

TOTAL AND PAIRED DOMINATION STABILITY IN PRISMS

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Abstract

A set D of vertices in an isolate-free graph is a total dominating set if every vertex is adjacent to a vertex in D . If the set D has the additional property that the subgraph induced by D contains a perfect matching, then D is a paired dominating set of G . The total domination number $\gamma_t(G)$ and the paired domination number $\gamma_{pr}(G)$ of a graph G are the minimum cardinalities of a total dominating set and a paired dominating set of G , respectively. The total domination stability (respectively, paired domination stability) of G , denoted $st_{\gamma_t}(G)$ (respectively, $st_{\gamma_{pr}}(G)$), is the minimum size of a non-isolating set of vertices in G whose removal changes the total domination number (respectively, paired domination number). In this paper, we study total and paired domination stability in prisms.

Keywords: total domination stability, paired domination stability, prism, hypercube.

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1. INTRODUCTION

A *dominating set* of a graph G with vertex set $V(G)$ is a set D of vertices of G such that every vertex in $V(G) \setminus D$ is adjacent to a vertex in D . The *domination number* of G , denoted by $\gamma(G)$, is the minimum cardinality of a dominating set. A dominating set of G having cardinality $\gamma(G)$ is called a γ -*set* of G .

The concept of domination stability in graphs was introduced in 1983 by Bauer, Harary, Nieminen and Suffel [5] and has been studied, for example, in [18]. Stability for other domination type parameters has been studied in the literature. For example, total domination stability, paired domination stability, 2-rainbow domination stability, exponential domination stability, and Roman domination stability are studied in [1, 4, 7, 11, 14].

An *isolate-free graph* is a graph with no isolated vertex. A *total dominating set*, abbreviated TD-set, of an isolate-free graph G is a set D of vertices of G such that every vertex in $V(G)$ is adjacent to at least one vertex in D . The *total domination number* of G , denoted by $\gamma_t(G)$, is the minimum cardinality of a TD-set of G . A TD-set of G having cardinality $\gamma_t(G)$ is called a γ_t -*set* of G . A vertex v is *totally dominated* by a set D in G if the vertex v has a neighbor in D . Total domination in graphs is well studied in the literature [13].

The total version of domination stability was first studied by Henning and Krzywkowski [11]. To define this formally, we call a set $S \subseteq V(G)$ of vertices in G a *non-isolating set* if the graph $G - S$ is an isolate-free graph, where $G - S$ denotes the graph obtained from G by removing S and all edges incident with vertices in S . Let $\text{NI}(G)$ denote the set of all non-isolating sets of vertices of G . The γ_t^- -stability of G , denoted $\text{st}_{\gamma_t}^-(G)$, is the minimum size of a non-isolating set S of vertices in G whose removal decreases the total domination number. The γ_t^+ -stability of G , denoted $\text{st}_{\gamma_t}^+(G)$, is the minimum size of a non-isolating set of vertices in G whose removal increases the total domination number, if such a set exists. If no such non-isolating set exists whose removal increases the total domination number, we define $\text{st}_{\gamma_t}^+(G) = \infty$. As a trivial example, $\text{st}_{\gamma_t}^-(P_7) = 2$ while $\text{st}_{\gamma_t}^+(P_7) = \infty$. The *total domination stability* of G (or the γ_t -*stability*), denoted $\text{st}_{\gamma_t}(G)$, is the minimum size of a non-isolating set S of vertices in G whose removal changes the total domination number. Thus,

$$\text{st}_{\gamma_t}(G) = \min_{S \in \text{NI}(G)} \{|S| : \gamma_t(G - S) \neq \gamma_t(G)\} = \min \{\text{st}_{\gamma_t}^-(G), \text{st}_{\gamma_t}^+(G)\}.$$

A *paired dominating set*, abbreviated PD-set, of an isolate-free graph $G = (V, E)$ is a set $D \subseteq V$ such that every vertex of G is adjacent to some vertex in D and the induced subgraph $G[D]$ contains a perfect matching M (not necessarily induced). Two vertices joined by an edge of M are said to be *paired*. The *paired domination number* of G , denoted by $\gamma_{\text{pr}}(G)$, is the minimum cardinality of a PD-set of G , and a PD-set of G having cardinality $\gamma_{\text{pr}}(G)$ is called a γ_{pr} -*set* of G .

Necessarily, the paired domination number of a graph is an even integer. Since every PD-set is a TD-set, we note that $\gamma_t(G) \leq \gamma_{\text{pr}}(G)$ for all graphs G without isolated vertices. The concept of paired domination was first introduced and studied by Haynes and Slater in [10].

The paired version of domination stability was first studied by the authors in [7]. Unless otherwise stated, let G be an isolate-free graph. The γ_{pr}^- -stability of G , denoted $\text{st}_{\gamma_{\text{pr}}}^-(G)$, and the γ_{pr}^+ -stability of G , denoted $\text{st}_{\gamma_{\text{pr}}}^+(G)$, respectively, are defined analogously to the total versions. The *paired domination stability* of G (or the γ_{pr} -stability), denoted $\text{st}_{\gamma_{\text{pr}}}(G)$, is the minimum size of a non-isolating set S of vertices in G whose removal changes the paired domination number. Thus,

$$\text{st}_{\gamma_{\text{pr}}}(G) = \min_{S \in \text{NI}(G)} \{ |S| : \gamma_{\text{pr}}(G - S) \neq \gamma_{\text{pr}}(G) \} = \min \{ \text{st}_{\gamma_{\text{pr}}}^-(G), \text{st}_{\gamma_{\text{pr}}}^+(G) \}.$$

Note that if no such non-isolating set exists whose removal increases the paired domination number, we define $\text{st}_{\gamma_{\text{pr}}}^+(G) = \infty$. As a trivial example, $\text{st}_{\gamma_{\text{pr}}}^-(P_5) = 1$ while $\text{st}_{\gamma_{\text{pr}}}^+(P_5) = \infty$. Following the original paper of Bauer *et al.* [5], we consider the *null graph* K_0 (also called the *order-zero graph*), which is the unique graph having no vertices and hence has order zero, as a graph. As observed in [7], considering the null graph, the total domination and paired domination stability of a non-trivial graph is always defined. If G is a graph of order n and $\gamma_t(G) = 2$, then $\text{st}_{\gamma_t}^-(G) = n$ since removing all vertices from the graph G produces the null graph with total domination number zero. Analogously, if G is a graph of order n and $\gamma_{\text{pr}}(G) = 2$, then $\text{st}_{\gamma_{\text{pr}}}^-(G) = n$. Hence, it is only of interest for us to consider isolate-free graphs G with $\gamma_t(G) \geq 3$ when determining $\text{st}_{\gamma_t}^-(G)$, and with $\gamma_{\text{pr}}(G) \geq 4$ when determining $\text{st}_{\gamma_{\text{pr}}}^-(G)$.

A *perfect dominating set* D in a graph G is a dominating set of G in which every vertex of G is dominated by exactly one vertex in D . Thus, if a graph G has perfect dominating set, then the set $N_G[v]$ for all $v \in D$, partition the set $V(G)$. It is clear that a perfect dominating set for a graph G is necessarily a γ -set of G .

The *Cartesian product* $G \square H$ of graphs G and H is the graph whose vertex set is $V(G) \times V(H)$ and two vertices (g_1, h_1) and (g_2, h_2) are adjacent in $G \square H$ if either $g_1 = g_2$ and $h_1 h_2$ is an edge in H , or $h_1 = h_2$ and $g_1 g_2$ is an edge in G .

The *prism* of a graph G is the graph $G \square K_2$. Thus, it is defined by taking two disjoint copies G_1 and G_2 of G , called *layers*, and adding an edge between each pair of corresponding vertices. For each vertex v in G we denote its equivalent in G_i by v_i for $i \in [2]$, and refer to the vertices v_1 and v_2 as *partners*. If G is a bipartite graph, then we call the prism $G \square K_2$ the *bipartite prism* of G . If G is a cycle, then we call the prism $G \square K_2$ a *cycle prism* of G . It should be mentioned that the Cartesian products of graphs have wide applications to

numerous problems of theoretical computer science, where the information nets are very often modelled by prisms.

The relationship between domination parameters in the graph and its prism have been studied extensively. In particular, we note that total domination and paired domination in prisms have been studied, for example, in [3, 6, 9, 16].

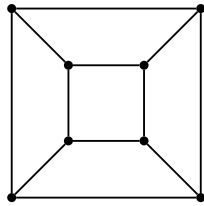


Figure 1. The hypercube Q_3 .

We denote by Q_n the n -dimensional hypercube, and so Q_n can be represented as the n^{th} power of K_2 with respect to the Cartesian product operation \square , that is, $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$ for $n \geq 2$. The vertices are represented by binary sequences of length n and two vertices are adjacent if the corresponding sequences differ in exactly one coordinate. Note that the hypercube Q_3 , illustrated in Figure 1, is the bipartite cycle prism of C_4 .

For notation and graph theory terminology we generally follow [13].

2. KNOWN RESULTS AND MOTIVATION

We recall first trivial lower bounds on the total and paired domination numbers of a graph in terms of the maximum degree of the graph.

Observation 1. *If G is an isolate-free graph of order n and maximum degree Δ , then $\gamma_t(G) \geq \lceil \frac{n}{\Delta} \rceil$ and $\gamma_{\text{pr}}(G) \geq 2 \lceil \frac{n}{2\Delta} \rceil$.*

Let G be a graph, and consider the prism $G \square K_2$ formed by taking two disjoint copies G_1 and G_2 of G . Let D be a γ -set of G , and let D_i be the set of vertices in G_i corresponding to D for $i \in [2]$. The set $D_1 \cup D_2$ is a PD-set of G , with each vertex of D_1 paired with its neighbor in D_2 . Thus, $\gamma_{\text{pr}}(G) \leq |D_1| + |D_2| = 2|D| = 2\gamma(G)$. We state this observation formally.

Observation 2. *If G is a graph, then $\gamma_t(G \square K_2) \leq \gamma_{\text{pr}}(G \square K_2) \leq 2\gamma(G)$.*

Azarija, Henning, and Klavžar [3] proved that if G is a bipartite graph, then we have equality throughout the inequality chain in Observation 2. A simple proof of this result was given in [6].

Theorem 3 [3]. *If G is a bipartite graph, then $\gamma_t(G \square K_2) = \gamma_{\text{pr}}(G \square K_2) = 2\gamma(G)$.*

It is also shown in [3] that the bipartite condition in the statement of Theorem 3 is essential.

The γ_t^- -stability and the total domination stability of paths and cycles is computed in [11], while the γ_{pr}^- -stability and the paired domination stability of paths and cycles is computed in [7]. In the introductory paper [7] on paired domination stability, it is shown that the paired domination stability of a graph can be very different from its domination or total domination stability.

Our aim in this paper is to study total and paired domination stability in prisms. We show that the difference between γ_{pr} -stability and γ_t -stability is a small constant for some special but useful class of the Cartesian products. Additionally, γ_{pr} is very close to γ_t for such graphs. In Section 3 we determine γ_t^- -stability of cycle prisms (Theorem 6) and establish upper bounds for γ_t^+ -stability of these graphs (Theorems 7 and 8).

