r$ -regular graph with order  $n_0(r, g)$  and fixed girth  $g$ . Because of the later, it is natural that most known cages, including biregular ones, are not planar.

### 3. BIREGULAR PLANAR CAGES

Now, we turn our attention to planar  $(\{r, m\}; g)$ -cages. We may assume without loss of generality that  $2 \leq r < m$  and  $3 \leq g$ , as we only consider simple graphs. Also, as we have argued before, a planar graph must have a vertex of degree less than or equal to 5, thus,  $2 \leq r \leq 5$ .

Here we start again by noticing that if  $y$  is the number of vertices of  $G$  with degree  $r$  and  $x$  is the number of vertices of degree  $m$ , then, by the Handshaking lemma,  $2e = yr + xm$ . Also, as before, we have  $2e \geq gf$  and  $f = 2 - v + e$ . Combining these three relations together we have

$$(3) \quad y[r(2 - g) + 2g] + x[m(2 - g) + 2g] - 4g \geq 0.$$

Using this equation, we may prove the following lemma.

**Lemma 5.** *An  $(\{r, m\}, g)$ -graph can be planar if and only if the triplet  $(\{r, m\}, g)$  is in one of the sets*

$$\begin{aligned} & \{(\{r, m\}, 3) \mid 2 \leq r \leq 5, r < m\}, \quad \{(\{r, m\}, 4) \mid 2 \leq r \leq 3, r < m\}, \\ & \{(\{r, m\}, 5) \mid 2 \leq r \leq 3, r < m\}, \quad \{(\{2, m\}, g) \mid 2 < m, 6 \leq g\}. \end{aligned}$$

**Proof.** Notice that, for equation (3) to hold, we need at least one of the following:  $r(2 - g) + 2g > 0$  or  $m(2 - g) + 2g > 0$ . This is, an inequality of the form  $t < \frac{2g}{g-2} = 2 + \frac{4}{g-2}$  must be satisfied for either  $t = r$  or  $t = m$ . As  $r < m$ , if the inequality is satisfied by  $m$ , then it is automatically satisfied by  $r$ . Hence,  $r$  satisfies the inequality.

From these inequalities, it is easy to compute that the triplets  $(\{r, m\}, g) \in \mathbb{N}^3$  for which  $r < 2 + \frac{4}{g-2}$ , with  $2 \leq r$  and  $2 \leq g$ , are as stated (see Figure 2). ■

#### 3.0.1. Lower bounds for the orders of planar biregular cages

Recall that  $n_p(\{r, m\}; g)$  denotes the order of an  $(\{r, m\}; g)$ -planar cage. In this section we will provide some easy lower bounds on  $n_p(\{r, m\}; g)$ .

**Lemma 6.** *For each fixed triad  $(\{r, m\}; g)$  listed in one of the sets in Lemma 5, the minimum number of vertices  $n_p(\{r, m\}; g)$  that an  $(\{r, m\}; g)$ -planar cage has to have increases as the number of vertices of degree  $m$  increases.*

**Proof.** Recall Equation 3,  $y[r(2 - g) + 2g] + x[m(2 - g) + 2g] - 4g \geq 0$ , where  $x$  is the number of vertices of degree  $m$  and  $y$  is the number of vertices of degree  $r$ . Here we may write  $y = n_p - x$ , where  $n_p = n_p(\{r, m\}; g)$ , to obtain

$$(4) \quad n_p \geq \frac{4g + ([r(2 - g) + 2g] - [m(2 - g) + 2g])x}{[r(2 - g) + 2g]}.$$

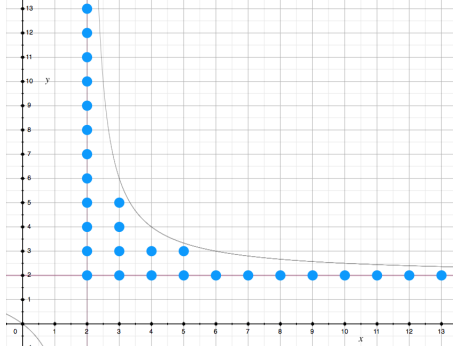


Figure 2. The curve is the graph of the function  $f(\alpha) = 2 + \frac{4}{\alpha-2}$ . The dots represent the integer pairs  $(\alpha, \beta)$  for which  $y < f(\alpha)$ ,  $\alpha \geq 2$ ,  $\beta \geq 2$ .

As we have mentioned before, since  $r < m$ , we have  $[m(2 - g) + 2g] < [r(2 - g) + 2g]$ . Also, from Lemma 5 we have  $[r(2 - g) + 2g] > 0$  for the triplets allowed. Given that  $x > 0$ , this implies that the right-hand side of Equation 4 is positive, furthermore  $n_p$  increases as  $x$  increases. ■

As an easy corollary of Lemma 6 we have the following general lower bound.

**Corollary 7.** *For all  $(\{r, m\}; g)$  triplets in Lemma 5,*

$$n_p(\{r, m\}; g) \geq 1 + \frac{m(g-2) + 2g}{r(2-g) + 2g}.$$

**Corollary 8.** *The following lower bounds for  $n_p(\{r, m\}; g)$  hold for the triplets  $(\{r, m\}; g)$  in Lemma 5.*

	$r$	$m$	$g$	$n_p(\{r, m\}; g)$
(a)	2	$m$	3	$\geq m + 1,$
(b)	3	$m$	3	$\geq m + 1,$
(c)	4	$m$	3	$\geq \max\{m + 1, \frac{m}{2} + 4\},$
(d)	5	$m$	3	$\geq m + 7,$
(e)	2	$m$	4	$\geq m + 2,$
(f)	3	$m$	4	$\geq m + 5,$
(g)	2	$m$	5	$\geq 2m + 1,$
(h)	3	$m$	5	$\geq 3m + 11,$
(i)	2	$m$	$6 \leq g \text{ even}$	$\geq \frac{m(g-2)+4}{2},$
(j)	2	$m$	$6 \leq g \text{ odd}$	$\geq \frac{m(g-1)+2}{2}.$

**Proof.** For each case, we will enlist the lemma or theorem that implies the better bound.

(a) Theorem 1, (b) Theorem 1, (c) Theorem 1 and Corollary 7, (d) Corollary 7, (e) Theorem 1, (f) Corollary 7, (g) Theorem 1, (h) Corollary 7, (i) Theorem 2, (j) Theorem 2. ■

### 3.1. Some properties of planar graphs

In this section we will state some technical properties of planar graphs, which will come in handy when presenting the biregular planar cages constructed in later sections. In particular, Lemmas 10 to 16 provide structural insight that will be used *systematically* for the constructions in Theorems 5 to 8.

We include the proofs to all properties in Section 4.

