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ℓ-COVERING k-HYPERGRAPHS ARE QUASI-EULERIAN

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Abstract

An *Euler tour* in a hypergraph H is a closed walk that traverses each edge of H exactly once, and an *Euler family* is a family of closed walks that jointly traverse each edge of H exactly once. An *ℓ-covering k-hypergraph*, for $2 \leq \ell < k$, is a k-uniform hypergraph in which every ℓ-subset of vertices lie together in at least one edge.

In this paper we prove that every ℓ-covering k-hypergraph, for $k \geq 3$, admits an Euler family.

Keywords: ℓ-covering k-hypergraph, Euler family, Euler tour, Lovász's (g, f)-factor theorem.

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1. Introduction

The complete characterization of graphs that admit an Euler tour is a classic result covered by any introductory graph theory course. The concept naturally extends to hypergraphs; that is, an Euler tour of a hypergraph is a closed walk that traverses every edge exactly once. However, the study of eulerian hypegraphs is a much newer and largely unexplored territory.

The first results on Euler tours in hypergraphs were obtained by Lonc and Naroski [4]. Most notably, they showed that the problem of existence of an Euler tour is NP-complete on the set of k-uniform hypergraphs, for any $k \geq 3$, as well as when restricted to a particular subclass of 3-uniform hypergraphs.

Bahmanian and Šajna [2] attempted a systematic study of eulerian properties of general hypergraphs; some of their techniques and results will be used in this paper. In particular, they introduced the notion of an *Euler family* — a collection of closed walks that jointly traverse each edge exactly once — and showed that the problem of existence of an Euler family is polynomial on the class of all hypergraphs.

In this paper, we define an *ℓ-covering k-hypergraph*, for $2 \leq \ell < k$, to be a non-empty k-uniform hypergraph in which every ℓ-subset of vertices appear together in at least one edge.

In [2], the authors proved that every 2-covering 3-hypergraph with at least two edges admits an Euler family, and the present authors gave a short proof [6] to show that every triple system — that is, a 3-uniform hypergraph in which every pair of vertices lie together in the same number of edges — admits an Euler tour as long as it has at least two edges. Most recently, the present authors proved the following result.

Theorem 1 [7]**.** *Let $k \geq 3$, and let H be a $(k-1)$-covering k-hypergraph. Then H admits an Euler tour if and only if it has at least two edges.*

In this paper, we aim to extend Theorem 1 to all ℓ-covering k-hypergraphs. Our main result is as follows.

Theorem 2. *Let ℓ and k be integers, $2 \leq \ell < k$, and let H be an ℓ-covering k-hypergraph. Then H admits an Euler family if and only if it has at least two edges.*

As the concept of an Euler family is a relaxation of the concept of an Euler tour, the conclusion of Theorem 2 is weaker than that of Theorem 1; however, it holds for a much larger class of hypergraphs.

We prove Theorem 2 by induction on ℓ. The base case $\ell = 2$ is stated as Theorem 12; its proof is essentially a counting argument and requires most of the work. The main part of the proof is presented in Section 5, while some special cases and technical details are handled in Sections 3 and 4. In particular, in Section 4, using the Lovász (g, f)-factor theorem, we develop a sufficient condition for a k-uniform hypergraph without cut edges to admit an Euler family.

We remark that we believe that Theorem 1 actually holds for all ℓ-covering k-hypergraphs. As in the proof of Theorem 2, we can see that it would suffice to prove it for $\ell = 2$ and $k \geq 4$. Hence we propose the following conjecture.

Conjecture 3. *Let k be an integer, $k \geq 4$, and let H be a 2-covering k-hypergraph with at least two edges. Then H admits an Euler tour.*

To examine the conjecture for small values of the parameters, we randomly (in a loose sense) generated over 10^5 examples of 2-covering k-hypergraphs of order n for each parameter pair (k, n) with $4 \leq k \leq 7$ and $2k-2 \leq n \leq 13$. (The case $n < 2k-2$ is confirmed using Lemma 7.) Our computer search shows that all of the generated hypergraphs admit Euler tours. Proving Conjecture 3, however, is presently beyond our reach.

2. Preliminaries

We use hypergraph terminology established in [1, 2], which applies to loopless graphs as well. Any graph theory terms not explained here can be found in [3].

