

THE ACHROMATIC NUMBER OF $K_6 \square K_q$
EQUALS $2q + 3$ IF $q \geq 41$ IS ODD

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Abstract

Let G be a graph and C be a finite set of colours. A vertex colouring $f : V(G) \rightarrow C$ is complete provided that for any two distinct colours $c_1, c_2 \in C$ there is $v_1v_2 \in E(G)$ such that $f(v_i) = c_i$, $i = 1, 2$. The achromatic number of G is the maximum number of colours in a proper complete vertex colouring of G . In the paper it is proved that if $q \geq 41$ is an odd integer, then the achromatic number of the Cartesian product of K_6 and K_q is $2q + 3$.

Keywords: complete vertex colouring, achromatic number, Cartesian product, complete graph.

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1. INTRODUCTION

Let G be a finite simple graph and C a finite set of colours. A vertex colouring $f : V(G) \rightarrow C$ is *complete* if for any pair of distinct colours $c_1, c_2 \in C$ one can find in G an edge $\{v_1, v_2\}$ (often shortened to v_1v_2) such that $f(v_i) = c_i$, $i = 1, 2$. The *achromatic number* of G , denoted by $\text{achr}(G)$, is the maximum cardinality of the colour set in a proper complete vertex colouring of G .

The concept was introduced quite a long ago in Harary, Hedetniemi and Prins [8], where the following was proved.

Theorem 1. *If G is a graph, and an integer k satisfies $\chi(G) \leq k \leq \text{achr}(G)$, then there exists a proper complete vertex colouring of G using k colours.*

There are still only a few graph classes \mathcal{G} such that $\text{achr}(G)$ is known for all $G \in \mathcal{G}$. This is certainly related to the fact that determining the achromatic number is an NP-complete problem even for trees, see Cairnie and Edwards [2].

Two surveys are available on the topic, namely Edwards [6] and Hughes and MacGillivray [10]; more generally, Chapter 12 in the book [3] by Chartrand and Zhang deals with complete vertex colourings. A comprehensive list of publications concerning the achromatic number is maintained by Edwards [7].

Some papers are devoted to the achromatic number of graphs constructed by graph operations. So, Hell and Miller [9] considered $\text{achr}(G_1 \times G_2)$, where $G_1 \times G_2$ stands for the categorical product of graphs G_1 and G_2 (we follow here the notation by Imrich and Klavžar [16]).

In this paper we are interested in the achromatic number of the Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2 , the graph with $V(G_1 \square G_2) = \{(v_1, v_2) : v_i \in V(G_i), i = 1, 2\}$, in which $(v_1^1, v_2^1)(v_1^2, v_2^2) \in E(G_1 \square G_2)$ if and only if there is $i \in \{1, 2\}$ such that $v_i^1 v_i^2 \in E(G_i)$ and $v_{3-i}^1 = v_{3-i}^2$. As observed by Chiang and Fu [4], $\text{achr}(G_1) = p$ and $\text{achr}(G_2) = q$ implies $\text{achr}(G_1 \square G_2) \geq \text{achr}(K_p \square K_q)$. This inequality motivates a special interest in the achromatic number of the Cartesian product of two complete graphs. From the obvious fact that $G_2 \square G_1$ is isomorphic to $G_1 \square G_2$ it is clear that when determining $\text{achr}(K_p \square K_q)$ we may suppose without loss of generality $p \leq q$.

The problem of determining $\text{achr}(K_p \square K_q)$ with $p \leq 4$ was solved in Horňák and Puntigán [15] (for $p \leq 3$ the result was rediscovered in [4]) and that for $p = 5$ in Horňák and Pćola [13, 14]. In [5] Chiang and Fu proved that if r is an odd projective plane order, then $\text{achr}(K_{(r^2+r)/2} \square K_{(r^2+r)/2}) = (r^3 + r^2)/2$. (For $r = 3$ the fact that $\text{achr}(K_6 \square K_6) = 18$ was known already to Bouchet [1].)

Here we show that $\text{achr}(K_6 \square K_q) = 2q + 3$ if q is an odd integer with $q \geq 41$. This is the first of three papers devoted to completely solve the problem of finding $\text{achr}(K_6 \square K_q)$. In Horňák [11] the cases, in which either $8 \leq q \leq 40$ or q is an even integer with $q \geq 42$, are analysed. Finally, $\text{achr}(K_6 \square K_7)$ is determined in Horňák [12].

For $k, l \in \mathbb{Z}$ we denote *integer intervals* by

$$[k, l] = \{z \in \mathbb{Z} : k \leq z \leq l\}, \quad [k, \infty) = \{z \in \mathbb{Z} : k \leq z\}.$$

Further, with a set A and $m \in [0, \infty)$ we use $\binom{A}{m}$ for the set of m -element subsets of A .

Now let $p, q \in [1, \infty)$. Under the assumption that $V(K_r) = [1, r]$, $r = p, q$, we have $V(K_p \square K_q) = [1, p] \times [1, q]$, while $E(K_p \square K_q)$ consists of edges $(i, j_1)(i, j_2)$ with $i \in [1, p]$, $j_1, j_2 \in [1, q]$, $j_1 \neq j_2$ and $(i_1, j)(i_2, j)$ with $i_1, i_2 \in [1, p]$, $i_1 \neq i_2$, $j \in [1, q]$.

A vertex colouring $f : [1, p] \times [1, q] \rightarrow C$ of the graph $K_p \square K_q$ can be conveniently described using the $p \times q$ matrix $M = M(f)$ whose entry in the i th row and the j th column is $(M)_{i,j} = f(i, j)$. Such a colouring is proper if any row of M consists of q distinct entries and any column of M consists of p distinct entries. Further, f is complete provided that any pair $\{\alpha, \beta\} \in \binom{C}{2}$ is *good*

in M in the following sense: there are $(i_1, j_1), (i_2, j_2) \in [1, p] \times [1, q]$ such that $\{(M)_{i_1, j_1}, (M)_{i_2, j_2}\} = \{\alpha, \beta\}$ and either $i_1 = i_2$, which we express by saying that the pair $\{\alpha, \beta\}$ is *row-based* (in M), or $j_1 = j_2$, i.e., the pair $\{\alpha, \beta\}$ is *column-based* (in M).

Let $\mathcal{M}(p, q, C)$ denote the set of $p \times q$ matrices M with entries from C such that all rows (columns) of M have q (p , respectively) distinct entries, and each pair $\{\alpha, \beta\} \in \binom{C}{2}$ is good in M . So, if $f : [1, p] \times [1, q] \rightarrow C$ is a proper complete vertex colouring of $K_p \square K_q$, then $M(f) \in \mathcal{M}(p, q, C)$.

Conversely, if $M \in \mathcal{M}(p, q, C)$, then the mapping $f_M : [1, p] \times [1, q] \rightarrow C$ with $f_M(i, j) = (M)_{i, j}$ is a proper complete vertex colouring of $K_p \square K_q$. Thus, we have proved:

Proposition 2. *If $p, q \in [1, \infty)$ and C is a finite set, then the following statements are equivalent.*

- (1) *There is a proper complete vertex colouring of $K_p \square K_q$ using as colours elements of C .*
- (2) $\mathcal{M}(p, q, C) \neq \emptyset$.

We have another evident result.

Proposition 3. *If $p, q \in [1, \infty)$, C, D are finite sets, $M \in \mathcal{M}(p, q, C)$, mappings $\rho : [1, p] \rightarrow [1, p]$, $\sigma : [1, q] \rightarrow [1, q]$, $\pi : C \rightarrow D$ are bijections, and $M_{\rho, \sigma}$, M_π are $p \times q$ matrices defined by $(M_{\rho, \sigma})_{i, j} = (M)_{\rho(i), \sigma(j)}$ and $(M_\pi)_{i, j} = \pi((M)_{i, j})$, then $M_{\rho, \sigma} \in \mathcal{M}(p, q, C)$ and $M_\pi \in \mathcal{M}(p, q, D)$.*

Let $M \in \mathcal{M}(p, q, C)$. The *frequency* of a colour $\gamma \in C$ is the number $\text{frq}(\gamma)$ of appearances of γ in M , and the *frequency* of M , denoted $\text{frq}(M)$, is the minimum of frequencies of colours in C . A colour of frequency l is an *l -colour*. C_l is the set of l -colours, $c_l = |C_l|$, and C_{l+} is the set of colours of frequency at least l , $c_{l+} = |C_{l+}|$. We denote by $\mathbb{R}(i)$ the set $\{(M)_{i, j} : j \in [1, q]\}$ of colours in the i th row of M and by $\mathbb{C}(j)$ the set $\{(M)_{i, j} : i \in [1, p]\}$ of colours in the j th column of M . Further, for $k \in \{l, l+\}$ let

$$\begin{aligned} \mathbb{R}_k(i) &= C_k \cap \mathbb{R}(i), & r_k(i) &= |\mathbb{R}_k(i)|, \\ \mathbb{C}_k(j) &= C_k \cap \mathbb{C}(j), & c_k(j) &= |\mathbb{C}_k(j)|. \end{aligned}$$