**Lemma 9.** *A planar graph with at least one vertex of degree  $m \geq 3$  that has exactly  $m + 1$  vertices satisfies one of the following.*

- (a) *It has four vertices of degree  $m$  and  $m = 3$ .*
- (b) *It has three vertices of degree  $m$  and  $m = 4$ .*
- (c) *It has at most two vertices of degree  $m$ .*

An *outerplanar graph* is a graph that has a planar drawing for which all vertices belong to the outer face of the drawing.

**Lemma 10.** *Let  $G$  be an  $(\{r, m\}; g)$ -planar graph. Then the subgraph of  $G$  induced by all the vertices in the faces incident to a vertex  $x$ ,  $link_G(x)$ , is an outerplanar graph consisting of a (not necessarily disjoint) union of cycles (with or without chords) and paths, with at least  $\deg(x)$  vertices.*

Let  $link_G(x) = (\bigcup_{i=1}^k C_i) \cup (\bigcup_{j=1}^{k'} P_j)$ , where the  $C_i$  are the maximal induced cycles of  $link_G(x)$  and  $P_j$  are its maximally induced trees in the sense that they cannot be extended further without acquiring edges of some other  $P_j$  or  $C_i$ . Define the *intersection graph* of  $link_G(x)$ ,  $I_x$ ; whose set of vertices is  $\{c_1, \dots, c_k, p_1, \dots, p_{k'}\}$  and where we put an edge  $c_i c_j$  or  $c_i p_j$  every time the cycles  $C_i, C_j$  or  $C_i, P_j$  intersect in a vertex. Here we note that we do not consider  $P_i$  to  $P_j$  incidences as it would be redundant by the definition of the  $P_j$ 's.

**Lemma 11.** *The intersection graph,  $I_x$ , where  $\deg(x) = m$  is a simple graph, furthermore it is a forest.*

**Lemma 12.** *If  $G$  is an outerplanar graph of order at least 4 such that all of its vertices have degree at least 2, then it has at least two non-adjacent vertices of degree exactly 2.*

### 3.2. Girth 3

The next theorem states bounds for the  $(\{r, m\}; 3)$ -cages and introduces the record graphs. The proofs and descriptions of each graph family are distributed in the following subsections.

**Theorem 13.** *The  $(\{r, m\}; 3)$ -cages are as follows.*



$r$	$m$	$n_p(\{r, m\}, 3)$	Graphs	Full list
2	$m$	$= m + 1$	$T'_{m-1}, T_{\frac{m}{2}}$	yes
3	$m$	$= m + 1$	$W_m, M_m, \mathbf{W}_3$ for $m = 4$	yes
4	$m$	$= m + 2$	$\mathbf{W}_m$	-
5	$m = 6, 7$	$= 2m + 2$	$I_m$	-
5	$8 \leq m \leq 13$	$m + 7 \leq n_p(\{r, m\}, 3) \leq 2m + 2$	$I_m$	-
5	$14 \leq m$	$m + 7 \leq n_p(\{r, m\}, 3) \leq m + 15$	$A_m$	-

### 3.2.1. Planar $(\{2, m\}; 3)$ -cages

First note that  $(\{2, m\}; 3)$ -cage has at least  $m + 1$  vertices. Let  $m = 2l$ , and let  $T_l$  be the graph consisting of  $l$  triangles sharing a common vertex. We will refer to it as the *windmill graph* (see Figure 3). Then  $V(T_l) = 2l + 1 = m + 1$ , and  $T_l$  has exactly one vertex of degree  $m$ ,  $m$  vertices of degree 2 and by construction it has girth 3.

Let  $l$  be even or odd, we denote by  $T'_l$  the graph consisting of  $l - 1$  triangles sharing a common edge, which we will refer to as the *pinwheel graph* (see Figure 3). Then  $V(T'_l) = l + 1$ , and  $T'_l$  has exactly two vertices of degree  $l$ ,  $l - 1$  vertices of degree 2 and by construction it has girth 3.

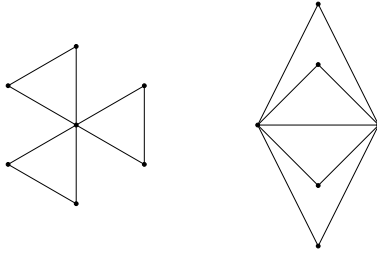


Figure 3. The graphs  $T_3$  and  $T'_4$ , respectively.

Both  $T'_l$  and  $T_l$  are clearly planar graphs, thus, they are clearly  $(\{2, m\}; 3)$ -cages. The following lemma proves that they are the only planar cages.

**Lemma 14.** *Let  $m \geq 3$ . The graphs  $T'_{m-1}$  and  $T_{m/2}$ , for  $m$  even, are the only planar  $(\{2, m\}; 3)$ -cages.*

**Proof.** The graphs are clearly planar  $(\{2, m\}; 3)$ -cages. By Lemma 9, we can assume that any planar  $(\{2, m\}; 3)$ -cage has at most two vertices of degree  $m$ .

Assume that  $G$  is a planar  $(\{2, m\}; 3)$ -cage with exactly one vertex of degree  $m$ , then  $G - \{v\}$  is a 1-regular graph with  $m$  vertices. Hence  $G - \{v\}$  is a perfect matching, and necessarily  $m$  is even. Clearly,  $G$  must be isomorphic to  $T_{m/2}$ .

Now, assume that  $G$  is a planar  $(\{2, m\}; 3)$ -cage that has two vertices of degree  $m$ , say  $u, v$ . Since  $G$  has order  $m + 1$ , then  $G - \{u, v\}$  is a 0-regular graph with  $m - 1$  vertices; that is, an independent set of size  $m - 1$ , where each vertex

is adjacent to both  $u$  and  $v$  in  $G$  and  $uv$  is also an edge of  $G$ . Clearly,  $G$  must be isomorphic to  $T'_{m-1}$ . ■

### 3.2.2. Planar $(\{3, m\}; 3)$ -cages

Let  $l \geq 3$ , we denote by  $W_l$  the *wheel* graph on  $l + 1$  vertices, see Figure 4. This graph has one vertex of degree  $l$  and  $l$  vertices of degree 3.

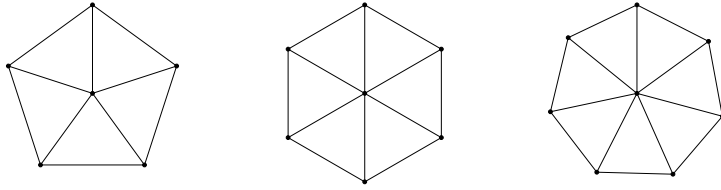


Figure 4. From left to right, wheels  $W_5$ ,  $W_6$  and  $W_7$ , respectively.