In Section 4 we determine γ_{pr}^- -stability of cycle prisms (Theorem 19) and we establish next upper bounds on the γ_{pr}^+ -stability of these graphs (Theorem 21).

On the other hand, the following result establishes an upper bound on the γ_t^- -stability and γ_{pr}^- -stability of a general graph in terms of its maximum degree.

Theorem 4. *If G is a connected graph with maximum degree Δ , then the following holds.*

- (a) ([11]) *If $\gamma_t(G) \geq 3$, then $\text{st}_{\gamma_t}^-(G) \leq 2\Delta - 1$, and this bound is sharp.*
- (b) ([7]) *If $\gamma_{\text{pr}}(G) \geq 4$, then $\text{st}_{\gamma_{\text{pr}}}^-(G) \leq 2\Delta$, and this bound is sharp.*

Both bounds in Theorem 4 are tight for an infinite family of trees, as shown in [7, 11]. In Section 4.1 we prove that the bound in Theorem 4(b) is tight for the family of hypercubes, Q_n , in the case when $n = 2^k$ for all $k \geq 1$ (Theorem 12).

3. TOTAL DOMINATION STABILITY IN CYCLE PRISMS

In this section, we investigate total domination stability of a cycle prism. Recall that for $n \geq 3$, $\gamma(C_n) = \lceil \frac{n}{3} \rceil$. We begin by establishing the total domination number of a cycle prism.

Proposition 5. *For $n \geq 3$,*

$$\gamma_t(C_n \square K_2) = \begin{cases} 2\gamma(C_n) - 1 & \text{if } n \equiv 1 \pmod{6}, \\ 2\gamma(C_n) & \text{otherwise.} \end{cases}$$

Proof. If n is even, then the result follows from Theorem 3. If $n \equiv 1 \pmod{6}$, then the result is known cf. [3, Proposition 16]. Suppose that $n \pmod{6} \in \{3, 5\}$. If $n = 6k + 3$ for some $k \geq 0$, then $4k + 2 = \lceil (12k + 6)/3 \rceil = \lceil 2n/\Delta \rceil \leq$

$\gamma_t(C_n \square K_2) \leq 2\gamma(C_n) = 2\lceil n/3 \rceil = 4k + 2$. If $n = 6k + 5$ for some $k \geq 0$, then $4k + 4 = \lceil (12k + 10)/3 \rceil = \lceil 2n/\Delta \rceil \leq \gamma_t(C_n \square K_2) \leq 2\gamma(C_n) = 2\lceil n/3 \rceil = 4k + 4$. In both cases, we must have equality throughout the above inequality chains, implying that $\gamma_t(C_n \square K_2) = 2\gamma(C_n)$ when $n \pmod 6 \in \{3, 5\}$. ■

We note that the vertices totally dominated by a set D of vertices in a graph G are precisely those vertices that have a neighbor in D . We are now in a position to present a proof of the following result determining the γ_t^- -stability of a cycle prism.

Theorem 6. For $n \geq 4$,

$$\text{st}_{\gamma_t}^-(C_n \square K_2) = \begin{cases} 1 & \text{if } n \equiv 4 \pmod 6, \\ 2 & \text{if } n \pmod 6 \in \{1, 2, 5\}, \\ 4 & \text{if } n \pmod 6 \in \{0, 3\}. \end{cases}$$

Proof. For $n \geq 4$, let $G = C_n \square K_2$. Let G_1 and G_2 be the two layers of the prism G , where G_1 is the cycle $v_1v_2 \cdots v_nv_1$ and G_2 is the cycle $u_1u_2 \cdots u_nu_1$. Let G be obtained from G_1 and G_2 by adding the edges u_iv_i for $i \in [n]$. Suppose that $n \equiv 4 \pmod 6$, and so $n = 6k + 4$ for some $k \geq 0$. By Proposition 5, $\gamma_t(G) = 4k + 4$. Letting $S = \{u_n\}$, the set $D = \{v_{n-1}\} \cup \bigcup_{i=0}^{2k} \{u_{3i+2}, v_{3i+2}\}$ is a TD-set of $G - S$, implying that S is a non-isolating set of vertices in G such that $\gamma_t(G - S) \leq |D| = 4k + 3 < \gamma_t(G)$. Hence, $\text{st}_{\gamma_t}^-(G) \leq |S| = 1$, implying that $\text{st}_{\gamma_t}^-(G) = 1$.

Suppose that $n \equiv 1 \pmod 6$, and so $n = 6k + 1$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k + 1$. If $|S| = 1$ and S is a non-isolating set of vertices in G , then $|V(G - S)| = 12k + 1$ and $\Delta(G - S) = 3$. Thus in this case, $\gamma_t(G - S) \geq \lceil (12k + 1)/3 \rceil = 4k + 1 = \gamma_t(G)$. Hence, if S is a non-isolating set of vertices in G such that $\gamma_t(G - S) \leq 4k$, then $|S| \geq 2$, implying that $\text{st}_{\gamma_t}^-(G) \geq 2$. Letting $S = \{u_n, v_n\}$, the set $D = \bigcup_{i=0}^{2k-1} \{u_{3i+2}, v_{3i+2}\}$ is a TD-set of $G - S$, implying that $\text{st}_{\gamma_t}^-(G) \leq 2$. Consequently, $\text{st}_{\gamma_t}^-(G) = 2$.

Suppose that $n \equiv 2 \pmod 6$, and so $n = 6k + 2$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k + 2$. Let S be a minimum non-isolating set of G such that $\gamma_t(G - S) \leq 4k + 1$. Let D be a γ_t -set of $G - S$, and so $|D| = \gamma_t(G - S) \leq 4k + 1$. Adding vertices to D if necessary, we may assume that $|D| = 4k + 1$. Thus, $G[D]$ consists of at least one component of odd order at least 3. This implies that at least two vertices in D have a common neighbor in D , and therefore that the number of vertices totally dominated by D is at most $3|D| - 1 = 3(4k + 1) - 1 = 12k + 2$. Since the number of vertices totally dominated by D is $|V(G - S)| = 12k + 4 - |S|$, we have $12k + 4 - |S| \leq 12k + 2$, implying that $\text{st}_{\gamma_t}^-(G) = |S| \geq 2$. We now consider the set $S = \{v_1, u_8\}$. Let $D_1 = \{u_2, u_3, u_4, v_6, v_7\}$. If $k = 1$, let $D = D_1$, while if $k \geq 2$, let $D_k = D_1 \cup \bigcup_{i=3}^{2k} \{u_{3i+1}, v_{3i+1}\}$. The resulting set D_k is a TD-set of

$G - S$, implying that $\gamma_t(G - S) \leq |D_k| = 4k + 1$. Hence, $\text{st}_{\gamma_t}^-(G) \leq 2$. Consequently, $\text{st}_{\gamma_t}^-(G) = 2$.

Suppose that $n \equiv 5 \pmod{6}$, and so $n = 6k + 5$ for some $k \geq 0$. By Proposition 5, $\gamma_t(G) = 4k + 4$. Let S be a minimum non-isolating set of G such that $\gamma_t(G - S) \leq 4k + 3$. Reasoning analogous to case when $n \equiv 2 \pmod{6}$ allows us to state that $\text{st}_{\gamma_t}^-(G) = |S| \geq 2$. We now consider the set $S = \{u_1, u_5\}$. Let $D_1 = \{v_2, v_3, v_4\}$. If $k = 0$, let $D = D_1$, while if $k \geq 1$, let $D_k = D_1 \cup \bigcup_{i=2}^{2k+1} \{u_{3i+1}, v_{3i+1}\}$. The resulting set D_k is a TD-set of $G - S$, implying that $\gamma_t(G - S) \leq |D_k| = 4k + 3$. Hence, $\text{st}_{\gamma_t}^-(G) \leq 2$. Consequently, $\text{st}_{\gamma_t}^-(G) = 2$.

Suppose that $n \equiv 0 \pmod{6}$, and so $n = 6k$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k$. Let S be a minimum non-isolating set of G such that $\gamma_t(G - S) \leq 4k - 1$. As previously, an easy calculation suffices to show that $\text{st}_{\gamma_t}^-(G) = |S| \geq 4$. We now consider the set $S = \{u_1, u_2, u_6, v_1\}$. Let $D_1 = \{v_3, v_4, v_5\}$. If $k = 1$, let $D = D_1$, while if $k \geq 2$, let $D_k = D_1 \cup \bigcup_{i=2}^{2k-1} \{u_{3i+2}, v_{3i+2}\}$. The resulting set D_k is a TD-set of $G - S$, implying that $\gamma_t(G - S) \leq |D_k| = 4k - 1$. Hence, $\text{st}_{\gamma_t}^-(G) \leq 4$. Consequently, $\text{st}_{\gamma_t}^-(G) = 4$.

Suppose that $n \equiv 3 \pmod{6}$, and so $n = 6k + 3$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k + 2$. Once more, we omit the calculation showing that $\text{st}_{\gamma_t}^-(G) = |S| \geq 4$. We now let $S = \{u_1, u_2, u_6, v_1\}$ and $D_k = \{v_3, v_4, v_5\} \cup \bigcup_{i=2}^{2k} \{u_{3i+2}, v_{3i+2}\}$. The resulting set D_k is a TD-set of $G - S$, implying that $\gamma_t(G - S) \leq |D_k| = 4k + 1$. Hence, $\text{st}_{\gamma_t}^-(G) \leq 4$. Consequently, $\text{st}_{\gamma_t}^-(G) = 4$. ■

We next establish upper bounds on the γ_t^+ -stability of a cycle prism. For small values of $n \in \{3, 4, 5\}$, we note that $\text{st}_{\gamma_t}^+(C_n \square K_2) = \infty$. Hence it is only of interest to consider values of $n \geq 6$. Firstly, we determine the exact value of the γ_t^+ -stability of a cycle prism $C_n \square K_2$ when $n \equiv 0 \pmod{3}$.

Theorem 7. For $n \geq 6$ and $n \equiv 0 \pmod{3}$,

$$\text{st}_{\gamma_t}^+(C_n \square K_2) = \begin{cases} 3 & \text{if } n \equiv 0 \pmod{6}, \\ 4 & \text{if } n \equiv 3 \pmod{6}. \end{cases}$$

Proof. For $n \geq 6$ and $n \equiv 0 \pmod{3}$, let $G = C_n \square K_2$. Let G_1 and G_2 be the two layers of the prism G , where G_1 is the cycle $v_1 v_2 \cdots v_n v_1$ and G_2 is the cycle $u_1 u_2 \cdots u_n u_1$. Let G be obtained from G_1 and G_2 by adding the edges $u_i v_i$ for $i \in [n]$. We show firstly that $\text{st}_{\gamma_t}^+(G) \geq 3$. By Proposition 5, $\gamma_t(G) = \frac{2}{3}n$. For $i \in [3]$, let $V_i = \{u_j, v_j : j \equiv i \pmod{3} \text{ and } j \in [n]\}$. Let S be a minimum non-isolating set of G such that $\gamma_t(G - S) > \frac{2}{3}n$. Since each of the sets V_1 , V_2 and V_3 is a TD-set of G of cardinality $\frac{2}{3}n = \gamma_t(G)$, we note that $|S \cap V_i| \geq 1$ for all $i \in [3]$, implying that $\text{st}_{\gamma_t}^+(G) = |S| \geq 3$.