So $\mathbb{R}_l(i)$ and $\mathbb{R}_{l+}(i)$ is the set of colours in the row i which occur exactly and at least l times altogether, respectively; the meaning of $\mathbb{C}_l(j)$ and $\mathbb{C}_{l+}(j)$ is similar. If $A \subseteq [1, p]$, $|A| \geq 2$, then

$$\mathbb{R}(A) = \bigcap_{l \in A} (C_{|A|} \cap \mathbb{R}(l)), \quad r(A) = |\mathbb{R}(A)|.$$

$\mathbb{R}(A)$ is the set of colours which appear precisely in the rows numbered by A . Provided that $A \in \{\{i, j\}, \{i, j, k\}, \{i, j, k, l\}\}$, instead of $\mathbb{R}(A)$ we write $\mathbb{R}(i, j)$, $\mathbb{R}(i, j, k)$, $\mathbb{R}(i, j, k, l)$, while $r(A)$ is simplified to $r(i, j)$, $r(i, j, k)$, $r(i, j, k, l)$, respectively. With $\{i, j\} \subseteq [1, p]$ and $\{m, n\} \subseteq [1, q]$ we set

$$\begin{aligned}\mathbb{R}_{3+}(i, j) &= C_{3+} \cap \mathbb{R}(i) \cap \mathbb{R}(j), & r_{3+}(i, j) &= |\mathbb{R}_{3+}(i, j)|, \\ \mathbb{C}(m, n) &= C_2 \cap \mathbb{C}(m) \cap \mathbb{C}(n), & c(m, n) &= |\mathbb{C}(m, n)|.\end{aligned}$$

$\mathbb{R}_{3+}(i, j)$ is the set of colours of frequency at least 3 occuring in both rows i and j , and $\mathbb{C}(m, n)$, an analogue of the notation $\mathbb{R}(i, j)$, stands for the set of 2-colours occuring exactly in the columns m and n . If $B \subseteq [1, p]$ and $3 \leq |B| \leq p-2$, then

$$\mathbb{R}^*(B) = \bigcup_{l=2}^{|B|} \bigcup_{A \in \binom{B}{l}} \mathbb{R}(A).$$

$\mathbb{R}^*(B)$ is the set of colours of frequency at least 2 occuring only in the rows numbered by B . Since $\{\mathbb{R}(A) : \exists l \in [2, |B|] A \in \binom{B}{l}\}$ is a set of pairwise disjoint sets, we have

$$r^*(B) = |\mathbb{R}^*(B)| = \sum_{l=2}^{|B|} \sum_{A \in \binom{B}{l}} r(A).$$

For $\gamma \in C$ let

$$\mathbb{R}(\gamma) = \{i \in [1, p] : \gamma \in \mathbb{R}(i)\}$$

be the set of (the numbers of) the rows containing the colour γ .

With $S \subseteq [1, p] \times [1, q]$ we say that a colour $\gamma \in C$ *occupies a position in S* if there is $(i, j) \in S$ such that $(M)_{i,j} = \gamma$. If $\emptyset \neq A \subseteq C$, the *set of columns covered by A* is

$$\text{Cov}(A) = \{j \in [1, q] : \mathbb{C}(j) \cap A \neq \emptyset\},$$

i.e., the set of columns containing an element of A . We define $\text{cov}(A) = |\text{Cov}(A)|$, and with $A \in \{\{\alpha\}, \{\alpha, \beta\}\}$ we use a simplified notation $\text{Cov}(\alpha)$, $\text{Cov}(\alpha, \beta)$ and $\text{cov}(\alpha)$, $\text{cov}(\alpha, \beta)$ instead of $\text{Cov}(A)$ and $\text{cov}(A)$.

2. LOWER BOUND

Proposition 4. *If $q \in [7, \infty)$ and $q \equiv 1 \pmod{2}$, then $\text{achr}(K_6 \square K_q) \geq 2q + 3$.*

Proof. Let $s = \frac{q-3}{2}$, and let M be the $6 \times q$ matrix below. We show that $M \in \mathcal{M}(6, q, C)$, where $C = [1, 9] \cup X_s \cup Y_s \cup Z_s \cup T_s$, $U_s = \{u_i : i \in [1, s]\}$ for

$U \in \{X, Y, Z, T\}$, and the sets $[1, 9]$, X_s, Y_s, Z_s, T_s are pairwise disjoint.

$$\begin{pmatrix} 1 & 2 & 3 & x_1 & x_2 & \dots & x_{s-1} & x_s & y_1 & y_2 & \dots & y_{s-1} & y_s \\ 4 & 5 & 6 & x_s & x_1 & \dots & x_{s-2} & x_{s-1} & z_1 & z_2 & \dots & z_{s-1} & z_s \\ 7 & 8 & 9 & t_1 & t_2 & \dots & t_{s-1} & t_s & x_1 & x_2 & \dots & x_{s-1} & x_s \\ 3 & 1 & 2 & z_1 & z_2 & \dots & z_{s-1} & z_s & t_1 & t_2 & \dots & t_{s-1} & t_s \\ 5 & 6 & 4 & t_s & t_1 & \dots & t_{s-2} & t_{s-1} & y_s & y_1 & \dots & y_{s-2} & y_{s-1} \\ 8 & 9 & 7 & y_1 & y_2 & \dots & y_{s-1} & y_s & z_s & z_1 & \dots & z_{s-2} & z_{s-1} \end{pmatrix}$$

Since $s \geq 2$, because of our assumptions on the structure of C it is clear that elements in lines (rows and columns) of M are pairwise distinct. Thus it is sufficient to show that each pair $\{\alpha, \beta\} \in \binom{C}{2}$ is good in M .

If $\alpha, \beta \in [1, 9]$, then both α and β appear twice in the columns 1, 2, 3, hence the pair $\{\alpha, \beta\}$ is column-based.

If $\alpha \in C$ and $\beta \in X_s \cup Y_s \cup Z_s \cup T_s$, realise that $\mathbb{R}(\alpha) \in \mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathbb{R}(\beta) \in \mathcal{R}_2$, where $\mathcal{R}_1 = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$ and $\mathcal{R}_2 = \{\{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}\}$. As $R \cap R_2 \neq \emptyset$ for any $R \in \mathcal{R}_1 \cup \mathcal{R}_2$ and any $R_2 \in \mathcal{R}_2$, the pair $\{\alpha, \beta\}$ is row-based.

So, Proposition 2 yields $\text{achr}(K_6 \square K_q) \geq |C| = 4s + 9 = 2q + 3$. \blacksquare

3. AUXILIARY RESULTS

Let $M \in \mathcal{M}(p, q, C)$ and let $\gamma \in C$. For the (complete) colouring f_M from the proof of Proposition 2 denote $V_\gamma = f_M^{-1}(\gamma) \subseteq [1, p] \times [1, q]$, and let $N(V_\gamma)$ be the neighbourhood of V_γ (the union of neighborhoods of vertices in V_γ). The *excess* of γ is defined to be the maximum number $\text{exc}(\gamma)$ of vertices in a set $S \subseteq N(V_\gamma)$ such that each pair $\{\gamma, \gamma'\} \in \binom{C}{2}$ is good even in the “partial matrix” corresponding to the restriction of f_M created by uncolouring the vertices of S .

Lemma 5. *If $p, q \in [1, \infty)$, C is a finite set, $M \in \mathcal{M}(p, q, C)$ and $\gamma \in C$, then the following hold.*

1. $\text{frq}(\gamma) \leq \min(p, q)$;
2. $\text{frq}(\gamma) = l$ implies $\text{exc}(\gamma) = l(p + q - l - 1) - (|C| - 1) \geq 0$;
3. $\text{frq}(M) = l$ implies $|C| \leq \lfloor \frac{pq}{l} \rfloor$.

Proof. 1. The assumption $\text{frq}(\gamma) = l > \min(p, q)$ would mean, by the pigeonhole principle, that the colouring f_M is not proper.

2. Because of Proposition 3 we may suppose without loss of generality $(M)_{i,i} = \gamma$ for all $i \in [1, l]$. For simplicity we use (w) to indicate that it is just Proposition 3, which enables us to restrict our attention to matrices with a special property.