A *hypergraph* H is a pair (V, E), where V is a non-empty set, and E is a multiset of elements from 2^V. The elements of $V = V(H)$ and $E = E(H)$ are called the *vertices* and *edges* of H, respectively. The *order* of H is $|V|$, and the *size* is $|E|$. A hypergraph of order 1 is called *trivial*, and a hypergraph with no edges is called *empty*.

Distinct vertices u and v in a hypergraph $H = (V, E)$ are called *adjacent* (or *neighbours*) if they lie in the same edge, while a vertex v and an edge e are said to be *incident* if $v \in e$. The *degree* of v in H, denoted $\deg_H(v)$, is the number of edges of H incident with v. An edge e is said to *cover* the vertex pair $\{u, v\}$ if $\{u, v\} \subseteq e$. A hypergraph H is called *k-uniform* if every edge of H has cardinality k.

Definition. Let ℓ and k be integers, $2 \leq \ell < k$. An *ℓ-covering k-hypergraph* is a k-uniform hypergraph in which every ℓ-subset of vertices lie together in at least one edge.

The *incidence graph* of a hypergraph $H = (V, E)$ is a bipartite simple graph G with vertex set $V \cup E$ and bipartition $\{V, E\}$ such that vertices $v \in V$ and $e \in E$ of G are adjacent if and only if v is incident with e in H. The elements of V and E are called *v-vertices* and *e-vertices* of G, respectively.

A hypergraph $H' = (V', E')$ is called a *subhypergraph* of the hypergraph $H = (V, E)$ if $V' \subseteq V$ and $E' = \{e \cap V' : e \in E''\}$ for some submultiset E'' of E. For $e \in E$, the symbol $H \backslash e$ denotes the subhypergraph $(V, E - \{e\})$ of H, and for $v \in V$, the symbol $H - v$ denotes the subhypergraph $(V - \{v\}, E')$ where $E' = \{e - \{v\} : e \in E, e - \{v\} \neq \emptyset\}$.

A (v_0, v_k)*-walk* in H is a sequence $W = v_0 e_1 v_1 e_2 \cdots e_k v_k$ such that $v_0, \ldots, v_k \in V$; $e_1, \ldots, e_k \in E$; and $v_{i-1}, v_i \in e_i$ with $v_{i-1} \neq v_i$ for all $i = 1, \ldots, k$. A walk is said to *traverse* each of the vertices and edges in the sequence. The vertices $v_0, v_1, \ldots, v_k$ are called the *anchors* of W. If $e_1, e_2, \ldots, e_k$ are pairwise distinct, then W is called a *trail* (*strict trail* in [1, 2]); if $v_0 = v_k$ and $k \geq 2$, then W is *closed*.

A hypergraph H is *connected* if every pair of vertices are connected in H; that is, if for any pair $u, v \in V(H)$, there exists a (u, v)-walk in H. A *connected* *component* of H is a maximal connected subhypergraph of H without empty edges. The number of connected components of H is denoted by $\mathrm{c}(H)$. We call $v \in V(H)$ a *cut vertex* of H, and $e \in E(H)$ a *cut edge* of H, if $\mathrm{c}(H - v) > \mathrm{c}(H)$ and $\mathrm{c}(H\backslash e) > \mathrm{c}(H)$, respectively.

An *Euler family* of a hypergraph H is a collection of pairwise anchor-disjoint and edge-disjoint closed trails that jointly traverse every edge of H, and an *Euler tour* is a closed trail that traverses every edge of H. A hypergraph that is either empty or admits an Euler tour (family) is called *eulerian (quasi-eulerian).* Note that an Euler tour corresponds to an Euler family of cardinality 1, so every eulerian hypergraph is also quasi-eulerian.

The following theorem allows us to determine whether a hypergraph is eulerian or quasi-eulerian from its incidence graph.