Suppose that $n \equiv 0 \pmod{6}$, and so $n = 6k$ for some $k \geq 1$. We show that in this case, $\text{st}_{\gamma_t}^+(G) \leq 3$. By Proposition 5, $\gamma_t(G) = 4k$. Let $S = \{u_1, u_3, u_5\}$ and

consider the graph $G - S$. If $k = 1$, then $\gamma_t(G - S) = 5 > 4 = \gamma_t(G)$, implying that $\text{st}_{\gamma_t}^+(G) \leq |S| = 3$. Hence we may assume that $k \geq 2$. Every TD-set of $G - S$ contains the two vertices v_2 and v_4 . Let D be a γ_t -set of $G - S$. We show that $|D| \geq 4k + 1$. Let $V' = \bigcup_{i=6}^{6k} \{u_i, v_i\}$ and let $H = G[V']$. Note that H is isomorphic to $P_{6(k-1)+1} \square K_2$. Suppose that $v_3 \in D$. If $v_5 \in D$, then we can replace v_5 in D with the vertex u_6 or v_7 . If $v_{6k} \in D$, then we can replace v_{6k} in D with the vertex u_{6k-1} . Hence, we may choose D so that $D \cap \{v_5, v_{6k}\} = \emptyset$, implying that $|D| \geq |\{v_2, v_3, v_4\}| + \gamma_t(H) = 3 + \gamma_t(P_{6(k-1)+1} \square K_2) = 3 + 2\gamma(P_{6(k-1)+1}) = 3 + 2(2(k-1) + 1) = 4k + 1$.

Assume now that $v_3 \notin D$. With this assumption, we can choose D so that $\{v_1, v_2, v_4, v_5\} \subset D$. If $v_6 \in D$, then we can replace v_6 in D with the vertex u_7 . If $v_{6k} \in D$, then we can replace v_{6k} in D with the vertex u_{6k-1} . Hence, we may choose D so that $D \cap \{v_6, v_{6k}\} = \emptyset$. In order to totally dominate the vertices u_6 and u_{6k} , this implies that $u_7 \in D$ and $u_{6k-1} \in D$, respectively. Let $V'' = V' \setminus \{v_6, v_{6k}\}$, and let $D' = D \cap V'$. Thus, $\{u_7, u_{6k-1}\} \subset D'$ and the set D' totally dominates the set V'' . Since $|V''| = 12(k-1)$ and each vertex of D' totally dominates at most three vertices, we note that $|D'| \geq 4(k-1)$.

We show that $|D'| > 4(k-1)$. Suppose, to the contrary, that $|D'| = 4(k-1)$. This implies that each vertex of D' uniquely totally dominates three vertices of V' . By our earlier observations, $u_7 \in D'$. Since u_6 has only one neighbor in V'' , and since v_7 has only two neighbors in V'' , we note that $u_6 \notin D'$ and $v_7 \notin D'$, and therefore $u_8 \in D'$. This in turn implies that $D' \cap \{v_8, u_9, v_9, u_{10}\} = \emptyset$. Therefore, $\{v_{10}, v_{11}\} \subset D'$. If $k = 2$, then as observed earlier, $u_{11} \in D'$, and so $|D'| \geq 5 > 4(k-1)$, a contradiction. Hence, $k \geq 3$. Since each vertex of D' uniquely totally dominates three vertices of V'' , and since $\{v_{10}, v_{11}\} \subset D'$, we therefore have that $D' \cap \{u_{11}, u_{12}, v_{12}, v_{13}\} = \emptyset$. Therefore, $\{u_{13}, u_{14}\} \subset D'$. This in turn implies that $D' \cap \{v_{14}, u_{15}, v_{15}, u_{16}\} = \emptyset$ and $\{v_{16}, v_{17}\} \subset D'$. Continuing in this way, for each $i \in [k-1]$ we have $\{u_{6i+1}, u_{6i+2}, v_{6i+4}, v_{6i+5}\} \subseteq D'$. However as observed earlier, $u_{6k-1} = u_{6(k-1)+5} \in D'$, implying that $|D'| \geq 4(k-1) + 1$, a contradiction. Hence, $|D'| > 4(k-1)$. Therefore, $|D| \geq 4 + |D'| > 4k = \gamma_t(G)$, implying that $\text{st}_{\gamma_t}^+(G) \leq |S| = 3$.

Suppose that $n \equiv 3 \pmod{6}$, and so $n = 6k + 3$ for some $k \geq 1$. We show that in this case, $\text{st}_{\gamma_t}^+(G) \leq 4$. By Proposition 5, $\gamma_t(G) = 4k + 2$. Let $S = \{u_1, u_3, u_5, u_7\}$ and consider the graph $G - S$. If $k = 1$, then $\gamma_t(G - S) = 7 > 6 = \gamma_t(G)$, implying that $\text{st}_{\gamma_t}^+(G) \leq |S| = 4$. Hence we may assume that $k \geq 2$. Every TD-set of $G - S$ contains the three vertices v_2, v_4 and v_6 . Let D be a γ_t -set of $G - S$. In order to totally dominate the vertex v_4 , we can choose D so that $v_3 \in D$ or $v_5 \in D$. By symmetry, we may assume that $v_5 \in D$. With this assumption, we can choose D so that $v_1 \in D$ in order to totally dominate the vertex v_2 . Thus, $\{v_1, v_2, v_4, v_5, v_6\} \subset D$. If $v_7 \in D$, then we can replace v_7 in D with the vertex u_8 or the vertex v_9 . If $v_n = v_{6k+3} \in D$, then we can replace v_{6k+3}

in D with the vertex u_{6k+2} . Hence, we may choose D so that $D \cap \{v_7, v_{6k+3}\} = \emptyset$. Let $V' = \{u_{6k+3}\} \cup \bigcup_{i=8}^{6k+2} \{u_i, v_i\}$ and let $D' = D \cap V'$. Since each vertex of D' totally dominates at most three vertices, in order to totally dominate the $12(k-1) + 3$ vertices in V' we note that $|D'| \geq 4(k-1) + 1 = 4k - 3$. We show that $|D'| > 4k - 3$. Suppose, to the contrary, that $|D'| = 4k - 3$. This implies that each vertex of D' uniquely totally dominates three vertices of V' . Since each of u_8 and v_8 has only two neighbors in V' , this implies that $D' \cap \{u_8, v_8\} = \emptyset$ and therefore that $\{u_9, v_9\} \subset D'$. This in turn implies that $D' \cap \{u_{10}, v_{10}, u_{11}, v_{11}\} = \emptyset$, and therefore that $\{u_{12}, v_{12}\} \subset D'$. Continuing in this way, we have that $\{u_{3i}, v_{3i}\} \subset D'$ and $D' \cap \{u_{3i+1}, v_{3i+1}, u_{3i+2}, v_{3i+2}\} = \emptyset$ for all $i \in [2k] \setminus \{1, 2\}$. This implies that the vertex u_{6k+3} is not totally dominated by D , a contradiction. Hence, $|D'| > 4k - 3$. Therefore, $|D| = 5 + |D'| > 5 + (4k - 3) = 4k + 2 = \gamma_t(G)$, implying that $\text{st}_{\gamma_t}^+(G) \leq |S| = 4$.

It remains for us to show that $\text{st}_{\gamma_t}^+(G) \geq 4$ in this case when $n = 6k + 3$ for some $k \geq 1$. Let S be a minimum non-isolating set of G such that $\gamma_t(G - S) > \frac{2}{3}n$. As shown earlier, $|S| \geq 3$. We wish to show that $|S| \geq 4$. Suppose, to the contrary, that $|S| = 3$. If $k = 1$ (and so, $n = 9$), then this can be readily checked. Hence we may assume that $k \geq 2$. By our earlier observations, $|S \cap V_i| \geq 1$ for all $i \in [3]$ where recall V_i is the set of all vertices of G with subscript congruent to i modulo 3. Thus, $|S \cap V_i| = 1$ for all $i \in [3]$.

Suppose that S contains two adjacent vertices. Renaming vertices if necessary, we may assume that $\{u_1, u_2\} \subset S$. If $v_{6k+3} \notin S$, then $(V_1 \setminus \{u_1\}) \cup \{v_{6k+3}\}$ is a TD-set of $G - S$ of cardinality $|V_1| = \gamma_t(G)$. If $v_{6k+3} \in S$, then $(V_2 \setminus \{u_2\}) \cup \{v_3\}$ is a TD-set of $G - S$ of cardinality $|V_2| = \gamma_t(G)$. Hence, $\gamma_t(G - S) \leq \gamma_t(G)$, a contradiction. Thus, the set S is an independent set in G . A detailed case analysis, which we omit, shows that $\gamma_t(G - S) \leq \gamma_t(G)$. We remark that our case analysis relies heavily on the fact that $n \equiv 3 \pmod{6}$. To illustrate this, consider an arbitrary vertex v of G . For notational convenience we may assume $v = u_3$. In this case, the set $D' = \{v_1, v_2, v_4, v_5\} \cup \left(\bigcup_{i=1}^k \{u_{6i+1}, u_{6i+2}\}\right) \cup \left(\bigcup_{i=1}^{k-1} \{v_{6i+4}, v_{6i+5}\}\right)$ is a γ_t -set of $G - u_3$. Thus for example, if $S = \{u_1, u_3, u_5\}$, then the set D' is a TD-set of $G - S$, implying that $\gamma_t(G - S) \leq |D'| = \gamma_t(G)$, a contradiction. Indeed, the fact that $n \equiv 3 \pmod{6}$ implies that there are many γ_t -sets of G , in addition to the sets V_1 , V_2 and V_3 , and one can guarantee that at least one such set is always a TD-set of $G - S$ in this case when $|S| = 3$, producing a contradiction. Hence, $|S| \geq 4$, and so $\text{st}_{\gamma_t}^+(G) \geq 4$. By our earlier observation, $\text{st}_{\gamma_t}^+(G) \leq 4$. Consequently, $\text{st}_{\gamma_t}^+(G) = 4$ in this case when $n \equiv 3 \pmod{6}$. ■

We present next a proof of the following result establishing upper bounds on the γ_t^+ -stability of a cycle prism $C_n \square K_2$ when $n \geq 7$ and $n \not\equiv 0 \pmod{3}$.

Theorem 8. For $n \geq 7$ and $n \not\equiv 0 \pmod{3}$,

$$\text{st}_{\gamma_t}^+(C_n \square K_2) \leq \begin{cases} 5 & \text{if } n \equiv 1 \pmod{6}, \\ 7 & \text{if } n \pmod{6} \in \{2, 5\}, \\ 8 & \text{if } n \equiv 4 \pmod{6}. \end{cases}$$

Proof. For $n \geq 6$, let $G = C_n \square K_2$. Let G_1 and G_2 be the two layers of the prism G , where G_1 is the cycle $v_1 v_2 \cdots v_n v_1$ and G_2 is the cycle $u_1 u_2 \cdots u_n u_1$. Let G be obtained from G_1 and G_2 by adding the edges $u_i v_i$ for $i \in [n]$.

