Discussiones Mathematicae
Graph Theory 43 (2023) 1195–1202
https://doi.org/10.7151/dmgt.2419

SPANNING TREES WITH A BOUNDED NUMBER OF BRANCH VERTICES IN A $K_{1,4}$-FREE GRAPH

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Abstract

In 2008, it was conjectured that, for any positive integer k, a connected n-vertex graph G must contain a spanning tree with at most k branch vertices if $\sigma_{k+3}(G) \geq n-k$. In this paper, we resolve this conjecture in the affirmative for the graphs $K_{1,4}$-free.

Keywords: spanning tree, branch vertices, $K_{1,4}$-free.

2020 Mathematics Subject Classification: 05C05, 05C70, 05C07, 05C69.

1. Introduction and Main Result

In this paper, we are interested in finite simple graphs. Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. For any vertex $v \in V(G)$, we use $N_G(v)$ and $\deg_G(v)$ to denote the set of neighbors of v and the degree of v in G, respectively. We define $G-uv$ to be the graph obtained from G by deleting the edge $uv \in E(G)$, and $G+uv$ to be the graph obtained from G by adding an edge uv between two non-adjacent vertices u and v of G. For any $X \subseteq V(G)$, we denote by $|X|$ the cardinality of X. We use $G-X$ to denote the graph obtained from G by deleting the vertices in X together with their incident edges. The subgraph of G induced by X is denoted by $G[X]$.

A subset $X \subseteq V(G)$ is called an *independent set* of G if no two vertices of X are adjacent in G. For each positive integer k, we define

$$\sigma_k(G) = \min\left\{ \sum_{i=1}^{k} \deg_G(v_i) \mid \{v_1, \ldots, v_k\} \text{ is an independent set in } G \right\}.$$

Let T be a tree. A vertex of degree one is a *leaf* of T and a vertex of degree at least three is a *branch vertex* of T. We recall to [5] for terminology and notation not defined here.

Definition [5]**.** Denote by $L(T)$ and $B(T)$ the set of leaves and the set of branch vertices of a tree T, respectively. Let $B_n(T)$ denote the set of branch vertices of T with degree exactly n, and let $B_{\leq n}(T)$, $(B_{\geq n}(T))$ denote the set branch vertices of T with degree at most (at least) n. Any two vertices of T, say u and v, are joined by a unique path, denoted uTv. Now if $e \in E(T)$, then eTv denotes the unique shortest path containing v and one of the vertices incident to e, but not edge e. We also denote $\{u_v\} = V(uTv) \cap N_T(u)$ and e_v as the vertex incident to e in the direction toward v. We call the set $S_T = \bigcup_{u,v \in B(T)} uTv$ the *internal subtree* of T.

Definition [5]**.** Let T be a spanning tree of a graph G and let $v \in V(G)$ and $e \in E(T)$. Denote $g(e, v)$ as the vertex incident to e farthest away from v in T. We say v is an *oblique neighbor* of e with respect to T if $vg(e, v) \in E(G)$.

Definition [5]**.** Let T be a spanning tree of a graph G. Two vertices are *pseudoadjacent* with respect to T if there is some $e \in E(T)$ which has them both as oblique neighbors. Similarly, a vertex set is *pseudoindependent* with respect to T if no two vertices in the set are pseudoadjacent with respect to T.

For positive integer r, a graph is said to be $K_{1,r}$*-free* if it does not contain $K_{1,r}$ as an induced subgraph. A $K_{1,3}$-free graph is also called a *claw-free* graph. We use K_n to denote the complete graph on n vertices. There are several well-known conditions (such as the independence number conditions and the degree sum conditions) ensuring that a graph G contains a spanning tree with a bounded number of leaves or branch vertices.

Theorem 1 [3, Gargano *et al.*]**.** *Let k be a non-negative integer and let G be a connected claw-free graph of order n. If $\sigma_{k+3} \geq n - k - 2$, then G has a spanning tree with at most k branch vertices.*

Theorem 2 [7, Kano *et al.*]**.** *Let k be a non-negative integer and let G be a connected claw-free graph of order n. If $\sigma_{k+3} \geq n - k - 2$, then G has a spanning tree with at most $k + 2$ leaves.*

For connected $K_{1,4}$-free graphs, Kyaw [8, 9] obtained the following two sharp results.