Theorem 4 [2, Theorem 2.18]**.** *Let H be a hypergraph and G its incidence graph. Then the following hold.*

(1) *H is quasi-eulerian if and only if G has a spanning subgraph G' such that $\deg_{G'}(e) = 2$ for all $e \in E(H)$, and $\deg_{G'}(v)$ is even for all $v \in V(H)$.*

(2) *H is eulerian if and only if G has a spanning subgraph G' with at most one non-trivial connected component such that $\deg_{G'}(e) = 2$ for all $e \in E(H)$, and $\deg_{G'}(v)$ is even for all $v \in V(H)$.*

3. Technical Lemmas

In this section, we take care of some special cases and prove some technical results that will aid in the proof of our base case, Theorem 12.

Lemma 5. *Let $k \geq 4$, and let H be a 2-covering k-hypergraph with at least 2 edges. Then H has no cut edges.*

Proof. Suppose e is a cut edge of H. Then there exist vertices $u, v \in e$ that are disconnected in $H\backslash e$. Since H has at least 2 edges, it must be that $k \neq |V(H)|$ and $e \neq V(H)$. Hence there exists $w \in V(H) - e$. Let e_1, e_2 be edges of H containing u and w, and v and w, respectively. As $e \notin \{e_1, e_2\}$, we can see that ue_1we_2v is a (u, v)-walk in $H\backslash e$, a contradiction. ∎

Lemma 6. *Let $k \geq 4$, and let H be a 2-covering k-hypergraph of order $n > \frac{3k}{2}$ and size $m \geq 2$. Then $m \geq 2\left\lfloor \frac{n+3}{k} \right\rfloor$.*

Proof. If $n \leq 2k - 4$, then $2\left\lfloor \frac{n+3}{k} \right\rfloor \leq 2 \leq m$. Hence assume $n \geq 2k - 3$.

Suppose first that $n \geq 3k - 3$. Since there are $\binom{n}{2}$ pairs of vertices to cover, and each edge covers $\binom{k}{2}$ pairs, we know that $m \geq \frac{n(n-1)}{k(k-1)}$. As $k \geq 4$, we have

$$\begin{aligned} m &\geq \frac{n(n-1)}{k(k-1)} \geq \frac{(3k-3)(n-1)}{k(k-1)} = \frac{3(n-1)}{k} = \frac{2n+n-3}{k} \\ &\geq \frac{2n+3k-6}{k} \geq \frac{2n+6}{k} \geq 2\left\lfloor \frac{n+3}{k} \right\rfloor. \end{aligned}$$

Finally, assume $2k-3 \leq n \leq 3k-4$. As $2\left\lfloor \frac{n+3}{k} \right\rfloor \leq 4$, it suffices to show that $m \geq 4$. Suppose $m \leq 3$. Since H is a 2-covering k-hypergraph with $n > k$ and $m \geq 2$, every vertex has degree at least 2. Thus

$$2n \leq \sum_{v \in V(H)} \deg(v) = km \leq 3k$$

and $n \leq \frac{3k}{2}$, contradicting the assumption that $n > \frac{3k}{2}$.

Therefore, in all cases we have $m \geq 2\left\lfloor \frac{n+3}{k} \right\rfloor$. ■

Lemma 7. *Let H be a hypergraph with $|E(H)| \geq 2$ satisfying the following.*

- *For all $e, f \in E(H)$, we have $|e \cap f| \geq 2$; and*
- *there exist distinct $e, f \in E(H)$ such that $|e \cap f| \geq 3$.*

Then H is eulerian.

Proof. Let $E(H) = \{e_1, \ldots, e_m\}$ and assume e_1 and e_m are distinct edges such that $|e_1 \cap e_m| \geq 3$. Take any $v_1 \in e_1 \cap e_2$. For $i = 2, \ldots, m-1$, let v_i be a vertex in $(e_i \cap e_{i+1}) - \{v_{i-1}\}$, and let $v_0 \in (e_1 \cap e_m) - \{v_1, v_{m-1}\}$. It is easy to verify that $v_0 e_1 v_1 \cdots v_{m-1} e_m v_0$ is an Euler tour of H. ■

Corollary 8. *Let H be a 2-covering k-hypergraph of order n. If $n \leq 2k-3$ or $(k, n) = (4, 6)$, then H is eulerian.*

Proof. If $n \leq 2k-3$, then every pair of edges $e, f \in E(H)$ satisfies $|e \cap f| \geq 3$, so H is eulerian by Lemma 7.