Suppose that $n \equiv 1 \pmod{6}$, and so $n = 6k + 1$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k + 1$. We let $S = \{u_1, u_3, v_1, v_3, v_5\}$ and D be a γ_t -set of $G - S$. Necessarily, $\{u_2, u_4, v_2\} \subset D$ and we can choose the set D so that $u_5 \in D$. Let $V' = \{v_6\} \cup \bigcup_{i=7}^{6k+1} \{u_i, v_i\}$.

In order to totally dominate the $12k - 9$ vertices in the set V' , the set D contains at least $4k - 3$ vertices in addition to the four vertices in the set $\{u_2, u_4, u_5, v_2\}$. Suppose that D contains exactly $4k - 3$ additional vertices. In this case, each additional vertex uniquely totally dominates three new vertices. Hence neither u_{6k+1} nor v_{6k+1} belong to D since both these vertices have degree 2 in $G - S$. Hence, $\{u_{6k}, v_{6k}\} \subset D$ in order to totally dominate the vertices u_{6k+1} and v_{6k+1} . Since each of u_{6k} and v_{6k} uniquely totally dominates three vertices, this implies that $\{u_{6k-2}, u_{6k-1}, v_{6k-2}, v_{6k-1}\} \cap D = \emptyset$ and that $\{u_{6k-3}, v_{6k-3}\} \subset D$. Continuing this argument, we have that $\bigcup_{i=3}^{2k} \{u_{3i}, v_{3i}\} \subset D$.

At least two additional vertices are needed to totally dominate the three vertices v_6, u_7 and v_7 , implying that in addition to the vertices in $\{u_2, u_4, u_5, v_2\}$, the set D contains at least $2 + 4(k - 1) = 4k - 2$ additional vertices to totally dominate the vertices in V' , a contradiction. Therefore, the set D contains at least $4k - 2$ vertices in addition to the vertices in $\{u_2, u_4, u_5, v_2\}$, implying that $\gamma_t(G - S) = |D| \geq 4k + 2 > \gamma_t(G)$. Thus, $\text{st}_{\gamma_t}^+(G) \leq |S| = 5$.

Suppose that $n \equiv 2 \pmod{6}$, and so $n = 6k + 2$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k + 2$. Let $S = \{u_1, v_1, u_3, v_3, u_5, v_5, v_7\}$ and consider the graph $G - S$. Let D be a γ_t -set of $G - S$. Necessarily, $\{u_2, v_2, u_4, v_4, u_6\} \subset D$ and we can choose the set D so that $u_7 \in D$. If $k = 1$, then the vertex u_8 is needed in D to dominate the vertex v_8 , implying that $\gamma_t(G - S) = |D| = 7 > 6 = \gamma_t(G)$. Suppose that $k \geq 2$ and let $V' = \{v_8\} \cup \bigcup_{i=9}^{6k+2} \{u_i, v_i\}$.

In order to totally dominate the $12k - 11$ vertices in the set V' , the set D contains at least $4k - 3$ vertices in addition to the six vertices in the set $\{u_2, v_2, u_4, v_4, u_6, u_7\}$. Thus, $|D| \geq 6 + (4k - 3) = 4k + 3 > \gamma_t(G)$. Hence, $\text{st}_{\gamma_t}^+(G) \leq |S| = 7$.

Suppose that $n \equiv 5 \pmod{6}$, and so $n = 6k + 5$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k + 4$. As in the previous case when $n \equiv 2 \pmod{6}$, we let $S = \{u_1, v_1, u_3, v_3, u_5, v_5, v_7\}$ and consider the graph $G - S$. Let D be a γ_t -set of

$G - S$. Necessarily, $\{u_2, v_2, u_4, v_4, u_6\} \subset D$ and we can choose the set D so that $u_7 \in D$. Let $V' = \{v_8\} \cup \bigcup_{i=9}^{6k+5} \{u_i, v_i\}$.

In order to totally dominate the $12k - 5$ vertices in the set V' , the set D contains at least $4k - 1$ vertices in addition to the six vertices in the set $\{u_2, v_2, u_4, v_4, u_6, u_7\}$. Thus, $|D| \geq 6 + (4k - 1) = 4k + 5 > \gamma_t(G)$. Hence, $\text{st}_{\gamma_t}^+(G) \leq |S| = 7$.

Suppose that $n \equiv 4 \pmod{6}$, and so $n = 6k + 4$ for some $k \geq 1$. By Proposition 5, $\gamma_t(G) = 4k + 4$. Let $S = \{u_1, v_1, u_3, v_3, u_5, v_5, v_7, v_9\}$ and let D be a γ_t -set of $G - S$. We show that $|D| \geq 4k + 5$. Necessarily, $\{u_2, v_2, u_4, v_4, u_6, u_8\} \subset D$ and we can choose the set D so that $u_7 \in D$. If $k = 1$, then $\{u_9, u_{10}\} \subseteq D$ or $\{u_{10}, v_{10}\} \subseteq D$, and so $|D| \geq 9 = 4k + 5$, as claimed. Hence we may assume that $k \geq 2$. Suppose that $u_9 \notin D$. In this case, $|D| = 7 + \gamma_t(P_{6k-5} \square K_2) = 7 + 2\gamma(P_{6k-5}) = 7 + 2(2k - 1) = 4k + 5 > \gamma_t(G)$. Assume now that $u_9 \in D$ and let $V' = \{v_{10}\} \cup \bigcup_{i=11}^{6k+4} \{u_i, v_i\}$.

In order to totally dominate the $12k - 11$ vertices in the set V' , the set D contains at least $4k - 3$ vertices in addition to the eight vertices in the set $\{u_2, v_2, u_4, v_4, u_6, u_7, u_8, u_9\}$. Thus, $|D| \geq 8 + (4k - 3) = 4k + 5$. Hence in all cases, $|D| \geq 4k + 5 > \gamma_t(G)$, implying that $\text{st}_{\gamma_t}^+(G) \leq |S| = 8$. ■

As an immediate consequence of Theorems 6, 7 and 8 we have the following result on the total domination stability of a cycle prism.

Corollary 9. For $n \geq 4$,

$$\text{st}_{\gamma_t}(C_n \square K_2) = \begin{cases} 1 & \text{if } n \equiv 4 \pmod{6}, \\ 2 & \text{if } n \pmod{6} \in \{1, 2, 5\}, \\ 3 & \text{if } n \equiv 0 \pmod{6}, \\ 4 & \text{if } n \equiv 3 \pmod{6}. \end{cases}$$

Proof. By Theorems 6, 7 and 8, the only remaining case that needs to be covered now is the fact that the γ_t^+ -stability of $C_n \square K_2$ is greater than 1 in cases when $n \pmod{6} \in \{1, 2, 5\}$. Let v be an arbitrary vertex of $C_n \square K_2$. We remark that removing one vertex from $C_n \square K_2$ does not produce isolated vertices. It is easily seen that there is a γ_t -set D of $C_n \square K_2$ such that $v \notin D$. Hence, D is a TD-set for $(C_n \square K_2) - \{v\}$, so we conclude that $\text{st}_{\gamma_t}^+(C_n \square K_2) > 1$. ■

It remains an open problem to determine the exact value of the γ_t^+ -stability of a cycle prism $C_n \square K_2$ for $n \geq 7$ and $n \not\equiv 0 \pmod{3}$.

4. PAIRED DOMINATION STABILITY IN PRISMS

In this section, we investigate paired domination stability of a prism. We consider three types of prisms, namely hypercubes, bipartite prisms, and cycle prisms.

4.1. Hypercubes

In this subsection, we consider the class of prisms called *hypercubes* Q_n for $n \geq 1$. Recall that $Q_1 = K_2$ and $Q_n = Q_{n-1} \square K_2$ for $n \geq 2$. To determine $\gamma(Q_n)$ turns out to be an intrinsically difficult problem. To date, exact values are only known for $n \leq 9$, and for two infinite families of hypercubes. These results are summarized in Table 1 and Theorem 10. The result $\gamma(Q_9) = 62$ in Table 1 due to Östergård and Blass [17] actually presented a breakthrough back in 2001.

n	1	2	3	4	5	6	7	8	9
$\gamma(Q_n)$	1	2	2	4	7	12	16	32	62

Table 1. Domination numbers of hypercubes up to dimension 9.

Theorem 10 [3]. *If $k \geq 1$, then $\gamma(Q_{2^k-1}) = 2^{2^k-k-1}$ and $\gamma(Q_{2^k}) = 2^{2^k-k}$.*

A *code* in a graph $G = (V, E)$ is a subset $C \subset V$ such that any two vertices of C are at distance at least 3 in G . A *perfect code* is a code C with the property that C is a dominating set in G , cf. [15]. The first assertion of Theorem 10 is based on the fact that hypercubes Q_{2^k-1} contain perfect codes, cf. [8], and the domination number of a graph with a perfect code is equal to the size of such a code. We note that Q_n contains a perfect code if and only if $n = 2^k - 1$ for some $k \geq 1$. The second assertion of Theorem 10 is due to van Wee [19].

As a consequence of Table 1 and Theorem 3, the exact values of $\gamma_{\text{pr}}(Q_n)$ for $n \leq 10$ are given in Table 2.

n	2	3	4	5	6	7	8	9	10
$\gamma_{\text{pr}}(Q_n)$	2	4	4	8	14	24	32	64	124

Table 2. Paired domination numbers of hypercubes up to dimension 10.

Arumugam and Kala [2] established the following upper bound on the domination number of a hypercube.

Theorem 11 [2]. *For all $n \geq 7$, we have $\gamma(Q_n) \leq 2^{n-3}$.*

Total domination in Cartesian products has been studied in [12]. We focus on the paired domination and paired domination stability in hypercubes.

We are now in a position to prove the following theorem, that establishes the existence of a 2^k -regular connected graph G , where k can be chosen arbitrarily large, satisfying $\text{st}_{\gamma_{\text{pr}}}^-(G) = 2\Delta(G)$. Thus, the family of hypercubes Q_n , where $n = 2^k$ for some $k \geq 1$, shows that the upper bound in Theorem 4(b) is tight.