Theorem 3 [8, Kyaw]**.** *Let G be a connected $K_{1,4}$-free graph with n vertices. If $\sigma_4(G) \geq n - 1$, then G contains a spanning tree with at most 3 leaves.*

Theorem 4 [9, Kyaw]**.** *Let G be a connected $K_{1,4}$-free graph with n vertices.*

(i) *If $\sigma_3(G) \geq n$, then G has a Hamiltonian path.*

(ii) *If $\sigma_{m+1}(G) \geq n - \frac{m}{2}$ for some integer $m \geq 3$, then G has a spanning tree with at most m leaves.*

For the graph $K_{1,5}$-free, some results were obtained in 2019.

Theorem 5 [1, Chen, Ha and Hanh]. *Let G be a connected $K_{1,5}$-free graph with n vertices. If $\sigma_5(G) \geq n-1$, then G contains a spanning tree with at most 4 leaves.*

Theorem 6 [6, Hu and Sun]. *Let G be a connected $K_{1,5}$-free graph with n vertices. If $\sigma_6(G) \geq n-1$, then G contains a spanning tree with at most 5 leaves.*

In [10], Matsuda *et al.* gave a conjecture of conditions on connected claw-free graph which ensures the existence of a spanning tree with at most k branch vertices. As they mentioned, it is best possible.

Conjecture 7 [10, Matsuda *et al.*]. *Let k be a non-negative interger and let G be a connected claw-free graph of order n. If $\sigma_{2k+3}(G) \geq n-2$, then G has a spanning tree with at most k branch vertices.*

This conjecture was proved for $k=1$ in [10], $k=0$ in [11] and $k=2$ in [4]. Very recently, the authors in [5] have completely solved Conjecture 7 for $k \geq 0$. The technique used in [5] is to control the total order condition of each independent set by counting the oblique neighbors of the edges in a spanning tree T. Regarding the existence of a spanning tree with a number of branched vertices bounded in a connected graph, Flandrin *et al.* [2] proposed the following conjecture.

Conjecture 8 [2, Flandrin *et al.*]. *Let k be a positive interger and let G be a connected graph of order n. If $\sigma_{k+3}(G) \geq n-k$, then G has a spanning tree with at most k branch vertices.*

In this paper, we will prove the conjecture for the case of the graph is $K_{1,4}$-free.

Theorem 9. *Let k be a positive interger and let G be a connected $K_{1,4}$-free graph of order n. If $\sigma_{k+3}(G) \geq n-k$, then G has a spanning tree with at most k branch vertices.*

We end this section by constructing an example to show that the conditions of Theorem 9 is sharp. Let k, m be positive integers. Let $P = x_1x_2\cdots x_{k+1}$ be a path. Let $D_0, D_1, \ldots, D_{k+1}, D_{k+2}$ be copies of the graph K_m. For each $i \in \{1, 2, \ldots, k+1\}$, join x_i to all vertices of the graph D_i, join x_1 to all vertices of the graph D_0 and join x_{k+1} to all vertices of the graph D_{k+2}. Then the resulting graph G is a $K_{1,4}-$free graph. On the other hand, we have $|G| = n = k+1+(k+3)m$ and $\sigma_{k+3}(G) = n-k-1$, but G has no spanning tree with at most k branch vertices.

2. Proof of Theorem 9

Suppose that G has no spanning tree with at most k branch vertices. Choose some spanning T of G such that the following conditions are satisfied.

(T1) $|B(T)|$ is as small as possible.
(T2) $|L(T)|$ is as small as possible, subject to (T1).
(T3) $|S_T|$ is as small as possible, subject to (T1), (T2).

Note that T must have at least $k+1$ branch vertices.

We have the following claims.