Assume now that $(k, n) = (4, 6)$. For all $e, f \in E(H)$, we have $|e \cap f| \geq 2$. If there exist distinct edges $e, f \in E(H)$ such that $|e \cap f| \geq 3$, then H is eulerian by Lemma 7. Hence assume $|e \cap f| = 2$ for all $e, f \in E(H)$, and let $V(H) = \{v_1, \ldots, v_6\}$. It is not difficult to see that we must have $E(H) = \{e_1, e_2, e_3\}$ where, without loss of generality, the edges are $e_1 = v_1v_2v_3v_4$, $e_2 = v_1v_2v_5v_6$, and $e_3 = v_3v_4v_5v_6$. It follows that $W = v_3e_1v_2e_2v_5e_3v_3$ is an Euler tour of H. ■

Lemma 9. *Let $n, k, q \in \mathbb{Z}^+$ be such that $n \geq qk$. Let*

$$S = \big\{(x_1, \ldots, x_q) \in (\mathbb{Z}^+)^q : x_1 + \cdots + x_q = n, x_i \geq k \text{ for all } i\big\},$$

and define $f : S \to \mathbb{Z}^+$ by $f(x_1, \ldots, x_q) = \binom{x_1}{2} + \cdots + \binom{x_q}{2}$. Then f attains its maximum on S at the point $\big(k, \ldots, k, n-k(q-1)\big)$.

Proof. Since the domain S is finite, function f indeed attains a maximum on S.

Let $\mathbf{x} = (x_1, \dots, x_q) \in S$ be such that $f(\mathbf{x})$ is maximum. By symmetry of f, we may assume that $x_1 \le x_2 \le \dots \le x_q$. As $x_1 \ge k$ and $x_q = n-(x_1+\dots+x_{q-1})$, we observe that $x_q \le n - k(q-1)$.

Suppose that $x_q < n - k(q-1)$. Then there exists $i \in \{1, \dots, q-1\}$ such that $x_i > k$. Let i be the smallest index with this property, and let

$$\mathbf{y} = \big(x_1, \dots, x_{i-1}, x_i - 1, x_{i+1}, \dots, x_{q-1}, x_q + 1\big).$$

Then $\mathbf{y} \in S$ and

$$\begin{aligned}
f(\mathbf{y}) &= \sum_{\substack{j=1\\ j\neq i}}^{q-1} \binom{x_j}{2} + \binom{x_i - 1}{2} + \binom{x_q + 1}{2} \\
&= \sum_{\substack{j=1\\ j\neq i}}^{q-1} \binom{x_j}{2} + \frac{x_i(x_i-1)}{2} - \frac{2(x_i-1)}{2} + \frac{x_q(x_q-1)}{2} + \frac{2x_q}{2} \\
&= \sum_{j=1}^{q} \binom{x_j}{2} + (x_q - x_i + 1) > f(\mathbf{x}),
\end{aligned}$$

contradicting the choice of $\mathbf{x}$.

Hence $x_q = n - k(q-1)$, and consequently $x_1 = \dots = x_{q-1} = k$. Thus f attains its maximum on S at the point $\mathbf{x} = \big(k, \dots, k, n - k(q-1)\big)$ as claimed. ■

4. A Sufficient Condition

In this section, we state and prove Proposition 11, which gives a sufficient condition for a k-uniform hypergraph to admit an Euler family. This sufficient condition will be our main tool in the proof of Theorem 12. It is based on the (g, f)-factor theorem by Lovász [5], stated below as Theorem 10.

For a graph G and functions $f, g : V(G) \to \mathbb{N}$, a *$(g, f)$-factor* of G is a spanning subgraph F of G such that $g(x) \le \deg_F(x) \le f(x)$ for all $x \in V(G)$. An *f-factor* is simply an (f, f)-factor. For any sets $U, W \subseteq V(G)$, let $\varepsilon_G(U, W)$ denote the number of edges of G with one endpoint in U and the other in W.