Theorem 12. *If Q_n is a hypercube such that $n = 2^k$, for $k \geq 1$, then*

$$\text{st}_{\gamma_{\text{pr}}}^-(Q_n) = 2\Delta(Q_n) = 2^{k+1}.$$

Proof. Let $n = 2^k$ for some integer $k \geq 1$, and consider the hypercube Q_n . We note that $|V(Q_n)| = 2^n$ and $\Delta(Q_n) = n = 2^k$. By Theorem 3 and Theorem 10, we have that $\gamma_{\text{pr}}(Q_n) = 2\gamma(Q_{n-1}) = 2 \cdot 2^{n-k-1} = 2^{n-k}$. Suppose that there exists a non-isolating subset S of vertices in the hypercube Q_n such that $\gamma_{\text{pr}}(Q_n - S) < \gamma_{\text{pr}}(Q_n)$ and $|S| \leq 2\Delta(Q_n) - 1 = 2^{k+1} - 1$. Thus, $|V(Q_n - S)| = |V(Q_n)| - |S| \geq 2^n - 2^{k+1} + 1$. We note that $\Delta(Q_n - S) \leq \Delta(Q_n) = 2^k$. By Observation 1, we therefore have

$$\gamma_{\text{pr}}(Q_n - S) \geq \frac{|V(Q_n - S)|}{\Delta(Q_n - S)} \geq \left\lceil \frac{2^n - 2^{k+1} + 1}{2^k} \right\rceil = 2^{n-k} - 1.$$

Since the paired domination number of a graph is an even integer, this implies that $\gamma_{\text{pr}}(Q_n - S) \geq 2^{n-k} = \gamma_{\text{pr}}(Q_n)$, a contradiction. Hence, every non-isolating subset S of vertices in the hypercube Q_n such that $\gamma_{\text{pr}}(Q_n - S) < \gamma_{\text{pr}}(Q_n)$ has cardinality at least $2\Delta(Q_n)$, that is, $\text{st}_{\gamma_{\text{pr}}}^-(Q_n) \geq 2\Delta(Q_n)$. By Theorem 4, $\text{st}_{\gamma_{\text{pr}}}^-(Q_n) \leq 2\Delta(Q_n)$. Consequently, $\text{st}_{\gamma_{\text{pr}}}^-(Q_n) = 2\Delta(Q_n) = 2^{k+1}$. ■

We establish next the upper bound on the γ_{pr}^+ -stability of a class of connected bipartite prisms.

Proposition 13. *If $G = (X, Y; E)$ is a connected bipartite graph, with $\gamma(G) < \min\{|X|, |Y|\}$, then $\text{st}_{\gamma_{\text{pr}}}^+(G \square K_2) \leq \min\{|X|, |Y|\}$.*

Proof. Let $G = (X, Y; E)$ be a connected bipartite graph with $\gamma(G) < \min\{|X|, |Y|\}$. Without the loss of generality, let $\min\{|X|, |Y|\} = |X| = k$ and let G_1 and G_2 be two copies of G that form a graph $G \square K_2$ by adding a perfect matching between corresponding vertices in G_1 and G_2 . Let X_i and Y_i be the two partite sets of G_i for $i \in [2]$. Renaming the sets if necessary, we may assume that there is a perfect matching between the vertices of X_1 and X_2 (respectively, between Y_1 and Y_2) in $G \square K_2$. Note that $\gamma_{\text{pr}}(G \square K_2) < 2k$ since we can choose a dominating set in G_i and its corresponding set in G_{3-i} . Let H be the (connected) graph obtained from $G \square K_2$ by removing the vertices in the set X_1 , and so $H = G \square K_2 - X_1$. We note that each vertex in Y_1 has degree 1 in H . Further, the (unique) neighbor in H of each vertex in Y_1 belongs to the set Y_2 . Thus, each vertex of Y_2 is a support vertex in H , and therefore belongs to every PD-set of H . Since the set Y_2 is an independent set, this implies that every PD-set of H has cardinality at least $2|Y_2| \geq 2|X_2| = 2k > \gamma_{\text{pr}}(G \square K_2)$. Hence, the set X_1 is a non-isolating set of vertices of $G \square K_2$ such that $\gamma_{\text{pr}}(G \square K_2 - X_1) > \gamma_{\text{pr}}(G \square K_2)$, implying that $\text{st}_{\gamma_{\text{pr}}}^+(G \square K_2) \leq k = |X_1|$. ■

From Proposition 13 we can immediately conclude the following result.

Corollary 14. *For $n \geq 4$, we have $\text{st}_{\gamma_{\text{pr}}}^+(Q_n) \leq 2^{n-2}$.*

Proof. Let $G = Q_n$ for some $n \geq 4$. We show that $\gamma_{\text{pr}}(G) < 2^{n-1}$ for all $n \geq 4$. By Table 2, this is true for small $n \in \{4, 5, \dots, 10\}$. For $n \geq 11$, by Theorems 3 and 11, we have that $\gamma_{\text{pr}}(G) = 2\gamma(Q_{n-1}) \leq 2 \cdot 2^{n-4} < 2^{n-1}$. Thus, $\gamma_{\text{pr}}(G) < 2^{n-1}$ for all $n \geq 4$. Further, with reasoning as in the proof of the Proposition 13 we conclude that the result follows. ■

It remains an open problem, however, to determine the exact value of the γ_{pr}^+ -stability of a hypercube Q_n for $n \geq 4$.

4.2. Bipartite prisms

In this section, we study bipartite prisms. For this purpose, we first determine the γ^- -stability of a regular graph.

Lemma 15. *For $r \geq 2$, if G is an r -regular graph that contains a perfect dominating set, then $\text{st}_{\gamma}^-(G) = r + 1$.*

Proof. Let G be an r -regular graph of order n that contains a perfect dominating set. Since any perfect dominating set is necessarily a minimum dominating set, we have $n = k(r + 1)$ for some integer $k \geq 1$, and $\gamma(G) = k$. Let S be a set of vertices of G such that $\gamma(G - S) \leq \gamma(G) - 1 = k - 1$. We note that $|V(G - S)| = n - |S|$ and $\Delta(G - S) \leq \Delta(G) = r$. Thus,

$$k - 1 \geq \gamma(G - S) \geq \frac{|V(G - S)|}{\Delta(G - S) + 1} \geq \frac{n - |S|}{r + 1},$$

and so, $|S| \geq n - (k - 1)(r + 1) = k(r + 1) - (k - 1)(r + 1) = r + 1$, implying that $\text{st}_{\gamma}^-(G) \geq r + 1$. However, if D is a perfect dominating set of G and $v \in D$, then removing v and all its neighbors from G produces a graph with domination number $|D| - 1 = \gamma(G) - 1$, implying that $\text{st}_{\gamma}^-(G) \leq r + 1$. Consequently, $\text{st}_{\gamma}^-(G) = r + 1$. ■

We are now ready to prove a sharp upper bound on the γ_{pr}^- -stability of a bipartite prism.

Theorem 16. *If G is a bipartite graph, then*

$$\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) \leq 2\text{st}_{\gamma}^-(G),$$

and this bound is sharp.

Proof. Let G be a bipartite graph, and consider the prism $G \square K_2$ formed by taking two disjoint copies G_1 and G_2 of G . Let S be a non-isolating set of vertices in G such that $\gamma(G - S) < \gamma(G)$ and $|S| = \text{st}_{\gamma}^-(G)$. Let D be a γ -set of $G - S$. Let D_i and S_i be the set of vertices in G_i corresponding to the sets D and S , respectively, in G for $i \in [2]$.

We note that the set $S_1 \cup S_2$ is a non-isolating set of vertices in $G \square K_2$. Further, we note that the graph $(G \square K_2) - (S_1 \cup S_2)$ is isomorphic to $(G - S) \square K_2$. Additionally, in the graph $(G \square K_2) - (S_1 \cup S_2)$ every vertex of the set D_1 is adjacent to its partner in D_2 . Hence, $D_1 \cup D_2$ is a PD-set of $(G - S) \square K_2$. Therefore, $S_1 \cup S_2$ is a non-isolating set in $G \square K_2$ such that $\gamma_{\text{pr}}((G \square K_2) - (S_1 \cup S_2)) \leq |D_1 \cup D_2| = 2|D| = 2\gamma(G - S) < 2\gamma(G) = \gamma_{\text{pr}}(G \square K_2)$. Consequently, $\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) \leq |S_1 \cup S_2| = 2|S| = 2\text{st}_{\gamma}^-(G)$.

It remains to show that this bound is tight. This may be seen by taking, for example, G to be a hypercube Q_n , where $n = 2^k - 1$ for some $k \geq 1$. In this case the graph $G \square K_2$ is isomorphic to Q_{n+1} , where $n + 1 = 2^k$. Thus by Theorem 12, we have $\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) = \text{st}_{\gamma_{\text{pr}}}^-(Q_{n+1}) = 2\Delta(Q_{n+1}) = 2(n + 1)$. As observed earlier, the hypercube $G = Q_{2^k-1}$ contain a perfect code, that is, the graph G contains a perfect dominating set. This implies by Lemma 15 that $\text{st}_{\gamma}^-(G) = \text{st}_{\gamma}^-(Q_n) = \Delta(Q_n) + 1 = n + 1$. Thus, $\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) = 2\text{st}_{\gamma}^-(G)$. ■

We present next an additional example of a class of graphs G achieving equality in the upper bound of Theorem 16. For $k \geq 1$, let G_{2k} be the graph constructed as follows. Consider two copies of the path P_{4k} with respective vertex sequences $a_1b_1a_2b_2 \cdots a_{2k}b_{2k}$ and $c_1d_1c_2d_2 \cdots c_{2k}d_{2k}$. For each $i \in [2k]$, join a_i to d_i and b_i to c_i . To complete the construction of the graph G_{2k} , add the two edges a_1b_{2k} and c_1d_{2k} . Let $\mathcal{G} = \{G_{2k} : k \geq 1\}$. The graph $G_4 \in \mathcal{G}$ is illustrated in Figure 2, where the framed vertices form a γ -set of G_4 . We note that G_{2k} is a cubic graph of order $8k$ for all $k \geq 1$, and that G_{2k} contains a perfect dominating set. In particular, $\gamma(G_{2k}) = 2k$.

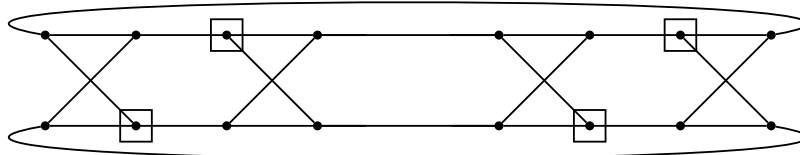


Figure 2. A graph $G_4 \in \mathcal{G}$.

Proposition 17. *If $G \in \mathcal{G}$, then $\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) = 2\text{st}_{\gamma}^-(G)$.*

Proof. Let G be an arbitrary graph in the family \mathcal{G} , and so $G = G_{2k}$ for some $k \geq 1$. Thus, $\gamma(G) = 2k$. The graph G is a 3-regular graph that contains a perfect dominating set, and so by Lemma 15, we have $\text{st}_{\gamma}^-(G) = 4$.