The colouring f_M is complete, hence each of $|C| - 1$ colours in $C \setminus \{\gamma\}$ must occupy a position in the set $N(V_\gamma) = \{(i, j) : (i \leq l \vee j \leq l) \wedge i \neq j\}$. Thus, $|N(V_\gamma)| = ql + (p - l)l - l \geq |C| - 1$ and $\text{exc}(\gamma) = |N(V_\gamma)| - (|C| - 1) = l(p + q - l - 1) - (|C| - 1) \geq 0$.

3. If $\alpha \in C$ is such that $\text{frq}(\alpha) = \text{frq}(M) = l$ and $\beta \in C$, then $\text{frq}(\beta) \geq \text{frq}(\alpha) = l$. Therefore, the total number of entries of the matrix M is $pq \geq l|C|$, and the desired inequality follows. ■

The *excess* of a matrix $M \in \mathcal{M}(p, q, C)$, denoted by $\text{exc}(M)$, is the minimum of excesses of colours in C .

Lemma 6. *If $p, q \in [1, \infty)$, C is a finite set and $M \in \mathcal{M}(p, q, C)$, then $\text{exc}(M) = \text{exc}(\gamma)$, where $\gamma \in C$ and $\text{frq}(\gamma) = \text{frq}(M)$.*

Proof. Let $m = \min(p, q)$, and let $\gamma \in C$ be such that $l = \text{frq}(\gamma) = \text{frq}(M)$. For any $\alpha \in C$ then $k = \text{frq}(\alpha) \geq \text{frq}(\gamma) = l$, and, by Lemma 5.2, $\text{exc}(\alpha) = k(p + q - k - 1) - |C| + 1 \geq 0$. Therefore, $\text{exc}(\alpha) - \text{exc}(\gamma) = k(p + q - k - 1) - l(p + q - l - 1) \geq 0$, since $h(x) = x(p + q - x - 1)$ is increasing in the interval $\langle 1, \frac{p+q-1}{2} \rangle \supseteq \langle 1, m - 1 \rangle$, and $p = q = m$ implies $h(m - 1) = h(m)$. Consequently, $\text{exc}(M) = \text{exc}(\gamma)$. ■

Lemma 7 (see [15] and [4]). *If $p, q \in [1, \infty)$ and $p \leq q$, then*

$$\text{achr}(K_p \square K_q) \leq \max \left(\min (l(p + q - l - 1) + 1, \lfloor pq/l \rfloor) : l \in [1, p] \right).$$

Corollary 8. *If $q \in [7, \infty)$, then $\text{achr}(K_6 \square K_q) \leq 2q + 7$.*

Proof. By Lemma 7 with $p = 6$ we obtain $\text{achr}(K_6 \square K_q) \leq \max (q + 5, 2q + 7, 2q, \lfloor \frac{6q}{4} \rfloor, \lfloor \frac{6q}{5} \rfloor, q) = 2q + 7$. ■

4. PROPERTIES OF MATRICES IN $\mathcal{M}(6, q, C)$

Suppose we know that $\text{achr}(K_6 \square K_q) \geq 2q + s - 1$ for a pair (q, s) with $q \in [7, \infty)$ and $s \in [1, \infty)$, and we want to prove that $\text{achr}(K_6 \square K_q) = 2q + s - 1$; clearly, because of Corollary 8 it is sufficient to work with $s \leq 7$. Proceeding by the way of contradiction let s satisfy $\text{achr}(K_6 \square K_q) = 2q + s$. By Theorem 1 and Proposition 2 there is a $(2q + s)$ -element set C and a matrix $M \in \mathcal{M}(6, q, C)$. Our task will be accomplished by showing that the existence of M leads to a contradiction. For that purpose we shall need properties of M . So in all claims of the present section we suppose that the notation corresponds to a matrix $M \in \mathcal{M}(6, q, C)$ with $q \in [7, \infty)$ and $|C| = 2q + s \leq 2q + 7$. We associate with M an auxiliary graph G with $V(G) = [1, 6]$, in which $\{i, k\} \in E(G)$ if and only if $r(i, k) \geq 1$ (so that there is a 2-colour appearing in both rows i and k).

Claim 9. *The following statements are true:*

1. $c_1 = 0$;
2. $c_l = 0$ for $l \in [7, \infty)$;
3. $c_2 \geq 3s$;
4. $c_{3+} \leq 2q - 2s$;
5. $\sum_{i=3}^6 ic_i \leq 6q - 6s$;
6. $\text{frq}(M) = 2$;
7. $\text{exc}(M) = 7 - s$;
8. $c_{4+} \leq c_2 - 3s$;
9. if $\{i, k\} \in \binom{[1,6]}{2}$, then $r(i, k) \leq 8 - s$.

Proof. 1. If $c_1 > 0$, a 1-colour $\gamma \in C$ satisfies $\text{exc}(\gamma) = q + 4 - (2q + s - 1) = 5 - q - s < 0$ in contradiction to Lemma 5.2.

2. Use Lemma 5.1.

3. By Claims 9.1 and 9.2, counting the number of vertices of $K_6 \square K_q$ we get $6q = \sum_{i=2}^6 ic_i$. Therefore, $3(2q + s) = 3|C| = 3(c_2 + c_{3+}) \leq c_2 + \sum_{i=2}^6 ic_i = c_2 + 6q$, which yields $c_2 \geq 3s$.

4. From $2(2q + s) + c_{3+} = 2|C| + c_{3+} = 2c_2 + 3c_{3+} \leq \sum_{i=2}^6 ic_i = 6q$ we obtain $c_{3+} \leq 2q - 2s$.

5. The assertion of Claim 9.3 leads to $\sum_{i=3}^6 ic_i = \sum_{i=2}^6 ic_i - 2c_2 = 6q - 2c_2 \leq 6q - 6s$.

6. A consequence of Claims 9.1, 9.3 and the assumption $s \in [1, 7]$.

7. Since $\text{frq}(M) = 2$ (Claim 9.6), by Lemma 6 we get $\text{exc}(M) = 2q + 6 - (2q + s - 1) = 7 - s$.

8. We have $3(2q + s) - c_2 + c_{4+} = 3(c_2 + c_3 + c_{4+}) - c_2 + c_{4+} \leq \sum_{i=2}^6 ic_i = 6q$ and $c_{4+} \leq c_2 - 3s$.

9. The inequality is trivial if $r(i, k) = 0$. If $\gamma \in \mathbb{R}(i, k)$, then each colour of $\mathbb{R}(i, k) \setminus \{\gamma\}$ contributes one to the excess of γ , hence, by Claims 9.6 and 9.7, $r(i, k) - 1 \leq \text{exc}(\gamma) = \text{exc}(M) = 7 - s$ and $r(i, k) \leq 8 - s$. ■

Claim 10. *If $\{i, k\} \in \binom{[1,6]}{2}$ and $r(i, k) \geq 1$, then $r(i, k) + r_{3+}(i, k) \leq 8 - s$.*

Proof. With $\gamma \in \mathbb{R}(i, k)$ each colour of $(\mathbb{R}(i, k) \setminus \{\gamma\}) \cup \mathbb{R}_{3+}(i, k)$ makes a contribution of one to the excess of γ , hence $r(i, k) - 1 + r_{3+}(i, k) \leq \text{exc}(\gamma) = \text{exc}(M) \leq 7 - s$, and the claim follows. ■

Claim 11. *If $\{i, k\} \in \binom{[1,6]}{2}$, $r(i, k) \geq 1$, $B \subseteq [1, 6]$, $3 \leq |B| \leq 4$ and $B \cap \{i, k\} = \emptyset$, then $r^*(B) \leq 2|B|$.*

Proof. Consider a colour $\gamma \in \mathbb{R}(i, k)$ with $(M)_{i,j} = (M)_{k,l} = \gamma$ (where, of course, $j \neq l$). If $\beta \in \mathbb{R}^*(B)$, there is $A \subseteq B$ with $|A| \geq 2$ and $\beta \in \mathbb{R}(A)$. The colour β

appears in neither of the rows i, k , hence the pair $\{\beta, \gamma\}$ is good in M only if β occupies a position in the set $\bigcup_{m \in A} \{(m, j), (m, l)\} \subseteq \bigcup_{m \in B} \{(m, j), (m, l)\}$. As a consequence, $r^*(B) = |\mathbb{R}^*(B)| \leq |\bigcup_{m \in B} \{(m, j), (m, l)\}| = 2|B|$. ■

Claim 12. *If $\{i, j, k, l, m, n\} = [1, 6]$ and $r(i, j, k) \geq 1$, then $r(l, m, n) \leq 9$.*

