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THE NICHE GRAPHS OF MULTIPARTITE TOURNAMENTS

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Abstract

The niche graph of a digraph D has $V(D)$ as the vertex set and an edge uv if and only if $(u,w) \in A(D)$ and $(v,w) \in A(D)$, or $(w,u) \in A(D)$ and $(w,v) \in A(D)$ for some $w \in V(D)$. The notion of niche graphs was introduced by Cable *et al.* [*Niche graphs*, Discrete Appl. Math. 23 (1989), 231–241] as a variant of competition graphs. If a graph is the niche graph of a digraph D, it is said to be niche-realizable through D. If a graph G is niche-realizable through a k-partite tournament for an integer $k \geq 2$, then we say that the pair (G,k) is niche-realizable. Bowser *et al.* [*Niche graphs and mixed pair graphs of tournaments*, J. Graph Theory 31 (1999) 319–332] studied the graphs that are niche-realizable through a tournament and Eoh *et al.* [*The niche graphs of bipartite tournaments*, Discrete Appl. Math. 282 (2020) 86–95] recently studied niche-realizable pairs (G,k) for $k=2$. In this paper, we extend their work for $k \geq 3$. We show that the niche graph of a k-partite tournament has at most three components if $k \geq 3$ and is connected if $k \geq 4$. Then we find all the niche-realizable pairs (G,k) in each case: G is disconnected; G is a complete graph; G is connected and triangle-free.

Keywords: niche graph, multipartite tournament, niche-realizable pair, true twins, triangle-free graph.

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1. Introduction

In this paper, a graph means a simple graph. For all undefined graph theory terminology, see [2].

Cohen [7] introduced the notion of competition graph while studying predator-prey concepts in ecological food webs. The *competition graph* of a digraph D is the graph having the vertex set $V(D)$ and an edge uv if and only if $(u, w) \in A(D)$ and $(v, w) \in A(D)$ for some $w \in V(D)$. Cohen's empirical observation that real-world competition graphs are usually interval graphs has led to a great deal of research on the structure of competition graphs and on the relation between the structure of digraphs and their corresponding competition graphs. In the same vein, various variants of competition graphs have been introduced and studied, one of which is the notion of niche graphs introduced by Cable *et al.* [5] (see [3,4,6,9–14] for various variants of competition graph).

The *niche graph* of a digraph D, denoted by $\mathcal{N}(D)$, has $V(D)$ as the vertex set and an edge uv if and only if $(u, w) \in A(D)$ and $(v, w) \in A(D)$, or $(w, u) \in A(D)$ and $(w, v) \in A(D)$ for some $w \in V(D)$. If a graph is the niche graph of a digraph D, then it is said to be *niche-realizable through* D. If a graph G is niche-realizable through a k-partite tournament for an integer $k \geq 2$, then we say that the pair (G, k) is *niche-realizable* for notational convenience.

Bowser *et al.* [4] studied the graphs that are niche-realizable through a tournament and Eoh *et al.* [8] studied the graphs that are niche-realizable through a bipartite tournament. We extend their work by studying niche-realizable pairs (G, k) for a graph G and an integer $k \geq 3$.

Multipartite tournaments have been actively studied by graph theorists (see survey work such as [1, 16], and [17]).

We first show that the niche graph of a k-partite tournament is connected if $k \geq 4$ and has at most three components if $k \geq 3$ (Theorem 2.6 and Corollary 2.8). Then we find all the niche-realizable pairs (G, k) when G is disconnected (Theorems 3.1 and 3.8). We show that the niche graph of a k-partite tournament contains no induced path of length 5 (Theorem 4.4). Finally, we find all the niche-realizable pairs (G, k) when G is a complete graph (Theorem 4.1) and when G is a connected triangle-free graph (Theorem 4.12).

2. Preliminaries

For a digraph D, a digraph is said to be the *converse* of D and denoted by $D^{\leftarrow}$ if its vertex set is $V(D)$ and its arc set is $\{(u, v) \mid (v, u) \in A(D)\}$.

By the definition of niche graphs, the following lemmas are immediately true.

Lemma 2.1. *For a digraph D, the niche graph of D and the niche graph of $D^{\leftarrow}$ are the same.*

Lemma 2.2. *Let D be a digraph and D' be a subdigraph of D. Then the niche graph of D' is a subgraph of the niche graph of D.*

Lemma 2.3. *For a digraph D, if the niche graph of D is K_m-free, then $d_D^+(u) \leq m-1$ and $d_D^-(u) \leq m-1$ for each vertex u in D.*

It is easy to check that the following lemma is true.

Lemma 2.4. *Let D be an orientation of K_3. Then the niche graph of D is isomorphic to*

$$\begin{cases} I_3 & \text{if } D \text{ is a directed cycle;} \\ P_3 & \text{otherwise.} \end{cases}$$

Bowser *et al.* [4] have shown that the complement of the niche graph of a tournament is one of the following: a cycle of odd order, a path of even order, a forest of odd order consisting of two paths, a forest of even order consisting of three paths, or a forest of four or more paths. By this result, we have the following lemma.

Lemma 2.5. *The niche graph of an orientation of K_4 is connected.*

Theorem 2.6. *For $k \geq 4$, the niche graph of a k-partite tournament is connected.*

Proof. Let G be the niche graph of the k-partite tournament D. We denote the partite sets of D by $(X_1, X_2, \ldots, X_k)$. Take two vertices x and y in G. It suffices to show that x and y are connected in G.

Suppose that x and y belong to different partite sets in D. Without loss of generality, we may assume that $x \in X_1$ and $y \in X_2$. Since $k \geq 4$, we may take $z \in X_3$ and $w \in X_4$. Let D_1 be the subdigraph of D induced by $\{x, y, z, w\}$. Then D_1 is an orientation of K_4. Thus, by Lemma 2.5, the niche graph of D_1 is connected. By Lemma 2.2, the niche graph of D_1 is a subgraph of G and so x and y are connected in G.

Now suppose that x and y belong to the same partite set in D. Then, without loss of generality, we may assume that $\{x, y\} \subset X_4$. Take a vertex z in X_i for some $i \in \{1, 2, 3\}$. Since x (respectively, y) and z belong to different partite set in D, x (respectively, y) and z are connected in G by the previous argument. Therefore x and y are connected in G. ■

A *stable set* of a graph is a set of vertices no two of which are adjacent. A stable set in a graph is *maximum* if the graph contains no larger stable set. The cardinality of a maximum stable set in a graph G is called the *stability number* of G, denoted by $\alpha(G)$.

Theorem 2.7. *For $k \geq 3$, the niche graph of a k-partite tournament has stability number at most* 3.

Proof. Let G be the niche graph of a k-partite tournament D. Suppose, to the contrary, $\alpha(G) \geq 4$. Then we may take a stable set of size 4 in G. We denote it by $\{x_1, x_2, x_3, x_4\}$.

Suppose that there exist partite sets X_1 and X_2 of D such that $\{x_1, x_2, x_3, x_4\} \subset X_1 \cup X_2$. Since $k \geq 3$, we may take a vertex x_5 in a partite set X_3 of D distinct from X_1 and X_2. Since D is a k-partite tournament, $\{x_1, x_2, x_3, x_4\} \subset N_D^+(x_5) \cup N_D^-(x_5)$. Therefore $|N_D^+(x_5) \cap \{x_1, x_2, x_3, x_4\}| \geq 2$ or $|N_D^-(x_5) \cap \{x_1, x_2, x_3, x_4\}| \geq 2$. Yet, each of $N_D^+(x_5) \cap \{x_1, x_2, x_3, x_4\}$ and $N_D^-(x_5) \cap \{x_1, x_2, x_3, x_4\}$ forms a clique in G, which is a contradiction to the assumption that $\{x_1, x_2, x_3, x_4\}$ is a stable set of G. Hence there are three elements in $\{x_1, x_2, x_3, x_4\}$ belonging to distinct partite sets. Then there is a partite set X satisfying $|X \cap \{x_1, x_2, x_3, x_4\}| = 1$. Without loss of generality, we may assume that $X \cap \{x_1, x_2, x_3, x_4\} = \{x_4\}$. Then $\{x_1, x_2, x_3\} \subset N_D^+(x_4) \cup N_D^-(x_4)$ and so $|N_D^+(x_4) \cap \{x_1, x_2, x_3\}| \geq 2$ or $|N_D^-(x_4) \cap \{x_1, x_2, x_3\}| \geq 2$. Since each of $N_D^+(x_4) \cap \{x_1, x_2, x_3\}$ and $N_D^-(x_4) \cap \{x_1, x_2, x_3\}$ forms a clique in G, $\{x_1, x_2, x_3\}$ cannot be a stable set of G, which is a contradiction. This completes the proof. ■

From the above theorem, the following corollary immediately follows.

Corollary 2.8. *For $k \geq 3$, the niche graph of a k-partite tournament has at most three components.*

3. Niche-Realizable Pairs (G, k) When G is Disconnected

In this section, we completely characterize the niche graphs of k-partite tournaments for $k \geq 3$ which are disconnected.

Theorem 2.6 tells us that, for a disconnected graph G and $k \geq 3$, if (G, k) is niche-realizable, then $k = 3$. In addition, the niche graph of a k-partite tournament has at most three components for $k \geq 3$ by Corollary 2.8.

We first characterize the niche-realizable pair (G, k) for a graph G with three components.

Theorem 3.1. *Let G be a graph with three components and k be an integer greater than or equal to 3. Then (G, k) is niche-realizable if and only if $k = 3$ and G is isomorphic to $K_p \cup K_q \cup K_r$ for positive integers p, q, and r.*

Proof. Suppose that (G, k) is niche-realizable. If there exists a component which is not isomorphic to a complete graph, then $\alpha(G) \geq 4$, which contradicts Theorem 2.7. Therefore G is isomorphic to $K_p \cup K_q \cup K_r$ for positive integers p, q, and r. Since $K_p \cup K_q \cup K_r$ is disconnected, $k \leq 3$ by Theorem 2.6. Therefore the "only if" part is true.

