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THE ACHROMATIC NUMBER OF $K_6 \square K_q$ EQUALS $2q+3$ IF $q \geq 41$ IS ODD

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Abstract

Let G be a graph and C be a finite set of colours. A vertex colouring $f : V(G) \to C$ is complete provided that for any two distinct colours $c_1, c_2 \in C$ there is $v_1v_2 \in E(G)$ such that $f(v_i) = c_i$, $i = 1, 2$. The achromatic number of G is the maximum number of colours in a proper complete vertex colouring of G. In the paper it is proved that if $q \geq 41$ is an odd integer, then the achromatic number of the Cartesian product of K_6 and K_q is $2q+3$.

Keywords: complete vertex colouring, achromatic number, Cartesian product, complete graph.

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1. Introduction

Let G be a finite simple graph and C a finite set of colours. A vertex colouring $f : V(G) \to C$ is *complete* if for any pair of distinct colours $c_1, c_2 \in C$ one can find in G an edge $\{v_1, v_2\}$ (often shortened to v_1v_2) such that $f(v_i) = c_i$, $i = 1, 2$. The *achromatic number* of G, denoted by $\operatorname{achr}(G)$, is the maximum cardinality of the colour set in a proper complete vertex colouring of G.

The concept was introduced quite a long ago in Harary, Hedetniemi and Prins [8], where the following was proved.

Theorem 1. *If G is a graph, and an integer k satisfies $\chi(G) \leq k \leq \operatorname{achr}(G)$, then there exists a proper complete vertex colouring of G using k colours.*

There are still only a few graph classes $\mathcal{G}$ such that $\operatorname{achr}(G)$ is known for all $G \in \mathcal{G}$. This is certainly related to the fact that determining the achromatic number is an NP-complete problem even for trees, see Cairnie and Edwards [2].

Two surveys are available on the topic, namely Edwards [6] and Hughes and MacGillivray [10]; more generally, Chapter 12 in the book [3] by Chartrand and Zhang deals with complete vertex colourings. A comprehensive list of publications concerning the achromatic number is maintained by Edwards [7].

Some papers are devoted to the achromatic number of graphs constructed by graph operations. So, Hell and Miller [9] considered $\mathrm{achr}(G_1 \times G_2)$, where $G_1 \times G_2$ stands for the categorical product of graphs G_1 and G_2 (we follow here the notation by Imrich and Klavžar [16]).

In this paper we are interested in the achromatic number of the Cartesian product $G_1 \square G_2$ of graphs G_1 and G_2, the graph with $V(G_1 \square G_2) = \{(v_1, v_2) : v_i \in V(G_i),\ i = 1, 2\}$, in which $\big(v_1^1, v_2^1\big)\big(v_1^2, v_2^2\big) \in E(G_1 \square G_2)$ if and only if there is $i \in \{1, 2\}$ such that $v_i^1 v_i^2 \in E(G_i)$ and $v_{3-i}^1 = v_{3-i}^2$. As observed by Chiang and Fu [4], $\mathrm{achr}(G_1) = p$ and $\mathrm{achr}(G_2) = q$ implies $\mathrm{achr}(G_1 \square G_2) \geq \mathrm{achr}(K_p \square K_q)$. This inequality motivates a special interest in the achromatic number of the Cartesian product of two complete graphs. From the obvious fact that $G_2 \square G_1$ is isomorphic to $G_1 \square G_2$ it is clear that when determining $\mathrm{achr}(K_p \square K_q)$ we may suppose without loss of generality $p \leq q$.

The problem of determining $\mathrm{achr}(K_p \square K_q)$ with $p \leq 4$ was solved in Horňák and Puntigán [15] (for $p \leq 3$ the result was rediscovered in [4]) and that for $p = 5$ in Horňák and Pčola [13, 14]. In [5] Chiang and Fu proved that if r is an odd projective plane order, then $\mathrm{achr}(K_{(r^2+r)/2} \square K_{(r^2+r)/2}) = (r^3 + r^2)/2$. (For $r = 3$ the fact that $\mathrm{achr}(K_6 \square K_6) = 18$ was known already to Bouchet [1].)

Here we show that $\mathrm{achr}(K_6 \square K_q) = 2q + 3$ if q is an odd integer with $q \geq 41$. This is the first of three papers devoted to completely solve the problem of finding $\mathrm{achr}(K_6 \square K_q)$. In Horňák [11] the cases, in which either $8 \leq q \leq 40$ or q is an even integer with $q \geq 42$, are analysed. Finally, $\mathrm{achr}(K_6 \square K_7)$ is determined in Horňák [12].

For $k, l \in \mathbb{Z}$ we denote *integer intervals* by

$$[k, l] = \{z \in \mathbb{Z} : k \leq z \leq l\}, \qquad [k, \infty) = \{z \in \mathbb{Z} : k \leq z\}.$$

Further, with a set A and $m \in [0, \infty)$ we use $\binom{A}{m}$ for the set of m-element subsets of A.

Now let $p, q \in [1, \infty)$. Under the assumption that $V(K_r) = [1, r]$, $r = p, q$, we have $V(K_p \square K_q) = [1, p] \times [1, q]$, while $E(K_p \square K_q)$ consists of edges $(i, j_1)(i, j_2)$ with $i \in [1, p]$, $j_1, j_2 \in [1, q]$, $j_1 \neq j_2$ and $(i_1, j)(i_2, j)$ with $i_1, i_2 \in [1, p]$, $i_1 \neq i_2$, $j \in [1, q]$.

A vertex colouring $f : [1, p] \times [1, q] \to C$ of the graph $K_p \square K_q$ can be conveniently described using the $p \times q$ matrix $M = M(f)$ whose entry in the ith row and the jth column is $(M)_{i,j} = f(i, j)$. Such a colouring is proper if any row of M consists of q distinct entries and any column of M consists of p distinct entries. Further, f is complete provided that any pair $\{\alpha, \beta\} \in \binom{C}{2}$ is *good* in M in the following sense: there are $(i_1, j_1), (i_2, j_2) \in [1, p] \times [1, q]$ such that $\{(M)_{i_1,j_1}, (M)_{i_2,j_2}\} = \{\alpha, \beta\}$ and either $i_1 = i_2$, which we express by saying that the pair $\{\alpha, \beta\}$ is *row-based* (in M), or $j_1 = j_2$, i.e., the pair $\{\alpha, \beta\}$ is *column-based* (in M).

Let $\mathcal{M}(p, q, C)$ denote the set of $p \times q$ matrices M with entries from C such that all rows (columns) of M have q (p, respectively) distinct entries, and each pair $\{\alpha, \beta\} \in \binom{C}{2}$ is good in M. So, if $f : [1, p] \times [1, q] \to C$ is a proper complete vertex colouring of $K_p\square K_q$, then $M(f) \in \mathcal{M}(p, q, C)$.

Conversely, if $M \in \mathcal{M}(p, q, C)$, then the mapping $f_M : [1, p] \times [1, q] \to C$ with $f_M(i, j) = (M)_{i,j}$ is a proper complete vertex colouring of $K_p\square K_q$. Thus, we have proved:

Proposition 2. *If $p, q \in [1, \infty)$ and C is a finite set, then the following statements are equivalent.*

(1) *There is a proper complete vertex colouring of $K_p\square K_q$ using as colours elements of C.*

(2) $\mathcal{M}(p, q, C) \neq \emptyset$.

We have another evident result.