Proof. There is nothing to prove if $\mathbb{R}(l, m, n) = \emptyset$. Further, with $\alpha \in \mathbb{R}(i, j, k)$ and $\beta \in \mathbb{R}(l, m, n)$ the pair $\{\alpha, \beta\}$ is good in M only if the colour β occupies a position in the 9-element set $\{l, m, n\} \times \text{Cov}(\alpha)$. ■

Claim 13. *If $\Delta(G) \geq 4$, then $q \leq 40 - 5s$.*

Proof. Suppose (w) $\Delta(G) = \deg_G(1) \geq 4$, and, moreover, let (w) the sequence $(r(1, k))_{k=2}^6$ be nondecreasing. The least $p \in [2, 6]$ with $r(1, p) \geq 1$ satisfies $p \leq 3$, and we have $\mathbb{R}_{3+}(1) = \bigcup_{k=p}^6 \mathbb{R}_{3+}(1, k)$. Then, by Claim 9.1, $q = |\mathbb{R}(1)| = r_2(1) + r_{3+}(1)$. The inequality $r(1, k) \geq 1$ for $k \in [p, 6]$ yields, by Claim 10, $r_{3+}(1, k) \leq 8 - s - r(1, k)$; therefore,

$$\begin{aligned} q - r_2(1) = r_{3+}(1) &= \left| \bigcup_{k=p}^6 \mathbb{R}_{3+}(1, k) \right| \leq \sum_{k=p}^6 r_{3+}(1, k) \\ &\leq \sum_{k=p}^6 [8 - s - r(1, k)] = (7 - p)(8 - s) - \sum_{k=p}^6 r(1, k), \end{aligned}$$

and then, since $r_2(1) = \sum_{k=p}^6 r(1, k)$, we finish with $q \leq (7 - p)(8 - s) \leq 40 - 5s$. ■

Claim 14. *If $\Delta(G) = 3$, $\{i, j, k, l, m, n\} = [1, 6]$, $r(i, l) \geq 1$, $r(j, l) \geq 1$ and $r(k, l) \geq 1$, then $r(l, m, n) \geq q + 3s - 24$.*

Proof. We have $\Delta(G) = \deg_G(l)$, $\mathbb{R}(l, m) = \mathbb{R}(l, n) = \emptyset$, $\mathbb{R}(l) = \mathbb{R}_2(l) \cup \mathbb{R}_{3+}(l)$, $\mathbb{R}_2(l) = \mathbb{R}(i, l) \cup \mathbb{R}(j, l) \cup \mathbb{R}(k, l)$ and $\mathbb{R}_{3+}(l) = \mathbb{R}(l, m, n) \cup \mathbb{R}_{3+}(i, l) \cup \mathbb{R}_{3+}(j, l) \cup \mathbb{R}_{3+}(k, l)$. Proceeding similarly as in the proof of Claim 13 leads to $q - r_2(l) = r_{3+}(l) \leq r(l, m, n) + [8 - s - r(i, l)] + [8 - s - r(j, l)] + [8 - s - r(k, l)] = r(l, m, n) + 3(8 - s) - r_2(l)$, which yields the desired result. ■

5. MAIN THEOREM

Theorem 15. *If $q \in [41, \infty)$ and $q \equiv 1 \pmod{2}$, then $\text{achr}(K_6 \square K_q) = 2q + 3$.*

Proof. We proceed by the way of contradiction. As mentioned in the beginning of Section 4, we have to show that the existence of a matrix $M \in \mathcal{M}(6, q, C)$, where C is a set of $2q + s = 2q + 4$ colours, leads to a contradiction. First notice that, by Claim 9, all colours of C are of frequency $l \in [2, 6]$, $c_2 \geq 12$,

$c_{3+} \leq 2q - 8$, $\sum_{i=3}^6 ic_i \leq 6q - 24$, $\text{frq}(M) = 2$, $\text{exc}(M) = 3$, and $\{i, k\} \in \binom{[1,6]}{2}$ implies $r(i, k) \leq 4$.

Since $q \geq 41$, from Claim 13 we know that $\Delta(G) \leq 3$ for the auxiliary graph G . Besides that, $\deg_G(i) = d_i$ for $i \in [1, 6]$ yields $r_2(i) = \sum_{\{i,k\} \in E(G)} r(i, k) \leq \sum_{\{i,k\} \in E(G)} 4 = 4d_i$.

Claim 16. $\Delta(G) \leq 2$.

Proof. If $\Delta(G) = 3$, (w) $\deg_G(1) = 3$, $r(1, 4) \geq 1$, $r(1, 5) \geq 1$ and $r(1, 6) \geq 1$. By Claim 14 we have $r(1, 2, 3) \geq q - 12 \geq 29$, and so Claim 11 yields $r(4, 5) = r(4, 6) = r(5, 6) = 0$ (if $r(i, k) \geq 1$ for $\{i, k\} \in \binom{[4,6]}{2}$, then $29 \leq r(1, 2, 3) \leq r^*([1, 3]) \leq 2|[1, 3]| = 6$, a contradiction). Moreover, $r(4, 5, 6) = 0$, for otherwise, by Claim 12, $r(1, 2, 3) \leq 9$, a contradiction.

There is no $i \in [4, 6]$ with $\deg_G(i) = 3$, because then, again by Claim 14, $r(4, 5, 6) \geq q - 12 \geq 29$ in contradiction to Claim 12. So, $\deg_G(i) \leq 2$, $i = 4, 5, 6$, $2c_2 = \sum_{i=1}^6 r_2(i) \leq 3 \cdot 12 + 3 \cdot 8 = 60$ and $c_2 \leq 30$. Since $r(1, i) \geq 1$, Claim 11 yields $r^*([2, 6] \setminus \{i\}) \leq 2|[2, 6] \setminus \{i\}| = 8$, $i = 4, 5, 6$, and then $\rho^* = \sum_{i=4}^6 r^*([2, 6] \setminus \{i\}) \leq 24$.

By inspection of summands of type $r(A)$ with $A \subseteq [2, 6]$, $2 \leq |A| \leq 3$, that appear when counting the three summands of ρ^* , one can see that each of $r(A)$ with $A \in \binom{[2,6]}{2} \setminus \{\{4, 5\}, \{4, 6\}, \{5, 6\}\}$ appears at least twice, and each of $r(A)$ with $A \in \binom{[2,6]}{3} \setminus \{\{4, 5, 6\}\}$ (which is a set belonging to $C_3 \setminus \mathbb{R}_3(1)$) appears at least once. Because of $r(4, 5) = r(4, 6) = r(5, 6) = r(4, 5, 6) = 0$ this leads to $2 \sum_{\{i,k\} \in \binom{[2,6]}{2}} r(i, k) + c_3 - r_3(1) \leq \rho^* \leq 24$, which, having in mind that $2 \sum_{\{i,k\} \in \binom{[2,6]}{2}} r(i, k) = 2c_2 - 2r_2(1)$, yields $2c_2 - 2r_2(1) + c_3 - r_3(1) \leq 24$. Together with the inequality $r_2(1) + r_3(1) \leq q$ then $c_2 + c_3 \leq q + 24 + r_2(1) - c_2 \leq q + 24$, $2q + 4 = |C| = c_2 + c_3 + c_{4+} \leq q + 24 + c_{4+}$, and so $c_{4+} \geq q - 20 \geq 21$ in contradiction to $c_{4+} \leq c_2 - 3s \leq 30 - 12 = 18$, which comes from Claim 9.8 and the above inequality for c_2 . \square

By Claim 16 each component of G is either a path or a cycle.

Claim 17. No component of the graph G is K_2 .

Proof. Let (w) G have a component K_2 with vertex set $[1, 2]$. Then $r(1, 2) \in [1, 4]$, $r(i, k) = 0$ for $(i, k) \in [1, 2] \times [3, 6]$, and so $c_2 = r(1, 2) + \rho$, where, by Claim 11, $\rho = \sum_{\{i,k\} \in \binom{[3,6]}{2}} r(i, k) \leq r^*([3, 6]) \leq 8$. Further, (w) $\text{Cov}(\mathbb{R}(1, 2)) = [1, n]$ with $n \in [2, 8]$.

If $r(1, 2) \in [1, 3]$, then $c_2 \leq 3 + 8 = 11$, a contradiction.

If $r(1, 2) = 4$, then $n \geq 4$, $8 \geq \rho = |C_2 \setminus \mathbb{R}(1, 2)| \geq 12 - 4 = 8$, $\rho = 8$ and $c_2 = 12$. In the case $n \in [5, 8]$ there is $j \in [1, n]$ such that $\mathbb{C}(j)$ contains at most $\lfloor \frac{2 \cdot 8}{n} \rfloor \leq 3$ colours of $C_2 \setminus \mathbb{R}(1, 2)$. Then, however, for a colour $\gamma \in \mathbb{R}(1, 2) \cap \mathbb{C}(j)$

the number of colours $\delta \in C_2 \setminus \mathbb{R}(1, 2)$, for which the pair $\{\gamma, \delta\}$ is good in M , is at most seven, a contradiction.