To show the "if" part, let D be a digraph with the vertex set

$$\{x_1, \ldots, x_p, y_1, \ldots, y_q, z_1, \ldots, z_r\}$$

and the arc set

$$\{(x_i, y_j) \mid i \in [p] \text{ and } j \in [q]\} \cup \{(y_j, z_l) \mid j \in [q] \text{ and } l \in [r]\} \\ \cup \{(z_l, x_i) \mid l \in [r] \text{ and } i \in [p]\}.$$

Then it is easy to check that D is a 3-partite tournament and the niche graph of D is isomorphic to $K_p \cup K_q \cup K_r$. Hence the "if" part is true. ■

Let G be a graph. Two vertices u and v of G are said to be *true twins* if they have the same closed neighborhood, and are denoted by $u \equiv_G v$. We may introduce an analogous notion for a digraph. Let D be a digraph. Two vertices u and v of D are said to be *true twins* if they have the same open out-neighborhood and open in-neighborhood, and are denoted by $u \equiv_D v$.

The following lemma is true by definitions of niche graphs and true twins.

Lemma 3.2. *Let D be a digraph without isolated vertices. If vertices u and v are true twins in D, then u and v are true twins in $\mathcal{N}(D)$.*

Proof. Suppose that two vertices u and v are true twins in D. Then $N_D^+(u) = N_D^+(v)$ and $N_D^-(u) = N_D^-(v)$. Therefore, by the definition of niche graphs, u and v have the same open neighborhood in $\mathcal{N}(D)$. Since D has no isolated vertices, $N_D^+(u) \neq \emptyset$ or $N_D^-(u) \neq \emptyset$. Thus u and v have a common out-neighbor or a common in-neighbor in D and so they are adjacent in $\mathcal{N}(D)$. Hence u and v have the same closed neighborhood in $\mathcal{N}(D)$. ■

Lemma 3.3. *Let D be a multipartite tournament. If vertices u and v are true twins in D, then u and v are in the same partite set.*

Proof. Suppose that vertices u and v are true twins in D. If u and v are not in the same partite set, then we may assume $(u, v) \in A(D)$ and so, by the definition of true twins, $(v, v) \in A(D)$, which contradicts the hypothesis that D is a multipartite tournament. ■

Proposition 3.4. *Given a graph G with at least four vertices, suppose that G is niche-realizable through a k-partite tournament D for $k \geq 3$, and vertices u and v are true twins in D. Then $D - v$ is a k-partite tournament whose niche graph is $G - v$.*

Proof. Let $D' = D - v$. By Lemma 3.3, D' is a k-partite tournament. Since D' is a subdigraph of D, $\mathcal{N}(D')$ is a subgraph of G by Lemma 2.2. Therefore $\mathcal{N}(D')$ is a subgraph of $G - v$. To show that $G - v$ is a subgraph of $\mathcal{N}(D')$, take an edge xy in $G - v$. Then xy is an edge in G, so $N_D^+(x) \cap N_D^+(y) \neq \emptyset$ or $N_D^-(x) \cap N_D^-(y) \neq \emptyset$. By symmetry, we assume that $N_D^+(x) \cap N_D^+(y) \neq \emptyset$. If $v \in N_D^+(x) \cap N_D^+(y)$, then $u \in N_D^+(x) \cap N_D^+(y)$ and so $u \in N_{D'}^+(x) \cap N_{D'}^+(y)$. If $v \notin N_D^+(x) \cap N_D^+(y)$, then $N_D^+(x) \cap N_D^+(y) = N_{D'}^+(x) \cap N_{D'}^+(y)$. Therefore we may conclude that xy is an edge in $\mathcal{N}(D')$. Thus $G - v$ is a subgraph of $\mathcal{N}(D')$ and so $\mathcal{N}(D') = G - v$. ∎

Lemma 3.5. *Let D be an orientation of $K_{2,1,1}$ with true twins. Then the niche graph of D either is connected or has three components.*

Proof. We denote the partite sets of D by (X_1, X_2, X_3). Then we may assume that $X_1 = \{x_1, x_2\}$, $X_2 = \{x_3\}$, and $X_3 = \{x_4\}$. By the hypothesis, D has true twins and so, by Lemma 3.3, x_1 and x_2 are true twins. By Lemma 2.1, there are two cases to consider: $d_D^+(x_1) = 2$; $d_D^+(x_1) = 1$. We first consider the case $d_D^+(x_1) = 2$. Then $N_D^+(x_1) = \{x_3, x_4\}$. Since x_1 and x_2 are true twins, $N_D^+(x_2) = \{x_3, x_4\}$. Therefore $N_D^-(x_3) \cap N_D^-(x_4) \neq \emptyset$. Thus x_3 is adjacent to x_4 in $\mathcal{N}(D)$. By symmetry, we may assume $N_D^+(x_3) = \{x_4\}$. Then $N_D^-(x_4) = \{x_1, x_2, x_3\}$, so $\{x_1, x_2, x_3\}$ forms a clique in $\mathcal{N}(D)$. Therefore $\mathcal{N}(D)$ is connected.

Now we consider the case $d_D^+(x_1) = 1$. Without loss of generality, we may assume that $N_D^+(x_1) = \{x_3\}$. Then $N_D^+(x_2) = \{x_3\}$ and $N_D^-(x_1) = N_D^-(x_2) = \{x_4\}$. If $(x_3, x_4) \in A(D)$, then $\mathcal{N}(D) \cong K_2 \cup K_1 \cup K_1$. Therefore $\mathcal{N}(D)$ has three components. Suppose that $(x_3, x_4) \notin A(D)$, i.e. $(x_4, x_3) \in A(D)$. Then $N_D^-(x_3) = \{x_1, x_2, x_4\}$, so $\{x_1, x_2, x_4\}$ forms a clique in $\mathcal{N}(D)$. Since $N_D^+(x_4) = \{x_1, x_2, x_3\}$, $\{x_1, x_2, x_3\}$ forms a clique in $\mathcal{N}(D)$. Therefore $\mathcal{N}(D)$ is connected. ∎

Lemma 3.6. *Let D be an orientation of $K_{2,1,1}$ whose niche graph is disconnected. Suppose that no two vertices are true twins in D. Then the niche graph of D is isomorphic to $P_3 \cup K_1$.*

Proof. Let $\{x_1, x_2\}$, $\{x_3\}$, and $\{x_4\}$ be the partite sets of D. First we consider the case $|N_D^+(x_1)| = 2$ or $|N_D^+(x_2)| = 2$, i.e., $N_D^+(x_1) = \{x_3, x_4\}$ or $N_D^+(x_2) = \{x_3, x_4\}$. By symmetry, we may assume that $N_D^+(x_1) = \{x_3, x_4\}$. Then x_3 and x_4 are adjacent in $\mathcal{N}(D)$. Since x_1 and x_2 are not true twins in D, at least one of x_3 and x_4 is an in-neighbor of x_2. We may assume that x_4 is an in-neighbor of x_2. Suppose, to the contrary, that x_1 and x_2 are adjacent in $\mathcal{N}(D)$. Then x_3 is a common out-neighbor of x_1 and x_2. If $(x_3, x_4) \in A(D)$ (respectively, $(x_4, x_3) \in A(D)$), then x_3 (respectively, x_4) is adjacent to x_1 in $\mathcal{N}(D)$. In either case, $\mathcal{N}(D)$ is connected and we reach a contradiction. Thus x_1 and x_2 are not adjacent in $\mathcal{N}(D)$ and so $N_D^-(x_2) = \{x_3, x_4\}$.

We denote D_1 the subdigraph of D induced by $\{x_1, x_3, x_4\}$. Since $N_D^+(x_1) = \{x_3, x_4\}$, D_1 is not a directed cycle. Thus, by Lemma 2.4, $\mathcal{N}(D_1)$ is connected, and so, by Lemma 2.2, the subgraph of $\mathcal{N}(D)$ induced by $\{x_1, x_3, x_4\}$ is connected. By applying a similar argument to the subdigraph induced by $\{x_2, x_3, x_4\}$, we may show that the subgraph of $\mathcal{N}(D)$ induced by $\{x_2, x_3, x_4\}$ is connected. Therefore $\mathcal{N}(D)$ is connected and we reach a contradiction. Thus $|N_D^+(x_1)| = 2$ or $|N_D^+(x_2)| = 2$ cannot happen. Then, by Lemma 2.1, the case $N_D^+(x_1) = \emptyset$ or $N_D^+(x_2) = \emptyset$ cannot happen. Thus $|N_D^+(x_1)| = |N_D^+(x_2)| = 1$. If $N_D^+(x_1) = N_D^+(x_2)$, then $N_D^-(x_1) = N_D^-(x_2)$ and so x_1 and x_2 are true twins, which is a contradiction. Therefore $N_D^+(x_1) \neq N_D^+(x_2)$. Thus either $N_D^+(x_1) = \{x_3\}$ and $N_D^+(x_2) = \{x_4\}$ or $N_D^+(x_1) = \{x_4\}$ and $N_D^+(x_2) = \{x_3\}$. By symmetry, we may assume $N_D^+(x_1) = \{x_3\}$ and $N_D^+(x_2) = \{x_4\}$. Since x_3 and x_4 belong to different partite sets in D, $(x_3, x_4) \in A(D)$ or $(x_4, x_3) \in A(D)$. By symmetry again, we may assume that $(x_3, x_4) \in A(D)$. Then D is isomorphic to the digraph given in Figure 1. Hence the niche graph of D is isomorphic to $P_3 \cup K_1$. ■

Figure 1. An orientation of $K_{2,1,1}$ and its niche graph isomorphic to $P_3 \cup K_1$.