Proposition 3. *If $p, q \in [1, \infty)$, C, D are finite sets, $M \in \mathcal{M}(p, q, C)$, mappings $\rho : [1, p] \to [1, p]$, $\sigma : [1, q] \to [1, q]$, $\pi : C \to D$ are bijections, and $M_{\rho,\sigma}$, M_π are $p \times q$ matrices defined by $(M_{\rho,\sigma})_{i,j} = (M)_{\rho(i),\sigma(j)}$ and $(M_\pi)_{i,j} = \pi((M)_{i,j})$, then $M_{\rho,\sigma} \in \mathcal{M}(p, q, C)$ and $M_\pi \in \mathcal{M}(p, q, D)$.*

Let $M \in \mathcal{M}(p, q, C)$. The *frequency* of a colour $\gamma \in C$ is the number $\mathrm{frq}(\gamma)$ of appearances of γ in M, and the *frequency* of M, denoted $\mathrm{frq}(M)$, is the minimum of frequencies of colours in C. A colour of frequency l is an *l-colour*. C_l is the set of l-colours, $c_l = |C_l|$, and C_{l+} is the set of colours of frequency at least l, $c_{l+} = |C_{l+}|$. We denote by $\mathbb{R}(i)$ the set $\{(M)_{i,j} : j \in [1, q]\}$ of colours in the ith row of M and by $\mathbb{C}(j)$ the set $\{(M)_{i,j} : i \in [1, p]\}$ of colours in the jth column of M. Further, for $k \in \{l, l+\}$ let

$$\mathbb{R}_k(i) = C_k \cap \mathbb{R}(i), \qquad r_k(i) = |\mathbb{R}_k(i)|,$$
$$\mathbb{C}_k(j) = C_k \cap \mathbb{C}(j), \qquad c_k(j) = |\mathbb{C}_k(j)|.$$

So $\mathbb{R}_l(i)$ and $\mathbb{R}_{l+}(i)$ is the set of colours in the row i which occur exactly and at least l times altogether, respectively; the meaning of $\mathbb{C}_l(j)$ and $\mathbb{C}_{l+}(j)$ is similar. If $A \subseteq [1, p]$, $|A| \geq 2$, then

$$\mathbb{R}(A) = \bigcap_{l \in A} (C_{|A|} \cap \mathbb{R}(l)), \qquad r(A) = |\mathbb{R}(A)|.$$

$\mathbb{R}(A)$ is the set of colours which appear precisely in the rows numbered by A. Provided that $A \in \{\{i,j\},\{i,j,k\},\{i,j,k,l\}\}$, instead of $\mathbb{R}(A)$ we write $\mathbb{R}(i,j)$, $\mathbb{R}(i,j,k)$, $\mathbb{R}(i,j,k,l)$, while $r(A)$ is simplified to $r(i,j)$, $r(i,j,k)$, $r(i,j,k,l)$, respectively. With $\{i,j\} \subseteq [1,p]$ and $\{m,n\} \subseteq [1,q]$ we set

$$\begin{aligned} \mathbb{R}_{3+}(i,j) &= C_{3+} \cap \mathbb{R}(i) \cap \mathbb{R}(j), & r_{3+}(i,j) &= |\mathbb{R}_{3+}(i,j)|,\\ \mathbb{C}(m,n) &= C_2 \cap \mathbb{C}(m) \cap \mathbb{C}(n), & c(m,n) &= |\mathbb{C}(m,n)|. \end{aligned}$$

$\mathbb{R}_{3+}(i,j)$ is the set of colours of frequency at least 3 occuring in both rows i and j, and $\mathbb{C}(m,n)$, an analogue of the notation $\mathbb{R}(i,j)$, stands for the set of 2-colours occuring exactly in the columns m and n. If $B \subseteq [1,p]$ and $3 \le |B| \le p-2$, then

$$\mathbb{R}^*(B) = \bigcup_{l=2}^{|B|} \bigcup_{A \in \binom{B}{l}} \mathbb{R}(A).$$

$\mathbb{R}^*(B)$ is the set of colours of frequency at least 2 occuring only in the rows numbered by B. Since $\left\{\mathbb{R}(A) : \exists_{l\in[2,|B|]}\, A \in \binom{B}{l}\right\}$ is a set of pairwise disjoint sets, we have

$$r^*(B) = |\mathbb{R}^*(B)| = \sum_{l=2}^{|B|} \sum_{A \in \binom{B}{l}} r(A).$$

For $\gamma \in C$ let

$$\mathbb{R}(\gamma) = \{i \in [1,p] : \gamma \in \mathbb{R}(i)\}$$

be the set of (the numbers of) the rows containing the colour γ.

With $S \subseteq [1,p] \times [1,q]$ we say that a colour $\gamma \in C$ *occupies a position in* S if there is $(i,j) \in S$ such that $(M)_{i,j} = \gamma$. If $\emptyset \neq A \subseteq C$, the *set of columns covered by* A is

$$\mathrm{Cov}(A) = \{j \in [1,q] : \mathbb{C}(j) \cap A \neq \emptyset\},$$

i.e., the set of columns containing an element of A. We define $\mathrm{cov}(A) = |\mathrm{Cov}(A)|$, and with $A \in \{\{\alpha\},\{\alpha,\beta\}\}$ we use a simplified notation $\mathrm{Cov}(\alpha)$, $\mathrm{Cov}(\alpha,\beta)$ and $\mathrm{cov}(\alpha)$, $\mathrm{cov}(\alpha,\beta)$ instead of $\mathrm{Cov}(A)$ and $\mathrm{cov}(A)$.

2. Lower Bound

Proposition 4. *If* $q \in [7,\infty)$ *and* $q \equiv 1 \pmod 2$, *then* $\mathrm{achr}(K_6 \square K_q) \ge 2q+3$.

Proof. Let $s = \frac{q-3}{2}$, and let M be the $6 \times q$ matrix below. We show that $M \in \mathcal{M}(6,q,C)$, where $C = [1,9] \cup X_s \cup Y_s \cup Z_s \cup T_s$, $U_s = \{u_i : i \in [1,s]\}$ for

$U \in \{X, Y, Z, T\}$, and the sets $[1, 9]$, X_s, Y_s, Z_s, T_s are pairwise disjoint.

$$\begin{pmatrix} 1 & 2 & 3 & x_1 & x_2 & \dots & x_{s-1} & x_s & y_1 & y_2 & \dots & y_{s-1} & y_s \\ 4 & 5 & 6 & x_s & x_1 & \dots & x_{s-2} & x_{s-1} & z_1 & z_2 & \dots & z_{s-1} & z_s \\ 7 & 8 & 9 & t_1 & t_2 & \dots & t_{s-1} & t_s & x_1 & x_2 & \dots & x_{s-1} & x_s \\ 3 & 1 & 2 & z_1 & z_2 & \dots & z_{s-1} & z_s & t_1 & t_2 & \dots & t_{s-1} & t_s \\ 5 & 6 & 4 & t_s & t_1 & \dots & t_{s-2} & t_{s-1} & y_s & y_1 & \dots & y_{s-2} & y_{s-1} \\ 8 & 9 & 7 & y_1 & y_2 & \dots & y_{s-1} & y_s & z_s & z_1 & \dots & z_{s-2} & z_{s-1} \end{pmatrix}$$

Since $s \geq 2$, because of our assumptions on the structure of C it is clear that elements in lines (rows and columns) of M are pairwise distinct. Thus it is sufficient to show that each pair $\{\alpha, \beta\} \in \binom{C}{2}$ is good in M.

If $\alpha, \beta \in [1, 9]$, then both α and β appear twice in the columns $1, 2, 3$, hence the pair $\{\alpha, \beta\}$ is column-based.

If $\alpha \in C$ and $\beta \in X_s \cup Y_s \cup Z_s \cup T_s$, realise that $\mathbb{R}(\alpha) \in \mathcal{R}_1 \cup \mathcal{R}_2$ and $\mathbb{R}(\beta) \in \mathcal{R}_2$, where $\mathcal{R}_1 = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$ and $\mathcal{R}_2 = \{\{1, 2, 3\}, \{1, 5, 6\}, \{2, 4, 6\}, \{3, 4, 5\}\}$. As $R \cap R_2 \neq \emptyset$ for any $R \in \mathcal{R}_1 \cup \mathcal{R}_2$ and any $R_2 \in \mathcal{R}_2$, the pair $\{\alpha, \beta\}$ is row-based.

So, Proposition 2 yields $\operatorname{achr}(K_6\square K_q) \geq |C| = 4s + 9 = 2q + 3$. ■

3. Auxiliary Results

Let $M \in \mathcal{M}(p, q, C)$ and let $\gamma \in C$. For the (complete) colouring f_M from the proof of Proposition 2 denote $V_\gamma = f_M^{-1}(\gamma) \subseteq [1, p] \times [1, q]$, and let $N(V_\gamma)$ be the neighbourhood of V_γ (the union of neighborhoods of vertices in V_γ). The *excess* of γ is defined to be the maximum number $\operatorname{exc}(\gamma)$ of vertices in a set $S \subseteq N(V_\gamma)$ such that each pair $\{\gamma, \gamma'\} \in \binom{C}{2}$ is good even in the "partial matrix" corresponding to the restriction of f_M created by uncolouring the vertices of S.

Lemma 5. *If $p, q \in [1, \infty)$, C is a finite set, $M \in \mathcal{M}(p, q, C)$ and $\gamma \in C$, then the following hold.*

1. $\operatorname{frq}(\gamma) \leq \min(p, q)$;
2. $\operatorname{frq}(\gamma) = l$ *implies* $\operatorname{exc}(\gamma) = l(p + q - l - 1) - (|C| - 1) \geq 0$;
3. $\operatorname{frq}(M) = l$ *implies* $|C| \leq \left\lfloor \frac{pq}{l} \right\rfloor$.