Let  $\mathbf{W}_l$  be the a graph obtained from the wheel graph  $W_l$  by duplicating its vertex of degree  $l$ ; we will refer to  $\mathbf{W}_l$  as a *biwheel*. Notice that,  $\mathbf{W}_3$  is isomorphic to  $K_5$  minus one edge.

Let  $M_l$ , the *double windmill*, be the graph obtained from the windmill graph  $T_{l-1}$  by duplicating its vertex of degree  $l - 1$  and making these two vertices adjacent.

**Lemma 15.** *Let  $m \geq 4$ . Then the only planar  $(\{3, m\}; 3)$ -cages are the double wheel  $\mathbf{W}_3$  when  $m = 4$ , the double windmill  $M_m$  for  $m$  odd and the wheels  $W_m$ , for all  $m$ .*

**Proof.** Let  $G$  be a planar  $(\{3, m\}; 3)$ -cage with  $m + 1$  vertices. By Lemma 9, we may have the following cases.

*Case 1.*  $G$  has three vertices of degree 4,  $m = 4$  and it has two vertices of degree 3. Hence  $G$  is the biwheel,  $\mathbf{W}_3$ .

*Case 2.*  $G$  has two vertices of degree  $m$ , say  $\{u, v\}$ , and  $m - 1$  vertices of degree 3. Then  $G - \{u, v\}$  is a 1-regular graph. This is only possible if  $G - \{u, v\}$  is a perfect matching, hence  $m - 1$  is even and  $G$  is the double windmill  $M_m$ .

*Case 3.*  $G$  has one vertex of degree  $m$ , say  $v$ , and  $m$  vertices of degree 3. Then  $G - \{v\}$  is a 2-regular graph with  $m$  vertices, this is an  $m$ -cycle or a union of cycles whose size adds to  $m$ . Then  $G$  is clearly the wheel  $W_m$ . ■

### 3.2.3. Planar $(\{4, m\}; 3)$ -cages

**Lemma 16.** *The graph  $\mathbf{W}_m$  is a planar  $(\{4, m\}; 3)$ -cage, with  $m \geq 5$ .*

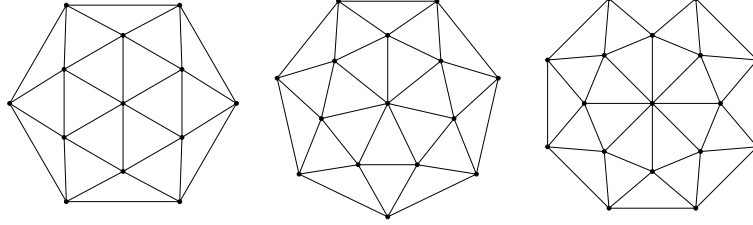


Figure 5. The family of planar  $(\{5, m\}; 3)$ -graphs,  $I_m$ . For each case, all vertices in the boundary are adjacent to an external vertex, to complete the graph.

**Proof.** It is clear that  $\mathbf{W}_m$  is a planar  $(\{4, m\}; 3)$ -graph on  $m+2$  vertices. Thus it remains to argue that  $m+2$  is indeed the minimum number of vertices that a planar  $(\{4, m\}; 3)$ -graph can have.

Suppose for a contradiction that there exists a planar  $(\{4, m\}; 3)$ -cage  $G$  that has fewer than  $m+2$  vertices. Thus,  $G$  has exactly  $m+1$  vertices. By Lemma 9, since  $m \geq 5$ , the graph  $G$  must either have two vertices of degree  $m$  and  $m-1$  vertices of degree 4, or one vertex of degree  $m$  and  $m$  vertices of degree 4. We will deal with each case separately.

*Case 1.* Suppose that  $G$  has two vertices of degree  $m$ , say  $u$  and  $v$ , and  $m-1$  vertices of degree 4, say  $x_1, \dots, x_{m-1}$ . Note that in this case  $uv \in E(G)$  and  $ux_i, vx_i \in E(G)$  for  $1 \leq i \leq m-1$ . Since  $G - \{u, v\}$  is a 2-regular graph, it must contain a cycle  $C$ , which is also a cycle in  $G$ . We successively contract edges of  $C$  in  $G$  until we obtain a triangle, say  $a, b, c$  in the graph  $G'$ . But now note that  $G'[u, v, a, b, c]$  is isomorphic to  $K_5$ , thus  $G$  contains  $K_5$  as a minor, which contradicts the planarity of  $G$ .

*Case 2.* Now assume that  $G$  has exactly one vertex  $u$  of degree  $m$ . Observe that  $H = G - u$  is a planar 3-regular graph. Furthermore,  $H$  is outerplanar since in any planar drawing of  $G$  every vertex is visible from  $u$ , thus all vertices of  $H$  are incident to the region of the plane that contained vertex  $u$ . Thus  $H$  is a 3-regular outerplanar graph and this contradicts Lemma 12. ■

### 3.2.4. Planar $(\{5, m\}; 3)$ -cages

We start by describing the family  $I_m$  of planar  $(\{5, m\}; 3)$ -graphs. In Figure 5 we show schematics for  $I_m$  for  $m = 6, 7, 8$ .

Let  $I_m$  be the graph on  $2m+2$  vertices,  $V(I_m) = \{x, x_0, \dots, x_{m-1}, x', x'_0, \dots, x'_{m-1}\}$  with the following set of edges, taking the subindices module  $m$ ,

$$\begin{aligned} E(I_m) = & \{x, x_i \mid 0 \leq i \leq m-1\} \cup \{x_i, x_{i+1} \mid 0 \leq i \leq m-1\} \\ & \cup \{x', x'_i \mid 0 \leq i \leq m-1\} \cup \{x'_i, x'_{i+1} \mid 0 \leq i \leq m-1\} \\ & \cup \{x_i, x'_i \mid 0 \leq i \leq m-1\} \cup \{x_i, x'_{i+1} \mid 0 \leq i \leq m-1\}. \end{aligned}$$

This construction proves the following.

**Proposition 17.** *For all  $6 \leq m$ ,  $n_p(\{5, m\}; 3) \leq 2m + 2$ .*

In order to explore the bound  $m + 7$  given in Corollary 8, we implement a brief but exhaustive computer search that proves the non-existence of planar  $(\{5, 6\}; 3)$ -graphs with 13 vertices and planar  $(\{5, 7\}; 3)$ -graphs with 14 vertices, showing that in both of these cases  $n_p(\{5, m\}; 3) = 2m + 2$ . These computations were performed using McKay's plantri graph generator [9] via Sagemath [20].

Additionally, consider the family of graphs depicted in Figure 6,  $A'_m$ , each of these graphs has 10 vertices at each end and  $l = m - 8$  vertices in between the ends. In total it has  $m + 14$  vertices of which  $m$  have degree 4 and 14 have degree 5. Let  $A_m$  be the graph constructed by adding a vertex to  $A'_m$  which is connected to all vertices of degree 4. This construction proves the following.