Lemma 3.7. *For positive integers n_1, n_2, and n_3 satisfying $n_1 + n_2 + n_3 \geq 5$, suppose that an orientation D of K_{n_1,n_2,n_3} has no true twins. Then the niche graph of D is connected.*

Proof. Without loss of generality, we may assume that $n_1 \geq n_2 \geq n_3$. We first consider the case $n_1 + n_2 + n_3 = 5$. Then $n_1 = 2$ or $n_1 = 3$. We will show that $\mathcal{N}(D)$ is connected in each of the following cases.

Case 1. $n_1 = 2$. Then $n_2 = 2$ and $n_3 = 1$. Let $\{u_1, u_2\}$, $\{v_1, v_2\}$, and $\{w\}$ be the partite sets of D. By Lemma 2.1, we may assume $d_D^+(w) \geq 2$. Suppose $d_D^+(w) = 4$. Then u_1, u_2, v_1, and v_2 form a clique in $\mathcal{N}(D)$. If $(u_1, v_1) \in A(D)$ (respectively, $(v_1, u_1) \in A(D)$), then v_1 (respectively, u_1) is a common out-neighbor of u_1 (respectively, v_1) and w and so $\mathcal{N}(D)$ is connected.

We consider the case $d_D^+(w) = 3$. Then $N_D^+(w) = \{u_2, v_1, v_2\}$, $\{u_1, v_1, v_2\}$, $\{u_1, u_2, v_2\}$, or $\{u_1, u_2, v_1\}$. By symmetry, we may assume $N_D^+(w) = \{u_1, u_2, v_1\}$. Then $N_D^-(w) = \{v_2\}$. Moreover, the subdigraphs D_1 and D_2 of D induced by $\{w, u_1, v_1\}$ and by $\{w, u_2, v_1\}$, respectively, are orientations of K_3 which are not directed cycles. Thus, by Lemma 2.4, $\mathcal{N}(D_1)$ and $\mathcal{N}(D_2)$ are connected.

Since D_1 and D_2 are subdigraphs of D, by Lemma 2.2, the subgraphs of $\mathcal{N}(D)$ induced by $\{w, u_1, v_1\}$ and by $\{w, u_2, v_1\}$ are connected respectively and so the subgraph of $\mathcal{N}(D)$ induced by $\{w, u_1, u_2, v_1\}$ is connected. If $(v_2, u_1) \in A(D)$ or $(v_2, u_2) \in A(D)$, then w and v_2 are adjacent in $\mathcal{N}(D)$ and we are done. Suppose that $(u_1, v_2) \in A(D)$ and $(u_2, v_2) \in A(D)$. If $(v_1, u_1) \in A(D)$ and $(v_1, u_2) \in A(D)$, then $N_D^+(u_1) = N_D^+(u_2)$ and $N_D^-(u_1) = N_D^-(u_2)$, which contradicts the hypothesis. Therefore $(u_1, v_1) \in A(D)$ or $(u_2, v_1) \in A(D)$, and so v_1 is adjacent to v_2 in $\mathcal{N}(D)$. Thus $\mathcal{N}(D)$ is connected.

We consider the case $d_D^+(w) = 2$. Then one of the following is true.

- $|N_D^+(w) \cap \{u_1, u_2\}| = 1$ and $|N_D^+(w) \cap \{v_1, v_2\}| = 1$;
- $N_D^+(w) = \{u_1, u_2\}$ or $N_D^+(w) = \{v_1, v_2\}$.

We first suppose that $|N_D^+(w) \cap \{u_1, u_2\}| = 1$ and $|N_D^+(w) \cap \{v_1, v_2\}| = 1$. By symmetry, we may assume that $N_D^+(w) = \{u_1, v_1\}$. Then $N_D^-(w) = \{u_2, v_2\}$. Therefore the subdigraphs D_3 and D_4 of D induced by $\{w, u_1, v_1\}$ and by $\{w, u_2, v_2\}$ are orientations of K_3 which are not directed cycles. Then, by Lemma 2.4, both $\mathcal{N}(D_3)$ and $\mathcal{N}(D_4)$ are connected. Therefore, by Lemma 2.2, the subgraphs of $\mathcal{N}(D)$ induced by $\{w, u_1, v_1\}$ and by $\{w, u_2, v_2\}$ are connected, respectively. Thus $\mathcal{N}(D)$ is connected. Now suppose $N_D^+(w) = \{u_1, u_2\}$ or $N_D^+(w) = \{v_1, v_2\}$. By symmetry, we may assume that $N_D^+(w) = \{u_1, u_2\}$. Then $N_D^-(w) = \{v_1, v_2\}$. Then u_1 and u_2 are adjacent and v_1 and v_2 are adjacent in $\mathcal{N}(D)$. If $(u_j, v_i) \in A(D)$ for all $1 \le i, j \le 2$, then $N_D^+(u_1) = N_D^+(u_2)$ and $N_D^-(u_1) = N_D^-(u_2)$, which contradicts the hypothesis. Thus $(v_i, u_j) \in A(D)$ for some i and j in $\{1, 2\}$. Then u_j (respectively, v_i) is a common out-neighbor (respectively, common in-neighbor) of v_i and w (respectively, u_j and w) in D. Thus each of v_i and u_j is adjacent to w in $\mathcal{N}(D)$ and so $\mathcal{N}(D)$ is connected. Hence we have shown that $\mathcal{N}(D)$ is connected if $n_1 = 2$.

Case 2. $n_1 = 3$. Then $n_2 = n_3 = 1$. Let $\{x_1, x_2, x_3\}$, $\{y\}$, and $\{z\}$ be the partite sets of D. We note that $N_D^+(x_i) = N_D^+(x_j)$ if and only if $N_D^-(x_i) = N_D^-(x_j)$ for each $1 \le i < j \le 3$. Therefore, by the hypothesis, $N_D^+(x_i) \ne N_D^+(x_j)$ for each $1 \le i < j \le 3$. Then, since $N_D^+(x_i)$ is one of $\emptyset$, $\{y\}$, $\{z\}$, and $\{y, z\}$ for each $i = 1$, 2, and 3, $d_D^+(x_i) = 1$ for some $i \in \{1, 2, 3\}$ and $d_D^+(x_j) \ne 1$ for some $j \in \{1, 2, 3\} \setminus \{i\}$. By symmetry, we may assume that $d_D^+(x_1) = 1$ and $d_D^+(x_2) \in \{0, 2\}$. In addition, by Lemma 2.1, we may assume that $d_D^+(x_2) = 2$, i.e., $N_D^+(x_2) = \{y, z\}$. Then x_1 and x_2 have a common out-neighbor in D, so x_1 and x_2 are adjacent in $\mathcal{N}(D)$. On the other hand, since y and z belong to different partite sets, there is an arc between y and z and so the subdigraph D_5 of D induced by $\{x_2, y, z\}$ is an orientation of K_3. Since $N_D^+(x_2) = \{y, z\}$, D_5 is not a directed cycle, and so, by Lemma 2.4, $\mathcal{N}(D_5)$ is connected. Thus, by Lemma 2.2, the subgraph of $\mathcal{N}(D)$ induced by $\{x_2, y, z\}$ is connected. Since x_1 and x_2 are adjacent in $\mathcal{N}(D)$, the subgraph of $\mathcal{N}(D)$ induced by $\{x_1, x_2, y, z\}$

is connected. We will show that x_3 is adjacent to a vertex in $\{x_1, x_2, y, z\}$ in $\mathcal{N}(D)$ to take care of this case. If x_3 has an out-neighbor in D, then x_2 and x_3 are adjacent in $\mathcal{N}(D)$ and so we are done. Suppose that $d_D^+(x_3) = 0$. Then the subdigraph of D induced by $\{x_3, y, z\}$ is an orientation of K_3 which is not a directed cycle. By applying the same argument for D_5, we may show that $\mathcal{N}(D)$ is connected. Hence we have shown that $\mathcal{N}(D)$ is connected in the case $n_1 + n_2 + n_3 = 5$.

Now suppose that $n_1 + n_2 + n_3 > 5$. To show that $\mathcal{N}(D)$ is connected, take two vertices w_1 and w_2 in D. Then we may take three vertices w_3, w_4, and w_5 in D such that the induced subdigraph D_6 of D induced by $\{w_1, w_2, w_3, w_4, w_5\}$ is a 3-partite tournament. By the above argument, $\mathcal{N}(D_6)$ is connected, so there is a (w_1, w_2)-path P in $\mathcal{N}(D_6)$. Since D_6 is a subdigraph of D, $\mathcal{N}(D_6)$ is a subgraph of $\mathcal{N}(D)$ by Lemma 2.2. Thus P is a (w_1, w_2)-path in $\mathcal{N}(D)$ and hence $\mathcal{N}(D)$ is connected. This completes the proof. ■

For a graph G, a vertex v of G, and a finite set K disjoint from $V(G)$, we say that v *is replaced with a clique* formed by K to obtain a new graph with the vertex set $(V(G) \cup K) \setminus \{v\}$ and the edge set

$$E(G - v) \cup \{wx \mid w \neq x, \{w, x\} \subset K\} \cup \{uw \mid uv \in E(G), w \in K\}.$$

See Figure 2 for an illustration. We call a graph an *expansion* of a graph G if it is obtained by replacing each vertex in G with a clique (possibly of size 1).

Figure 2. The vertex v of the graph on the left is replaced with a clique K of size 3 to yield the graph on the right.