Proof. 1. The assumption $\operatorname{frq}(\gamma) = l > \min(p, q)$ would mean, by the pigeonhole principle, that the colouring f_M is not proper.

2. Because of Proposition 3 we may suppose without loss of generality $(M)_{i,i} = \gamma$ for all $i \in [1, l]$. For simplicity we use (w) to indicate that it is just Proposition 3, which enables us to restrict our attention to matrices with a special property.

The colouring f_M is complete, hence each of $|C|-1$ colours in $C\setminus\{\gamma\}$ must occupy a position in the set $N(V_\gamma)=\{(i,j):(i\le l\vee j\le l)\wedge i\ne j\}$. Thus, $|N(V_\gamma)|=ql+(p-l)l-l\ge|C|-1$ and $\operatorname{exc}(\gamma)=|N(V_\gamma)|-(|C|-1)=l(p+q-l-1)-(|C|-1)\ge 0$.

3. If $\alpha\in C$ is such that $\operatorname{frq}(\alpha)=\operatorname{frq}(M)=l$ and $\beta\in C$, then $\operatorname{frq}(\beta)\ge\operatorname{frq}(\alpha)=l$. Therefore, the total number of entries of the matrix M is $pq\ge l|C|$, and the desired inequality follows. ∎

The *excess* of a matrix $M\in\mathcal{M}(p,q,C)$, denoted by $\operatorname{exc}(M)$, is the minimum of excesses of colours in C.

Lemma 6. *If* $p,q\in[1,\infty)$, C *is a finite set and* $M\in\mathcal{M}(p,q,C)$, *then* $\operatorname{exc}(M)=\operatorname{exc}(\gamma)$, *where* $\gamma\in C$ *and* $\operatorname{frq}(\gamma)=\operatorname{frq}(M)$.

Proof. Let $m=\min(p,q)$, and let $\gamma\in C$ be such that $l=\operatorname{frq}(\gamma)=\operatorname{frq}(M)$. For any $\alpha\in C$ then $k=\operatorname{frq}(\alpha)\ge\operatorname{frq}(\gamma)=l$, and, by Lemma 5.2, $\operatorname{exc}(\alpha)=k(p+q-k-1)-|C|+1\ge 0$. Therefore, $\operatorname{exc}(\alpha)-\operatorname{exc}(\gamma)=k(p+q-k-1)-l(p+q-l-1)\ge 0$, since $h(x)=x(p+q-x-1)$ is increasing in the interval $\left\langle 1,\frac{p+q-1}{2}\right\rangle\supsetneqq\langle 1,m-1\rangle$, and $p=q=m$ implies $h(m-1)=h(m)$. Consequently, $\operatorname{exc}(M)=\operatorname{exc}(\gamma)$. ∎

Lemma 7 (see [15] and [4]). *If* $p,q\in[1,\infty)$ *and* $p\le q$, *then*

$$\operatorname{achr}(K_p\square K_q)\le\max\big(\min\big(l(p+q-l-1)+1,\lfloor pq/l\rfloor\big):l\in[1,p]\big).$$

Corollary 8. *If* $q\in[7,\infty)$, *then* $\operatorname{achr}(K_6\square K_q)\le 2q+7$.

Proof. By Lemma 7 with $p=6$ we obtain $\operatorname{achr}(K_6\square K_q)\le\max\big(q+5,2q+7,2q,\lfloor\frac{6q}{4}\rfloor,\lfloor\frac{6q}{5}\rfloor,q\big)=2q+7$. ∎

4. Properties of Matrices in $\mathcal{M}(6,q,C)$

Suppose we know that $\operatorname{achr}(K_6\square K_q)\ge 2q+s-1$ for a pair (q,s) with $q\in[7,\infty)$ and $s\in[1,\infty)$, and we want to prove that $\operatorname{achr}(K_6\square K_q)=2q+s-1$; clearly, because of Corollary 8 it is sufficient to work with $s\le 7$. Proceeding by the way of contradiction let s satisfy $\operatorname{achr}(K_6\square K_q)=2q+s$. By Theorem 1 and Proposition 2 there is a $(2q+s)$-element set C and a matrix $M\in\mathcal{M}(6,q,C)$. Our task will be accomplished by showing that the existence of M leads to a contradiction. For that purpose we shall need properties of M. So in all claims of the present section we suppose that the notation corresponds to a matrix $M\in\mathcal{M}(6,q,C)$ with $q\in[7,\infty)$ and $|C|=2q+s\le 2q+7$. We associate with M an auxiliary graph G with $V(G)=[1,6]$, in which $\{i,k\}\in E(G)$ if and only if $r(i,k)\ge 1$ (so that there is a 2-colour appearing in both rows i and k).

Claim 9. *The following statements are true:*

1. $c_1 = 0$;
2. $c_l = 0$ *for* $l \in [7, \infty)$;
3. $c_2 \geq 3s$;
4. $c_{3+} \leq 2q - 2s$;
5. $\sum_{i=3}^{6} ic_i \leq 6q - 6s$;
6. $\mathrm{frq}(M) = 2$;
7. $\mathrm{exc}(M) = 7 - s$;
8. $c_{4+} \leq c_2 - 3s$;
9. *if* $\{i,k\} \in \binom{[1,6]}{2}$, *then* $r(i,k) \leq 8 - s$.

Proof. 1. If $c_1 > 0$, a 1-colour $\gamma \in C$ satisfies $\mathrm{exc}(\gamma) = q + 4 - (2q + s - 1) = 5 - q - s < 0$ in contradiction to Lemma 5.2.

2. Use Lemma 5.1.

3. By Claims 9.1 and 9.2, counting the number of vertices of $K_6\Box K_q$ we get $6q = \sum_{i=2}^{6} ic_i$. Therefore, $3(2q+s) = 3|C| = 3(c_2+c_{3+}) \leq c_2+\sum_{i=2}^{6} ic_i = c_2+6q$, which yields $c_2 \geq 3s$.

4. From $2(2q+s)+c_{3+} = 2|C|+c_{3+} = 2c_2+3c_{3+} \leq \sum_{i=2}^{6} ic_i = 6q$ we obtain $c_{3+} \leq 2q - 2s$.

5. The assertion of Claim 9.3 leads to $\sum_{i=3}^{6} ic_i = \sum_{i=2}^{6} ic_i - 2c_2 = 6q - 2c_2 \leq 6q - 6s$.

6. A consequence of Claims 9.1, 9.3 and the assumption $s \in [1, 7]$.

7. Since $\mathrm{frq}(M) = 2$ (Claim 9.6), by Lemma 6 we get $\mathrm{exc}(M) = 2q + 6 - (2q + s - 1) = 7 - s$.

8. We have $3(2q + s) - c_2 + c_{4+} = 3(c_2 + c_3 + c_{4+}) - c_2 + c_{4+} \leq \sum_{i=2}^{6} ic_i = 6q$ and $c_{4+} \leq c_2 - 3s$.

9. The inequality is trivial if $r(i,k) = 0$. If $\gamma \in \mathbb{R}(i,k)$, then each colour of $\mathbb{R}(i,k) \setminus \{\gamma\}$ contributes one to the excess of γ, hence, by Claims 9.6 and 9.7, $r(i,k) - 1 \leq \mathrm{exc}(\gamma) = \mathrm{exc}(M) = 7 - s$ and $r(i,k) \leq 8 - s$. ∎

Claim 10. *If* $\{i,k\} \in \binom{[1,6]}{2}$ *and* $r(i,k) \geq 1$, *then* $r(i,k) + r_{3+}(i,k) \leq 8 - s$.

Proof. With $\gamma \in \mathbb{R}(i,k)$ each colour of $(\mathbb{R}(i,k) \setminus \{\gamma\}) \cup \mathbb{R}_{3+}(i,k)$ makes a contribution of one to the excess of γ, hence $r(i,k) - 1 + r_{3+}(i,k) \leq \mathrm{exc}(\gamma) = \mathrm{exc}(M) \leq 7 - s$, and the claim follows. ∎

Claim 11. *If* $\{i,k\} \in \binom{[1,6]}{2}$, $r(i,k) \geq 1$, $B \subseteq [1,6]$, $3 \leq |B| \leq 4$ *and* $B \cap \{i,k\} = \emptyset$, *then* $r^*(B) \leq 2|B|$.