**Proposition 18.** *For  $m \geq 13$ ,  $n_p(\{5, m\}, 3) \leq m + 15$ .*

In addition  $2m + 2 \geq m + 15$  when  $m \geq 13$ . This completes the proof of the bounds. Finally, we remark that the families given in Propositions 17 and 18 give two non-isomorphic  $(\{5, 13\}; 3)$ -planar cages of order 28.

In order to compare these values with the values for usual cages, we may remark that in [18] the authors prove that the  $(\{r, m\}; 3)$ -cages have order  $m + 1$ . That is, the order coincides with the planar  $(\{r, m\}; 3)$ -cages only for  $r = \{2, 3\}$ .

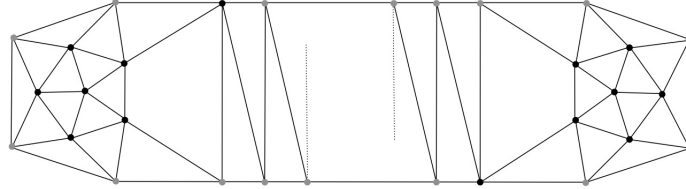


Figure 6. A depiction of the graphs  $A'_m$  with  $16 \leq m$ .

### 3.3. Girth 4

**Theorem 19.** *The planar  $(\{r, m\}; 4)$ -cages are as follows.*

$r$	$m$	$n_p(\{r, m\}, 4)$	Graphs	Full list
2	$m$	$m + 2$	$K_{2,m}$	yes
3	$4 \leq m \leq 13$	$n_p(\{3, m\}, 4) = 2m + 2$	$D_m$	-
3	$14 \leq m$	$m + \frac{4(m+1)}{5} + 3 \leq n_p(\{3, m\}, 4) \leq m + 4 \lceil \frac{m+1}{5} \rceil + 3$	-	-
3	$m = 5k - 1$ for $k \geq 3$	$n_p(\{3, m\}, 4) = m + \frac{4(m+1)}{5} + 3$	$Z_k$	-

The statement of the theorem is a summary of the results presented in the coming subsections. We may begin by remarking that Lemma 5 implies that  $2 \leq r \leq 3$ ,  $r < m$  and Corollary 8 yields that  $m + 2 \leq n_p(\{2, m\}, 4)$  and  $m + 5 \leq n_p(\{3, m\}, 4)$ .

### 3.3.1. Planar $(\{2, m\}; 4)$ -cages

**Lemma 20.** *The complete bipartite graph  $K_{2,m}$  is the  $(\{2, m\}, 4)$ -planar cage.*

**Proof.** By Corollary 8, we have that  $n_p(\{2, m\}, 4) \geq m+2$ . Therefore, we require at least  $m+2$  vertices and this bound is achieved by the complete bipartite graph  $K_{2,m}$ , which is clearly planar.

To prove uniqueness, it suffices to prove that if  $G$  is a  $(\{2, m\}, 4)$ -planar cage, then it must have two vertices of degree  $m$ . Assume, to the contrary, that it only has one vertex  $v$  of degree  $m$ . Then  $G \setminus \{v\}$  would be a graph with  $m$  vertices of degree 1 and one vertex of degree 2, where two vertices adjacent to  $v$  in  $G$  cannot be adjacent, or there would be a triangle. It is obvious that such a graph does not exist. Thus  $G$  has at least two vertices of degree  $m$ ,  $m$  vertices of degree 2, and it has no triangles, which characterizes  $K_{2,m}$ . ■

### 3.3.2. Planar $(\{3, m\}; 4)$ -cages

In this section we will first show the lower bound for all cases and then we will introduce families of graphs for which the lower bounds are attained for some values of  $m$ .

**Lemma 21.** *Let  $G$  be a  $(\{3, m\}, 4)$ -planar graph with only one vertex  $x$  of degree  $m$ , and let  $k$  be the number of cycles in the decomposition of  $\text{link}_G(x) = (\bigcup_{i=1}^k C_i) \cup (\bigcup_{j=1}^{k'} P_j)$ ,  $c$  the number of connected components of  $\text{link}_G(x)$  and let  $\text{ends}(I_x)$  be the number of ends of  $I_x$ . Then  $v(G) \geq 2m - k + c + \text{ends}(I_x) + 1$ .*

**Proof.** Notice that in  $\text{link}_G(x)$  two consecutive vertices cannot both be neighbours of  $x$ , this implies that each of the cycles and trees in the decomposition  $\text{link}_G(x) = (\bigcup_{i=1}^k C_i) \cup (\bigcup_{j=1}^{k'} P_j)$ , inherits this property. Hence, for each  $C_i$  and  $P_j$  we have that  $v(C_i) \geq 2m_i$  and  $v(P_j) \geq 2m_j + 1$ , respectively, where  $m_i, m_j$  represent the number of vertices adjacent to  $x$ . Here  $\sum_{i=1}^k m_i + \sum_{j=1}^{k'} m_j = m$ .

By Lemma 11 we know that  $I_x$  is a forest, this implies that the number of vertices belonging to the pairwise intersections  $C_i \cap P_j$  or  $C_i \cap C_j$  is exactly  $k + k' - c$ . Thus, we have  $v(\text{link}_G(x)) \geq \sum_{i=1}^k v(C_i) + \sum_{j=1}^{k'} v(P_j) - (k + k' - c) = 2m + k' - (k + k' - c) = 2m - k + c$ .

As  $v(G) = v(\text{link}_G(x)) + v(G \setminus \text{link}_G(x))$ , we now look at how the structure of  $I_x$  helps to bound  $v(G \setminus \text{link}_G(x))$ .

**Claim 1.** *The end vertices of  $I_x$  are vertices that represent a cycle-type component.*

Otherwise, there would be terminal vertices in  $\text{link}_G(x)$ , which would be vertices of degree 2 in  $G$ .

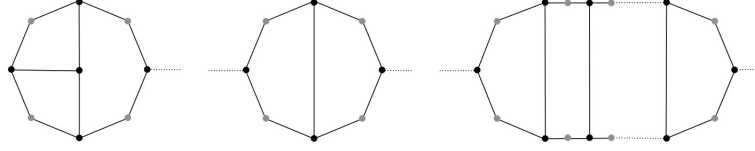


Figure 7. The graphs  $F$ ,  $E_4$  and a depiction of  $E_l$  for  $l \geq 8$ .