Theorem 10 [5]**.** *Let $G = (V, E)$ be a graph and $f, g : V \to \mathbb{N}$ be functions such that $g(x) \le f(x)$ and $g(x) \equiv f(x) \pmod 2$ for all $x \in V$. Then G has a (g, f)-factor F such that $\deg_F(x) \equiv f(x) \pmod 2$ for all $x \in V$ if and only if, for all disjoint $S, T \subseteq V$, we have*

$$\sum_{x\in S} f(x) + \sum_{x\in T} (\deg_G(x) - g(x)) - \varepsilon_G(S, T) - q(S, T) \ge 0, \tag{1}$$

where $q(S,T)$ is the number of connected components C of $G-(S\cup T)$ such that

$$\sum_{x\in V(C)} f(x)+\varepsilon_G(V(C),T)\equiv 1 \pmod 2.$$

Proposition 11. *Let $k\geq 3$, and let $H=(V,E)$ be a k-uniform hypergraph of order n and size m. Let G be the incidence graph of H, and G^* the graph obtained from G by appending $2(m+n)^2$ loops to every v-vertex.*

Assume that H has no cut edges and that for all $X\subseteq E$ with $|X|\geq 2$, we have that $|X|\geq 2\left\lfloor\frac{\mathrm{c}(G^*-X)+3}{k}\right\rfloor$. Then H is quasi-eulerian.

Proof. Let $r=2(m+n)^2$, and define $f:V(G^*)\to\mathbb{Z}$ by

$$f(x)=\begin{cases} r & \text{if } x\in V,\\ 2 & \text{if } x\in E.\end{cases}$$

We shall use Theorem 10 to show that G^* has an (f,f)-factor, so let $S,T\subseteq V(G^*)$ be disjoint sets, and denote

$$\gamma(S,T)=\sum_{x\in S} f(x)+\sum_{x\in T}(\deg_{G^*}(x)-f(x))-\varepsilon_{G^*}(S,T)-q(S,T),$$

where $q(S,T)$ is the number of connected components C of $G^*-(S\cup T)$ such that $\varepsilon_{G^*}(V(C),T)$ is odd. Observe that Condition (1) for G^* with $g=f$ is equivalent to $\gamma(S,T)\geq 0$.

Since G is a subgraph of $K_{n,m}$, we have $\varepsilon_{G^*}(S,T)\leq mn$ and $q(S,T)\leq m+n$, and therefore $\varepsilon_{G^*}(S,T)+q(S,T)\leq (m+n)^2=\frac{r}{2}$. In addition, we have $\deg_{G^*}(x)-f(x)\geq r$ for all $x\in V$, and $\deg_{G^*}(x)-f(x)\geq k-2$ for all $x\in E$.

Case 1. $(S\cup T)\cap V\neq\emptyset$. If $S\cap V\neq\emptyset$, then $\sum_{x\in S} f(x)\geq r$, and if $T\cap V\neq\emptyset$, then $\sum_{x\in T}(\deg_{G^*}(x)-f(x))\geq r$. Thus, in both cases

$$\begin{aligned}\gamma(S,T)&=\left(\sum_{x\in S} f(x)+\sum_{x\in T}\big(\deg_{G^*}(x)-f(x)\big)\right)-\Big(\varepsilon_{G^*}(S,T)+q(S,T)\Big)\\ &\geq r-\frac{r}{2}\geq 0.\end{aligned}$$

Case 2. $(S\cup T)\cap V=\emptyset$. Then $\varepsilon_{G^*}(S,T)=0$ since $S\cup T\subseteq E$. First, suppose $T=\emptyset$. Then $\varepsilon_{G^*}(V(C),T)=0$ for all connected components C of $G^*-(S\cup T)$, so $q(S,T)=0$. Hence $\gamma(S,T)=\sum_{x\in S} f(x)\geq 0$.

Next, suppose $S=\emptyset$ and $|T|=1$. Then $S\cup T=\{e\}$ for some $e\in E$. By assumption, edge e is not a cut edge of H and hence by [1, Theorem 3.23], e-vertex e is not a cut vertex of G^*, and $G^*-(S\cup T)$ is connected. It follows that $q(S,T)\leq 1$ and

$$\gamma(S,T)=\big(\deg_{G^*}(e)-f(e)\big)-q(S,T)\geq (k-2)-1\geq 0.$$

We may now assume that $T \neq \emptyset$ and $|S \cup T| \geq 2$. Since each connected component C of $G^* - (S \cup T)$ with $\varepsilon_{G^*}(V(C), T)$ odd corresponds to at least one edge incident with a vertex in T, the number of such components is at most $k|T|$. Hence $q(S,T) \leq \min\{\mathrm{c}(G^* - (S \cup T)), k|T|\}$, and

$$\begin{aligned} \gamma(S,T) &= 2|S| + (k-2)|T| - q(S,T) \\ &\geq 2|S \cup T| + (k-4)|T| - \min\{\mathrm{c}(G^* - (S \cup T)), k|T|\}. \end{aligned} \tag{2}$$