Proof. Consider a colour $\gamma \in \mathbb{R}(i,k)$ with $(M)_{i,j} = (M)_{k,l} = \gamma$ (where, of course, $j \neq l$). If $\beta \in \mathbb{R}^*(B)$, there is $A \subseteq B$ with $|A| \geq 2$ and $\beta \in \mathbb{R}(A)$. The colour β

appears in neither of the rows i, k, hence the pair $\{\beta, \gamma\}$ is good in M only if β occupies a position in the set $\bigcup_{m\in A}\{(m,j),\ (m,l)\} \subseteq \bigcup_{m\in B}\{(m,j),(m,l)\}$. As a consequence, $r^*(B) = |\mathbb{R}^*(B)| \le \left|\bigcup_{m\in B}\{(m,j),(m,l)\}\right| = 2|B|$. ■

Claim 12. *If* $\{i,j,k,l,m,n\} = [1,6]$ *and* $r(i,j,k) \ge 1$*, then* $r(l,m,n) \le 9$.

Proof. There is nothing to prove if $\mathbb{R}(l,m,n) = \emptyset$. Further, with $\alpha \in \mathbb{R}(i,j,k)$ and $\beta \in \mathbb{R}(l,m,n)$ the pair $\{\alpha, \beta\}$ is good in M only if the colour β occupies a position in the 9-element set $\{l,m,n\} \times \mathrm{Cov}(\alpha)$. ■

Claim 13. *If* $\Delta(G) \ge 4$*, then* $q \le 40 - 5s$.

Proof. Suppose (w) $\Delta(G) = \deg_G(1) \ge 4$, and, moreover, let (w) the sequence $(r(1,k))_{k=2}^6$ be nondecreasing. The least $p \in [2,6]$ with $r(1,p) \ge 1$ satisfies $p \le 3$, and we have $\mathbb{R}_{3+}(1) = \bigcup_{k=p}^6 \mathbb{R}_{3+}(1,k)$. Then, by Claim 9.1, $q = |\mathbb{R}(1)| = r_2(1) + r_{3+}(1)$. The inequality $r(1,k) \ge 1$ for $k \in [p,6]$ yields, by Claim 10, $r_{3+}(1,k) \le 8 - s - r(1,k)$; therefore,

$$\begin{aligned} q - r_2(1) = r_{3+}(1) &= \left|\bigcup_{k=p}^{6} \mathbb{R}_{3+}(1,k)\right| \le \sum_{k=p}^{6} r_{3+}(1,k) \\ &\le \sum_{k=p}^{6} [8 - s - r(1,k)] = (7-p)(8-s) - \sum_{k=p}^{6} r(1,k), \end{aligned}$$

and then, since $r_2(1) = \sum_{k=p}^6 r(1,k)$, we finish with $q \le (7-p)(8-s) \le 40-5s$. ■

Claim 14. *If* $\Delta(G) = 3$*,* $\{i,j,k,l,m,n\} = [1,6]$*,* $r(i,l) \ge 1$*,* $r(j,l) \ge 1$ *and* $r(k,l) \ge 1$*, then* $r(l,m,n) \ge q + 3s - 24$.

Proof. We have $\Delta(G) = \deg_G(l)$, $\mathbb{R}(l,m) = \mathbb{R}(l,n) = \emptyset$, $\mathbb{R}(l) = \mathbb{R}_2(l) \cup \mathbb{R}_{3+}(l)$, $\mathbb{R}_2(l) = \mathbb{R}(i,l) \cup \mathbb{R}(j,l) \cup \mathbb{R}(k,l)$ and $\mathbb{R}_{3+}(l) = \mathbb{R}(l,m,n) \cup \mathbb{R}_{3+}(i,l) \cup \mathbb{R}_{3+}(j,l) \cup \mathbb{R}_{3+}(k,l)$. Proceeding similarly as in the proof of Claim 13 leads to $q - r_2(l) = r_{3+}(l) \le r(l,m,n) + [8-s-r(i,l)] + [8-s-r(j,l)] + [8-s-r(k,l)] = r(l,m,n) + 3(8-s) - r_2(l)$, which yields the desired result. ■

5. Main Theorem

Theorem 15. *If* $q \in [41, \infty)$ *and* $q \equiv 1 \pmod 2$*, then* $\mathrm{achr}(K_6 \square K_q) = 2q + 3$.

Proof. We proceed by the way of contradiction. As mentioned in the beginning of Section 4, we have to show that the existence of a matrix $M \in \mathcal{M}(6, q, C)$, where C is a set of $2q + s = 2q + 4$ colours, leads to a contradiction. First notice that, by Claim 9, all colours of C are of frequency $l \in [2,6]$, $c_2 \ge 12$,

$c_{3+} \leq 2q - 8$, $\sum_{i=3}^{6} ic_i \leq 6q - 24$, $\mathrm{frq}(M) = 2$, $\mathrm{exc}(M) = 3$, and $\{i,k\} \in \binom{[1,6]}{2}$ implies $r(i,k) \leq 4$.

Since $q \geq 41$, from Claim 13 we know that $\Delta(G) \leq 3$ for the auxiliary graph G. Besides that, $\deg_G(i) = d_i$ for $i \in [1,6]$ yields $r_2(i) = \sum_{\{i,k\}\in E(G)} r(i,k) \leq \sum_{\{i,k\}\in E(G)} 4 = 4d_i$.

Claim 16. $\Delta(G) \leq 2$.

Proof. If $\Delta(G) = 3$, (w) $\deg_G(1) = 3$, $r(1,4) \geq 1$, $r(1,5) \geq 1$ and $r(1,6) \geq 1$. By Claim 14 we have $r(1,2,3) \geq q - 12 \geq 29$, and so Claim 11 yields $r(4,5) = r(4,6) = r(5,6) = 0$ (if $r(i,k) \geq 1$ for $\{i,k\} \in \binom{[4,6]}{2}$, then $29 \leq r(1,2,3) \leq r^*([1,3]) \leq 2|[1,3]| = 6$, a contradiction). Moreover, $r(4,5,6) = 0$, for otherwise, by Claim 12, $r(1,2,3) \leq 9$, a contradiction.

There is no $i \in [4,6]$ with $\deg_G(i) = 3$, because then, again by Claim 14, $r(4,5,6) \geq q - 12 \geq 29$ in contradiction to Claim 12. So, $\deg_G(i) \leq 2$, $i = 4,5,6$, $2c_2 = \sum_{i=1}^{6} r_2(i) \leq 3\cdot 12 + 3\cdot 8 = 60$ and $c_2 \leq 30$. Since $r(1,i) \geq 1$, Claim 11 yields $r^*([2,6] \setminus \{i\}) \leq 2|[2,6] \setminus \{i\}| = 8$, $i = 4,5,6$, and then $\rho^* = \sum_{i=4}^{6} r^*([2,6] \setminus \{i\}) \leq 24$.

By inspection of summands of type $r(A)$ with $A \subseteq [2,6]$, $2 \leq |A| \leq 3$, that appear when counting the three summands of ρ^*, one can see that each of $r(A)$ with $A \in \binom{[2,6]}{2} \setminus \{\{4,5\},\{4,6\},\{5,6\}\}$ appears at least twice, and each of $r(A)$ with $A \in \binom{[2,6]}{3} \setminus \{\{4,5,6\}\}$ (which is a set belonging to $C_3 \setminus \mathbb{R}_3(1)$) appears at least once. Because of $r(4,5) = r(4,6) = r(5,6) = r(4,5,6) = 0$ this leads to $2\sum_{\{i,k\}\in\binom{[2,6]}{2}} r(i,k) + c_3 - r_3(1) \leq \rho^* \leq 24$, which, having in mind that $2\sum_{\{i,k\}\in\binom{[2,6]}{2}} r(i,k) = 2c_2 - 2r_2(1)$, yields $2c_2 - 2r_2(1) + c_3 - r_3(1) \leq 24$. Together with the inequality $r_2(1) + r_3(1) \leq q$ then $c_2 + c_3 \leq q + 24 + r_2(1) - c_2 \leq q + 24$, $2q + 4 = |C| = c_2 + c_3 + c_{4+} \leq q + 24 + c_{4+}$, and so $c_{4+} \geq q - 20 \geq 21$ in contradiction to $c_{4+} \leq c_2 - 3s \leq 30 - 12 = 18$, which comes from Claim 9.8 and the above inequality for c_2. □

By Claim 16 each component of G is either a path or a cycle.