Claim 10. *$L(T)$ is independent.*

Proof. Suppose two leaves s and t are adjacent in G. Then s has some nearest branch vertex b. Let $T' = T - \{bb_s\} + \{st\}$. Then T' is a spanning in G. If $\deg_T(b) = 3$, then $|B(T')| < |B(T)|$ (by vertex b), which contradicts the condition (T1). If $\deg_T(b) \geq 4$, then $|B(T')| = |B(T)|$ and $|L(T')| < |L(T)|$ (since two leaves s and t are lost while b_s is gained), which contradicts the condition (T2). So the claim holds. □

Claim 11. *$L(T)$ is pseudoindependent with respect to T.*

Proof. Suppose two leaves s and t are pseudoadjacent with respect to T. Then there is some edge $e \in E(T)$ such that $sg(e,s), tg(e,t) \in E(G)$. Let b and u be the nearest branch vertices of s and t, respectively. Consider two cases.

Case 1. Suppose $g(e,s) \neq g(e,t)$. Then $e_s = g(e,t)$ and $e_t = g(e,s)$, so $se_t, te_s \in E(G)$. If $e \in E(P_T[u,t])$, we consider the tree

$$T' = T + \{te_s, se_t\} - \{e, bb_s\}.$$

If $\deg_T(b) = 3$, then $B(T') = B(T) \setminus \{b\}$, so $|B(T')| < |B(T)|$. Thus T' violates (T1). If $\deg_T(b) \geq 4$, then $B(T') = B(T), L(T') = (L(T) \cup \{b_s\}) \setminus \{s,t\}$, so $|B(T')| = |B(T)|$ and $|L(T')| < |L(T)|$. Thus T' violates (T2). So $e \notin E(P_T[u,t])$. Now, we consider the tree

$$T' = T - \{e, uu_t\} + \{se_t, te_s\}.$$

If $\deg_T(u) = 3$, then $B(T') = B(T) \setminus \{u\}$, so $|B(T')| < |B(T)|$. Thus T' violates (T1). If $\deg_T(u) \geq 4$, then $B(T') = B(T), L(T') = (L(T) \cup \{u_t\}) \setminus \{s,t\}$, so $|B(T')| = |B(T)|$ and $|L(T')| < |L(T)|$. Thus T' violates (T2). So Case 1 does not happen.

Case 2. Suppose $g(e,s) = g(e,t)$. Define $a := g(e,s) = g(e,t)$. Then $e_s = e_t$ and denoted by vertex z. We have $as, at \in E(G)$. By $s,t \in L(T)$ and $L(T)$ is independent, so we have $a \notin L(T)$.

If $sz \in E(T)$, then $T' = T - \{bb_s, e\} + \{sz, ta\}$ violates (T1) if $\deg_T(b) = 3$, and violates (T2) if $\deg_T(b) \geq 4$ (since two leaves s and t are lost while b_s is gained). So $sz \notin E(G)$. The same argument gives $tz \notin E(G)$.

If $\deg_T(a) = 2$, then we call $c = N_T(a) \setminus \{z\}$. Since $G[a, z, c, s, t]$ is not $K_{1,4}$-free and $st, zs, zt \notin E(G)$, we have $zc \in E(G)$ or $sc \in E(G)$ or $tc \in E(G)$. If $sc \in E(G)$, then the tree $T' = T - \{ac, uu_t\} + \{sc, ta\}$ violates either (T1) or (T2) depending on $\deg_T(u) = 3$ or $\deg_T(u) \geq 4$. So $sc \notin E(G)$. By the same argument, $tc \notin E(G)$. So $zc \in E(G)$. Then the tree $T' = T - \{e, ac\} + \{sa, zc\}$ violates (T3) (due to a). So we must have $\deg_T(a) \geq 3$.

Let c be any vertex in $N_T(a) \setminus \{z\}$. If $sc \in E(G)$, then $T' = T - \{ac\} + \{sc\}$ violates either (T1) or (T2) depending on $\deg_T(a) = 3$ or $\deg_T(a) \geq 4$. So $sc \notin E(G)$. The same argument yields $tc \notin E(G)$. Since $G[a, z, c, s, t]$ is not $K_{1,4}$-free, we have $zc \in E(G)$ for all $c \in N_T(a) \setminus \{z\}$. Then the tree

$$T' = T - \{e\} - \{ac \mid c \in N_T(a) \setminus \{z\}\} + \{sa\} + \{zc \mid c \in N_T(a) \setminus \{z\}\},$$

violates (T3) (due to a). So Case 2 does not happen. The Claim 11 has been proven. $\square$

A leaf $x \in L(T)$ is called *associated with branch vertex* b if b is the nearest branch vertex of x in T.