Theorem 3.8. *Let G be a graph having exactly two components. For $k \geq 3$, (G, k) is niche-realizable if and only if $k = 3$ and G is isomorphic to an expansion of $P_3 \cup K_1$.*

Proof. To show the "if" part, suppose that G is isomorphic to an expansion of $P_3 \cup K_1$. We will show that $(G, 3)$ is niche-realizable. Let D be the digraph given in Figure 1. Then $\mathcal{N}(D)$ is isomorphic to $P_3 \cup K_1$. Let X_i be the set of vertices of G which are true twins to the vertex corresponding to x_i in $\mathcal{N}(D)$ for each

$1 \leq i \leq 4$. We construct a digraph D^* from D in the following way:

$$V(D^*) = V(G);$$

$$A(D^*) = \{(v, w) \mid v \in X_i, w \in X_j, (i, j) \in \{(1, 3), (2, 4), (3, 2), (3, 4), (4, 1)\}\}.$$

Then D^* is a 3-partite tournament, and

- $N^+_{D^*}(u_1) = X_3$, $N^-_{D^*}(u_1) = X_4$;
- $N^+_{D^*}(u_2) = X_4$, $N^-_{D^*}(u_2) = X_3$;
- $N^+_{D^*}(u_3) = X_2 \cup X_4$, $N^-_{D^*}(u_3) = X_1$;
- $N^+_{D^*}(u_4) = X_1$, $N^-_{D^*}(u_4) = X_2 \cup X_3$

for each vertex $u_i \in X_i$; for each $1 \leq i \leq 4$. Thus X_i forms a clique in $\mathcal{N}(D^*)$ for each $1 \leq i \leq 4$. Take v and w in G. We first consider the case in which v and w are adjacent in G. Then v and w belong to X_i for some $i \in \{1, 2, 3, 4\}$ or exactly one of v and w belongs to X_2 and the other one belongs to $X_3 \cup X_4$. If the former is true, then v and w are adjacent in $\mathcal{N}(D^*)$ by above observation. Suppose the latter. Then, without loss of generality, we may assume that v belongs to X_2 and w belongs to $X_3 \cup X_4$. If w belongs to X_3 (respectively, X_4), then v and w have a common out-neighbor (respectively, common in-neighbor) in D^* by the above observation, and so they are adjacent in $\mathcal{N}(D^*)$.

Now we consider the case where v and w are not adjacent in G. Then, without loss of generality, we may assume that v belongs to X_1 and w does not belong to X_1 or v and w belong to X_3 and X_4, respectively. If the former is true, $N^+_{D^*}(v) = X_3$, $N^-_{D^*}(v) = X_4$, $N^+_{D^*}(w) \subset X_1 \cup X_2 \cup X_4$, and $N^-_{D^*}(w) \subset X_1 \cup X_2 \cup X_3$ by the above observation, and so v and w are not adjacent in $\mathcal{N}(D^*)$. If the latter is true, $N^+_{D^*}(v) = X_2 \cup X_4$, $N^-_{D^*}(v) = X_1$, $N^+_{D^*}(w) = X_1$, and $N^-_{D^*}(w) = X_2 \cup X_3$ by the above observation, and so v and w are not adjacent in $\mathcal{N}(D^*)$. Hence we have shown that G is isomorphic to $\mathcal{N}(D^*)$.

To show the "only if" part, suppose that (G, k) is a niche-realizable. Let D be a k-partite tournament whose niche graph is G. Since G is not connected, $k < 4$ by Theorem 2.6 and so $k = 3$. Thus D is an orientation of K_{n_1,n_2,n_3} for positive integers n_1, n_2, and n_3. If $|V(G)| = 3$, then D is an orientation of K_3 and so, by Lemma 2.4, G is connected or has three components, which contradicts the hypothesis that G has exactly two components. Therefore $|V(G)| \geq 4$. In the following, we show that G is isomorphic to an expansion of $P_3 \cup K_1$ by induction on $|V(G)|$. First we consider the case where $|V(G)| = 4$. Then D is an orientation of $K_{2,1,1}$. If D has true twins, then G is connected or has three components by Lemma 3.5, which is a contradiction. Therefore D has no true twins, so $G \cong P_3 \cup K_1$ by Lemma 3.6. Thus the basis step is true.

We assume that the statement is true for any niche-realizable graph on l vertices which has exactly two components for a positive integer $l \geq 4$. Now we assume $|V(G)| = l+1$. Then $n_1+n_2+n_3 = l+1 \geq 5$. Since G is not connected, D has true twins by Lemma 3.7. Let u and v be true twins in D. Then $D-v$ is a 3-partite tournament and $G-v$ is the niche graph of $D-v$ by Proposition 3.4. On the other hand, u and v are true twins in G by Lemma 3.2. Then, G, $G-u$, and $G-v$ have the same number of components. Since G has two components by the hypothesis, $G-v$ has exactly two components. Therefore, by the induction hypothesis, $G-v$ is an expansion of $P_3 \cup K_1$. Since v and u are true twins in G, G is an expansion of $P_3 \cup K_1$. ■

4. Niche-Realizable Pairs (G, k) When G is Connected

In this section, we study the niche graphs of k-partite tournaments for $k \geq 3$ which are connected. The niche graphs of multipartite tournaments come in many different forms, which makes it hard to give a general characterization, if they are connected. Yet, we identify niche-realizable pairs for complete graphs and connected triangle-free graphs. We first find all the niche-realizable pairs (K_n, k) for positive integers $n \geq k \geq 3$.

Theorem 4.1. *For positive integers $n \geq k \geq 3$, (K_n, k) is niche-realizable if and only if $(n, k) \in \{(4, 4)\} \cup \{(n, k) \mid n \geq 5\}$.*

Proof. To show the "if" part, we construct a digraph D in the following way. Let $V(D) = \{v_1, v_2, \ldots, v_n\}$. If $k = 3$ and $n \geq 5$, then let D be any 3-partite tournament with partite sets $\{v_1\}$, $\{v_2, v_3\}$, and $\{v_4, v_5, \ldots, v_n\}$ whose arc set includes the following arc set (the remaining arcs have an arbitrary orientation):

$$\{(v_1, v_i) \mid 2 \leq i \leq n\} \cup \{(v_2, v_4), (v_4, v_3), (v_3, v_5), (v_5, v_2)\} \cup \{(v_i, v_2) \mid 6 \leq i \leq n\}.$$

If $k \geq 4$ and $n \geq 4$, then let D be any k-partite tournament with partite sets $\{v_1\}, \{v_2\}, \ldots, \{v_{k-1}\}, \{v_k, v_{k+1}, \ldots, v_n\}$ whose arc set includes the following arc set (the remaining arcs have an arbitrary orientation):

$$\{(v_1, v_i) \mid 2 \leq i \leq n\} \cup \bigcup_{i=2}^{k-2} \{(v_i, v_{i+1})\} \cup \bigcup_{i=k}^{n} \{(v_{k-1}, v_i), (v_i, v_2)\}.$$

In both cases, v_1 is a common in-neighbor of the remaining vertices, so the set $\{v_2, v_3, \ldots, v_n\}$ forms a clique in $\mathcal{N}(D)$. Moreover, since v_i has at least one out-neighbor in $\{v_2, v_3, \ldots, v_n\}$ for each $2 \leq i \leq n$, v_1 and v_i have a common out-neighbor in D, and so they are adjacent in $\mathcal{N}(D)$. Therefore $\mathcal{N}(D)$ is a complete graph with n vertices.

Now we show the "only if" part. By Lemma 2.4, $(K_3, 3)$ is not niche-realizable. We only need to show that $(K_4, 3)$ is not niche-realizable. Suppose, to the

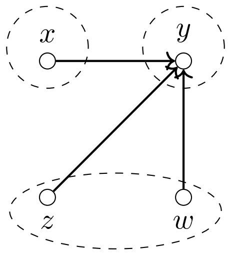

Figure 3. A subdigraph of D.

contrary, that $(K_4, 3)$ is niche-realizable. Then there is an orientation D of $K_{1,1,2}$ such that $\mathcal{N}(D)$ is isomorphic to K_4. Let $\{x\}$, $\{y\}$, and $\{z, w\}$ be the partite sets of D. Since $\mathcal{N}(D) \cong K_4$, z and w are adjacent in $\mathcal{N}(D)$, and so have a common out-neighbor or a common in-neighbor in D. By Lemma 2.1, we may assume that they have a common out-neighbor and, by symmetry, we may assume that y is a common out-neighbor of z and w. Then, since x and z are adjacent in $\mathcal{N}(D)$, $(x, y) \in A(D)$. Thus $N_D^-(y) = \{x, z, w\}$. On the other hand, since y and z (respectively, w) are adjacent in $\mathcal{N}(D)$, they have a common out-neighbor or a common in-neighbor in D. Yet, y has no out-neighbor in D, so y and z (respectively, w) have a common in-neighbor that must be x (see Figure 3). Then $A(D) = \{(x, y), (x, z), (x, w), (z, y), (w, y)\}$. Since x has only out-neighbors and y has only in-neighbors, they are not adjacent in $\mathcal{N}(D)$, which is a contradiction to the supposition that $\mathcal{N}(D) \cong K_4$. Hence the "only if" part is true. ∎

The rest of this paper will be devoted to finding all the niche-realizable pairs (G, k) when G is connected and triangle-free.

Lemma 4.2. *Let D be a digraph with at least three vertices whose niche graph $\mathcal{N}(D)$ is connected. If there are two distinct vertices which are true twins in D, then $\mathcal{N}(D)$ contains a triangle.*

Proof. Suppose that u and v are distinct vertices which are true twins in D. Since $\mathcal{N}(D)$ is connected and has at least three vertices, D contains a vertex w other than u and v that is adjacent to u or v in $\mathcal{N}(D)$. Without loss of generality, we may assume that w is adjacent to v in $\mathcal{N}(D)$. Since $\mathcal{N}(D)$ is connected, D has no isolated vertices. Then u and v are true twins in $\mathcal{N}(D)$ by Lemma 3.2. Thus $\{u, v, w\}$ forms a triangle in $\mathcal{N}(D)$. ∎

We make the following rather obvious observation.