Claim 17. *No component of the graph G is K_2.*

Proof. Let (w) G have a component K_2 with vertex set $[1,2]$. Then $r(1,2) \in [1,4]$, $r(i,k) = 0$ for $(i,k) \in [1,2] \times [3,6]$, and so $c_2 = r(1,2) + \rho$, where, by Claim 11, $\rho = \sum_{\{i,k\}\in\binom{[3,6]}{2}} r(i,k) \leq r^*([3,6]) \leq 8$. Further, (w) $\mathrm{Cov}(\mathbb{R}(1,2)) = [1,n]$ with $n \in [2,8]$.

If $r(1,2) \in [1,3]$, then $c_2 \leq 3 + 8 = 11$, a contradiction.

If $r(1,2) = 4$, then $n \geq 4$, $8 \geq \rho = |C_2 \setminus \mathbb{R}(1,2)| \geq 12 - 4 = 8$, $\rho = 8$ and $c_2 = 12$. In the case $n \in [5,8]$ there is $j \in [1,n]$ such that $\mathbb{C}(j)$ contains at most $\left\lfloor \frac{2\cdot 8}{n} \right\rfloor \leq 3$ colours of $C_2 \setminus \mathbb{R}(1,2)$. Then, however, for a colour $\gamma \in \mathbb{R}(1,2) \cap \mathbb{C}(j)$

the number of colours $\delta \in C_2 \setminus \mathbb{R}(1,2)$, for which the pair $\{\gamma, \delta\}$ is good in M, is at most seven, a contradiction.

Therefore $n = 4$, 2-colours occupy all positions in $[1,6] \times [1,4]$, Claim 9.8 yields $c_{4+} \le 12 - 3 \cdot 4 = 0$, $c_{4+} = 0$, and all positions in the set $[1,6] \times [5,q]$ are occupied by 3-colours. Among other things this means that

$$c_3 = \frac{6(q-4)}{3} = 2q - 8 \ge q + (41-8) = q + 33, \tag{1}$$

$r_2(i) = 4$ and $r_3(i) = q-4$ for $i \in [1,6]$, and $r(i,k) \in [0,4]$ for every $\{i,k\} \in \binom{[3,6]}{2}$.

Recall that, by Claim 16, $\Delta(G) \le 2$. Further, if $r(i,k) = 4$ for some $\{i,k\} \in \binom{[3,6]}{2}$, then $\deg_G(m) = 1$, $m = 3,4,5,6$, since with $\{i,j,k,l\} = [3,6]$ we have $r(j,l) = 4$. In such a case G is (isomorphic to) $3K_2$. Otherwise, if $r(i,k) < 4$ for each $\{i,k\} \in \binom{[3,6]}{2}$, then $\deg_G(m) = 2$, $m = 3,4,5,6$, and G is (isomorphic to) $K_2 \cup C_4$.

Let us first consider the case $G = 3K_2$, in which (w) $r(i,i+1) = 4$, $i = 3,5$. Clearly, a set $\mathbb{R}(i,j,k)$ with $\{i,j,k\} \in \binom{[1,6]}{3}$ can be nonempty only if $\{i,j,k\} \cap \{l,l+1\} \ne \emptyset$, $l = 1,3,5$. As a consequence the assumption $\mathbb{R}(i,j,k) \ne \emptyset$ with $i < j < k$ implies $(i,j,k) \in \{(1,3,5),(1,3,6),(1,4,5),(1,4,6),(2,3,5),(2,3,6),(2,4,5),(2,4,6)\}$.

Suppose that $\{(i_m,j_m) : m \in [1,4]\} = \{(3,5),(3,6),(4,5),(4,6)\} = \{(k_m,l_m) : m \in [1,4]\}$ and $\{i_m,j_m,k_m,l_m\} = [3,6]$ for $m \in [1,4]$. Then

$$c_3 = \sum_{m=1}^{4} [r(1,i_m,j_m) + r(2,k_m,l_m)]. \tag{2}$$

Further, for $m,n \in [1,4]$ the sets $\{i_m,j_m\}$ and $\{k_n,l_n\}$ are disjoint if and only if $m = n$. Put

$$\begin{aligned} T(1) &= \{m \in [1,4] : r(1,i_m,j_m) \ge 1\}, \qquad & t(1) &= |T(1)|,\\ T(2) &= \{m \in [1,4] : r(2,k_m,l_m) \ge 1\}, \qquad & t(2) &= |T(2)| \end{aligned}$$

and $T = T(1) \cap T(2)$. Let

$$\sigma(P) = \sum_{p \in P} [r(1,i_p,j_p) + r(2,k_p,l_p)]$$

for $P \subseteq [1,4]$. Using Claim 12 we see that $m \in T(1)$ implies $r(2,k_m,l_m) \le 9$, while $m \in T(2)$ means that $r(1,i_m,j_m) \le 9$. Therefore, with $m \in T$ we have $r(1,i_m,j_m) + r(2,k_m,l_m) \le 9 + 9 = 18$, and so $\sigma(T) \le 18|T|$.

If $t(1) = 4$, then $q - 4 = r_3(2) \le \sum_{x=1}^{4} r(2,k_x,l_x) \le 4 \cdot 9 = 36$, and $q \le 40$, a contradiction. Similarly, with $t(2) = 4$ we get $q - 4 = r_3(1) \le \sum_{x=1}^{4} r(1,i_x,j_x) \le 36$, and $q \le 40$ as well.

If there is $x \in [1,2]$ such that $t(x) = 1$ and $T(x) = \{m\}$, then $r_3(x) = r(x, i_m, j_m) = r_3(i_m) = r_3(j_m) = q - 4 \geq 37$, hence $r(3-x, k_m, l_m) = r_3(3-x) = r_3(k_m) = r_3(l_m) = q - 4 \geq 37$, which contradicts Claim 12.

We are left with the situation $t(1), t(2) \in [2,3]$ (and $|T| \leq \min(t(1), t(2))$). Suppose (w) $t(1) \geq t(2)$.

If $|T| = 3$, then $T(1) = T = T(2)$, $[1,4] \setminus T = \{p\}$ and $r(1, i_p, j_p) = r(2, k_p, l_p) = 0$, hence $2q - 8 = c_3 = \sigma([1,4]) = \sigma(T) \leq 18 \cdot 3 = 54$ and $q \leq 31$, a contradiction.

If $|T| = 2$, then n out of four summands that sum up to $\sigma([1,4] \setminus T)$ are positive, $n \leq 2$. Moreover, $\sigma([1,4] \setminus T) \leq q - 4$. The inequality is obvious provided that $n \leq 1$, while if $n = 2$, $a \in T(1) \setminus T(2)$ and $b \in T(2) \setminus T(1)$, then with $e \in \{i_a, j_a\} \cap \{k_b, l_b\}$ we have $\sigma([1,4] \setminus T) = r(1, i_a, j_a) + r(2, k_b, l_b) \leq r_3(e) = q - 4$. Thus $c_3 = \sigma(T) + \sigma([1,4] \setminus T) \leq 18 \cdot 2 + (q-4) = q + 32$ in contradiction to (1).

For $|T| = 1$ we get $t(2) = 2$. With $t(1) = 2$ we obtain, similarly as in the case $|T| = 2$, $c_3 \leq 18 + (q-4) = q + 14$. So, assume that $t(1) = 3$, $T = \{t\}$ and $T(2) \setminus T(1) = \{p\}$ (note that $p \neq t$).

Suppose first that $\{k_p, l_p\} \neq \{i_t, j_t\}$. In such a case $|\{k_p, l_p\} \cap \{i_t, j_t\}| = 1$; moreover, since $\{k_p, l_p\} \subseteq [3,6] = \{i_t, j_t, k_t, l_t\}$, we have $|\{k_p, l_p\} \cap \{k_t, l_t\}| = 1$, and there is $g \in \{k, l\}$ such that $\{k_p, l_p\} \cap \{k_t, l_t\} = \{g_t\}$. Then five from among eight summands in (2) are positive, namely $r(1, i_t, j_t)$, $r(2, k_t, l_t)$, $r(2, k_p, l_p)$ and $r(1, i_m, j_m)$ with $m \in [1,4] \setminus \{t, p\}$. Having in mind that $g_t \notin \{i_t, j_t\}$, $g_t \notin \{i_p, j_p\}$, and each element of $[3,6]$ is involved in exactly two of the ordered pairs $(3,5), (3,6), (4,5), (4,6)$, we see that except for $r(1, i_t, j_t)$ all mentioned positive summands correspond to colours of $\mathbb{R}_3(g_t)$. That is why $c_3 \leq r(1, i_t, j_t) + r_3(g_t) \leq 9 + (q-4) = q + 5$, a contradiction to (1) again.