Claim 12. *For each branch vertex $b \in B_{\geq 4}(T)$, there are at most $\deg_T(b) - 3$ leaves associated with vertex b such that they are adjacent to some vertex of $B_3(T)$.*

Proof. Put $q = \deg_T(b) - 2$. Suppose that $s_1, s_2, \ldots, s_j$, where $j \geq q$, are j leaves associate with b such that they are adjacent to some vertex in $B_3(T)$. Then there exists $w_i \in B_3(T)$ such that $s_i w_i \in E(G)$ with $i = 1, 2, \ldots, q$ (w_i may overlap). Therefore the tree

$$T' = T - \{bb_{s_i}\}_{i=1,\ldots,q} + \{s_i w_i\}_{i=1,\ldots,q}$$

violates (T1) due to b. The Claim 12 has been proven. $\square$

Let $e \in E(T)$ and $X \subseteq V(G)$. The edge e has an *oblique neighbor in the set* X if there exists a vertex of X which is an oblique neighbor of e with respect to T.

Claim 13. *In the graph G there exists an independent set X with $k+3$ elements and in the set $E(T)$ there exist at least k edges such that each of which has no oblique neighbor in the set X.*

Proof. Consider the case $B_3(T) = \emptyset$. Then we have $|B(T)| = |B_{\geq 4}(T)| \geq k+1$. So

$$|L(T)| = 2 + \sum_{v \in B(T)} (\deg_T(v) - 2) \geq 2 + 2(k+1) = 2k + 4.$$

Let X be a subset of $L(T)$ including $k+3$ elements. Set $Y = L(T) \setminus X$. We have

$$|Y| = |L(T)| - |X| \geq 2k + 4 - (k+3) = k+1.$$

Because $L(T)$ is an independent set in G, every edge of T which is adjacent to a vertex in the set Y has no oblique neighbor in the set X. Therefore, the number of edges of T without oblique neighbor in the set X is greater than or equal to $|Y| \geq k+1$.

Consider the case $|B_3(T)| = m \geq 1$. Let Z be the set of leaves associated with a branched vertex of $B_{\geq 4}(T)$ with neighbors in $B_3(T)$. According to Claim 12 we have

$$|Z| \leq \sum_{b \in B_{\geq 4}(T)} (\deg_T(b) - 3).$$

Put $X^* = L(T) \setminus Z$. We have

$$\begin{aligned} |X^*| &= |L(T)| - |Z| = 2 + m + \sum_{b \in B_{\geq 4}(T)} (\deg_T(b) - 2) - |Z| \\ &\geq 2 + m + \sum_{b \in B_{\geq 4}(T)} (\deg_T(b) - 2) - \sum_{b \in B_{\geq 4}(T)} (\deg_T(b) - 3) \\ &= 2 + m + |B_{\geq 4}(T)| = 2 + |B(T)| \geq k+3. \end{aligned}$$

Next take $e \in E(S_T)$ as an adjacent edge with a vertex of $B_3(T)$, we will show that e without oblique neighbor in X^*.

Indeed, suppose there exists $s \in X^*$ and s is a oblique neighbor of e with respect to T. Then $sg(e,s) \in E(G)$. Let b be the nearest branch vertex of s. Consider the case $g(e,s) \in B_3(T)$. According to the definition of the set X^*, we have $b \in B_3(T)$. Then $T' = T + \{sg(e,s)\} - \{bb_s\}$ violates (T1) due to the vertex b. So $g(e,s) \notin B_3(T)$. By the definition of the edge e, we infer $e_s \in B_3(T)$. Then the tree $T' = T - \{e\} + \{sg(e,s)\}$ violates (T1) due to the vertex e_s. So e has no oblique neighbor in the set X^*.

Let X be a subset of X^* with $k+3$ elements. Because $|B_3(T)| = m$, there must exist at least $m-1$ edges of S_T attached to vertices in $B_3(T)$ without oblique neighbor in X.