Define $t = \left\lfloor \frac{\mathrm{c}(G^* - (S \cup T)) + 3}{k} \right\rfloor$, so that

$$kt - 3 \leq \mathrm{c}(G^* - (S \cup T)) \leq kt + k - 4.$$

If $|T| \geq t + 1$, then $\min\{\mathrm{c}(G^* - (S \cup T)), k|T|\} = \mathrm{c}(G^* - (S \cup T)) \leq kt + k - 4$, so Inequality (2) yields

$$\gamma(S,T) \geq 2|S \cup T| + (k-4)(t+1) - (kt + k - 4) = 2|S \cup T| - 4t.$$

The same bound is obtained if $|T| \leq t$: in this case, we have $\min\{\mathrm{c}(G^* - (S \cup T)), k|T|\} \leq k|T|$, so that (2) yields

$$\gamma(S,T) \geq 2|S \cup T| + (k-4)|T| - k|T| = 2|S \cup T| - 4|T| \geq 2|S \cup T| - 4t.$$

In both cases, as $S \cup T \subseteq E$ and $|S \cup T| \geq 2$, the assumption of the proposition implies $|S \cup T| \geq 2 \left\lfloor \frac{\mathrm{c}(G^* - (S \cup T)) + 3}{k} \right\rfloor = 2t$, so that $\gamma(S,T) \geq 0$.

Therefore, $\gamma(S,T) \geq 0$ for all disjoint $S, T \subseteq V(G^*)$, and by Theorem 10, we conclude that G^* has an (f,f)-factor F. Deleting the loops of F, we obtain a spanning subgraph F' of G in which all v-vertices have even degree and all e-vertices have degree 2. Thus H admits an Euler family by Theorem 4. ■

5. Proof of the Main Result

We shall now prove our main result, Theorem 2. We use induction on ℓ, and most of the work is required to prove the basis of induction, which we state below as Theorem 12.

Theorem 12. *Let $k \geq 4$, and let H be a 2-covering k-hypergraph with at least two edges. Then H is quasi-eulerian.*

Proof. Let $H = (V, E)$ with $n = |V|$ and $m = |E|$. If $n \leq 2k - 3$, then H is eulerian by Corollary 8, so we may assume that $n \geq 2k - 2$.

If $n \leq \frac{3k}{2}$, it then follows that $(k, n) = (4, 6)$. Again, H is eulerian by Corollary 8. Hence $n > \frac{3k}{2}$, and Lemma 6 implies that $m \geq 2 \left\lfloor \frac{n+3}{k} \right\rfloor$.

In the rest of the proof we show that H satisfies the conditions of Proposition 11.

Let G^* be the graph obtained from the incidence graph of H by adjoining $2(m+n)^2$ loops to every v-vertex.

Fix any $X \subseteq E$ with $|X| \geq 2$, and denote $q = \mathrm{c}(G^* - X)$.

Suppose that $|X| < 2\left\lfloor \frac{q+3}{k} \right\rfloor$. If $q \leq 2k-4$, then this supposition implies that $|X| < 2$, a contradiction. Hence we may assume that $q \geq 2k-3$, and hence $q \geq 5$. Moreover, our supposition implies

$$|X| \leq 2\frac{q+3}{k} - 1. \tag{3}$$

Let ℓ denote the number of v-vertices that are isolated in $G^* - X$.

Case 1. $\ell \geq 1$. If $\ell = n$, then $X = E$, $q = n$, and $|X| = |E| \geq 2\left\lfloor \frac{n+3}{k} \right\rfloor = 2\left\lfloor \frac{q+3}{2} \right\rfloor$, contradicting our assumption on X. Thus we may assume $\ell < n$, and hence $\ell < q$.