Lemma 4.3. *Let D be a k-partite tournament for $k \geq 3$. Then, for each partite set X and each $x \in X$, $N_D^+(x) \cup N_D^-(x) = V(D) \setminus X$.*

Theorem 4.4. *Let D be a k-partite tournament for $k \geq 3$. Then $\mathcal{N}(D)$ contains no induced path of length 5, that is, $\mathcal{N}(D)$ is P_6-free.*

Proof. We denote the partite sets of D by $X_1, \ldots, X_{k-1}$, and X_k. If $\mathcal{N}(D)$ is disconnected, it contains no induced path of length 5 by Corollary 2.8 and Theorems 3.1 and 3.8.

Suppose that $\mathcal{N}(D)$ is connected. To reach a contradiction, suppose that $\mathcal{N}(D)$ contains an induced path P of length 5. Let $P = x_1x_2x_3x_4x_5x_6$. Suppose that $|X_i \cap V(P)| \leq 1$ for some $i \in [k]$. Without loss of generality, we may assume that $|X_1 \cap V(P)| \leq 1$. Take a vertex $x \in X_1$. Then $N^+_D(x) \cup N^-_D(x)$ contains at least five vertices in $V(P)$ by Lemma 4.3. Therefore $N^+_D(x)$ or $N^-_D(x)$ contains at least three vertices in $V(P)$. Since each of $N^+_D(x)$ and $N^-_D(x)$ forms a clique in $\mathcal{N}(D)$, the subgraph of $\mathcal{N}(D)$ induced by $V(P)$ contains a triangle, which contradicts the choice of P as an induced path of $\mathcal{N}(D)$. Thus $|X_i \cap V(P)| \geq 2$ for each $1 \leq i \leq k$. Since $|V(P)| = 6$, $k \geq 3$, and $X_1, \ldots, X_k$ are mutually disjoint, we obtain $k = 3$ and

$$|X_i \cap V(P)| = 2 \tag{1}$$

for each $i = 1$, 2, and 3. Now let D_1 be the subdigraph of D induced by $V(P)$. Then D_1 is a 3-partite tournament. By Lemma 2.2, $\mathcal{N}(D_1)$ is a subgraph of P. Thus $\mathcal{N}(D_1)$ is triangle-free and so, by Lemma 2.3, $d^+_{D_1}(x) \leq 2$ and $d^-_{D_1}(x) \leq 2$ for all $x \in V(D_1)$. By (1), $d^+_{D_1}(x) + d^-_{D_1}(x) = 4$, so

$$d^+_{D_1}(x) = 2 \quad \text{and} \quad d^-_{D_1}(x) = 2 \tag{2}$$

for all $x \in V(D_1)$.

Suppose that $\mathcal{N}(D_1)$ is disconnected. Then x_j and x_{j+1} are not adjacent in $\mathcal{N}(D_1)$ for some $j \in \{1, 2, 3, 4, 5\}$, so

$$N^+_{D_1}(x) \neq \{x_j, x_{j+1}\} \quad \text{and} \quad N^-_{D_1}(x) \neq \{x_j, x_{j+1}\} \tag{3}$$

for all $x \in V(D_1)$. Yet, since x_j and x_{j+1} are adjacent in $\mathcal{N}(D)$, they have a common in-neighbor or a common out-neighbor in D. By Lemma 2.1, we may assume that x_j and x_{j+1} have a common out-neighbor y in D. Obviously $y \notin V(D_1)$. Without loss of generality, we may assume that $y \in X_1$. Then x_j and x_{j+1} do not belong to X_1. By (1), $|V(P) \setminus X_1| = 4$, so $|(N^+_D(y) \cup N^-_D(y)) \cap V(P)| = 4$ by Lemma 4.3. Since P is an induced path of D, $|N^-_D(y) \cap V(P)| = 2$ and $|N^+_D(y) \cap V(P)| = 2$. Thus $N^-_D(y) \cap V(P) = \{x_j, x_{j+1}\}$. Since $|N^+_D(y) \cap V(P)| = 2$, $N^+_D(y) \cap V(P)$ also forms an edge in $\mathcal{N}(D)$, that is, $N^+_D(y) \cap V(P) = \{x_k, x_{k+1}\}$ for some $k \in \{1, 2, 3, 4, 5\} \setminus \{j-1, j, j+1\}$. Therefore $V(P) \setminus X_1 = \{x_j, x_{j+1}, x_k, x_{k+1}\}$. Let z be one of the two vertices in $X_1 \cap V(D_1)$. Then $z \neq y$. By Lemma 4.3,

$N^+_{D_1}(z) \cup N^-_{D_1}(z) = \{x_j, x_{j+1}, x_k, x_{k+1}\}$. By (2), $d^+_{D_1}(z) = d^-_{D_1}(z) = 2$. Then, by (3),

$$\{N^+_{D_1}(z), N^-_{D_1}(z)\} = \{\{x_j, x_k\}, \{x_{j+1}, x_{k+1}\}\}$$

or

$$\{N^+_{D_1}(z), N^-_{D_1}(z)\} = \{\{x_j, x_{k+1}\}, \{x_{j+1}, x_k\}\}.$$

In the former case, x_j and x_k are adjacent in $\mathcal{N}(D_1)$ and so in $\mathcal{N}(D)$, which is impossible as P is an induced path in $\mathcal{N}(D)$. In the latter case, x_j and x_{k+1} are adjacent and x_{j+1} and x_k are adjacent in $\mathcal{N}(D)$. However, either x_j and x_{k+1} or x_{j+1} and x_k are not consecutive on P and we reach a contradiction. Thus $\mathcal{N}(D_1)$ is connected. Since P is an induced path of $\mathcal{N}(D)$ and $\mathcal{N}(D_1)$ is a spanning subgraph of P, we may conclude that $\mathcal{N}(D_1) = P$.

Let $D_2 = D_1 - x_2$. Then D_2 is a 3-partite tournament by (1) and, by Lemma 2.2, $\mathcal{N}(D_2)$ is a subgraph of $\mathcal{N}(D_1) = P$. Since $P - x_2$ is disconnected, $\mathcal{N}(D_2)$ is disconnected. Without loss of generality, we may assume that $x_2 \in X_1$. Then, by (1),

$$|V(D_2) \cap X_1| = 1 \quad \text{and} \quad |V(D_2) \cap X_2| = |V(D_2) \cap X_3| = 2. \tag{4}$$

Suppose that u and v are true twins in D_2 for some distinct vertices u and v in $V(D_2)$, that is, $N^+_{D_2}(u) = N^+_{D_2}(v)$ and $N^-_{D_2}(u) = N^-_{D_2}(v)$. Then both u and v belong to the same partite set by Lemma 3.3. Thus, by (4), u and v belong to X_2 or X_3. By (2), either $d^+_{D_2}(u) = d^+_{D_2}(v) = 2$ and $d^-_{D_2}(u) = d^-_{D_2}(v) = 1$ or $d^+_{D_2}(u) = d^+_{D_2}(v) = 1$ and $d^-_{D_2}(u) = d^-_{D_2}(v) = 2$. By Lemma 2.1, we may assume that $d^+_{D_2}(u) = d^+_{D_2}(v) = 2$ and $d^-_{D_2}(u) = d^-_{D_2}(v) = 1$. Then x_2 is a common in-neighbor of u and v in D_1 by (2). Thus $N^+_{D_1}(u) = N^+_{D_1}(v)$ and $N^-_{D_1}(u) = N^-_{D_1}(v)$, that is, u and v are true twins in D_1. Since $|V(D_1)| \geq 3$ and $\mathcal{N}(D_1)$ is connected, $\mathcal{N}(D_1)$ contains a triangle by Lemma 4.2. Yet, $\mathcal{N}(D_1) = P$ and we reach a contradiction. Therefore there is no pair of vertices which are true twins in D_2. Thus, by Lemma 3.7, $\mathcal{N}(D_2)$ is connected and we reach a contradiction. Hence $\mathcal{N}(D)$ contains no induced path of length 5 and we are done. ∎

From the above theorem, the following corollary immediately follows.

Corollary 4.5. *Let D be a k-partite tournament for $k \geq 3$. Then each component of $\mathcal{N}(D)$ has diameter at most 4.*

A graph is said to be *triangle extended complete bipartite* if it is obtained from a complete bipartite graph by possibly attaching some P_3s to a common edge of the bipartite graph. A set $U \subseteq V$ *dominates* a set $U' \subseteq V$ if any vertex $v \in U'$ either lies in U or has a neighbor in U. We also say that U dominates $G[U']$. A subgraph H of G is a *dominating subgraph* of G if $V(H)$ dominates G.

Hof *et al.* [15] showed that a graph G is P_6-free if and only if each connected induced subgraph of G has a dominating (not necessarily induced) triangle extended complete bipartite graph or an induced dominating C_6. Thus the following result immediately follows.

Corollary 4.6. *Let D be a k-partite tournament for $k \geq 3$. Then each connected induced subgraph of $\mathcal{N}(D)$ has a dominating* (*not necessarily induced*) *triangle extended complete bipartite graph or an induced dominating C_6.*

By using Theorem 4.4, we may find all the niche-realizable pairs (P_n, k) and all the niche-realizable pairs (C_n, k) for positive integers $n \geq k \geq 3$.

Lemma 4.7. *For positive integers $n \geq k \geq 3$, (P_n, k) is niche-realizable if and only if $(n, k) \in \{(3,3), (4,3), (4,4), (5,3)\}$.*

Proof. Let D_1, D_2, D_3, and D_4 be the digraphs in Figure 4 which are isomorphic to some orientations of $K_{1,1,1}$, $K_{1,1,2}$, $K_{1,1,1,1}$, and $K_{1,2,2}$, respectively. It is easy to check that $\mathcal{N}(D_1) \cong P_3$, $\mathcal{N}(D_2) \cong P_4$, $\mathcal{N}(D_3) \cong P_4$, and $\mathcal{N}(D_4) \cong P_5$. Hence the "if" part is true.