Put $H = L(T) \setminus X$. We have

$$\begin{aligned} |H| &= |L(T)| - |X| = 2 + m + \sum_{b \in B_{\geq 4}(T)} (\deg_T(b) - 2) - |X| \\ &= 2 + m + \sum_{b \in B_{\geq 4}(T)} (\deg_T(b) - 2) - (k+3) \\ &\geq 2 + m + 2(k+1-m) - k - 3 = k - m + 1. \end{aligned}$$

Since the set $L(T)$ is independent in G, every adjacent edge with a vertex of H has no oblique neighbor in X. So there are at least $k-m+1$ edges of T adjacent to the set H without oblique neighbor in X. Note that the edges adjacent to the set H do not belong to $E(S_T)$. Hence, there are at least $(k-m+1)+(m-1)=k$ edges of T which are not oblique neighbor in X.

So in both cases $B_3(T)=\emptyset$ and $B_3(T)\neq\emptyset$, we always find an independent set X with $k+3$ elements and in the set $E(T)$, there are at least k edges without oblique neighbor in X. Claim 13 is proved. □

For any $v, x \in E(G)$, we have $vx \in E(G)$ if and only if v is an oblique neighbor of xx_v. Therefore, the number of edges of T with v as an oblique neighbor equals the degree of v in G. Combining with Claims 11 and 13, we obtain that

$$\sigma_{k+3}(G) \leq |E(T)| - k = |V(T)| - 1 - k = n - 1 - k,$$

which contradicts the assumption of Theorem 9. The proof of Theorem 9 is completed. ■

Acknowledgements

This research is funded by Vietnam National Foundation for Science and Technology Development (NAFOSTED) under grant number 101.04-2021.19.

References

[1] Y. Chen, P.H. Ha and D.D. Hanh, *Spanning trees with at most* 4 *leaves in* $K_{1,5}$*-free graphs*, Discrete Math. **342** (2019) 2342–2349.
https://doi.org/10.1016/j.disc.2019.05.005

[2] E. Flandrin, T. Kaiser, R. Kužel, H. Li and Z. Ryjáček, *Neighborhood unions and extremal spanning trees*, Discrete Math. **308** (2008) 2343–2350.
https://doi.org/10.1016/j.disc.2007.04.071

[3] L. Gargano, M. Hammar, P. Hell, L. Stacho and U. Vaccaro, *Spanning spiders and light-splitting switches*, Discrete Math. **285** (2004) 83–95.
https://doi.org/10.1016/j.disc.2004.04.005

[4] R. Gould and W. Shull, *On a conjecture on spanning trees with few branch vertices*, J. Combin. Math. Combin. Comput. **108** (2019) 259–283.

[5] R. Gould and W. Shull, *On spanning trees with few branch vertices*, Discrete Math. **343** (2020) 111581.
https://doi.org/10.1016/j.disc.2019.06.037

[6] Z. Hu and P. Sun, *Spanning* 5*-ended trees in* $K_{1,5}$*-free graphs*, Bull. Malays. Math. Sci. Soc. **43** (2020) 2565–2586.
https://doi.org/10.1007/s40840-019-00821-w

[7] M. Kano, A. Kyaw, H. Matsuda, K. Ozeki, A. Saito and T. Yamashita, *Spanning trees with a bounded number of leaves in a claw-free graph*, Ars Combin. **103** (2012) 137–154.

[8] A. Kyaw, *Spanning trees with at most* 3 *leaves in* $K_{1,4}$*-free graphs*, Discrete Math. **309** (2009) 6146–6148.
https://doi.org/10.1016/j.disc.2009.04.023

[9] A. Kyaw, *Spanning trees with at most* k *leaves in* $K_{1,4}$*-free graphs*, Discrete Math. **311** (2011) 2135–2142.
https://doi.org/10.1016/j.disc.2011.06.025

[10] H. Matsuda, K. Ozeki and T. Yamashita, *Spanning trees with a bounded number of branch vertices in a claw-free graph*, Graphs Combin. **30** (2014) 429–437.
https://doi.org/10.1007/s00373-012-1277-5

[11] M.M. Matthews and D.P. Sumner, *Longest paths and cycles in* $K_{1,3}$*-free graphs*, J. Graph Theory **9** (1985) 269–277.
https://doi.org/10.1002/jgt.3190090208

Received 1 February 2020
Revised 4 July 2021
Accepted 4 July 2021
Available online 13 August 2021