Therefore $n = 4$, 2-colours occupy all positions in $[1, 6] \times [1, 4]$, Claim 9.8 yields $c_{4+} \leq 12 - 3 \cdot 4 = 0$, $c_{4+} = 0$, and all positions in the set $[1, 6] \times [5, q]$ are occupied by 3-colours. Among other things this means that

$$(1) \quad c_3 = \frac{6(q-4)}{3} = 2q - 8 \geq q + (41 - 8) = q + 33,$$

$r_2(i) = 4$ and $r_3(i) = q - 4$ for $i \in [1, 6]$, and $r(i, k) \in [0, 4]$ for every $\{i, k\} \in \binom{[3, 6]}{2}$.

Recall that, by Claim 16, $\Delta(G) \leq 2$. Further, if $r(i, k) = 4$ for some $\{i, k\} \in \binom{[3, 6]}{2}$, then $\deg_G(m) = 1$, $m = 3, 4, 5, 6$, since with $\{i, j, k, l\} = [3, 6]$ we have $r(j, l) = 4$. In such a case G is (isomorphic to) $3K_2$. Otherwise, if $r(i, k) < 4$ for each $\{i, k\} \in \binom{[3, 6]}{2}$, then $\deg_G(m) = 2$, $m = 3, 4, 5, 6$, and G is (isomorphic to) $K_2 \cup C_4$.

Let us first consider the case $G = 3K_2$, in which (w) $r(i, i+1) = 4$, $i = 3, 5$. Clearly, a set $\mathbb{R}(i, j, k)$ with $\{i, j, k\} \in \binom{[1, 6]}{3}$ can be nonempty only if $\{i, j, k\} \cap \{l, l+1\} \neq \emptyset$, $l = 1, 3, 5$. As a consequence the assumption $\mathbb{R}(i, j, k) \neq \emptyset$ with $i < j < k$ implies $(i, j, k) \in \{(1, 3, 5), (1, 3, 6), (1, 4, 5), (1, 4, 6), (2, 3, 5), (2, 3, 6), (2, 4, 5), (2, 4, 6)\}$.

Suppose that $\{(i_m, j_m) : m \in [1, 4]\} = \{(3, 5), (3, 6), (4, 5), (4, 6)\} = \{(k_m, l_m) : m \in [1, 4]\}$ and $\{i_m, j_m, k_m, l_m\} = [3, 6]$ for $m \in [1, 4]$. Then

$$(2) \quad c_3 = \sum_{m=1}^4 [r(1, i_m, j_m) + r(2, k_m, l_m)].$$

Further, for $m, n \in [1, 4]$ the sets $\{i_m, j_m\}$ and $\{k_n, l_n\}$ are disjoint if and only if $m = n$. Put

$$\begin{aligned} T(1) &= \{m \in [1, 4] : r(1, i_m, j_m) \geq 1\}, & t(1) &= |T(1)|, \\ T(2) &= \{m \in [1, 4] : r(2, k_m, l_m) \geq 1\}, & t(2) &= |T(2)| \end{aligned}$$

and $T = T(1) \cap T(2)$. Let

$$\sigma(P) = \sum_{p \in P} [r(1, i_p, j_p) + r(2, k_p, l_p)]$$

for $P \subseteq [1, 4]$. Using Claim 12 we see that $m \in T(1)$ implies $r(2, k_m, l_m) \leq 9$, while $m \in T(2)$ means that $r(1, i_m, j_m) \leq 9$. Therefore, with $m \in T$ we have $r(1, i_m, j_m) + r(2, k_m, l_m) \leq 9 + 9 = 18$, and so $\sigma(T) \leq 18|T|$.

If $t(1) = 4$, then $q - 4 = r_3(2) \leq \sum_{x=1}^4 r(2, k_x, l_x) \leq 4 \cdot 9 = 36$, and $q \leq 40$, a contradiction. Similarly, with $t(2) = 4$ we get $q - 4 = r_3(1) \leq \sum_{x=1}^4 r(1, i_x, j_x) \leq 36$, and $q \leq 40$ as well.

If there is $x \in [1, 2]$ such that $t(x) = 1$ and $T(x) = \{m\}$, then $r_3(x) = r(x, i_m, j_m) = r_3(i_m) = r_3(j_m) = q - 4 \geq 37$, hence $r(3 - x, k_m, l_m) = r_3(3 - x) = r_3(k_m) = r_3(l_m) = q - 4 \geq 37$, which contradicts Claim 12.

We are left with the situation $t(1), t(2) \in [2, 3]$ (and $|T| \leq \min(t(1), t(2))$). Suppose (w) $t(1) \geq t(2)$.

If $|T| = 3$, then $T(1) = T = T(2)$, $[1, 4] \setminus T = \{p\}$ and $r(1, i_p, j_p) = r(2, k_p, l_p) = 0$, hence $2q - 8 = c_3 = \sigma([1, 4]) = \sigma(T) \leq 18 \cdot 3 = 54$ and $q \leq 31$, a contradiction.

If $|T| = 2$, then n out of four summands that sum up to $\sigma([1, 4] \setminus T)$ are positive, $n \leq 2$. Moreover, $\sigma([1, 4] \setminus T) \leq q - 4$. The inequality is obvious provided that $n \leq 1$, while if $n = 2$, $a \in T(1) \setminus T(2)$ and $b \in T(2) \setminus T(1)$, then with $e \in \{i_a, j_a\} \cap \{k_b, l_b\}$ we have $\sigma([1, 4] \setminus T) = r(1, i_a, j_a) + r(2, k_b, l_b) \leq r_3(e) = q - 4$. Thus $c_3 = \sigma(T) + \sigma([1, 4] \setminus T) \leq 18 \cdot 2 + (q - 4) = q + 32$ in contradiction to (1).

For $|T| = 1$ we get $t(2) = 2$. With $t(1) = 2$ we obtain, similarly as in the case $|T| = 2$, $c_3 \leq 18 + (q - 4) = q + 14$. So, assume that $t(1) = 3$, $T = \{t\}$ and $T(2) \setminus T(1) = \{p\}$ (note that $p \neq t$).

Suppose first that $\{k_p, l_p\} \neq \{i_t, j_t\}$. In such a case $|\{k_p, l_p\} \cap \{i_t, j_t\}| = 1$; moreover, since $\{k_p, l_p\} \subseteq [3, 6] = \{i_t, j_t, k_t, l_t\}$, we have $|\{k_p, l_p\} \cap \{k_t, l_t\}| = 1$, and there is $g \in \{k, l\}$ such that $\{k_p, l_p\} \cap \{k_t, l_t\} = \{g_t\}$. Then five from among eight summands in (2) are positive, namely $r(1, i_t, j_t)$, $r(2, k_t, l_t)$, $r(2, k_p, l_p)$ and $r(1, i_m, j_m)$ with $m \in [1, 4] \setminus \{t, p\}$. Having in mind that $g_t \notin \{i_t, j_t\}$, $g_t \notin \{i_p, j_p\}$, and each element of $[3, 6]$ is involved in exactly two of the ordered pairs $(3, 5)$, $(3, 6)$, $(4, 5)$, $(4, 6)$, we see that except for $r(1, i_t, j_t)$ all mentioned positive summands correspond to colours of $\mathbb{R}_3(g_t)$. That is why $c_3 \leq r(1, i_t, j_t) + r_3(g_t) \leq 9 + (q - 4) = q + 5$, a contradiction to (1) again.

On the other hand, if $\{k_p, l_p\} = \{i_t, j_t\}$, then $|\{i_p, j_p\} \cap \{i_t, j_t\}| = |\{i_p, j_p\} \cap \{k_p, l_p\}| = 0$, hence for $m \in [1, 4] \setminus \{t, p\}$ we have $|\{i_m, j_m\} \cap \{i_t, j_t\}| = 1$, and so positive summands in (2) are $r(1, i_t, j_t)$, $r(2, k_t, l_t)$, $r(2, k_p, l_p) = r(2, i_t, j_t)$, $r(1, g_t, k_t)$ and $r(1, h_t, l_t)$, where $\{g, h\} = \{i, j\}$. Then

$$\begin{aligned} q - 4 &= r_3(2) = r(2, k_t, l_t) + r(2, i_t, j_t), \\ q - 4 &= r_3(k_t) = r(2, k_t, l_t) + r(1, g_t, k_t), \end{aligned}$$

which yields

$$r(2, i_t, j_t) = r(1, g_t, k_t) = q - 4 - r(2, k_t, l_t) \geq (q - 4) - 9 = q - 13$$

so that

$$q - 4 = r_3(g_t) = r(1, i_t, j_t) + r(1, g_t, k_t) + r(2, i_t, j_t) \geq 1 + 2(q - 13)$$

and $q \leq 21$, a contradiction.