**Claim 2.** *In  $I_x$  edges only exist among pairs of vertices representing one tree and one cycle.*

If two components representing cycles in the decomposition of  $\text{link}_G(x)$  intersect, the intersecting vertex would have degree four, but the vertices in  $\text{link}_G(x)$  have degree 3 or 2. Recall that, by Lemma 11, cycles cannot intersect in more than one vertex.

**Claim 3.**  $v(G \setminus \text{link}_G(x)) \geq \text{ends}(I_x) + 1$ .

Notice that for each  $C_i$  we have at least  $m_i$  vertices of degree 2 which are not adjacent to  $x$ . This implies that in the embedding of the graph there must be some edges emanating from said vertices, either reaching some other vertices in  $C_i$ , vertices in the components of  $\text{link}_G(x)$  adjacent to  $C_i$  or some vertices not in  $C_i$  (lying in the area enclosed by  $C_i$  in the embedding).

Suppose there are some chords (non-crossing in the embedding) among the vertices of  $C_i$  non adjacent to  $x$ . As there are no triangles in  $G$ , these vertices have to be at distance at least three in the cycle. So, there would have to be at least two vertices in  $C_i$  not adjacent to  $x$  and not adjacent to any other vertex in  $C_i$ . Thus, if there are no additional vertices of  $G$  in the area enclosed by  $C_i$ , then  $C_i$  has exactly  $2m_i$  vertices,  $m_i - 2$  which are paired by non-crossing chords and two which have to be connected to other parts of  $\text{link}_G(x)$ . (See the graphs in the center and right side in Figure 7.)

Thus, for  $C_i$  to represent an end vertex of  $I_x$  there is no remedy but to have one additional vertex of  $G$  in the area enclosed by  $C_i$ , connected to three vertices of  $C_i$  non adjacent to  $x$  and exactly one vertex of  $C_i$  non adjacent to  $x$  which connects to another part of  $\text{link}_G(x)$ . (See the leftmost graph in Figure 7.) The result follows from Claim 3.  $\blacksquare$

**Corollary 22.** *Let  $G$  be a  $(\{3, m\}, 4)$ -planar graph. Then for  $4 \leq m \leq 13$  we have that  $2m + 2 \leq v(G)$ , and for  $14 \leq m$  we have the following lower bound  $\frac{9m+19}{5} \leq v(G)$ .*

**Proof.** As we have proved in Lemma 6, for a fixed triad  $(\{r, m\}, g)$ , the value of  $v(G)$  increases as the number of vertices of degree  $m$  increases, then we will assume for this lower bound that  $G$  has exactly one vertex of degree  $m$  called  $x$ .

Let  $G$  be a  $(\{3, m\}, 4)$ -planar graph and  $x$  be a vertex of degree  $m$  in  $G$ . From the equation  $v(G) \geq 2m - k + c + \text{ends}(I_x) + 1$  in the previous lemma, we can observe that, when  $G$  is the smallest possible  $(\{3, m\}, 4)$ -planar graph, then  $G$  is connected, has big  $k$  and small  $\text{ends}(I_x)$ .

As  $\text{ends}(I_x) \geq 1$ , we have that  $v(G) \geq 2m - k + \text{ends}(I_x) + 2$ . Obviously this bound will decrease as  $k$  increases.

For  $I_x$  to have few ends the optimal case is when it is a path where each  $P_j$  represents a path of length two or one. Thus, we observe that, for the extremally small  $(\{3, m\}, 4)$ -planar graphs,  $\text{ends}(I_x) = 1$  or  $2$ .

**Claim 1.** For  $4 \leq m \leq 13$ ,  $2m + 2 \leq v(G)$ .

From the proof of Lemma 21 we can observe that the smallest  $C_i$  that can represent an end vertex of  $I_x$  has 9 vertices and  $m_i = 4$  and the smallest  $C_i$  that can represent a non-end of  $I_x$  has 8 vertices and  $m_i = 4$  (see Figure 7). That is, if  $m$  is small enough, we have no alternative but to have  $k = 1$ ,  $\text{ends}(I_x) = 1$ , and  $v(G) \geq 2m + 2$ , this also holds if  $k = 2$ ,  $\text{ends}(I_x) = 2$ , for example.

On the other hand, for  $2m - k + \text{ends}(I_x) + 2 < 2m + 1$  to be satisfied we need  $k \geq 3$ . As each  $C_i$  is followed by  $P_j$  then, for a given  $m$ , the best possible situation is to have as many  $P_j$  of length 2 between each  $C_i$  of size 8. This implies that for  $k \geq 3$  we must be able to have at least three cycles of size 8 and two paths of length two in the decomposition of  $\text{link}_G(x)$ . Thus, the total degree of  $x$  is  $m \geq 4(3) + 2 = 14$ . This implies the claim.

**Claim 2.** For  $14 \leq m$ ,  $\frac{9m+19}{5} \leq v(G)$ .

Here, by Claim 1, we may have  $k \geq 3$  and  $\text{ends}(I_x) = 2$ . Thus, we need to find the greatest  $k$  such that  $m \geq 4k + (k - 1) = 5k - 1$  or, equivalently, the greatest  $k \leq \frac{m+1}{5}$  and the result follows by plugging in this values into the bound's equation. ■

Let  $D_m$  be the graph whose set of vertices is  $\{x_0, \dots, x_{2m-1}, y_0, y_1\}$  and whose set of edges is  $\{x_i x_{i+1} \mid i = 0, \dots, 2m-2\} \cup \{x_1, x_{2m-1}\} \cup \{y_0 x_{2j} \mid j = 0, \dots, m\} \cup \{y_1 x_{2j-1} \mid j = 0, \dots, m\}$ , see Figure 8. This graph is clearly a  $(\{3, m\}, 4)$ -planar graph with  $2m + 2$  vertices, implying the following result.

**Proposition 23.** For  $4 \leq m \leq 13$ ,  $n_p(\{3, m\}, 4) = 2m + 2$ .

Let  $F$  be the graph whose set of vertices is  $\{x, x_0, \dots, x_7\}$  and whose set of edges is  $\{x_i x_{i+1} \mid i = 0, \dots, 6\} \cup \{x_0 x_7\} \cup \{x x_{2i} \mid i = 0, 1, 2\}$ . Let  $E_4$  be the graph whose set of vertices is  $\{x_0, \dots, x_7\}$  and whose set of edges is  $\{x_i x_{i+1} \mid i = 0, \dots, 6\} \cup \{x_0 x_7\} \cup \{x_0 x_4\}$ . A depiction of these graphs can be found in Figure 7.