On the other hand, if $\{k_p, l_p\} = \{i_t, j_t\}$, then $|\{i_p, j_p\} \cap \{i_t, j_t\}| = |\{i_p, j_p\} \cap \{k_p, l_p\}| = 0$, hence for $m \in [1,4] \setminus \{t, p\}$ we have $|\{i_m, j_m\} \cap \{i_t, j_t\}| = 1$, and so positive summands in (2) are $r(1, i_t, j_t)$, $r(2, k_t, l_t)$, $r(2, k_p, l_p) = r(2, i_t, j_t)$, $r(1, g_t, k_t)$ and $r(1, h_t, l_t)$, where $\{g, h\} = \{i, j\}$. Then

$$\begin{aligned} q - 4 &= r_3(2) = r(2, k_t, l_t) + r(2, i_t, j_t), \\ q - 4 &= r_3(k_t) = r(2, k_t, l_t) + r(1, g_t, k_t), \end{aligned}$$

which yields

$$r(2, i_t, j_t) = r(1, g_t, k_t) = q - 4 - r(2, k_t, l_t) \geq (q-4) - 9 = q - 13$$

so that

$$q - 4 = r_3(g_t) = r(1, i_t, j_t) + r(1, g_t, k_t) + r(2, i_t, j_t) \geq 1 + 2(q - 13)$$

and $q \leq 21$, a contradiction.

If $|T| = 0$, then $t(1) = t(2) = 2$, and, by symmetry, $T(1) = [1,2]$, $T(2) = [3,4]$. We have $\{i_1, j_1\} \cup \{i_2, j_2\} = [3,6]$, for otherwise there is $e \in [3,6] \setminus (\{i_1, j_1\} \cup \{i_2, j_2\})$, hence $e \in \{i_3, j_3\} \cap \{i_4, j_4\}$, $e \notin \{k_3, l_3\} \cup \{k_4, l_4\}$ and $r_3(e) = 0 \neq q - 4$, a contradiction. Thus, by symmetry we may assume that $i_1 = 3 < i_2 = 4$.

Therefore, (w) $(i_1, j_1) = (3,5)$ and $(i_2, j_2) = (4,6)$, which means that $r(m, n, p)$ with $\{m, n, p\} \in \binom{[1,6]}{3}$ and $m < n < p$ is positive if and only if $(m, n, p) \in \{(1,3,5), (1,4,6), (2,3,6), (2,4,5)\}$. We have $r_3(6) = r(1,4,6) + r(2,3,6) = q - 4 \equiv 1 \pmod 2$, hence

$$r(1,4,6) \not\equiv r(2,3,6) \pmod 2. \tag{3}$$

Similarly, from $r_3(1) = q - 4 = r_3(3)$ it follows that $r(1,3,5) \not\equiv r(1,4,6) \pmod 2$ and $r(1,3,5) \not\equiv r(2,3,6) \pmod 2$ so that $r(1,4,6) \equiv r(2,3,6) \pmod 2$, which contradicts (3).

It remains to analyse the case $G = K_2 \cup C_4$, in which (w) $\{\{3,4\}, \{4,5\}, \{5,6\}, \{6,3\}\} \subseteq E(G)$. The completeness of f_M implies $C_3 \subseteq \mathbb{R}(1,3,5) \cup \mathbb{R}(1,4,6) \cup \mathbb{R}(2,3,5) \cup \mathbb{R}(2,4,6)$. So,

$$c_3 = [r(1,3,5) + r(2,4,6)] + [r(1,4,6) + r(2,3,5)].$$

Let $\bar{\mathcal{T}} = \{(1,3,5), (1,4,6), (2,3,5), (2,4,6)\}$,

$$\mathcal{T} = \{(i,j,k) \in \bar{\mathcal{T}} : r(i,j,k) \geq 1\}$$

and $t = |\mathcal{T}|$. From $r_3(i) = q - 4 > 0$ for $i \in [1,6]$ it follows that $t \geq 2$. If $(i,j,k), (l,m,n) \in \mathcal{T}$, $(i,j,k) \neq (l,m,n)$, then either $r(i,j,k) + r(l,m,n) \leq 9 + 9 = 18$ (by Claim 12, if $\{i,j,k\} \cap \{l,m,n\} = \emptyset$) or $r(i,j,k) + r(l,m,n) \leq r_3(p) = q - 4$ (if $p \in \{i,j,k\} \cap \{l,m,n\}$). As a consequence then $c_3 \leq \max(b_2, b_3, b_4)$, where $b_2 = q - 4$, $b_3 = 18 + (q - 4) = q + 14$ and $b_4 = 18 + 18 = 36$. Thus $c_3 \leq q + 14$, which contradicts (1). □

Claim 18. *No component of the graph G is K_3.*

Proof. Let (w) G have the component K_3 with the vertex set $[1,3]$.

If $G = K_3 \cup 3K_1$, then $r(1,2) = r(1,3) = r(2,3) = 4$, $c_2 = 12$ and $c_{4+} = 0$. From Claim 10 it follows that $r_{3+}(1,3) = 0 = r_{3+}(2,3)$, hence $r(3,i,k) \geq 1$ with $\{i,k\} \in \binom{[1,6] \setminus \{3\}}{2}$ implies $\{i,k\} \in \{\{4,5\}, \{4,6\}, \{5,6\}\}$. By Claim 11 then $r_3(3) = r(3,4,5) + r(3,4,6) + r(3,5,6) \leq r^*([3,6]) \leq 8$, hence $q = r_2(3) + r_3(3) \leq (4+4) + 8 = 16$, a contradiction.

If G has besides the above K_3 another nontrivial component (of order at least 2) and $r(i,k) \geq 1$ with $\{i,k\} \in \binom{[4,6]}{2}$, then, by Claim 11, $r(1,2) + r(1,3) + r(2,3) + r(1,2,3) = r^*([1,3]) \leq 6$ and $r(4,5) + r(4,6) + r(5,6) + r(4,5,6) = r^*([4,6]) \leq 6$, hence $12 \leq c_2 \leq 6 + 6$, $c_2 = 12$, $r(1,2) + r(1,3) + r(2,3) = r(4,5) + r(4,6) + r(5,6) =$

6, $r(1,2,3) = r(4,5,6) = 0$, $c_{4+} = 0$, and $c_3 = 2q-8 > 0$. So, among other things, with $C_2^1 = \mathbb{R}(1,2)\cup\mathbb{R}(1,3)\cup\mathbb{R}(2,3)$ and $C_2^2 = \mathbb{R}(4,5)\cup\mathbb{R}(4,6)\cup\mathbb{R}(5,6)$ we have $|C_2^1| = |C_2^2| = 6$. If $\alpha \in C_2^1$ and $\beta \in C_2^2$, then the pair $\{\alpha,\beta\}$ is column-based, hence all six positions in $\mathrm{Cov}(\alpha)\times[4,6]$ are occupied by colours of C_2^2, and for any $j \in \mathrm{Cov}(C_2^1)$ all three positions in $\{j\}\times[4,6]$ are occupied by colours of C_2^2. Consequently, $\mathrm{cov}(C_2^1) = \frac{2|C_2^2|}{3} = 4$, and (w) all positions in $[1,6]\times[1,4]$ are occupied by colours of $C_2 = C_2^1\cup C_2^2$. If $\{i,k\} \in \binom{J}{2}$, where $J \in \{[1,3],[4,6]\}$, then $r(i,k) > 0$, since otherwise with $\{i,j,k\} = J$ we get $r_2(j) = r(i,j)+r(j,k) = 6$, which contradicts Claim 9.9. For a colour $\gamma \in C_3$ there are $I, J \in \{[1,3],[4,6]\}$, $I \neq J$, such that $|\mathbb{R}(\gamma)\cap I| = 2$ and $|\mathbb{R}(\gamma)\cap J| = 1$. Then $\mathbb{R}(\gamma)\cap J = \{j\}$, and with $\{i,j,k\} = J$ there exists a colour $\alpha \in \mathbb{R}(i,k)$; in such a case, however, the pair $\{\alpha,\gamma\}$ is not good, a contradiction. □

Claim 19. *No component of the graph G is a path of order at least* 3.

Proof. Suppose that G has a path component P of order at least 3.