If $|T| = 0$, then $t(1) = t(2) = 2$, and, by symmetry, $T(1) = [1, 2]$, $T(2) = [3, 4]$. We have $\{i_1, j_1\} \cup \{i_2, j_2\} = [3, 6]$, for otherwise there is $e \in [3, 6] \setminus (\{i_1, j_1\} \cup \{i_2, j_2\})$, hence $e \in \{i_3, j_3\} \cap \{i_4, j_4\}$, $e \notin \{k_3, l_3\} \cup \{k_4, l_4\}$ and $r_3(e) = 0 \neq q - 4$, a contradiction. Thus, by symmetry we may assume that $i_1 = 3 < i_2 = 4$.

Therefore, (w) $(i_1, j_1) = (3, 5)$ and $(i_2, j_2) = (4, 6)$, which means that $r(m, n, p)$ with $\{m, n, p\} \in \binom{[1, 6]}{3}$ and $m < n < p$ is positive if and only if $(m, n, p) \in \{(1, 3, 5), (1, 4, 6), (2, 3, 6), (2, 4, 5)\}$. We have $r_3(6) = r(1, 4, 6) + r(2, 3, 6) = q - 4 \equiv 1 \pmod{2}$, hence

$$(3) \quad r(1, 4, 6) \not\equiv r(2, 3, 6) \pmod{2}.$$

Similarly, from $r_3(1) = q - 4 = r_3(3)$ it follows that $r(1, 3, 5) \not\equiv r(1, 4, 6) \pmod{2}$ and $r(1, 3, 5) \not\equiv r(2, 3, 6) \pmod{2}$ so that $r(1, 4, 6) \equiv r(2, 3, 6) \pmod{2}$, which contradicts (3).

It remains to analyse the case $G = K_2 \cup C_4$, in which (w) $\{\{3, 4\}, \{4, 5\}, \{5, 6\}, \{6, 3\}\} \subseteq E(G)$. The completeness of f_M implies $C_3 \subseteq \mathbb{R}(1, 3, 5) \cup \mathbb{R}(1, 4, 6) \cup \mathbb{R}(2, 3, 5) \cup \mathbb{R}(2, 4, 6)$. So,

$$c_3 = [r(1, 3, 5) + r(2, 4, 6)] + [r(1, 4, 6) + r(2, 3, 5)].$$

Let $\bar{\mathcal{T}} = \{(1, 3, 5), (1, 4, 6), (2, 3, 5), (2, 4, 6)\}$,

$$\mathcal{T} = \{(i, j, k) \in \bar{\mathcal{T}} : r(i, j, k) \geq 1\}$$

and $t = |\mathcal{T}|$. From $r_3(i) = q - 4 > 0$ for $i \in [1, 6]$ it follows that $t \geq 2$. If $(i, j, k), (l, m, n) \in \mathcal{T}$, $(i, j, k) \neq (l, m, n)$, then either $r(i, j, k) + r(l, m, n) \leq 9 + 9 = 18$ (by Claim 12, if $\{i, j, k\} \cap \{l, m, n\} = \emptyset$) or $r(i, j, k) + r(l, m, n) \leq r_3(p) = q - 4$ (if $p \in \{i, j, k\} \cap \{l, m, n\}$). As a consequence then $c_3 \leq \max(b_2, b_3, b_4)$, where $b_2 = q - 4$, $b_3 = 18 + (q - 4) = q + 14$ and $b_4 = 18 + 18 = 36$. Thus $c_3 \leq q + 14$, which contradicts (1). \square

Claim 18. *No component of the graph G is K_3 .*

Proof. Let (w) G have the component K_3 with the vertex set $[1, 3]$.

If $G = K_3 \cup 3K_1$, then $r(1, 2) = r(1, 3) = r(2, 3) = 4$, $c_2 = 12$ and $c_{4+} = 0$. From Claim 10 it follows that $r_{3+}(1, 3) = 0 = r_{3+}(2, 3)$, hence $r(3, i, k) \geq 1$ with $\{i, k\} \in \binom{[1, 6] \setminus \{3\}}{2}$ implies $\{i, k\} \in \{\{4, 5\}, \{4, 6\}, \{5, 6\}\}$. By Claim 11 then $r_3(3) = r(3, 4, 5) + r(3, 4, 6) + r(3, 5, 6) \leq r^*([3, 6]) \leq 8$, hence $q = r_2(3) + r_3(3) \leq (4 + 4) + 8 = 16$, a contradiction.

If G has besides the above K_3 another nontrivial component (of order at least 2) and $r(i, k) \geq 1$ with $\{i, k\} \in \binom{[4, 6]}{2}$, then, by Claim 11, $r(1, 2) + r(1, 3) + r(2, 3) + r(1, 2, 3) = r^*([1, 3]) \leq 6$ and $r(4, 5) + r(4, 6) + r(5, 6) + r(4, 5, 6) = r^*([4, 6]) \leq 6$, hence $12 \leq c_2 \leq 6 + 6$, $c_2 = 12$, $r(1, 2) + r(1, 3) + r(2, 3) = r(4, 5) + r(4, 6) + r(5, 6) =$

6, $r(1, 2, 3) = r(4, 5, 6) = 0$, $c_{4+} = 0$, and $c_3 = 2q - 8 > 0$. So, among other things, with $C_2^1 = \mathbb{R}(1, 2) \cup \mathbb{R}(1, 3) \cup \mathbb{R}(2, 3)$ and $C_2^2 = \mathbb{R}(4, 5) \cup \mathbb{R}(4, 6) \cup \mathbb{R}(5, 6)$ we have $|C_2^1| = |C_2^2| = 6$. If $\alpha \in C_2^1$ and $\beta \in C_2^2$, then the pair $\{\alpha, \beta\}$ is column-based, hence all six positions in $\text{Cov}(\alpha) \times [4, 6]$ are occupied by colours of C_2^2 , and for any $j \in \text{Cov}(C_2^1)$ all three positions in $\{j\} \times [4, 6]$ are occupied by colours of C_2^2 . Consequently, $\text{cov}(C_2^1) = \frac{2|C_2^2|}{3} = 4$, and (w) all positions in $[1, 6] \times [1, 4]$ are occupied by colours of $C_2 = C_2^1 \cup C_2^2$. If $\{i, k\} \in \binom{J}{2}$, where $J \in \{[1, 3], [4, 6]\}$, then $r(i, k) > 0$, since otherwise with $\{i, j, k\} = J$ we get $r_2(j) = r(i, j) + r(j, k) = 6$, which contradicts Claim 9.9. For a colour $\gamma \in C_3$ there are $I, J \in \{[1, 3], [4, 6]\}$, $I \neq J$, such that $|\mathbb{R}(\gamma) \cap I| = 2$ and $|\mathbb{R}(\gamma) \cap J| = 1$. Then $\mathbb{R}(\gamma) \cap J = \{j\}$, and with $\{i, j, k\} = J$ there exists a colour $\alpha \in \mathbb{R}(i, k)$; in such a case, however, the pair $\{\alpha, \gamma\}$ is not good, a contradiction. \square

Claim 19. *No component of the graph G is a path of order at least 3.*

Proof. Suppose that G has a path component P of order at least 3.

If G has besides P another nontrivial component (of order at least 2), then $G = P \cup P'$, where, by Claims 17 and 18, both P and P' are paths of order 3, (w) $V(P) = [1, 3]$, $V(P') = [4, 6]$ and $r(i, i+1) \geq 1$ for $i = 1, 2, 4, 5$. Similarly as in the proof of Claim 18 it is easy to see that $r(1, 2) + r(2, 3) = r(4, 5) + r(5, 6) = 6$. Since $r_2(2) = 6$, there are colours $\alpha, \beta \in \mathbb{R}(1, 2) \cup \mathbb{R}(2, 3)$ such that $\text{Cov}(\alpha) \cap \text{Cov}(\beta) = \emptyset$. Then each colour of $\mathbb{R}(4, 5) \cup \mathbb{R}(5, 6)$ occupies a position in $[4, 6] \times \text{Cov}(\alpha)$ and a position in $[4, 6] \times \text{Cov}(\beta)$ as well so that $\text{Cov}(\mathbb{R}(4, 5) \cup \mathbb{R}(5, 6)) \subseteq \text{Cov}(\alpha, \beta)$; this leads to a contradiction since $\text{cov}(\mathbb{R}(4, 5) \cup \mathbb{R}(5, 6)) \geq r_2(5) = 6$ and $\text{cov}(\alpha, \beta) = 4$.