For  $k \geq 3$ , let  $Z'_k$  be the graph resulting from joining two copies of  $F$ ,  $k - 2$  copies of  $E_4$  and  $k - 1$  paths of length 2,  $P$  as follows. Start with a copy of  $F$ , and join it to a copy of the path  $P$  by identifying its only even labeled vertex

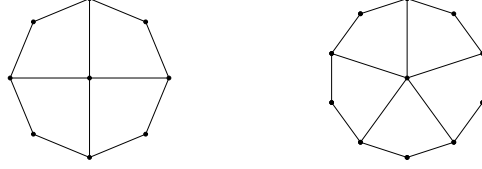


Figure 8.  $D_m$  graphs for  $m = 4, 5$ . In each case, all vertices of degree two in the boundary are adjacent to an external vertex.

of degree two with an end of  $P$ , join the remaining end of  $P$  to a copy of  $E_4$  by one of its two even labeled vertices of degree two. Now repeat this procedure subsequently joining copies of  $E_4$  to copies of  $P$  and ending with the second copy of  $F$  (the dotted lines indicate the subsequently join between copies of  $E_4$  and copies of  $P$ ). This graph has  $9k + 1$  vertices, out of which  $5k - 1$  have degree two and the remaining vertices have degree three. A depiction of  $Z'_k$  can be found in Figure 9.

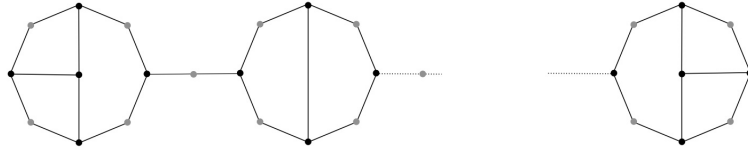


Figure 9. A graph  $Z'_k$  with  $k = 4$ . In order to build  $Z_k$  we may join all degree two vertices in  $Z'_k$  to the same vertex of degree  $m = 9k + 1$ .

Let  $Z_k$  be the graph whose set of vertices is  $V(Z'_k) \cup \{x^*\}$  and whose set of edges is  $E(Z'_k) \cup \{x^*y \mid y \in V(Z'_k) \text{ and } \deg(y) = 2\}$ . Observe that  $Z_k$  is clearly a planar graph that has  $9k + 2$  vertices, out of which one has degree  $5k - 1$  and the remaining vertices have degree 3. This construction proves the next result.

**Proposition 24.**  $n_p(\{3, m\}, 4) \leq \frac{9m+19}{5}$ , for  $m = 5k - 1$  and  $k \geq 3$ .

Finally, note that in the previous construction we may delete up to two degree 2 (diametrically opposite) vertices from each  $E_4$  or  $F$ , without violating the girth condition in  $Z'_k$ . Thus, for  $m > 14$  we may find the smallest  $m^*$  such that  $m < m^*$  and  $m^* = 5k^* - 1$  for some  $k^*$ , make the construction  $Z'_{k^*}$  and then remove  $m^* - m$  degree two vertices from such construction to obtain an improved upper bound for any  $m$ . A simple computation proves  $m^* = 5 \lceil \frac{m+1}{5} \rceil - 1$  and the number of vertices of such construction will be  $m + 4 \lceil \frac{m+1}{5} \rceil + 3$ . Hence, we have the following.

**Proposition 25.**  $n_p(\{3, m\}, 4) \leq m + 4 \lceil \frac{m+1}{5} \rceil + 3$ , for  $m > 14$ .



In order to compare these values with the values for usual cages, we remark that in [10] the authors prove that  $n_0(\{r, m\}; 4) = r + m$  and this bound is attained by the biregular bipartite complete graphs  $K_{r, m}$ . Clearly, these graphs are planar only for  $r = 1$  or  $r = 2$ .

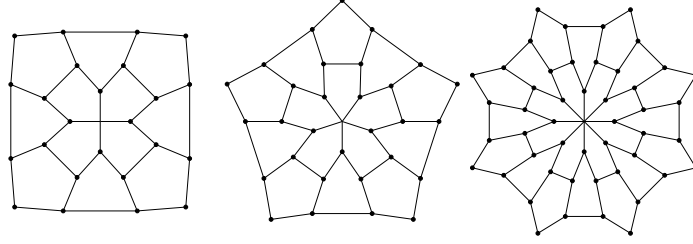


Figure 10. A family of planar  $(\{3, m\}; 5)$ -graphs. In all cases, the vertices of degree two in the boundary are adjacent to an external vertex.

### 3.4. Girth 5

**Theorem 26.** *The  $(\{r, m\}; 5)$ -cages are as follows.*

$r$	$m$	$n_p(\{r, m\}, 5)$	Graphs	Full list
2	$m$	$2m + 1$	$O_{m,5}$	yes
2	$m$ even	$2m + 1$	$F_{m,5}$	yes
3	$4 \leq m \leq 5$	$3m + 11 \leq n_p(\{3, m\}, 5) \leq 6m + 2$	$P_m$	-
3	$6 \leq m, m$ even	$3m + 11 \leq n_p(\{3, m\}, 5) \leq 3m + 2 \left\lfloor \frac{m-6}{4} \right\rfloor + 21$	$B_m$	-
3	$6 \leq m, m$ odd	$3m + 11 \leq n_p(\{3, m\}, 5) \leq 3m + 2 \left\lfloor \frac{m-5}{4} \right\rfloor + 22$	$B_m$	-

The proof of the theorem and description of each graph family is split in the following subsections. However, we may remark that Lemma 5 implies that  $2 \leq r \leq 3, r < m$  and Corollary 8 points that  $n_p(\{2, m\}, 4) \geq 2m + 1$  and  $n_p(\{3, m\}, 4) \geq 3m + 11$ .

#### 3.4.1. Planar $(\{2, m\}; 5)$ -cages

This case will follow as a consequence of a more general construction in the Subsection 3.5. We will define the graphs  $O_{m,g}$  and  $F_{m,g}$  for  $g \geq 3$ , and prove that they are the only planar cages in Lemma 31.

#### 3.4.2. Planar $(\{3, m\}; 5)$ -cages

Note that, if we allow two vertices of degree  $m$  in  $(\{3, m\}; 5)$ -graphs, from Lemma 6 we obtain that such graphs must have at least  $6m + 2$  vertices. Here we present an infinite family of  $(\{3, m\}; 5)$ -graphs,  $P_m$ , meeting that bound, see Figure 10. This proves that indeed.

**Proposition 27.** *For  $4 \leq m$ ,  $n_p(\{3, m\}, 5) \leq 6m + 2$ .*

Furthermore, consider the family of graphs  $B'_m$  depicted in Figure 11. This family has  $v = 3m + 2 \lfloor \frac{m-6}{4} \rfloor + 20$  of which  $m = 2l + 6$  vertices and  $l \geq 0$ , have degree two and the rest have degree 3. Let  $B_m$  be the graph in which we add a vertex to  $B'_m$  joined to all its vertices of degree two. This construction proves the following.