If G has besides P another nontrivial component (of order at least 2), then $G = P\cup P'$, where, by Claims 17 and 18, both P and P' are paths of order 3, (w) $V(P) = [1,3]$, $V(P') = [4,6]$ and $r(i,i+1) \geq 1$ for $i = 1,2,4,5$. Similarly as in the proof of Claim 18 it is easy to see that $r(1,2)+r(2,3) = r(4,5)+r(5,6) = 6$. Since $r_2(2) = 6$, there are colours $\alpha,\beta \in \mathbb{R}(1,2)\cup\mathbb{R}(2,3)$ such that $\mathrm{Cov}(\alpha)\cap\mathrm{Cov}(\beta) = \emptyset$. Then each colour of $\mathbb{R}(4,5)\cup\mathbb{R}(5,6)$ occupies a position in $[4,6]\times\mathrm{Cov}(\alpha)$ and a position in $[4,6]\times\mathrm{Cov}(\beta)$ as well so that $\mathrm{Cov}(\mathbb{R}(4,5)\cup\mathbb{R}(5,6)) \subseteq \mathrm{Cov}(\alpha,\beta)$; this leads to a contradiction since $\mathrm{cov}(\mathbb{R}(4,5)\cup\mathbb{R}(5,6)) \geq r_2(5) = 6$ and $\mathrm{cov}(\alpha,\beta) = 4$.

So, P is the unique nontrivial component of G, (w) $V(P) = [1,p]$ and $E(P) = \{\{i,i+1\} : i \in [1,p-1]\}$. Since $12 \leq c_2 = \sum_{i=1}^{p-1} r(i,i+1) \leq 4(p-1)$, we have $p \in [4,6]$.

If $p = 4$, then $r(i,i+1) = 4$, $i = 1,2,3$, $c_2 = 12$, $c_{4+} = 0$ and

$$i \in [1,6] \Rightarrow r_3(i) = q - r_2(i) \geq q-8 \geq 33. \tag{4}$$

If $\alpha \in \mathbb{R}(1,2)$ and $\beta \in \mathbb{R}(3,4)$, then the pair $\{\alpha,\beta\}$ is column-based, hence all four positions in $\mathrm{Cov}(\alpha)\times[3,4]$ are occupied by colours of $\mathbb{R}(3,4)$, and with $j \in \mathrm{Cov}(\mathbb{R}(1,2))\times[3,4]$, both positions in $\{j\}\times[3,4]$ are occupied by colours of $\mathbb{R}(3,4)$. Therefore, $\mathrm{cov}(\mathbb{R}(1,2)) = \frac{2|\mathbb{R}(3,4)|}{2} = 4$, and (w) all positions in $[1,4]\times[1,4]$ are occupied by colours of $\mathbb{R}(1,2)\cup\mathbb{R}(3,4)$. Thus, (w) $\mathrm{Cov}(\mathbb{R}(2,3)) = [5,n]$, where $n \in [8,12]$.

By Claim 10 we know that $r(i,j,k) = 0$ if there is $l \in [1,3]$ such that $\{l,l+1\} \subseteq \{i,j,k\}$.

Suppose that $\gamma \in \mathbb{R}(1,5,6)$. Since all pairs $\{\gamma,\delta\}$ with $\delta \in \mathbb{R}(3,4)$ are good in M, two positions in $[5,6]\times[1,4]$ must be occupied by γ. Then, however, the number of pairs $\{\gamma,\varepsilon\}$ with $\varepsilon \in \mathbb{R}(2,3)$ that are good in M is at most two, a contradiction. So, $r(1,5,6)=0$, and an analogous reasoning shows that $r(4,5,6)=0$.

Claim 11 yields $r(1,4,5)+r(1,4,6) \le r^*(\{1,4,5,6\}) \le 8$. A colour $\gamma \in \mathbb{R}(2,5,6)$ as well as a colour $\delta \in \mathbb{R}(3,5,6)$ occupies two positions in $[5,6]\times[1,4]$ (each pair $\{\gamma,\varepsilon\}$ with $\varepsilon \in \mathbb{R}(3,4)$ and each pair $\{\delta,\zeta\}$ with $\zeta \in \mathbb{R}(1,2)$ is good in M). Thus $r(2,5,6)+r(3,5,6) \le 4$ and

$$r(1,4,5)+r(1,4,6)+r(2,5,6)+r(3,5,6) \le 12. \tag{5}$$

The set of remaining triples $(i,j,k) \in [1,6]^3$ with $i<j<k$ such that $r(i,j,k)$ can be positive is $\mathcal{T}=\{(1,3,5),(1,3,6),(2,4,5),(2,4,6)\}$. Suppose that $r(i,j,k) \ge 1$ with $(i,j,k) \in \mathcal{T}$ if and only if $(i,j,k) \in \{(i_l,j_l,k_l) : l \in [1,t]\}$. We show that there is $m \in [1,6]$ with $r_3(m) \le 30$ in contradiction to (4).

If $t=4$, then, by Claim 12, $r(i_l,j_l,k_l) \le 9$, $l=1,2,3,4$, and so, using (5), $r_3(m) \le 12+2\cdot 9=30$ for (any) $m \in [1,6]$. If $t=3$ and $r(i,j,k)=0$ for $(i,j,k) \in \mathcal{T}$, then $r_3(m) \le 12+9=21$ for $m \in \{i,j,k\}$. The same upper bound applies for $m \in [1,6]$ if $t=2$ and $\{i_1,j_1,k_1\} \cap \{i_2,j_2,k_2\}=\emptyset$.

If $t=2$ and $\{i_1,j_1,k_1\} \cap \{i_2,j_2,k_2\} \ne \emptyset$, for $m \in [1,6] \setminus (\{i_1,j_1,k_1\} \cup \{i_2,j_2,k_2\})$ we obtain $r_3(m) \le 12$. The same inequality is available for $m \in [1,6] \setminus \{i_1,j_1,k_1\}$ if $t=1$ and for $m \in [1,6]$ if $t=0$.

If $p=5$, then, by Claim 11, from $r(4,5) \ge 1$ it follows that $r(1,2)+r(2,3) \le r^*([1,3]) \le 6$; similarly, $r(1,2) \ge 1$ yields $r(3,4)+r(4,5) \le r^*([3,5]) \le 6$. Then $12 \le c_2 = \sum_{i=1}^{4} r(i,i+1) \le 6+6$, $c_2=12$ and $r(1,2)+r(2,3)=6=r(3,4)+r(4,5)$. If $(M)_{1,j}=\gamma \in \mathbb{R}(1,2)$, then all positions in $[3,5]\times \mathrm{Cov}(\gamma)$ are occupied by six distinct colours of $\mathbb{R}(3,4)\cup\mathbb{R}(4,5)$, hence $(M)_{3,j} \in \mathbb{R}(3,4)$ and $\delta=(M)_{5,j} \in \mathbb{R}(4,5)$. Analogously, all positions in $[1,3]\times\mathrm{Cov}(\delta)$ are occupied by six distinct colours of $\mathbb{R}(1,2)\cup\mathbb{R}(2,3)$, which implies $(M)_{3,j} \in \mathbb{R}(2,3)$, a contradiction.

If $p=6$, then, by Claim 11, $r^*([1,6]\setminus[l,l+1]) \le 8$ for $l \in [1,5]$, hence

$$\rho^* = \sum_{l=1}^{5} r^*([1,6]\setminus[l,l+1]) \le 40. \tag{6}$$

It is easy to see that in the sum ρ^* each of the summands $r(i,i+1)$ with $i \in [1,5]$ appears in the expression of $r^*([1,6]\setminus[l,l+1])$ for at least two l's, while each of the summands $r(i,j,k)$ satisfying $\{i,j,k\} \in \binom{[1,6]}{3} \setminus \{\{1,3,5\},\{2,3,5\},\{2,4,5\},\{2,4,6\}\}$ and $i<j<k$ appears for at least one l. Since $c_2=\sum_{l=1}^{5} r(l,l+1)$, with

$$\rho = r(1,3,5)+r(2,3,5)+r(2,4,5)+r(2,4,6)$$

the inequality (6) leads to

$$2c_2+c_3-\rho \le \rho^* \le 40. \tag{7}$$

Moreover, $r_2(l)+r_3(l)\le q$ implies

$$r_3(l)\le q-r_2(l)=q-[r(l-1,l)+r(l,l+1)]\le q-2,\ l=2,5,$$

and so, having in mind that, by Claim 12, $\min(r(1,3,5),r(2,4,6))\le 9$,

$$\begin{aligned}\rho &\le \min(r_3(2)+r(1,3,5),r_3(5)+r(2,4,6))\\ &\le q-2+\min(r(1,3,5),r(2,4,6))\le q+7.\end{aligned}\tag{8}$$

Since, by Claim 9.8, $c_{4+}-c_2\le -12$, using (7) and (8) we obtain

$$2q+4=c_2+c_3+c_{4+}\le 40+\rho+c_{4+}-c_2\le 40+(q+7)-12=q+35,$$

and, finally, $q\le 31$, a contradiction. □

With $l\in\mathbb{Z}$ and $m\in[2,\infty)$ we use $(l)_m$ to denote the unique $n\in[1,m]$ satisfying $n\equiv l \pmod m$.