So, P is the unique nontrivial component of G , (w) $V(P) = [1, p]$ and $E(P) = \{\{i, i+1\} : i \in [1, p-1]\}$. Since $12 \leq c_2 = \sum_{i=1}^{p-1} r(i, i+1) \leq 4(p-1)$, we have $p \in [4, 6]$.

If $p = 4$, then $r(i, i+1) = 4$, $i = 1, 2, 3$, $c_2 = 12$, $c_{4+} = 0$ and

$$(4) \quad i \in [1, 6] \Rightarrow r_3(i) = q - r_2(i) \geq q - 8 \geq 33.$$

If $\alpha \in \mathbb{R}(1, 2)$ and $\beta \in \mathbb{R}(3, 4)$, then the pair $\{\alpha, \beta\}$ is column-based, hence all four positions in $\text{Cov}(\alpha) \times [3, 4]$ are occupied by colours of $\mathbb{R}(3, 4)$, and with $j \in \text{Cov}(\mathbb{R}(1, 2)) \times [3, 4]$, both positions in $\{j\} \times [3, 4]$ are occupied by colours of $\mathbb{R}(3, 4)$. Therefore, $\text{cov}(\mathbb{R}(1, 2)) = \frac{2|\mathbb{R}(3, 4)|}{2} = 4$, and (w) all positions in $[1, 4] \times [1, 4]$ are occupied by colours of $\mathbb{R}(1, 2) \cup \mathbb{R}(3, 4)$. Thus, (w) $\text{Cov}(\mathbb{R}(2, 3)) = [5, n]$, where $n \in [8, 12]$.

By Claim 10 we know that $r(i, j, k) = 0$ if there is $l \in [1, 3]$ such that $\{l, l+1\} \subseteq \{i, j, k\}$.

Suppose that $\gamma \in \mathbb{R}(1, 5, 6)$. Since all pairs $\{\gamma, \delta\}$ with $\delta \in \mathbb{R}(3, 4)$ are good in M , two positions in $[5, 6] \times [1, 4]$ must be occupied by γ . Then, however, the number of pairs $\{\gamma, \varepsilon\}$ with $\varepsilon \in \mathbb{R}(2, 3)$ that are good in M is at most

two, a contradiction. So, $r(1, 5, 6) = 0$, and an analogous reasoning shows that $r(4, 5, 6) = 0$.

Claim 11 yields $r(1, 4, 5) + r(1, 4, 6) \leq r^*(\{1, 4, 5, 6\}) \leq 8$. A colour $\gamma \in \mathbb{R}(2, 5, 6)$ as well as a colour $\delta \in \mathbb{R}(3, 5, 6)$ occupies two positions in $[5, 6] \times [1, 4]$ (each pair $\{\gamma, \varepsilon\}$ with $\varepsilon \in \mathbb{R}(3, 4)$ and each pair $\{\delta, \zeta\}$ with $\zeta \in \mathbb{R}(1, 2)$ is good in M). Thus $r(2, 5, 6) + r(3, 5, 6) \leq 4$ and

$$(5) \quad r(1, 4, 5) + r(1, 4, 6) + r(2, 5, 6) + r(3, 5, 6) \leq 12.$$

The set of remaining triples $(i, j, k) \in [1, 6]^3$ with $i < j < k$ such that $r(i, j, k)$ can be positive is $\mathcal{T} = \{(1, 3, 5), (1, 3, 6), (2, 4, 5), (2, 4, 6)\}$. Suppose that $r(i, j, k) \geq 1$ with $(i, j, k) \in \mathcal{T}$ if and only if $(i, j, k) \in \{(i_l, j_l, k_l) : l \in [1, t]\}$. We show that there is $m \in [1, 6]$ with $r_3(m) \leq 30$ in contradiction to (4).

If $t = 4$, then, by Claim 12, $r(i_l, j_l, k_l) \leq 9$, $l = 1, 2, 3, 4$, and so, using (5), $r_3(m) \leq 12 + 2 \cdot 9 = 30$ for (any) $m \in [1, 6]$. If $t = 3$ and $r(i, j, k) = 0$ for $(i, j, k) \in \mathcal{T}$, then $r_3(m) \leq 12 + 9 = 21$ for $m \in \{i, j, k\}$. The same upper bound applies for $m \in [1, 6]$ if $t = 2$ and $\{i_1, j_1, k_1\} \cap \{i_2, j_2, k_2\} = \emptyset$.

If $t = 2$ and $\{i_1, j_1, k_1\} \cap \{i_2, j_2, k_2\} \neq \emptyset$, for $m \in [1, 6] \setminus (\{i_1, j_1, k_1\} \cup \{i_2, j_2, k_2\})$ we obtain $r_3(m) \leq 12$. The same inequality is available for $m \in [1, 6] \setminus \{i_1, j_1, k_1\}$ if $t = 1$ and for $m \in [1, 6]$ if $t = 0$.

If $p = 5$, then, by Claim 11, from $r(4, 5) \geq 1$ it follows that $r(1, 2) + r(2, 3) \leq r^*([1, 3]) \leq 6$; similarly, $r(1, 2) \geq 1$ yields $r(3, 4) + r(4, 5) \leq r^*([3, 5]) \leq 6$. Then $12 \leq c_2 = \sum_{i=1}^4 r(i, i+1) \leq 6 + 6$, $c_2 = 12$ and $r(1, 2) + r(2, 3) = 6 = r(3, 4) + r(4, 5)$. If $(M)_{1,j} = \gamma \in \mathbb{R}(1, 2)$, then all positions in $[3, 5] \times \text{Cov}(\gamma)$ are occupied by six distinct colours of $\mathbb{R}(3, 4) \cup \mathbb{R}(4, 5)$, hence $(M)_{3,j} \in \mathbb{R}(3, 4)$ and $\delta = (M)_{5,j} \in \mathbb{R}(4, 5)$. Analogously, all positions in $[1, 3] \times \text{Cov}(\delta)$ are occupied by six distinct colours of $\mathbb{R}(1, 2) \cup \mathbb{R}(2, 3)$, which implies $(M)_{3,j} \in \mathbb{R}(2, 3)$, a contradiction.

If $p = 6$, then, by Claim 11, $r^*([1, 6] \setminus [l, l+1]) \leq 8$ for $l \in [1, 5]$, hence

$$(6) \quad \rho^* = \sum_{l=1}^5 r^*([1, 6] \setminus [l, l+1]) \leq 40.$$

It is easy to see that in the sum ρ^* each of the summands $r(i, i+1)$ with $i \in [1, 5]$ appears in the expression of $r^*([1, 6] \setminus [l, l+1])$ for at least two l 's, while each of the summands $r(i, j, k)$ satisfying $\{i, j, k\} \in \binom{[1, 6]}{3} \setminus \{\{1, 3, 5\}, \{2, 3, 5\}, \{2, 4, 5\}, \{2, 4, 6\}\}$ and $i < j < k$ appears for at least one l . Since $c_2 = \sum_{l=1}^5 r(l, l+1)$, with

$$\rho = r(1, 3, 5) + r(2, 3, 5) + r(2, 4, 5) + r(2, 4, 6)$$

the inequality (6) leads to

$$(7) \quad 2c_2 + c_3 - \rho \leq \rho^* \leq 40.$$

Moreover, $r_2(l) + r_3(l) \leq q$ implies

$$r_3(l) \leq q - r_2(l) = q - [r(l-1, l) + r(l, l+1)] \leq q - 2, \quad l = 2, 5,$$

and so, having in mind that, by Claim 12, $\min(r(1, 3, 5), r(2, 4, 6)) \leq 9$,

$$(8) \quad \begin{aligned} \rho &\leq \min(r_3(2) + r(1, 3, 5), r_3(5) + r(2, 4, 6)) \\ &\leq q - 2 + \min(r(1, 3, 5), r(2, 4, 6)) \leq q + 7. \end{aligned}$$

Since, by Claim 9.8, $c_{4+} - c_2 \leq -12$, using (7) and (8) we obtain

$$2q + 4 = c_2 + c_3 + c_{4+} \leq 40 + \rho + c_{4+} - c_2 \leq 40 + (q + 7) - 12 = q + 35,$$

and, finally, $q \leq 31$, a contradiction. \square

With $l \in \mathbb{Z}$ and $m \in [2, \infty)$ we use $(l)_m$ to denote the unique $n \in [1, m]$ satisfying $n \equiv l \pmod{m}$.