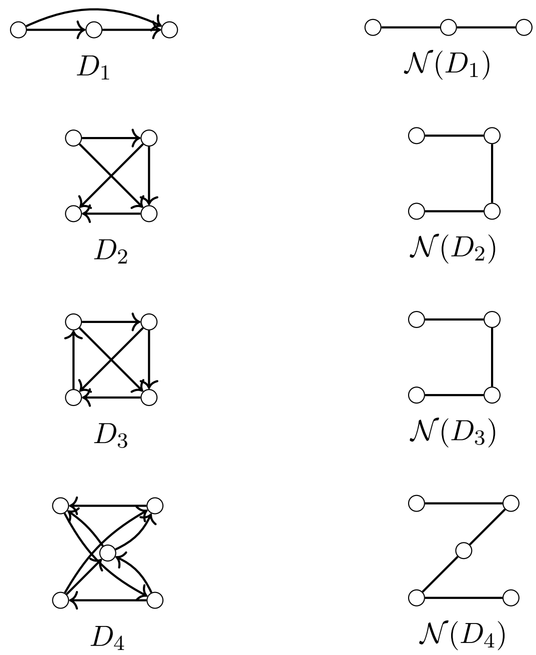

Figure 4. The digraphs D_1, D_2, D_3, and D_4 which are isomorphic to some orientations of $K_{1,1,1}$, $K_{1,1,2}$, $K_{1,1,1,1}$, and $K_{1,2,2}$, respectively, and their niche graphs.

Now suppose that (P_n, k) is niche-realizable. By Theorem 4.4, $n \leq 5$. Thus we only need to show that (n, k) is neither $(5,4)$ nor $(5,5)$. Let D be a k-partite tournament such that $\mathcal{N}(D) \cong P_5$. We denote P_5 by $x_1x_2x_3x_4x_5$. Since

$\mathcal{N}(D) \cong P_5$, $\mathcal{N}(D)$ is triangle-free and so, by Lemma 2.3, every vertex of D has indegree at most two and outdegree at most two in D. Suppose that $\{x_2\}$ is one of the partite sets of D. Then $N_D^+(x_2) \cup N_D^-(x_2) = V(D) \setminus \{x_2\}$ by Lemma 4.3, so $d_D^+(x_2) = 2$ and $d_D^-(x_2) = 2$. By Lemma 2.1, we may assume that x_1 is a out-neighbor of x_2 in D. Since $N_D^+(x_2)$ forms an edge in $\mathcal{N}(D)$, x_1 is adjacent to a vertex in P_5 other than x_2 and we reach a contradiction. Therefore $\{x_2\}$ is properly contained in a partite set of D. Thus $k \neq 5$. By symmetry, $\{x_4\}$ is properly contained in a partite set of D. Now suppose that $k = 4$. Then $\{x_1\}$, $\{x_3\}$, $\{x_5\}$, and $\{x_2, x_4\}$ are the partite sets of D. Therefore $d_D^+(x_2)+d_D^-(x_2) = 3$ by Lemma 4.3 and so $d_D^+(x_2) = 2$ or $d_D^-(x_2) = 2$. By Lemma 2.1, we may assume that $d_D^+(x_2) = 2$. Then the out-neighbors of x_2 in D are adjacent in $\mathcal{N}(D)$. However, the possible out-neighbors of x_2 in D are x_1, x_3, x_5 no two of which are consecutive on P_5. Hence we have reached a contradiction and so $k = 3$. This completes the proof. ■

Lemma 4.8. *For a k-partite tournament D with n vertices for some integers $n \geq k \geq 3$, suppose that $\mathcal{N}(D)$ is a connected triangle-free graph. Then $k \in \{3, 4, 5\}$ and*

$$\begin{cases} 3 \leq n \leq 6 & \text{if } k = 3; \\ 4 \leq n \leq 5 & \text{if } k = 4; \\ n = 5 & \text{if } k = 5. \end{cases} \tag{5}$$

Proof. If $k \geq 6$, then $5 \leq d_D^+(v) + d_D^-(v)$ for each vertex v in D by Lemma 4.3, which contradicts Lemma 2.3. Thus $k \leq 5$. Let X_i be a partite set of D for each $1 \leq i \leq k$. Without loss of generality, we may assume that X_1 is a partite set with the smallest size among the partite sets. Then $|X_1| \leq \left\lfloor \frac{n}{k} \right\rfloor$. Take a vertex u in X_1. By Lemma 4.3, $n - |X_1| = d_D^+(u) + d_D^-(u)$. Since $d_D^+(u) + d_D^-(u) \leq 4$ by Lemma 2.3, $n - |X_1| \leq 4$ and so

$$n - \left\lfloor \frac{n}{k} \right\rfloor \leq 4.$$

It is easy to check that (5) is an immediate consequence of this inequality. ■

Lemma 4.9. *For positive integers $n \geq k \geq 3$, (C_n, k) is niche-realizable if and only if $(n, k) \in \{(5, 3), (5, 4), (5, 5), (6, 3)\}$.*

Proof. Let D_1, D_2, and D_3 be the digraphs given in Figure 5. Clearly, D_1, D_2, and D_3 are orientations of $K_{1,1,3}$, $K_{1,1,1,2}$, and $K_{1,1,1,1,1}$, respectively. In addition, $\mathcal{N}(D_i) \cong C_5$ for each $i = 1$, 2, and 3. Thus $(C_5, 3)$, $(C_5, 4)$, and $(C_5, 5)$ are niche-realizable. Now let D_4 be a digraph with the vertex set $V(D_4) = \{v_0, v_1, v_2, v_3, v_4, v_5\}$ and the arc set

$$A(D_4) = \{(v_{i-2}, v_i), (v_{i-1}, v_i), (v_i, v_{i+1}), (v_i, v_{i+2}) \mid i \in \{0, 1, 2, 3, 4, 5\}\}$$

where all the subscripts are reduced to modulo 6 (see Figure 5 for an illustration). Since each vertex v_i takes v_{i+1} and v_{i+2} as its out-neighbors and v_{i-1} and v_{i-2} as its in-neighbors, D_4 is an orientation of $K_{2,2,2}$ with partite sets $\{v_0, v_3\}$, $\{v_1, v_4\}$, and $\{v_2, v_5\}$. Furthermore, it is easy to see that $\mathcal{N}(D_4) \cong C_6$. Hence the "if" part is true.

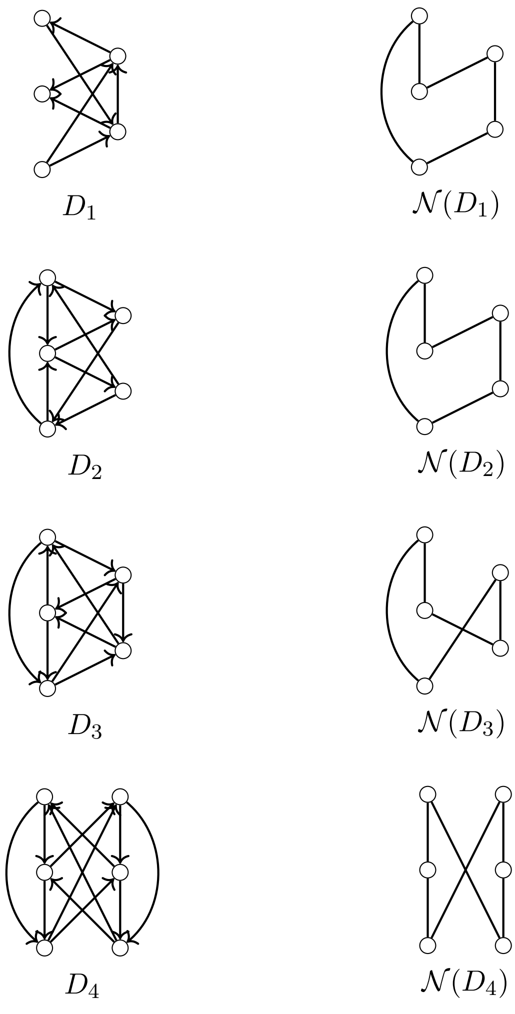

Figure 5. The digraphs D_1, D_2, D_3, and D_4 which are isomorphic to some orientations of $K_{1,1,3}$, $K_{1,1,1,2}$, $K_{1,1,1,1,1}$, and $K_{2,2,2}$, respectively, and their niche graphs.

Suppose that (C_n, k) is niche-realizable. By Theorem 4.4, $n \leq 6$. Thus we need to show that $(n, k) \notin \{(3,3),(4,3),(4,4),(6,4),(6,5),(6,6)\}$. By Lemma 2.4, $(n, k) \neq (3, 3)$. In addition, by Lemma 4.8, $(n, k) \notin \{(6,4),(6,5),(6,6)\}$.