Claim 20. *No component of the graph G is a 4-cycle.*

Proof. If G has a 4-cycle component, (w) $r(i,(i+1)_4)\ge 1$ for $i\in[1,4]$. Note that, by Claim 17, $r(5,6)=0=r_2(5)=r_2(6)$. Let $i\in[1,4]$ and

$$P_i=\{((i+2)_4,5),((i+2)_4,6),(5,6)\}.$$

If $\gamma\in\mathbb{R}_{3+}(i)$, there is $A\subseteq[1,6]$ such that $|A|\ge 3$, $\{i\}\subseteq A$ and $\gamma\in\mathbb{R}(A)$. Provided that $A\cap\{(i-1)_4,(i+1)_4\}\ne\emptyset$, we get $\gamma\in\bigcup_{j\in\{(i-1)_4,i\}}\mathbb{R}_{3+}(j,(j+1)_4)$. On the other hand, $A\cap\{(i-1)_4,(i+1)_4\}=\emptyset$ implies either $|A|=3$ and $\gamma\in\bigcup_{(j,k)\in P_i}\mathbb{R}(i,j,k)$ or $|A|=4$ and $\gamma\in\mathbb{R}(i,(i+2)_4,5,6)$. As a consequence,

$$\mathbb{R}_{3+}(i)\subseteq\mathbb{R}(i,(i+2)_4,5,6)\cup\bigcup_{j\in\{(i-1)_4,i\}}\mathbb{R}_{3+}(j,(j+1)_4)\cup\bigcup_{(j,k)\in P_i}\mathbb{R}(i,j,k),$$

and by Claim 10 we have

$$\begin{aligned}r_{3+}(i)&\le r(i,(i+2)_4,5,6)+\sum_{j\in\{(i-1)_4,i\}}r_{3+}(j,(j+1)_4)+\sum_{(j,k)\in P_i}r(i,j,k)\\ &\le r(i,(i+2)_4,5,6)+\sum_{j\in\{(i-1)_4,i\}}[4-r(j,(j+1)_4)]+\sum_{(j,k)\in P_i}r(i,j,k).\end{aligned}$$

Therefore, realising that

$$\sum_{i=1}^{4}\sum_{j\in\{(i-1)_4,i\}}r(j,(j+1)_4)=\sum_{i=1}^{4}r_2(i),$$

we obtain

$$(9)\qquad \sum_{i=1}^{4} r_{3+}(i) \leq 32 + \sum_{i=1}^{4}\left[r(i,(i+2)_4,5,6) + \sum_{(j,k)\in P_i} r(i,j,k)\right] - \sum_{i=1}^{4} r_2(i).$$

On the right-hand side of the inequality (9) there are among others (partial) summands $r(A)$ with $A \in \binom{[1,6]}{3} \cup \binom{[1,6]}{4}$, each such summand appears there with the frequency 0, 1 or 2, and the frequency is 2 if and only if $A \in \{\{1,3,5\},\{1,3,6\},\{2,4,5\},\{2,4,6\},\{1,3,5,6\},\{2,4,5,6\}\}$. Thus, with

$$\rho_3 = r(1,3,5) + r(1,3,6) + r(2,4,5) + r(2,4,6),$$
$$\rho_4 = r(1,3,5,6) + r(2,4,5,6)$$

the inequality (9) leads to

$$(10)\qquad 4q = \sum_{i=1}^{4}[r_2(i) + r_{3+}(i)] \leq 32 + \rho_3 + \rho_4 + c_3 + c_4.$$

Let us show that

$$(11)\qquad \rho_3 \leq q + 10.$$

To see it let a be the number of sets A belonging to

$$\mathcal{A} = \{\{1,3,5\},\{1,3,6\},\{2,4,5\},\{2,4,6\}\}$$

with $r(A) \geq 1$.

If $a = 4$, by Claim 12 we have $\rho_3 \leq 4\cdot 9 = 36 \leq q + 10$.

In the case $a = 3$ there is $i \in [1,4]$ such that $\rho_3 \leq 2\cdot 9 + r_3(i)$. Evidently, $r_3(i) \leq q - r_2(i) = q - [r((i-1)_4, i) + r(i,(i+1)_4)] \leq q - (4+4) = q - 8$, hence $\rho_3 \leq q + 10$.

If $a = 2$, let the positive summands of ρ_3 be $r(i,j,k)$ and $r(l,m,n)$, $\{i,j,k\} \neq \{l,m,n\}$. If $\{i,j,k\} \cap \{l,m,n\} = \emptyset$, then $\rho_3 \leq 2\cdot 9 \leq q + 10$, and otherwise, with $p \in \{i,j,k\} \cap \{l,m,n\}$, we have $\rho_3 \leq r_3(p) \leq q$.

If $a \in [0,1]$, then $\rho_3 \leq q - 2$, since $r(i,j,k) \geq 1$ with $\{i,j,k\} \in \mathcal{A}$ and $i < j < k$ implies $i \in [1,2]$, while $r_2(i) \geq 1 + 1$.

Now realise that, by Claim 9.8, $\rho_4 + c_3 + c_4 \leq c_3 + 2c_{4+} = |C| + (c_{4+} - c_2) \leq (2q+4) - 12 = 2q - 8$, and so, using (10) and (11), $4q \leq 32 + (q+10) + (2q-8) = 3q + 34$ and $q \leq 34$, a contradiction. □

Claim 21. *No component of the graph G is a 5-cycle.*

Proof. If G has a 5-cycle component, (w) $r(i,(i+1)_5) \geq 1$ for $i \in [1,5]$. Similarly as in the proof of Claim 20 for $i \in [1,5]$ we get

$$r_{3+}(i) \leq r((i-2)_5, i, (i+2)_5, 6) + \sum_{j \in \{(i-1)_5, i\}} r_{3+}(j,(j+1)_5) + \sum_{(j,k) \in P_i} r(i,j,k),$$

this time with

$$P_i = \{((i-2)_5, 6), ((i+2)_5, 6), ((i-2)_5, (i+2)_5)\},$$

which yields

$$5q = \sum_{i=1}^{5} [r_2(i) + r_{3+}(i)] \leq 40 + \rho_3 + c_3 + c_4, \tag{12}$$

where

$$\rho_3 = r(1,3,6) + r(1,4,6) + r(2,4,6) + r(2,5,6) + r(3,5,6) \leq r_3(6) \leq q. \tag{13}$$

Moreover, $c_3 + c_4 \leq 2q + 4 - c_2 \leq 2q - 8$, and so, using (12) and (13), $5q \leq 40 + q + (2q-8) = 3q + 32$ and $q \leq 16$, a contradiction. □

Claim 22. *The graph G is not a 6-cycle.*

Proof. If G is a 6-cycle, (w) $r(i,(i+1)_6) \geq 1$ for $i \in [1,6]$. In this case $r_{3+}(i)$ is upper bounded by

$$r((i-2)_6, i, (i+2)_6, (i+3)_6) + \sum_{j \in \{(i-1)_6, i\}} [4 - r(j,(j+1)_6)] + \sum_{(j,k) \in P_i} r(i,j,k)$$

with

$$P_i = \{((i-2)_6, (i+2)_6), ((i-2)_6, (i+3)_6), ((i+2)_6, (i+3)_6)\},$$

and one can see that

$$6q \leq 48 + \rho_3 + c_3 + c_4, \tag{14}$$

where $\rho_3 = 2r(1,3,5) + 2r(2,4,6)$. We can bound ρ_3 from above by $2q-4$. Indeed, if both $r(1,3,5)$ and $r(2,4,6)$ are positive, then Claim 12 yields $\rho_3 \leq 18 + 18 = 36 \leq 2q - 4$. On the other hand, if $r(i,j,k) = 0$ with $(i,j,k) \in \{(1,3,5),(2,4,6)\}$, then $\rho_3 \leq 2r_3(i+1) \leq 2[q - r_2(i+1)] = 2q - 2[r_2(i,i+1) + r_2(i+1,i+2)] \leq 2q - 4$. Therefore, similarly as in the proof of Claim 21, from (14) we obtain $6q \leq 48 + (2q-4) + (2q-8) = 4q + 36$ and $q \leq 18$, a contradiction. □

Thus, by Claims 16–22, we conclude that $G = 6K_1$ and $c_2 = 0$, which contradicts Claim 9.3. Therefore, Theorem 15 is proved. ∎

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