Since $G^* - X$ has $q - \ell$ non-trivial connected components, each with at least k v-vertices, we have

$$n \geq \ell + k(q - \ell). \tag{4}$$

Since $q > \ell$, this inequality also implies

$$n \geq \ell + k. \tag{5}$$

Let S be the set of pairs $\{u, v\}$ of v-vertices such that u is isolated in $G^* - X$, and v is not. Then $|S| = \ell(n - \ell)$. Observe that every edge of H covers at most $\frac{k^2}{4}$ pairs from S, which implies that $|X| \geq \frac{\ell(n-\ell)}{\frac{k^2}{4}}$. Combining this inequality with (3), we obtain

$$\frac{4\ell(n-\ell)}{k^2} \leq \frac{2q+6-k}{k}. \tag{6}$$

Substituting $q \leq \ell + \frac{n-\ell}{k}$ from Inequality (4) and rearranging yields

$$n(4\ell - 2) \leq 4\ell^2 - k^2 + 2\ell k - 2\ell + 6k.$$

Further substituting $n \geq \ell + k$ from (5) and isolating ℓ, we obtain $\ell \leq 4 - \frac{k}{2}$, which implies $\ell \in \{1, 2\}$ as $k \geq 4$.

However, if on the left-hand side of Inequality (6) we apply $\frac{n-\ell}{k} \geq q - \ell$ from (4) and simplify, then we obtain

$$(4\ell - 2)q - 4\ell^2 \leq 6 - k \leq 2.$$

Now substituting either $\ell = 1$ or $\ell = 2$ yields $q \le 3$, a contradiction.

Case 2. $\ell = 0$. Let $C_1, C_2, \ldots, C_q$ be the connected components of $G^* - X$, and let n_i denote the number of v-vertices of C_i. Note that $n_i \ge k$ for all i.

The number of pairs of v-vertices that lie in distinct connected components of $G^* - X$ is $\binom{n}{2} - \sum_{i=1}^{q} \binom{n_i}{2}$, and these pairs must all be covered by the edges of X. As $n \ge qk$, $n_1 + \cdots + n_q = n$, and $n_i \ge k$, for all i, we know that $\sum_{i=1}^{q} \binom{n_i}{2} \le (q-1)\binom{k}{2} + \binom{n-k(q-1)}{2}$ by Lemma 9. Therefore,

$$\binom{n}{2} - \sum_{i=1}^{q} \binom{n_i}{2} \ge \binom{n}{2} - (q-1)\binom{k}{2} - \binom{n-k(q-1)}{2}.$$

Since each edge of X covers up to $\binom{k}{2}$ pairs of v-vertices in distinct connected components, we deduce that

$$|X| \ge \frac{\binom{n}{2} - (q-1)\binom{k}{2} - \binom{n-k(q-1)}{2}}{\binom{k}{2}}.$$

On the other hand, by (3), we have $|X| \le \frac{2q+6-k}{k}$, so

$$\frac{\binom{n}{2} - (q-1)\binom{k}{2} - \binom{n-k(q-1)}{2}}{\binom{k}{2}} \le \frac{2q+6-k}{k}. \tag{7}$$

We now substitute $x = q - 1$, noting that $x \ge 4$ as $q \ge 5$. Rearranging Inequality (7), we then obtain

$$2kxn \le k^2x^2 + (k^2 + 2k - 2)x - (k-8)(k-1).$$

Applying $n \ge qk = (x+1)k$ further yields

$$k^2x^2 + (k^2 - 2k + 2)x + (k-8)(k-1) \le 0.$$

Denote the left-hand side by $f(x) = ax^2 + bx + c$, where $a = k^2$, $b = k^2 - 2k + 2$, and $c = (k-8)(k-1)$, and observe that $a, b > 0$ as $k \ge 4$. If $b^2 - 4ac < 0$, then $f(x) > 0$ for all x, a contradiction. Hence assume $b^2 - 4ac \ge 0$. Let x_2 be the larger of the two roots of $f(x) = 0$. If $x_2 < 4$, then $f(x) > 0$ for all $x \ge 4$, a contradiction. Hence we must have

$$4 \le \frac{-b + \sqrt{b^2 - 4ac}}{2a}.$$

Since $a, b > 0$, it is straightforward to show that $16a + 4b + c \le 0$ follows. However,

$$16a + 4b + c = k(21k - 17) + 16 > 0,$$

a contradiction.