We show next that $\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) = 8$. Let G_1 and G_2 be the two layers of the prism $G \square K_2$. Let D be a γ -set of G , and let D_i be the set of vertices in G_i corresponding to the set D in G for $i \in [2]$. The set $D_1 \cup D_2$ is a PD-set of $G \square K_2$, and so $\gamma_{\text{pr}}(G \square K_2) \leq |D_1| + |D_2| = 2|D| = 2\gamma(G) = 4k$. Since $G \square K_2$ is a 4-regular graph of order $16k$, by Observation 1 we have $\gamma_{\text{pr}}(G \square K_2) \geq$

$|V(G \square K_2)|/\Delta(G \square K_2) = 16k/4 = 4k$. Consequently, $\gamma_{\text{pr}}(G \square K_2) = 4k$. Let S be a non-isolating set of vertices in $G \square K_2$ such that $\gamma_{\text{pr}}((G \square K_2) - S) < \gamma_{\text{pr}}(G \square K_2)$, and so $\gamma_{\text{pr}}((G \square K_2) - S) \leq 4k - 2$. We note that $|V((G \square K_2) - S)| = 16k - |S|$ and $\Delta((G \square K_2) - S) \leq \Delta(G \square K_2) = 4$. By Observation 1,

$$4k - 2 \geq \gamma_{\text{pr}}((G \square K_2) - S) \geq \frac{|V((G \square K_2) - S)|}{\Delta((G \square K_2) - S)} \geq \frac{16k - |S|}{4},$$

and so, $|S| \geq 8$, implying that $\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) \geq 8$. However, removing from the γ_{pr} -set $D_1 \cup D_2$ of $G \square K_2$ a vertex $v_1 \in D_1$ and its partner $v_2 \in D_2$, and all their neighbors in $G \square K_2$, produces a graph with PD-set $(D_1 \cup D_2) \setminus \{v_1, v_2\}$. Thus, there exists a set of eight vertices whose removal from $G \square K_2$ decreases the paired domination number, implying that $\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) \leq 8$. Consequently, $\text{st}_{\gamma_{\text{pr}}}^-(G \square K_2) = 8$. ■

4.3. Cycle prisms

As observed earlier, for $n \geq 3$, $\gamma(C_n) = \lceil n/3 \rceil$. We show first that the result of Theorem 3 can be extended to (odd) cycles.

Proposition 18. *If G is a cycle, then $\gamma_{\text{pr}}(G \square K_2) = 2\gamma(G)$.*

Proof. For $n \geq 3$, let G be a cycle C_n and consider the cycle prism of G , namely $G \square K_2$. We note that $|V(G \square K_2)| = 2n$ and that $G \square K_2$ is a cubic graph. By Observations 1 and 2, we have

$$2 \left\lceil \frac{n}{3} \right\rceil = 2 \left\lceil \frac{2n}{2 \cdot 3} \right\rceil = 2 \left\lceil \frac{|V(G \square K_2)|}{2 \cdot \Delta(G \square K_2)} \right\rceil \leq \gamma_{\text{pr}}(G \square K_2) \leq 2\gamma(G) = 2 \left\lceil \frac{n}{3} \right\rceil.$$

Consequently, we must have equality throughout the above inequality chain. In particular, $\gamma_{\text{pr}}(G \square K_2) = 2\gamma(G)$. ■

We are now in a position to determine the γ_{pr}^- -stability of a cycle prism.

Theorem 19. *For $n \geq 4$,*

$$\text{st}_{\gamma_{\text{pr}}}^-(C_n \square K_2) = \begin{cases} 2 & \text{if } n \pmod{6} \in \{1, 4\}, \\ 4 & \text{if } n \pmod{6} \in \{2, 5\}, \\ 6 & \text{if } n \pmod{6} \in \{0, 3\}. \end{cases}$$

Proof. For $n \geq 4$, let $G = C_n \square K_2$. Let G_1 and G_2 be the two layers of the prism G used to build the graph G , where G_1 is the cycle $v_1 v_2 \cdots v_n v_1$ and G_2 is the cycle $u_1 u_2 \cdots u_n u_1$. Let G be obtained from G_1 and G_2 by adding the edges $u_i v_i$ for $i \in [n]$.

Suppose that $n \equiv 1 \pmod{6}$, and so $n = 6k + 1$ for some $k \geq 1$. By Proposition 18, $\gamma_{\text{pr}}(G) = 2\gamma(C_{6k+1}) = 2 \left\lceil \frac{6k+1}{3} \right\rceil = 4k + 2$. If $|S| = 1$, then $|V(G - S)|$

$= 12k + 1$ and $\Delta(G - S) = 3$. Thus in this case, $\gamma_{\text{pr}}(G - S) \geq 2 \lceil \frac{12k+1}{2.3} \rceil = 4k + 2 = \gamma_{\text{pr}}(G)$, a contradiction. Hence, $|S| \geq 2$, implying that $\text{st}_{\gamma_{\text{pr}}}^-(G) \geq 2$. Letting $S = \{u_1, v_1\}$, the graph $G - S$ is isomorphic to $P_{6k} \square K_2$. In this case, by Theorem 3 we have $\gamma_{\text{pr}}(G - S) = 2\gamma(P_{6k}) = 2 \lceil \frac{6k}{3} \rceil = 4k < 4k + 2 = \gamma_{\text{pr}}(G)$. Thus, S is a non-isolating set of vertices in G such that $\gamma_{\text{pr}}(G - S) < \gamma_{\text{pr}}(G)$, implying that $\text{st}_{\gamma_{\text{pr}}}^-(G) \leq 2$. Consequently, $\text{st}_{\gamma_{\text{pr}}}^-(G) = 2$.

Suppose that $n \equiv 4 \pmod{6}$. With computations analogous to the case when $n \equiv 1 \pmod{6}$ we obtain the desired result.

Suppose that $n \equiv 2 \pmod{6}$, and so $n = 6k + 2$ for some $k \geq 1$. By Proposition 18, $\gamma_{\text{pr}}(G) = 2\gamma(C_{6k+2}) = 2 \lceil \frac{6k+2}{3} \rceil = 4k + 2$. If $|S| \leq 3$, then $|V(G - S)| \geq 12k + 1$ and $\Delta(G - S) = 3$. Thus in this case, $\gamma_{\text{pr}}(G - S) \geq 2 \lceil \frac{12k+1}{2.3} \rceil = 4k + 2 = \gamma_{\text{pr}}(G)$, a contradiction. Hence, $|S| \geq 4$, implying that $\text{st}_{\gamma_{\text{pr}}}^-(G) \geq 4$. Letting $S = \{u_1, v_1, u_2, v_2\}$, the graph $G - S$ is isomorphic to $P_{6k} \square K_2$. Thus, by Theorem 3 we have $\gamma_{\text{pr}}(G - S) = 2\gamma(P_{6k}) = 2 \lceil \frac{6k}{3} \rceil = 4k < 4k + 2 = \gamma_{\text{pr}}(G)$. Hence, S is a non-isolating set of vertices in G such that $\gamma_{\text{pr}}(G - S) < \gamma_{\text{pr}}(G)$, implying that $\text{st}_{\gamma_{\text{pr}}}^-(G) \leq 4$. Consequently, $\text{st}_{\gamma_{\text{pr}}}^-(G) = 4$.

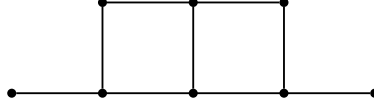
Suppose that $n \equiv 5 \pmod{6}$. The result is obtained by analogous reasoning as in the case when $n \equiv 2 \pmod{6}$.

Suppose that $n \equiv 0 \pmod{6}$, and so $n = 6k$ for some $k \geq 1$. By Proposition 18, $\gamma_{\text{pr}}(G) = 2\gamma(C_{6k}) = 2 \lceil \frac{6k}{3} \rceil = 4k$. If $|S| \leq 5$, then $|V(G - S)| \geq 12k - 5$ and $\Delta(G - S) = 3$. Thus in this case, $\gamma_{\text{pr}}(G - S) \geq 2 \lceil \frac{12k-5}{2.3} \rceil = 4k = \gamma_{\text{pr}}(G)$, a contradiction. Hence, $|S| \geq 6$, implying that $\text{st}_{\gamma_{\text{pr}}}^-(G) \geq 6$. Letting $S = \{u_1, v_1, u_2, v_2, u_3, v_3\}$, the graph $G - S$ is isomorphic to $P_{6k-3} \square K_2$. Thus, by Theorem 3 we have $\gamma_{\text{pr}}(G - S) = 2\gamma(P_{6k-3}) = 2 \lceil \frac{6k-3}{3} \rceil = 4k - 2 < 4k = \gamma_{\text{pr}}(G)$. Hence, S is a non-isolating set of vertices in G such that $\gamma_{\text{pr}}(G - S) < \gamma_{\text{pr}}(G)$, implying that $\text{st}_{\gamma_{\text{pr}}}^-(G) \leq 6$. Consequently, $\text{st}_{\gamma_{\text{pr}}}^-(G) = 6$.

Suppose that $n \equiv 3 \pmod{6}$. The calculations are analogous to the case when $n \equiv 0 \pmod{6}$. ■

We next consider the γ_{pr}^+ -stability of a cycle prism $C_n \square K_2$. For small values of $n \in \{3, 4, 5, 7\}$, we note that $\text{st}_{\gamma_{\text{pr}}}^+(C_n \square K_2) = \infty$. Hence it is only of interest to consider values of n where $n \geq 6$ and $n \neq 7$. For this purpose, we prove an additional lemma.

Let G be a path P_k for some $k \geq 1$, and consider the prism $G \square K_2$ formed by taking two disjoint copies G_1 and G_2 of G . Thus, G_1 and G_2 are the two layers of the prism, $G \square K_2$. If $k = 1$, then G_1 is isomorphic to K_1 , and we let L_k be obtained from $G \square K_2$ by attaching two leaf neighbors to the vertex of G_1 . Thus, L_1 is isomorphic to $K_{1,3}$. For $k \geq 2$, let L_k be a graph of order $2k + 2$ obtained from $G \square K_2$ by attaching a leaf neighbor to each end-vertex of the path G_1 . Let \mathcal{L} be the family of all such graphs L_k , and so $\mathcal{L} = \{L_k : k \geq 1\}$. We proceed further with the following lemma that computes the paired domination number of a graph that belongs to the family \mathcal{L} .

Figure 3. The graph L_3 from the family \mathcal{L} .

Lemma 20. *If G is a graph in the family \mathcal{L} with order n , then*

$$\gamma_{\text{pr}}(G) = \begin{cases} 2 \lceil \frac{n}{6} \rceil & \text{if } n \not\equiv 0 \pmod{12}, \\ \frac{n}{3} + 2 & \text{if } n \equiv 0 \pmod{12}. \end{cases}$$

Proof. Let $G \in \mathcal{L}$ have order n , and so $G = L_k$ for some $k \geq 1$. Let G_1 and G_2 be the two layers of the prism $P_k \square K_2$ used to build the graph G , where G_1 is the path $v_1 v_2 \cdots v_k$ and G_2 is the path $u_1 u_2 \cdots u_k$. Let G be obtained from G_1 and G_2 by adding the two new vertices v_0 and v_{k+1} , and adding the edges $v_0 v_1$, $v_k v_{k+1}$, and the edges $u_i v_i$ for $i \in [k]$. We note that $\Delta(G) = 3$ and $|V(G)| = n = 2k + 2$. Thus, by Observation 1, we have $\gamma_{\text{pr}}(G) \geq 2 \lceil \frac{n}{6} \rceil = 2 \lceil \frac{k+1}{3} \rceil$.