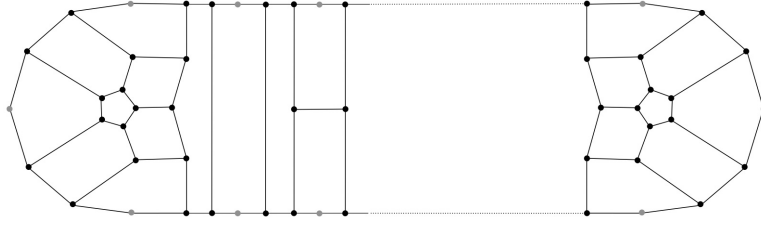


Figure 11. A family of planar  $(\{3, m\}; 5)$ -graphs. In all cases, the vertices of degree two in the boundary are adjacent to an external vertex.

**Proposition 28.** For  $6 \leq m$ , even,  $n_p(\{3, m\}, 5) \leq 3m + 2 \lfloor \frac{m-6}{4} \rfloor + 21$ .

Finally, note that in the construction of  $B_m$  there are several degree two vertices that can be deleted without decreasing the girth. Thus for even  $m \geq 7$  we may construct  $B_{m+1}$  and then delete one vertex to obtain a graph with  $3m + 2 \lfloor \frac{m-5}{4} \rfloor + 22$  vertices.

**Proposition 29.** For  $7 \leq m$ , odd,  $n_p(\{3, m\}, 5) \leq 3m + 2 \lfloor \frac{m-5}{4} \rfloor + 22$ .

As in the previous cases, in order to compare these values with the values for usual cages, we highlight that in [10] the authors prove that  $n_0(\{r, m\}; 5) = rm + 1$ . Here the bounds only coincide for  $r = 2$ .

### 3.5. Girth $g \geq 6$

**Theorem 30.** The  $(\{r, m\}; g)$ -planar cages with  $g \geq 6$  are as follows.

$r$	$g$	$n_p(\{r, m\}, g)$	Graphs	Full list
2	even	$\frac{m(g-2)+4}{2}$	$O_{m,g}$	yes
2	odd	$\frac{m(g-1)+2}{2}$	$O_{m,g}, F_{m,g}$ for even $m$	yes

By Lemma 5, we know that the only possible cases are when  $r = 2$  and  $m \geq 3$ .

For any  $g$  and  $m \geq 3$ , let  $O_{m,g}$  be the graph on  $(m-1) \left\lceil \frac{g-2}{2} \right\rceil + \left\lfloor \frac{g-2}{2} \right\rfloor + 2$  vertices that consists of  $m$  independent paths between two vertices  $u, v$ ,  $(m-1)$  being of length  $\left\lceil \frac{g}{2} \right\rceil$  and one of length  $\left\lfloor \frac{g}{2} \right\rfloor$ . It is easy to see that this is a  $(\{2, m\}; g)$ -graph which is embeddable in the plane.

For any  $g$  and  $m$  even, let  $F_{m,g}$  be the graph formed by  $\frac{m}{2}$  cycles of length  $g$  all incident in one vertex. This graph has  $\frac{m}{2}(g-1) + 1$  vertices and is clearly planar.

These graphs are also given in [10] as cages, that is, in this case the values coincide because the usual cages are planar.

**Lemma 31.** *The only possible  $(\{2, m\}; g)$ -planar cages are  $O_{m,g}$  for any  $2 < m$  and  $g \geq 6$ , and  $F_{m,g}$  for any  $2 < m$  even and  $g \geq 5$  odd.*

**Proof.** It is not difficult to check that the numbers of vertices of  $O_{m,g}$  and  $F_{m,g}$  reach the lower bounds in Corollary 8, for the cases mentioned.

Now, for the characterization part, we will first argue that the number of vertices  $x$  of degree  $m$  is at most two. We will proceed by contradiction. By Corollary 8, if  $x \geq 3$ , then the number of vertices  $v$  necessary to have a planar  $(\{2, m\}; g)$ -graph, is such that

$$v \geq \frac{3(g-2)m - 2g}{4} + 3.$$

It is easy to check that this number is always higher than the number of vertices of  $O_{m,g}$  and  $F_{m,g}$ . Thus, a  $(\{2, m\}; g)$ -planar cage has at most 2 vertices of degree  $m$ .

Let  $x = 1$ ,  $G$  be a planar cage and  $v$  be the vertex in  $G$  of degree  $m$ . Then  $G \setminus \{v\}$  is a graph with  $m$  vertices of degree 1 and  $V(G) - m - 1$  vertices of degree two. Hence,  $G \setminus \{v\}$  has to be exactly the union of  $\frac{m}{2}$  disjoint paths. This clearly implies that  $G$  is  $F_{m,g}$ .

Let  $x = 2$ ,  $G$  be a planar cage and  $v, u$  be the vertices in  $G$  of degree  $m$ . Then  $G \setminus \{v, u\}$  is a graph with  $2m$  vertices of degree 1 and  $V(G) - 2m - 1$  vertices of degree two. Hence,  $G \setminus \{v, u\}$  has to be exactly the union of  $m$  disjoint paths. This clearly implies that  $G$  is  $O_{m,g}$ . ■

#### 4. PROOFS OF LEMMAS IN SECTION 3.1

In this section we prove Lemmas 9–12. For convenient we also restate them first.

##### 4.1. Proof of Lemma 9

**Lemma 9.** *A planar graph with at least one vertex of degree  $m \geq 3$  that has exactly  $m + 1$  vertices satisfies one of the following.*

- (a) *It has four vertices of degree  $m$  and  $m = 3$ .*
- (b) *It has three vertices of degree  $m$  and  $m = 4$ .*
- (c) *It has at most two vertices of degree  $m$ .*

**Proof.** Assume  $G$  is such a graph.

(a) Assume that  $G$  has four vertices, say  $\{v_1, v_2, v_3, v_4\}$ , of degree  $m$ . The graph induced by this four vertices is  $K_4$ . If  $m = 3$ ,  $G$  is the tetrahedron. If  $m \geq 4$ , then every vertex in  $v \in V(G - \{v_1, v_2, v_3, v_4\})$  is connected to all of  $\{v_1, v_2, v_3, v_4\}$ . Hence  $\{v, v_1, v_2, v_3, v_4\}$  induces a  $K_5$ , making  $G$  non planar. Thus this is only possible when  $m = 3$ .

(b) Assume that  $G$  has exactly three vertices of degree  $m$ , say  $\{v_1, v_2, v_3\}$ . Then the graph induced by these three vertices is  $K_3$ . Also, every vertex in  $v \in V(G - \{v_1, v_2, v_3\})$  is connected to all of  $\{v_1, v_2, v_3\}$ . If  $|V(G - \{v_1, v_2, v_3\})| = 1$ , then  $m = 3$  and we would have the previous case, where all of the four vertices have degree  $m$ . Hence  $m \geq 4$ . We will argue that  $m = 4$ .