Claim 20. *No component of the graph G is a 4-cycle.*

Proof. If G has a 4-cycle component, (w) $r(i, (i+1)_4) \geq 1$ for $i \in [1, 4]$. Note that, by Claim 17, $r(5, 6) = 0 = r_2(5) = r_2(6)$. Let $i \in [1, 4]$ and

$$P_i = \{((i+2)_4, 5), ((i+2)_4, 6), (5, 6)\}.$$

If $\gamma \in \mathbb{R}_{3+}(i)$, there is $A \subseteq [1, 6]$ such that $|A| \geq 3$, $\{i\} \subseteq A$ and $\gamma \in \mathbb{R}(A)$. Provided that $A \cap \{(i-1)_4, (i+1)_4\} \neq \emptyset$, we get $\gamma \in \bigcup_{j \in \{(i-1)_4, i\}} \mathbb{R}_{3+}(j, (j+1)_4)$. On the other hand, $A \cap \{(i-1)_4, (i+1)_4\} = \emptyset$ implies either $|A| = 3$ and $\gamma \in \bigcup_{(j,k) \in P_i} \mathbb{R}(i, j, k)$ or $|A| = 4$ and $\gamma \in \mathbb{R}(i, (i+2)_4, 5, 6)$. As a consequence,

$$\mathbb{R}_{3+}(i) \subseteq \mathbb{R}(i, (i+2)_4, 5, 6) \cup \bigcup_{j \in \{(i-1)_4, i\}} \mathbb{R}_{3+}(j, (j+1)_4) \cup \bigcup_{(j,k) \in P_i} \mathbb{R}(i, j, k),$$

and by Claim 10 we have

$$\begin{aligned} r_{3+}(i) &\leq r(i, (i+2)_4, 5, 6) + \sum_{j \in \{(i-1)_4, i\}} r_{3+}(j, (j+1)_4) + \sum_{(j,k) \in P_i} r(i, j, k) \\ &\leq r(i, (i+2)_4, 5, 6) + \sum_{j \in \{(i-1)_4, i\}} [4 - r(j, (j+1)_4)] + \sum_{(j,k) \in P_i} r(i, j, k). \end{aligned}$$

Therefore, realising that

$$\sum_{i=1}^4 \sum_{j \in \{(i-1)_4, i\}} r(j, (j+1)_4) = \sum_{i=1}^4 r_2(i),$$

we obtain

$$(9) \quad \sum_{i=1}^4 r_{3+}(i) \leq 32 + \sum_{i=1}^4 \left[r(i, (i+2)_4, 5, 6) + \sum_{(j,k) \in P_i} r(i, j, k) \right] - \sum_{i=1}^4 r_2(i).$$

On the right-hand side of the inequality (9) there are among others (partial) summands $r(A)$ with $A \in \binom{[1,6]}{3} \cup \binom{[1,6]}{4}$, each such summand appears there with the frequency 0, 1 or 2, and the frequency is 2 if and only if $A \in \{\{1, 3, 5\}, \{1, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}, \{1, 3, 5, 6\}, \{2, 4, 5, 6\}\}$. Thus, with

$$\begin{aligned} \rho_3 &= r(1, 3, 5) + r(1, 3, 6) + r(2, 4, 5) + r(2, 4, 6), \\ \rho_4 &= r(1, 3, 5, 6) + r(2, 4, 5, 6) \end{aligned}$$

the inequality (9) leads to

$$(10) \quad 4q = \sum_{i=1}^4 [r_2(i) + r_{3+}(i)] \leq 32 + \rho_3 + \rho_4 + c_3 + c_4.$$

Let us show that

$$(11) \quad \rho_3 \leq q + 10.$$

To see it let a be the number of sets A belonging to

$$\mathcal{A} = \{\{1, 3, 5\}, \{1, 3, 6\}, \{2, 4, 5\}, \{2, 4, 6\}\}$$

with $r(A) \geq 1$.

If $a = 4$, by Claim 12 we have $\rho_3 \leq 4 \cdot 9 = 36 \leq q + 10$.

In the case $a = 3$ there is $i \in [1, 4]$ such that $\rho_3 \leq 2 \cdot 9 + r_3(i)$. Evidently, $r_3(i) \leq q - r_2(i) = q - [r((i-1)_4, i) + r(i, (i+1)_4)] \leq q - (4 + 4) = q - 8$, hence $\rho_3 \leq q + 10$.

If $a = 2$, let the positive summands of ρ_3 be $r(i, j, k)$ and $r(l, m, n)$, $\{i, j, k\} \neq \{l, m, n\}$. If $\{i, j, k\} \cap \{l, m, n\} = \emptyset$, then $\rho_3 \leq 2 \cdot 9 \leq q + 10$, and otherwise, with $p \in \{i, j, k\} \cap \{l, m, n\}$, we have $\rho_3 \leq r_3(p) \leq q$.

If $a \in [0, 1]$, then $\rho_3 \leq q - 2$, since $r(i, j, k) \geq 1$ with $\{i, j, k\} \in \mathcal{A}$ and $i < j < k$ implies $i \in [1, 2]$, while $r_2(i) \geq 1 + 1$.

Now realise that, by Claim 9.8, $\rho_4 + c_3 + c_4 \leq c_3 + 2c_{4+} = |C| + (c_{4+} - c_2) \leq (2q + 4) - 12 = 2q - 8$, and so, using (10) and (11), $4q \leq 32 + (q + 10) + (2q - 8) = 3q + 34$ and $q \leq 34$, a contradiction. \square

Claim 21. *No component of the graph G is a 5-cycle.*

Proof. If G has a 5-cycle component, (w) $r(i, (i+1)_5) \geq 1$ for $i \in [1, 5]$. Similarly as in the proof of Claim 20 for $i \in [1, 5]$ we get

$$r_{3+}(i) \leq r((i-2)_5, i, (i+2)_5, 6) + \sum_{j \in \{(i-1)_5, i\}} r_{3+}(j, (j+1)_5) + \sum_{(j,k) \in P_i} r(i, j, k),$$

this time with

$$P_i = \{((i-2)_5, 6), ((i+2)_5, 6), ((i-2)_5, (i+2)_5)\},$$

which yields

$$(12) \quad 5q = \sum_{i=1}^5 [r_2(i) + r_{3+}(i)] \leq 40 + \rho_3 + c_3 + c_4,$$

where

$$(13) \quad \rho_3 = r(1, 3, 6) + r(1, 4, 6) + r(2, 4, 6) + r(2, 5, 6) + r(3, 5, 6) \leq r_3(6) \leq q.$$

Moreover, $c_3 + c_4 \leq 2q + 4 - c_2 \leq 2q - 8$, and so, using (12) and (13), $5q \leq 40 + q + (2q - 8) = 3q + 32$ and $q \leq 16$, a contradiction. \square

Claim 22. *The graph G is not a 6-cycle.*

Proof. If G is a 6-cycle, (w) $r(i, (i+1)_6) \geq 1$ for $i \in [1, 6]$. In this case $r_{3+}(i)$ is upper bounded by

$$r((i-2)_6, i, (i+2)_6, (i+3)_6) + \sum_{j \in \{(i-1)_6, i\}} [4 - r(j, (j+1)_6)] + \sum_{(j,k) \in P_i} r(i, j, k)$$

with

$$P_i = \{((i-2)_6, (i+2)_6), ((i-2)_6, (i+3)_6), ((i+2)_6, (i+3)_6)\},$$

and one can see that

$$(14) \quad 6q \leq 48 + \rho_3 + c_3 + c_4,$$

where $\rho_3 = 2r(1, 3, 5) + 2r(2, 4, 6)$. We can bound ρ_3 from above by $2q - 4$. Indeed, if both $r(1, 3, 5)$ and $r(2, 4, 6)$ are positive, then Claim 12 yields $\rho_3 \leq 18 + 18 = 36 \leq 2q - 4$. On the other hand, if $r(i, j, k) = 0$ with $(i, j, k) \in \{(1, 3, 5), (2, 4, 6)\}$, then $\rho_3 \leq 2r_3(i+1) \leq 2[q - r_2(i+1)] = 2q - 2[r_2(i, i+1) + r_2(i+1, i+2)] \leq 2q - 4$. Therefore, similarly as in the proof of Claim 21, from (14) we obtain $6q \leq 48 + (2q - 4) + (2q - 8) = 4q + 36$ and $q \leq 18$, a contradiction. \square

Thus, by Claims 16–22, we conclude that $G = 6K_1$ and $c_2 = 0$, which contradicts Claim 9.3. Therefore, Theorem 15 is proved. ■

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