Claim 1. *If $n \not\equiv 0 \pmod{12}$, then $\gamma_{\text{pr}}(G) = 2 \lceil \frac{k+1}{3} \rceil$.*

Proof. Suppose that $n \not\equiv 0 \pmod{12}$, implying that $k \not\equiv 5 \pmod{6}$. We show that $\gamma_{\text{pr}}(G) \leq 2 \lceil \frac{k+1}{3} \rceil$. We proceed by induction on $k \geq 1$. If $k = 1$ or $k = 2$, then let $D = \{v_1, v_2\}$. If $k = 3$, then let $D = \{u_1, v_1, u_3, v_3\}$. If $k = 4$, then let $D = \{u_1, v_1, u_4, v_4\}$. If $k = 6$, then let $D = \{u_1, v_1, u_4, v_4, u_6, v_6\}$. In all four cases, the resulting set D is a PD-set of G and satisfies $|D| = 2 \lceil \frac{k+1}{3} \rceil$. This establishes the base cases. Let $k \geq 7$, and assume that if $G' = L_{k'}$ for some k' where $1 \leq k' < k$ and $k' \not\equiv 5 \pmod{6}$, then $\gamma_{\text{pr}}(G') = 2 \lceil \frac{k'+1}{3} \rceil$. We now consider the graph $G = L_k$. Let $S = \{v_0, v_1, v_2, v_3, v_{k-2}, v_{k-1}, v_k, v_{k+1}\} \cup \{u_1, u_2, u_{k-1}, u_k\}$.

We note that the set $D = \{v_1, v_2, v_{k-1}, v_k\}$ is a PD-set of the subgraph, $G[S]$, of G induced by the set S . Let $G' = G - S$, and note that G' is isomorphic to $L_{k'}$, where $k' = k - 6$. Thus, $1 \leq k' < k$ and $k' \not\equiv 5 \pmod{6}$. Applying the inductive hypothesis to G' , we have $\gamma_{\text{pr}}(G') = 2 \lceil \frac{k'+1}{3} \rceil = 2 \lceil \frac{k+1}{3} \rceil - 4$. Thus, $\gamma_{\text{pr}}(G) \leq \gamma_{\text{pr}}(G') + \gamma_{\text{pr}}(G[S]) \leq (2 \lceil \frac{k+1}{3} \rceil - 4) + 4 = 2 \lceil \frac{k+1}{3} \rceil$, as desired. Hence, by induction, $\gamma_{\text{pr}}(G) \leq 2 \lceil \frac{k+1}{3} \rceil$. As observed earlier, we have $\gamma_{\text{pr}}(G) \geq 2 \lceil \frac{k+1}{3} \rceil$. Consequently, $\gamma_{\text{pr}}(G) = 2 \lceil \frac{k+1}{3} \rceil$. This completes the proof of Claim 1. \square

By Claim 1, we may assume that $n \equiv 0 \pmod{12}$, for otherwise the desired result follows. With this assumption, $k \geq 5$ and $k \equiv 5 \pmod{6}$.

Suppose that $k = 5$, and so $G = L_5$. Let D be a γ_{pr} -set of G . Since D contains all support vertices of G , we have $\{v_1, v_5\} \subseteq D$. In order to dominate the vertex u_3 , the set D contains a vertex $v \in N_G[u_3]$. Thus, letting $A = \{v, v_1, v_5\}$, we note that the independent set A is a subset of D , implying that $\gamma_{\text{pr}}(G) \geq 2|A| = 6$.

Conversely, the set $\{v_1, v_2, u_3, v_3, v_4, v_5\}$ is a PD-set of G (with v_1 and v_2 paired, u_3 and v_3 paired, and v_4 and v_5 paired), and so $\gamma_{\text{pr}}(G) \leq 6$. Consequently, $\gamma_{\text{pr}}(G) = 6 = \frac{n}{3} + 2$ and this establishes the base case as we will further proceed with induction on k .

We show firstly that when $k \geq 5$ and $k \equiv 5 \pmod{6}$, we have $\gamma_{\text{pr}}(G) \leq \frac{n}{3} + 2$. Let $k \geq 11$, and assume that if $5 \leq k' < k$ where $k' \equiv 5 \pmod{6}$, then $\gamma_{\text{pr}}(L_{k'}) \leq \frac{2k'+2}{3} + 2$. Let the set S be defined as in the proof of Claim 1. As before, we note that $\gamma_{\text{pr}}(G[S]) \leq 4$. Let $G' = G - S$, and note that G' is isomorphic to $L_{k'}$, where $k' = k - 6$. Thus, $5 \leq k' < k$ and $k' \equiv 5 \pmod{6}$. Applying the inductive hypothesis to G' , we have $\gamma_{\text{pr}}(G') \leq \frac{2k'+2}{3} + 2 = \frac{2k+2}{3} - 2 = \frac{n}{3} - 2$. Thus, $\gamma_{\text{pr}}(G) \leq \gamma_{\text{pr}}(G') + \gamma_{\text{pr}}(G[S]) \leq (\frac{n}{3} - 2) + 4 = \frac{n}{3} + 2$, as desired.

We show next that when $k \geq 5$ and $k \equiv 5 \pmod{6}$, we have $\gamma_{\text{pr}}(G) \geq \frac{n}{3} + 2$. Let $k \geq 11$, and assume that if $5 \leq k' < k$ where $k' \equiv 5 \pmod{6}$, then $\gamma_{\text{pr}}(L_{k'}) \geq \frac{2k'+2}{3} + 2$. Let D be a γ_{pr} -set of G . Since D contains all support vertices of G , we have $\{v_1, v_k\} \subseteq D$.

We show that we can choose D so that $v_2 \in D$ and the vertices v_1 and v_2 are paired in D . Suppose that the vertex v_1 is paired with u_1 in D . (The case when v_1 is paired with v_0 is analogous.) If $v_2 \notin D$, then we can simply replace u_1 in D with the vertex v_2 , and in the resulting set pair v_1 and v_2 , as desired. Hence, we may assume that $v_2 \in D$. If v_2 is paired with u_2 , then we can simply replace the pairs v_1 and u_1 , and v_2 and u_2 , with the new pairing v_1 and v_2 , and u_1 and u_2 , to yield the desired result. Hence, we may assume that v_2 is paired with v_3 . If now $u_3 \in D$, then we contradict the minimality of the set D . Hence, $u_3 \notin D$. In this case, the set $D' = (D \setminus \{u_1\}) \cup \{u_3\}$ is a new γ_{pr} -set of G , with v_1 and v_2 paired, and u_3 and v_3 paired in D' , once again yielding a γ_{pr} -set of G with the desired property that v_1 and v_2 are paired in the set. Hence, we may choose D so that $v_2 \in D$ and the vertices v_1 and v_2 are paired in D .

With this choice of D , we note that $v_0 \notin D$. If $u_1 \in D$, then $u_2 \in D$ and u_1 and u_2 are partners in D . In this case, the vertex u_1 is only needed to partner the vertex u_2 , and we can replace u_1 in D with the vertex u_3 . Hence, we may assume that $u_1 \notin D$. If $v_3 \in D$, then we can replace v_3 in D with the vertex u_4 . Hence, we may further assume that $v_3 \notin D$. If $u_2 \in D$, then $u_3 \in D$ with u_2 and u_3 paired, and we can replace u_2 in D with the vertex u_4 . Hence, we can choose the set D so that $D \cap \{v_0, v_1, v_2, v_3, u_1, u_2\} = \{v_1, v_2\}$. Analogously, we can choose the set D so that $D \cap \{v_{k-2}, v_{k-1}, v_k, v_{k+1}, u_{k-1}, u_k\} = \{v_{k-1}, v_k\}$.

Let the set S be defined as in the proof of Claim 1, and consider the graph $G' = G - S$. We note that G' is isomorphic to $L_{k'}$, where $k' = k - 6$. Thus, $5 \leq k' < k$ and $k' \equiv 5 \pmod{6}$. By our choice of the set D , we note that the set $D' = D \setminus \{v_1, v_2, v_{k-1}, v_k\}$ is a PD-set of G' , and so $\gamma_{\text{pr}}(G') \leq |D'| = |D| - 4$. Applying the inductive hypothesis to G' , we have $|D| - 4 \geq \gamma_{\text{pr}}(G') \geq \frac{2k'+2}{3} + 2 = \frac{2k+2}{3} - 2 = \frac{n}{3} - 2$. Thus, $\gamma_{\text{pr}}(G) = |D| \geq \frac{n}{3} + 2$, as desired. As proven earlier,

$\gamma_{\text{pr}}(G) \leq \frac{n}{3} + 2$. Consequently, $\gamma_{\text{pr}}(G) = \frac{n}{3} + 2$. This completes the proof of Lemma 20. ■

We are now in a position to establish upper bounds on the γ_{pr}^+ -stability of a cycle prism $C_n \square K_2$ when $n \geq 6$ and $n \neq 7$.

Theorem 21. *For $n \geq 6$ and $n \neq 7$,*

$$\text{st}_{\gamma_{\text{pr}}}^+(C_n \square K_2) \leq \begin{cases} 3 & \text{if } n \equiv 0 \pmod{6}, \\ 4 & \text{if } n \pmod{6} \in \{2, 3\}, \\ 5 & \text{if } n \pmod{6} \in \{4, 5\}, \\ 6 & \text{if } n \equiv 1 \pmod{6}. \end{cases}$$

Proof. For $n \geq 6$ and $n \neq 7$, let $G = C_n \square K_2$. Let G_1 and G_2 be the two layers of the prism G used to build the graph G , where G_1 is the cycle $v_1 v_2 \cdots v_n v_1$ and G_2 is the cycle $u_1 u_2 \cdots u_n u_1$. Let G be obtained from G_1 and G_2 by adding the edges $u_i v_i$ for $i \in [n]$.

Suppose that $n \equiv 0 \pmod{6}$. In this case, $n = 6k$ for some $k \geq 1$, and $\gamma_{\text{pr}}(G) = 4k$. We consider the non-isolating set $S = \{u_1, u_3, u_5\}$. Let D be a γ_{pr} -set of $G - S$. Since D contains all support vertices of $G - S$, we have $\{v_2, v_4\} \subset D$. We can clearly choose the partner of v_2 in D as the vertex v_1 , and the partner of v_4 in D as the vertex v_5 . Thus, $D \cap \{u_2, v_3, u_4\} = \emptyset$. If $k = 1$, then $v_6 \in D$ in order to dominate the vertex u_6 , implying that $\gamma_{\text{pr}}(G - S) = 6 > 4 = 4k = \gamma_{\text{pr}}(G)$, and so $\text{st}_{\gamma_{\text{pr}}}^+(G) \leq |S| = 3$. Hence, we may assume now that $k \geq 2$.

If $v_n \in D$, then we can replace v_n in D with the vertex u_{n-1} . If $v_6 \in D$, then we can replace v_6 in D with the vertex u_7 . Hence, we may further choose D to contain neither v_6 nor v_n . Let $D_1 = \{v_1, v_2, v_4, v_5\}$ and let $G' = G - (S \cup N_G[D_1])$. We note that G' is isomorphic to $L_{6k-7} \in \mathcal{L}$, and so G' has order $n' = 12(k-1)$. By Lemma 20, $\gamma_{\text{pr}}(G') = 2 \left\lceil \frac{n'}{6} \right\rceil + 2 = 2 \left\lceil \frac{12(k-1)}{6} \right\rceil + 2 = 4k - 2$. By our choice of the set D , the set $D' = D \setminus D_1$ is a PD-set of G' , and so $\gamma_{\text{pr}}(G') \leq |D'| = |D| - 4 = \gamma_{\text{pr}}(G - S) - 4$. Thus, $\gamma_{\text{pr}}(G - S) \geq \gamma_{\text{pr}}(G') + 4 = 4k + 2 > 4k = \gamma_{\text{pr}}(G)$. This implies that $\text{st}_{\gamma_{\text{pr}}}^+(G) \leq |S| = 3$.