Suppose that $(n, k) \in \{(4,3),(4,4)\}$. Then there is a k-partite tournament

D_5 such that $\mathcal{N}(D_5) \cong C_4$ and so $\mathcal{N}(D_5)$ is triangle-free. Therefore

$$d^+_{D_5}(x) \leq 2 \quad \text{and} \quad d^-_{D_5}(x) \leq 2 \tag{6}$$

for all $x \in V(D_5)$. Let $X_1, \ldots, X_k$ be the partite sets of D_5. We take $x_i \in X_i$ for each $i = 1$, 2, and 3. Let x_4 be the vertex of D_5 that does not belong to $\{x_1, x_2, x_3\}$. Suppose that the subdigraph of D_5 induced by $\{x_1, x_2, x_3\}$ is a directed cycle. Then, by Lemma 2.1, (6), and the symmetry of the directed cycle, we may assume that

$$A(D_5) \subset \{(x_1, x_2), (x_2, x_3), (x_3, x_1), (x_1, x_4), (x_2, x_4), (x_4, x_3)\}.$$

Then, by Lemma 2.2, $\mathcal{N}(D_5)$ is a subgraph of P_4 and we reach a contradiction. Thus the subdigraph of D_5 induced by $\{x_1, x_2, x_3\}$ is not a directed cycle. Then, without loss of generality, we may assume that $(x_1, x_2), (x_1, x_3), (x_2, x_3) \in A(D_5)$. By (6), $(x_1, x_4) \notin A(D_5)$ and $(x_4, x_3) \notin A(D_5)$. Thus

$$A(D_5) \subset \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_4, x_1), (x_3, x_4), (x_2, x_4)\}$$

or

$$A(D_5) \subset \{(x_1, x_2), (x_1, x_3), (x_2, x_3), (x_4, x_1), (x_3, x_4), (x_4, x_2)\}.$$

In both cases, $\mathcal{N}(D_5)$ is a subgraph of P_4 by Lemma 2.2 and we reach a contradiction. Thus $(n, k) \notin \{(4, 3), (4, 4)\}$. This completes the proof. ■

Lemma 4.10. *Let G be a connected triangle-free graph with $3 \leq |V(G)| \leq 5$, stability number at most 3, and diameter at most 4. Then the following are true.*

(1) *Each vertex in G has degree at most 3;*

(2) *G is isomorphic to a path P_i for some $i \in \{3, 4, 5\}$ or cycle C_j for some $j \in \{4, 5\}$ or the graph G_k for some $k \in \{1, 2, 3, 4\}$ given in Figure 6.*

Proof. To show the statement (1) by contradiction, suppose that there exists a vertex x in G of degree at least 4. Then there exist four distinct vertices x_1, x_2, x_3, and x_4 which are adjacent to x in G. Since G is triangle-free, x_i and x_j are not adjacent if $i \neq j$. Therefore $\{x_1, x_2, x_3, x_4\}$ is a stable set, which contradicts the hypothesis that G has stability number at most 3. Thus the statement (1) is true.

To show the statement (2), we first consider the case where G is a tree. If G is isomorphic to a path, then $G \cong P_i$ for some $i \in \{3, 4, 5\}$ by the hypothesis. Suppose that G is not a path graph. Let t be the diameter of G. Then $t \leq 4$ by the hypothesis and there exists an induced path $P := x_1 \cdots x_{t+1}$ of length t in G. Since G is not a path graph, there exist a vertex of degree at least 3 on P. Let x_i be a vertex of degree at least 3. Then x_i has degree 3 by the statement (1). By the choice of P, $i \neq 1$ and $i \neq t + 1$. If $t = 1$, then G is a complete, which is contradiction. Therefore $t \geq 2$. If $t = 2$, then $i = 2$ and so G is isomorphic to G_1 given in Figure 6. Suppose $t = 3$. Then $i = 2$ or $i = 3$. By symmetry, we may assume $i = 2$. Then there exists a vertex x_5 not on P which is adjacent to x_2. Since $|V(G)| \leq 5$ by the hypothesis, $V(G) = \{x_1, x_2, x_3, x_4, x_5\}$. Then, since G is a tree, x_2 is the only vertex adjacent to x_5 in G. Thus G isomorphic to G_2 given in Figure 6. If $t = 4$, then $G = P$, which is a contradiction.

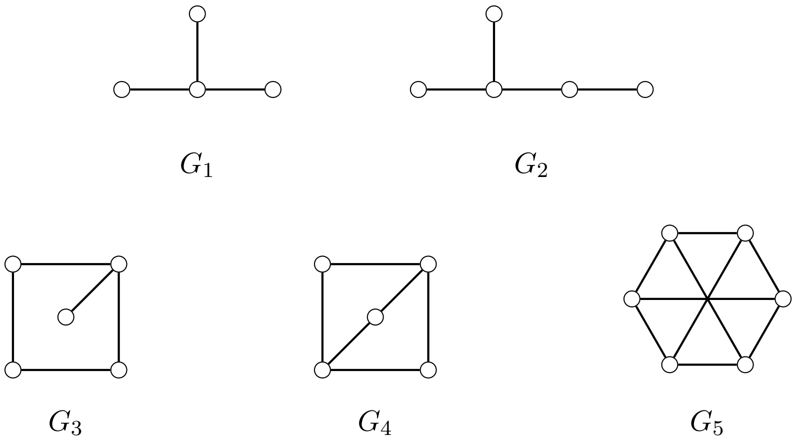

Figure 6. Connected triangle-free graphs mentioned in Lemmas 4.10 and 4.11.

Now we consider the case where G is not a tree. Then G has a cycle C of length at least 4 since G is triangle-free and connected. Then $4 \leq |V(G)|$. If $|V(G)| = 4$, then $G = C$, so G is isomorphic to a cycle C_4 by the hypothesis that G is triangle-free. Suppose that $|V(G)| = 5$. If G is a cycle, then G is isomorphic to a cycle C_5 by the hypothesis. Now we suppose that G is not a cycle. If $|V(C)| = 5$, then C is a spanning subgraph of G and so C has a chord, which contradicts the hypothesis that G is triangle-free. Therefore $|V(C)| = 4$. Let y be the vertex in $V(G) \setminus V(C)$. Then there exists a vertex y' on C which is adjacent to y by the hypothesis that G is connected. Therefore y' has degree 3 by the statement (1). If y has degree 3, then it is easy to check that G contains a triangle, which is a contradiction. Therefore y has degree 1 or 2. If y has degree 1, then G is isomorphic to a graph G_3 given in Figure 6. If y has degree 2, then G is isomorphic to a graph G_4 given in Figure 6. Therefore we have shown that the statement (2) is true. ■

Lemma 4.11. *Let G be a connected triangle-free graph with six vertices. Then (G, k) is niche-realizable for some integer $k \geq 3$ if and only if $k = 3$ and G is isomorphic to the cycle C_6 or the graph G_5 given in Figure* 6.

Proof. Suppose that (G, k) is niche-realizable for some integer $k \geq 3$. Then there exists a k-partite tournament D such that $\mathcal{N}(D) \cong G$. Since $|V(G)| = 6$,

$k = 3$ by Lemma 4.8. We denote the partite sets of D by (X_1, X_2, X_3). If $|X_l| = 1$ for some $l \in \{1, 2, 3\}$, then $d_D^+(x) + d_D^-(x) = 5$ for the vertex x in X_l by Lemma 4.3, which contradicts Lemma 2.3. Therefore each partite set in D has at least size 2. Since $|V(G)| = 6$ and $k = 3$, each partite set in D has size 2. Therefore $d_D^+(v) + d_D^-(v) = 4$ by Lemma 4.3 and so, by Lemma 2.3,

$$d_D^+(v) = d_D^-(v) = 2 \tag{7}$$

for all $v \in V(D)$. Now let $X_1 = \{v_1, v_2\}$, $X_2 = \{v_3, v_4\}$, and $X_3 = \{v_5, v_6\}$.

Case 1. The two vertices in X_i are not adjacent in G for each $i = 1$, 2, and 3. Then the out-neighbors (respectively, in-neighbors) of each vertex belong to distinct partite sets. Now, without loss of generality, we may assume $N_D^+(v_1) = \{v_3, v_5\}$ and $N_D^-(v_1) = \{v_4, v_6\}$. By symmetry, we may assume that $(v_3, v_5) \in A(D)$. Then $N_D^-(v_5) = \{v_1, v_3\}$, so $N_D^+(v_5) = \{v_2, v_4\}$. By the case assumption, $(v_3, v_6) \notin A(D)$, so $(v_6, v_3) \in A(D)$. Then $N_D^-(v_3) = \{v_1, v_6\}$, so $N_D^+(v_3) = \{v_2, v_5\}$. Therefore $N_D^-(v_2) = \{v_3, v_5\}$ and $N_D^+(v_2) = \{v_4, v_6\}$. Thus $N_D^-(v_4) = \{v_2, v_5\}$ and $N_D^+(v_4) = \{v_1, v_6\}$. Hence $N_D^-(v_6) = \{v_2, v_4\}$ and $N_D^+(v_6) = \{v_1, v_3\}$. Now D is uniquely determined and isomorphic to D_4 given in Figure 5 whose niche graph is a cycle of length 6.

Case 2. The two vertices in X_j are adjacent in G for some $j \in \{1, 2, 3\}$. Without loss of generality, we may assume that $j = 2$. By symmetry and Lemma 2.1, we may assume $\{v_3, v_4\} \subset N_D^+(v_1)$. Then

$$N_D^+(v_1) = \{v_3, v_4\} \tag{8}$$

and $N_D^-(v_1) = \{v_5, v_6\}$ by (7). If $N_D^+(v_2) = \{v_3, v_4\}$, then v_1 and v_2 are true twins and so, by Lemma 4.2, G contains a triangle, which contradicts the hypothesis that G is triangle-free. Therefore $N_D^+(v_2) \neq \{v_3, v_4\}$ and so $N_D^-(v_2) \cap \{v_3, v_4\} \neq \emptyset$. Then, there are two subcases to consider: $N_D^-(v_2) \cap \{v_3, v_4\} = \{v_3, v_4\}$; $|N_D^-(v_2) \cap \{v_3, v_4\}| = 1$.