Since each case leads to a contradiction, we conclude that $|X| \geq \left\lfloor \frac{c(G^*-X)+3}{k} \right\rfloor$. By Lemma 5, hypergraph H has no cut edges, so we may apply Proposition 11 to conclude that H is quasi-eulerian. ∎

We are now ready to prove our main result, restated below.

Theorem 2. *Let ℓ and k be integers, $2 \leq \ell < k$, and let H be an ℓ-covering k-hypergraph. Then H is quasi-eulerian if and only if it has at least two edges.*

Proof. Since H is non-empty, and since a hypergraph with a single edge does not admit a closed trail, necessity is easy to see.

To prove sufficiency, for $s \geq 1$ and $\ell \geq 2$, define the proposition $P_s(\ell)$ as "Every ℓ-covering $(\ell+s)$-hypergraph with at least two edges is quasi-eulerian". Theorem 1 implies that $P_1(\ell)$ holds for all $\ell \geq 2$. Hence fix any $s \geq 2$.

We prove $P_s(\ell)$ by induction on ℓ. As $\ell + s \geq 4$, the basis of induction, $P_s(2)$, follows from Theorem 12. Suppose that, for some $\ell \geq 2$, the proposition $P_s(\ell)$ holds; that is, every ℓ-covering $(\ell+s)$-hypergraph with at least two edges is quasi-eulerian.

Let $H = (V, E)$ be an $(\ell+1)$-covering $\big((\ell+1)+s\big)$-hypergraph with $|E| \geq 2$. Fix any $v \in V$ and let $V^* = V - \{v\}$. Define a mapping $\varphi : E \to 2^{V^*}$ by

$$\varphi(e) = e - \{v\} \quad \text{if } v \in e,$$

and otherwise,

$$\varphi(e) = e - \{u\} \quad \text{for any } u \in e.$$

Then let $E^* = \{\varphi(e) : e \in E\}$ and $H^* = (V^*, E^*)$, so that φ is a bijection from E to E^*. It is straightforward to verify that H^* is an ℓ-covering $(\ell+s)$-hypergraph. As $|E^*| = |E| \geq 2$, by induction hypothesis, hypergraph H^* admits an Euler family $\mathcal{F}^*$. In each closed trail in $\mathcal{F}^*$, replace each $e \in E^*$ with $\varphi^{-1}(e)$ to obtain a set $\mathcal{F}$ of closed trails of H. It is not difficult to verify that $\mathcal{F}$ is an Euler family of H, so $P_s(\ell+1)$ follows.

By induction, we conclude that $P_s(\ell)$ holds for all $\ell \geq 2$, and any $s \geq 1$. Therefore, every ℓ-covering k-hypergraph with at least two edges is quasi-eulerian. ∎

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References

[1] M.A. Bahmanian and M. Šajna, *Connection and separation in hypergraphs*, Theory Appl. Graphs **2** (2015) Art. 5.
https://doi.org/10.20429/tag.2015.020205

[2] M.A. Bahmanian and M. Šajna, *Quasi-eulerian hypergraphs*, Electron. J. Combin. **24** (2017) #P3.30.
https://doi.org/10.37236/6361

[3] J.A. Bondy and U.S.R. Murty, Graph Theory (Springer-Verlag, London, 2008).

[4] Z. Lonc and P. Naroski, *On tours that contain all edges of a hypergraph*, Electron. J. Combin. **17** (2010) #R144.
https://doi.org/10.37236/416

[5] L. Lovász, *The factorization of graphs* II, Acta Math. Acad. Sci. Hungar. **23** (1972) 223–246.
https://doi.org/10.1007/BF01889919

[6] M. Šajna and A. Wagner, *Triple systems are eulerian*, J. Combin. Des. **25** (2017) 185–191.
https://doi.org/10.1002/jcd.21536

[7] M. Šajna and A. Wagner, *Covering hypergraphs are eulerian*.
arXiv:2101.04561

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