First assume that  $m \geq 5$ . Then  $|V(G - \{v_1, v_2, v_3\})| \geq 3$ , and for all triplets  $\{u_1, u_2, u_3\} \in V(G - \{v_1, v_2, v_3\})$  we have that the graph induced by  $\{u_1, u_2, u_3\} \cup \{v_1, v_2, v_3\}$  contains  $K_{6,6}$ , hence  $G$  would not be planar.

Finally, if  $m = 4$ , then  $|V(G - \{v_1, v_2, v_3\})| = 2$ , and  $G$  would be the 1-skeleton of a double pyramid with triangular base. Clearly, a planar graph.

(c) Clearly, the graph must have at most two vertices of degree  $m$ . ■

#### 4.2. Proof of Lemma 10

**Lemma 10.** *Let  $G$  be an  $(\{r, m\}; g)$ -planar graph. Then the subgraph of  $G$  induced by all the vertices in the faces incident to a vertex  $x$ ,  $link_G(x)$ , is an outerplanar graph consisting of a (not necessarily disjoint) union of cycles (with or without chords) and paths, with at least  $\deg(x)$  vertices.*

**Proof.** We may assume without loss of generality that  $G \setminus x$  has only one connected component.

Let  $X = \{x_1, \dots, x_l\}$  be an ordered set that labels the vertices in  $link_G(x)$ , in the order they appear around  $x$  in the embedding. Here, by assumption, we need to have  $\deg(x) \leq l$  where there may be some repetitions of vertices.

Assume that there are indeed some repetitions in  $X$ , say  $x_i = x_j$  and  $x_{i'} = x_{j'}$ . Then we can never have  $i < i' < j < j'$ , where all subindices are taken mod  $l$ , otherwise we would violate the planarity of  $G$ . This proves the lemma. ■

#### 4.3. Proof of Lemma 11

**Lemma 11.** *The intersection graph,  $I_x$ , with  $\deg(x) = m$  is a simple graph, furthermore it is a forest.*

**Proof.** We may assume without loss of generality that  $link_G(x)$  is connected, as the result easily generalizes from the connected case to the disconnected case. Hence, we need to prove that  $I_x$  is a simple graph which is a tree.

(a) We will begin by arguing that there are no multiple edges. Suppose  $c_i c_j$  is a multiple edge, then  $C_i, C_j$  intersect in more than one vertex, say they are  $u$  and  $v$ .

- If these vertices are adjacent in both cycles, then we could have considered the union of  $C_i, C_j$  as a single cycle with a chord.
- Hence, we may assume that  $u$  and  $v$  are non adjacent in at least one of the two cycles. This is, there are at least three paths of length at least two between  $u$  and  $v$ . These paths divide the plane in at least three regions, thus the union of  $C_i, C_j$  would not be outerplanar, contradicting the outerplanarity of  $link_G(x)$ .

The case where we suppose  $c_i p_j$  is a multiple edge is proved similarly.

(b) We will now argue that there are no cycles in  $I_x$ . Suppose, to the contrary, that there is a cycle  $\mathcal{C}$  in  $I_x$ . Then there is a cycle of  $G$  contained in the union of the cycles and paths corresponding to each of the vertices in  $\mathcal{C}$ . This cycle divides the plane in two connected components, say  $\mathcal{C}^+$  and  $\mathcal{C}^-$ . We may assume without loss of generality that  $x \in \mathcal{C}^+$ , by construction there can be no vertices of  $link_G(x)$  in  $\mathcal{C}^-$  and all vertices of  $link_G(x)$  have to be visible from  $x$  in  $\mathcal{C}^+$ . Hence there is a cycle induced by the cycles and paths corresponding to the vertices of  $\mathcal{C}$  which necessarily contains all the vertices of the cycles corresponding to vertices in  $\mathcal{C}$ , otherwise outerplanarity would be violated, but this contradicts the maximality of the cycles represented in  $I_x$ . ■

#### 4.4. Proof of Lemma 12

**Lemma 12.** *If  $G$  is an outerplanar graph of order at least 4 such that all of its vertices have degree at least 2, then it has at least two non-consecutive vertices of degree exactly 2.*

**Proof.** We will denote the number of vertices of  $G$  as  $n$ . We will proceed by induction on  $n$ . Note that the graph may have different connected components and that the result holds trivially for cycles.

(a)  $n = 4$ . If  $G$  is a cycle the result follows. Assume that  $G$  is not a cycle and let  $(v_1, v_2, v_3, v_4)$  be the cycle in  $G$  that bounds the outer face of the drawing. Then the only additional edge of  $G$  not in  $(v_1, v_2, v_3, v_4)$  is either  $v_1 v_3$  or  $v_2 v_4$ . In the first instance the two non consecutive vertices of degree two are  $v_2, v_4$ , and in the second instance they are  $v_1, v_3$ .

(b)  $n \leq k$ . Assume that any outerplanar graph of order at most  $k$  such that all of its vertices have degree at least 2, contains at least two non consecutive vertices of degree 2.

(c)  $n = k + 1$ . If  $G$  is a cycle, the result follows trivially. Also, if  $G$  has at least two connected components, the result follows.

Thus, we may assume that  $G$  has a unique connected component. Let  $(v_1, v_2, \dots, v_n)$  be the cycle that bounds the outer face of the drawing. As  $G$  is not a cycle, there is an edge  $v_i v_j$  that splits the graph in to two smaller outer-planar subgraphs  $G_1$  and  $G_2$ , which both contain a copy of the edge  $v_i v_j$ .

If both  $G_1$  and  $G_2$  are of order at least 4, then each graph contains a vertex of degree 2, different from  $v_i$  and  $v_j$ , and the result follows.

Thus we only have to prove it for the case when either  $G_1$  or  $G_2$  is of order 3. We can assume without loss of generality that  $G_1$  is of order 3. Here the vertex in  $G_1$  different from  $v_i$  and  $v_j$ , has degree two. As for  $G_2$ , if its order is at least 4, then the result follows, by the induction hypothesis. Otherwise, the vertex in  $G_2$  different from  $v_i$  and  $v_j$ , has degree two. ■

## 5. CONCLUSIONS

For the  $(\{3, m\}; 4)$ -graphs where we have not reached the lower bound, we believe it to be unlikely that other constructions can improve the bounds provided. For all the other cases where there is still room for improvement it would be nice to see such improvements, either in the form of improved lower bounds or constructions.

Finally we consider that studying the *regular or biregular embedding cage problem* for other surfaces, oriented or non-oriented, will lead to nice discoveries.

The question is: Which is the minimum order of a regular or biregular graph embedding in some given surface?

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