Subcase 1. $N_D^-(v_2) \cap \{v_3, v_4\} = \{v_3, v_4\}$. Then $N_D^-(v_2) = \{v_3, v_4\}$ and $N_D^+(v_2) = \{v_5, v_6\}$ by (7), so

$$v_5 v_6 \in E(G). \tag{9}$$

Moreover, $|N_D^-(v_3) \cap \{v_5, v_6\}| = |N_D^-(v_4) \cap \{v_5, v_6\}| = 1$ by (7). If $N_D^-(v_3) \cap N_D^-(v_4) \cap \{v_5, v_6\} \neq \emptyset$, then v_3 and v_4 are true twins, which is a contradiction. Therefore $N_D^-(v_3) \cap N_D^-(v_4) \cap \{v_5, v_6\} = \emptyset$. By symmetry, we may assume that $N_D^-(v_3) \cap \{v_5, v_6\} = \{v_5\}$. Then $N_D^-(v_4) \cap \{v_5, v_6\} = \{v_6\}$. Therefore $\{v_1 v_5, v_1 v_6\} \subset E(G)$ and so, by (9), $v_1 v_5 v_6 v_1$ is a triangle in G, which contradicts the hypothesis.

Subcase 2. $|N_D^-(v_2) \cap \{v_3, v_4\}| = 1$. By symmetry, we may assume $N_D^-(v_2) \cap \{v_3, v_4\} = \{v_4\}$. Then $(v_2, v_3) \in A(D)$. Therefore $N_D^-(v_3) = \{v_1, v_2\}$ by (8) and so, by (7), $N_D^+(v_3) = \{v_5, v_6\}$. Moreover, $|N_D^+(v_2) \cap \{v_5, v_6\}| = 1$. By symmetry, we may assume $(v_2, v_5) \in A(D)$. Then $N_D^+(v_2) = \{v_3, v_5\}$. Therefore $N_D^-(v_2) = \{v_4, v_6\}$ and $N_D^-(v_5) = \{v_2, v_3\}$. Thus $N_D^+(v_5) = \{v_1, v_4\}$. Hence $N_D^-(v_4) = \{v_1, v_5\}$ and $N_D^+(v_4) = \{v_2, v_6\}$. Then $N_D^-(v_6) = \{v_3, v_4\}$ and $N_D^+(v_6) = \{v_1, v_2\}$. Now D is uniquely determined. It is easy to check that $\mathcal{N}(D)$ is isomorphic to the graph G_5 given in Figure 6. Therefore the "only if" part is true.

By the way, 3-partite tournaments whose niche graphs are isomorphic to the cycle C_6 and the graph G_5 were constructed in Cases 1 and 2, respectively. Thus the "if" part is true and this completes the proof. ■

Now we are ready to characterize connected triangle-free niche-realizable graphs.

Theorem 4.12. *Let G be a connected triangle-free graph with at least three vertices. Then (G, k) is niche-realizable for some integer $k \geq 3$ if and only if $k \in \{3, 4, 5\}$ and G is isomorphic to a graph belonging to the following set*

$$\begin{cases} \{P_3, P_4, P_5, C_5, C_6, G_4, G_5\} & \text{if } k = 3; \\ \{P_4, C_5\} & \text{if } k = 4; \\ \{C_5\} & \text{if } k = 5; \end{cases}$$

where G_4 and G_5 are the graphs given in Figure 6.

Proof. Let n denote the number of vertices in G. To show the "only if" part, suppose that (G, k) is niche-realizable for some integer $k \geq 3$. Then there exists a k-partite tournament D such that $\mathcal{N}(D) \cong G$. By Lemma 4.8, $k \leq 5$ and $n \leq 6$. If $n = 6$, then $k = 3$ and G is isomorphic to a cycle C_6 or the graph G_5 given in Figure 6 by Lemma 4.11. Now we suppose that $n \leq 5$. If G is a path or a cycle, then, by Lemmas 4.7 and 4.9, G is isomorphic to P_3, P_4, P_5, or C_5 when $k = 3$; G is isomorphic to P_4 or C_5 when $k = 4$; G is isomorphic to C_5 when $k = 5$.

Now we suppose that G is neither a path nor a cycle. By Theorem 2.7 and Corollary 4.5, G has stability number at most 3 and diameter at most 4. Therefore, by Lemma 4.10, G is isomorphic to the graph G_j given in Figure 6 for some $j \in \{1, 2, 3, 4\}$. Thus it remains to show that $k = 3$ and $G \cong G_4$. Since G is neither a path nor a cycle, there exists a vertex v_1 of degree at least 3 in G. If v_1 has degree at least 4, then $G \ncong G_i$ for each $1 \leq i \leq 4$. Therefore v_1 has degree 3. Since each of v_1 and its neighbors has indegree at most 2 and outdegree at most 2 by Lemma 2.3, v_1 is adjacent to at most two vertices if $d_D^+(v_1) = 0$ or $d_D^-(v_1) = 0$, which is a contradiction. Therefore $d_D^+(v_1) \geq 1$ and $d_D^-(v_1) \geq 1$. If $d_D^+(v_1) = 1$ and $d_D^-(v_1) = 1$, then v_1 has degree at most 2 for the same reason as the previous one, which is a contradiction. Therefore $d_D^+(v_1) \geq 2$ or $d_D^-(v_1) \geq 2$

and so $3 \leq d_D^+(v_1) + d_D^-(v_1)$. By Lemma 2.1, we may assume $d_D^+(v_1) \geq 2$ and then, by Lemma 2.3, $d_D^+(v_1) = 2$. Now we let

$$N_D^+(v_1) = \{v_3, v_4\} \tag{10}$$

and v_5 be an in-neighbor of v_1 in D. Suppose $n \leq 4$. Then $n = 4$ since degree of v_1 is 3. Therefore G is isomorphic to the graph G_1. However, two neighbors v_3 and v_4 of v_1 are adjacent in G by (10), which is a contradiction. Thus $n = 5$ and so

$$G \cong G_2,\ G \cong G_3,\ \text{or } G \cong G_4.$$

Let v_2 to be a vertex of G other than v_1, v_3, v_4, and v_5. Let X_i be the partite sets of D for each $1 \leq i \leq k$. We may assume that $v_1 \in X_1$. Since $k \geq 3$ and $n = 5$, $|X_1| = 1$, $|X_1| = 2$, or $|X_1| = 3$. Since $d_D^-(v_1) \geq 1$ and $d_D^+(v_1) \geq 2$, $|X_1| = 1$ or $|X_1| = 2$. Suppose, to the contrary, that $|X_1| = 1$. Then $X_1 = \{v_1\}$, so $N_D^-(v_1) = \{v_2, v_5\}$ and then $v_2v_5 \in E(G)$. By (10), $v_3v_4 \in E(G)$, so $G - v_1$ has at least two edges v_3v_4 and v_2v_5 not sharing end points in G, which cannot happen in any of G_2, G_3 and G_4. Thus $|X_1| = 2$ and

$$X_1 = \{v_1, v_2\}.$$

Then, since v_1 has three neighbors which form a stable set, each of v_3, v_4, and v_5 should be a common out-neighbor or in-neighbor of v_1 and a vertex adjacent to v_1. By the way, v_3 and v_4 are common out-neighbors and v_5 is a common in-neighbor by (10). Therefore $N_D^-(v_3)$, $N_D^-(v_4)$, and $N_D^+(v_5)$ are 2-element sets which differ from each other. Furthermore, since $v_3v_4 \in E(G)$, one of v_3 and v_4 is not adjacent to v_1. Without loss of generality, we may assume that v_3 is the vertex not adjacent to v_1.

Suppose, to the contrary, that v_3 and v_4 are in different partite sets. Since v_1 and v_3 are not adjacent in G, $(v_4, v_3) \in A(D)$ by (10). Then $N_D^-(v_3) = \{v_1, v_4\}$ by Lemma 2.3. Since v_5 is adjacent to v_1, v_1 and v_5 have common in-neighbor or out-neighbor. Since $N_D^-(v_1) = \{v_5\}$, v_1 and v_5 cannot have any common in-neighbor and so have a common out-neighbor. Since $N_D^+(v_1) = \{v_3, v_4\}$, v_3 and v_4 are possible common out-neighbors of v_1 and v_5. However, v_3 already has two in-neighbors distinct from v_5. Therefore v_4 must be a common out-neighbor of v_1 and v_5. Thus $(v_5, v_4) \in A(D)$ and so, by Lemma 2.3, $N_D^+(v_5) = \{v_1, v_4\}$. Therefore $N_D^-(v_3) = N_D^+(v_5)$, which is a contradiction. Thus v_3 and v_4 belong to the same partite set and $k = 3$. Let $X_2 = \{v_3, v_4\}$ and $X_3 = \{v_5\}$. Then, since v_1 and v_3 are not adjacent in G, $(v_3, v_5) \in A(D)$. Since $d_D^-(v_3) = 2$, $(v_2, v_3) \in A(D)$ and so $N_D^-(v_3) = \{v_1, v_2\}$. Since $N_D^-(v_3) \neq N_D^-(v_4)$ and $d_D^-(v_4) = 2$, $(v_5, v_4) \in A(D)$. Therefore $N_D^-(v_4) = \{v_1, v_5\}$ and so $N_D^+(v_4) = \{v_2\}$. Moreover, since $N_D^+(v_5) = \{v_1, v_4\}$, $(v_2, v_5) \in A(D)$. Now D is uniquely determined. Then, it is easy to check that $\mathcal{N}(D) \cong G_4$. Therefore the "only if" part is true.

The pairs $(P_3, 3)$, $(P_4, 3)$, $(P_5, 3)$ and $(P_4, 4)$ are niche-realizable by Lemma 4.7. The pairs $(C_5, 3)$, $(C_5, 4)$, $(C_5, 5)$, and $(C_6, 3)$ are niche-realizable by Lemma 4.9. The pair $(G_5, 3)$ is niche-realizable by Lemma 4.11. The pair $(G_4, 3)$ is niche-realizable as we have constructed a 3-partite tournament D whose niche graph is isomorphic to G_4 while showing the "only if" part of the statement. Hence the "if" part is true. ■

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