Suppose that $n \equiv 2 \pmod{6}$. In this case, $n = 6k + 2$ for some $k \geq 1$, and $\gamma_{\text{pr}}(G) = 4k + 2$. We consider the non-isolating set $S = \{u_1, u_3, u_5, u_7\}$. Let D be a γ_{pr} -set of $G - S$. Since D contains all support vertices of $G - S$, we have $\{v_2, v_4, v_6\} \subset D$. We can choose D so that the partner of v_2 in D as the vertex v_1 , the partner of v_6 in D as the vertex v_7 , and the partner of v_4 in D as either the vertex v_3 or v_5 , say v_3 . If $k = 1$, then $v_8 \in D$ in order to dominate the vertex u_8 , implying that $\gamma_{\text{pr}}(G - S) = 8 > 6 = 4k + 2 = \gamma_{\text{pr}}(G)$, and so $\text{st}_{\gamma_{\text{pr}}}^+(G) \leq |S| = 4$. Hence, we may assume now that $k \geq 2$.

If $v_n \in D$, then we can replace v_n in D with the vertex u_{n-1} . If $v_8 \in D$, then we can replace v_8 in D with the vertex u_9 . Hence, we may further choose

D to contain neither v_8 nor v_n . Let $D_1 = \{v_1, v_2, v_3, v_4, v_6, v_7\}$ and let $G' = G - (S \cup N_G[D_1])$. We note that G' is isomorphic to $L_{6k-7} \in \mathcal{L}$, and so G' has order $n' = 12(k-1)$. By Lemma 20, $\gamma_{\text{pr}}(G') = 4k-2$. By our choice of the set D , the set $D' = D \setminus D_1$ is a PD-set of G' , and so $\gamma_{\text{pr}}(G') \leq |D'| = |D| - 6 = \gamma_{\text{pr}}(G - S) - 6$. Thus, $\gamma_{\text{pr}}(G - S) \geq \gamma_{\text{pr}}(G') + 6 = 4k + 4 > 4k + 2 = \gamma_{\text{pr}}(G)$. This implies that $\text{st}_{\gamma_{\text{pr}}}^+(G) \leq |S| = 4$.

Suppose that $n \equiv 3 \pmod{6}$. The reasoning is analogous as in the case when $n \equiv 2 \pmod{6}$.

Suppose that $n \equiv 4 \pmod{6}$. In this case, $n = 6k + 4$ for some $k \geq 1$, and $\gamma_{\text{pr}}(G) = 4k + 4$. We consider the non-isolating set $S = \{u_1, u_3, u_5, u_7, u_9\}$. Let D be a γ_{pr} -set of $G - S$. Since D contains all support vertices of $G - S$, we have $\{v_2, v_4, v_6, v_8\} \subset D$. Let $D_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9\}$. We can choose D so that $D_1 \subset D$, where v_1 and v_2 are paired, v_3 and v_4 are paired, v_5 and v_6 are paired, and v_8 and v_9 are paired. If $k = 1$, then $v_{10} \in D$ in order to dominate the vertex u_{10} , implying that $\gamma_{\text{pr}}(G - S) = 10 > 8 = 4k + 4 = \gamma_{\text{pr}}(G)$, and so $\text{st}_{\gamma_{\text{pr}}}^+(G) \leq |S| = 5$. Hence, we may assume now that $k \geq 2$. We can now further choose D to contain neither v_{10} nor v_n . Let $G' = G - (S \cup N_G[D_1])$. We note that G' is isomorphic to $L_{6k-7} \in \mathcal{L}$, and so G' has order $n' = 12(k-1)$. By Lemma 20, $\gamma_{\text{pr}}(G') = 4k-2$. By our choice of the set D , the set $D' = D \setminus D_1$ is a PD-set of G' , and so $\gamma_{\text{pr}}(G') \leq |D'| = |D| - 8 = \gamma_{\text{pr}}(G - S) - 8$. Thus, $\gamma_{\text{pr}}(G - S) \geq \gamma_{\text{pr}}(G') + 8 = 4k + 6 > 4k + 4 = \gamma_{\text{pr}}(G)$. This implies that $\text{st}_{\gamma_{\text{pr}}}^+(G) \leq |S| = 5$.

Suppose that $n \equiv 5 \pmod{6}$. The case is analogous to the case when $n \equiv 4 \pmod{6}$.

Suppose that $n \equiv 1 \pmod{6}$. In this case, $n = 6k + 1$ for some $k \geq 2$, and $\gamma_{\text{pr}}(G) = 4k + 2$. We consider the non-isolating set $S = \{u_1, u_3, u_5, u_7, u_9, u_{11}\}$. Let D be a γ_{pr} -set of $G - S$. Let $D_1 = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7, v_8, v_{10}, v_{11}\}$. We can choose D so that $D_1 \subseteq D$, where v_1 and v_2 are paired, v_3 and v_4 are paired, v_5 and v_6 are paired, v_7 and v_8 are paired, and v_{10} and v_{11} are paired. If $k = 2$, then two additional vertices are needed in D , implying that $\gamma_{\text{pr}}(G - S) \geq 12 > 10 = 4k + 2 = \gamma_{\text{pr}}(G)$, and so $\text{st}_{\gamma_{\text{pr}}}^+(G) \leq |S| = 6$. Hence, we may assume now that $k \geq 3$. We can now further choose D to contain neither v_{12} nor v_n . We let $G' = G - (S \cup N_G[D_1])$, and note that G' is isomorphic to $L_{6(k-2)} \in \mathcal{L}$, and so G' has order $n' = 12k - 22$. By Lemma 20, $\gamma_{\text{pr}}(G') = 2 \left\lceil \frac{n'}{6} \right\rceil = 2 \left\lceil \frac{12k-22}{6} \right\rceil = 4k - 6$. By our choice of the set D , the set $D' = D \setminus D_1$ is a PD-set of G' , and so $\gamma_{\text{pr}}(G') \leq |D'| = |D| - 10 = \gamma_{\text{pr}}(G - S) - 10$. Thus, $\gamma_{\text{pr}}(G - S) \geq \gamma_{\text{pr}}(G') + 10 = 4k + 4 > 4k + 2 = \gamma_{\text{pr}}(G)$. This implies that $\text{st}_{\gamma_{\text{pr}}}^+(G) \leq 6$. ■

As an immediate consequence of Theorems 19 and 21, we have the following result on the paired domination stability of a cycle prism.

Corollary 22. *For $n \geq 4$, $\text{st}_{\gamma_{\text{pr}}}(C_n \square K_2) \leq 4$, with strict inequality if $n \pmod 6 \in \{0, 1, 4\}$.*

It remains an open problem to determine the exact value of the γ_{pr}^+ -stability of a cycle prism $C_n \square K_2$ for $n \geq 6$ and $n \neq 7$.

REFERENCES

- [1] M. Amraee, N. Jafari Rad and M. Maghasedi, *Roman domination stability in graphs*, Math. Rep. (Bucur.) **21(71)** (2019) 193–204.
- [2] S. Arumugam and R. Kala, *Domination parameters of hypercubes*, J. Indian Math. Soc. (N.S.) **65** (1998) 31–38.
- [3] J. Azarija, M.A. Henning and S. Klavžar, *(Total) domination in prisms*, Electron. J. Combin. **24(1)** (2017) #P1.19.
<https://doi.org/10.37236/6288>
- [4] A. Aytaç and B. Atay Atakul, *Exponential domination critical and stability in some graphs*, Internat. J. Found. Comput. Sci. **30** (2019) 781–791.
<https://doi.org/10.1142/S0129054119500217>
- [5] D. Bauer, F. Harary, J. Nieminen and C.L. Suffel, *Domination alternation sets in graphs*, Discrete Math. **47** (1983) 153–161.
[https://doi.org/10.1016/0012-365X\(83\)90085-7](https://doi.org/10.1016/0012-365X(83)90085-7)
- [6] W. Goddard and M.A. Henning, *A note on domination and total domination in prisms*, J. Comb. Optim. **35** (2018) 14–20.
<https://doi.org/10.1007/s10878-017-0150-0>
- [7] A. Gorzkowska, M.A. Henning, M. Pilśniak and E. Tumidajewicz, *Paired domination stability in graphs*, (2020), manuscript.
- [8] F. Harary and M. Livingston, *Independent domination in hypercubes*, Appl. Math. Lett. **6** (1993) 27–28.
[https://doi.org/10.1016/0893-9659\(93\)90027-K](https://doi.org/10.1016/0893-9659(93)90027-K)
- [9] T.W. Haynes, M.A. Henning and L.C. van der Merwe, *Domination and total domination in complementary prisms*, J. Comb. Optim. **18** (2009) 23–37.
<https://doi.org/10.1007/s10878-007-9135-8>
- [10] T.W. Haynes and P.J. Slater, *Paired domination in graphs*, Networks **32** (1998) 199–206.
[https://doi.org/10.1002/\(SICI\)1097-0037\(199810\)32:3<199::AID-NET4>3.0.CO;2-F](https://doi.org/10.1002/(SICI)1097-0037(199810)32:3<199::AID-NET4>3.0.CO;2-F)
- [11] M.A. Henning and M. Krzywkowski, *Total domination stability in graphs*, Discrete Appl. Math. **236** (2018) 246–255.
<https://doi.org/10.1016/j.dam.2017.07.022>
- [12] M.A. Henning and D.F. Rall, *On the total domination number of Cartesian products of graphs*, Graphs Combin. **21** (2005) 63–69.
<https://doi.org/10.1007/s00373-004-0586-8>

- [13] M.A. Henning and A. Yeo, *Total Domination in Graphs* (Springer Monographs in Mathematics, 2013).
<https://doi.org/10.1007/978-1-4614-6525-6>
- [14] Z. Li, Z. Shao and S.J. Xu, *2-rainbow domination stability of graphs*, *J. Comb. Optim.* **38** (2019) 836–845.
<https://doi.org/10.1007/s10878-019-00414-0>
- [15] M. Mollard, *On perfect codes in Cartesian product of graphs*, *European J. Combin.* **32** (2011) 398–403.
<https://doi.org/10.1016/j.ejc.2010.11.007>
- [16] C.M. Mynhardt and M. Schurch, *Paired domination in prisms of graphs*, *Discuss. Math. Graph Theory* **31** (2011) 5–23.
<https://doi.org/10.7151/dmgt.1526>
- [17] P.R.J. Östergård and U. Blass, *On the size of optimal binary codes of length 9 and covering radius 1*, *IEEE Trans. Inform. Theory* **47** (2001) 2556–2557.
<https://doi.org/10.1109/18.945268>
- [18] N. Jafari Rad, E. Sharifi and M. Krzywkowski, *Domination stability in graphs*, *Discrete Math.* **339** (2016) 1909–1914.
<https://doi.org/10.1016/j.disc.2015.12.026>
- [19] G.J.M. van Wee, *Improved sphere bounds on the covering radius of codes*, *IEEE Trans. Inform. Theory* **34** (1988) 237–245.
<https://doi.org/10.1109/18.2632>

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