

ON P_5 -FREE LOCALLY SPLIT GRAPHS

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Abstract

In this paper we study a graph which contains no induced path of five vertices which is known as the P_5 -free graph. We prove that every prime P_5 -free locally split graph has either a bounded number of vertices, or is a subclass of a $(2, 1)$ split graph, or is a split graph. Then we show that the Minimum Coloring problem (MC) and the maximum independent set problem (MIS) for P_5 -free locally split graphs can be both solved in polynomial time.

Keywords: SP_5 -free graphs, modular decomposition, recognition, maximum independent set, minimum coloring.

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1. INTRODUCTION

In this study, we consider P_5 -free locally split graphs, together with two NP-hard problems, namely, the Minimum Coloring problem (MC) and the maximum independent set problem (MIS). A graph is P_5 -free if it contains no induced path of five vertices. A locally split graph is a graph in which the open neighborhood of each vertex induces a split graph. The class of P_5 -free locally split graphs extends that of $(P_5, \text{triangle})$ -free graphs. From one hand, $(P_5, \text{triangle})$ -free graphs have

bounded clique with [5], and thus (MC) and (MIS) for $(P_5, \text{triangle})$ -free graphs can be solved in polynomial time. From the other hand, P_5 -free locally split graphs have unbounded clique width, and the complexity of (MC) and (MIS) for P_5 -free locally split graphs is open. Modular decomposition is a technique that applies to graphs [8, 17, 21, 34]. The classical combinatorial optimization problems are in general NP-complete, computing the modular decomposition tree when the prime graphs appearing on the tree nodes have better properties is an important preprocessing step when solving a large number of combinatorial optimization problems [21]. For instance, several researchers have studied the maximum independent set (MIS) of a subclass of P_5 -free graph [5, 6, 19]. Later, this problem has been solved in [28]. In the same perspective, we have also chosen the P_5 -free locally split graphs, in which the (MC) and (MIS) are open. Furthermore, the split graph is generalized by many special classes [1, 2]. These include the locally split graphs that contain interesting subclasses like the triangle-free graphs and subcubic graphs for which certain combinatorial optimization problems are NP-complete.

First, we studied the structure of P_5 -free locally split graphs which we denoted by SP_5 -free graph. We do this by switching between SP_5 -free graphs and their complement \overline{S} house-free graphs. G is a \overline{S} house-free graph if G is house-free and the non neighbors of each vertex induce a split graph. The objective of using this structure is to show that the (MC) problem and the (MIS) problem are polynomials for SP_5 -free graphs. We then deduce the same results for subcubic P_5 -free graphs.

The rest of the paper is organised as follows. In Section 2, we first give some notations followed by definitions and some theorems. In Section 3, we switch between SP_5 -free graphs and their complement \overline{S} house-free graphs to give the structure of prime SP_5 -free graphs. Here we also provide interesting lemmas to support our results. In Section 4, we show that the recognition, (MIS) problem and the (MC) problems of SP_5 -free graphs are polynomial.

2. NOTATIONS AND DEFINITIONS

We consider every graph $G = (V, E)$ to be finite, undirected and simple. A *set of vertices* of the graph G is denoted by $V(G)$ and *the edge set* by $E(G)$. For a graph $G = (V, E)$, let $|V| = n \geq 3$ and $|E| = m$. Let $N_G(v) = \{u : u \in V, u \neq v, uv \in E\}$ denote the *open neighborhood* of the vertex v . For $U \subseteq V$, we denote by $G[U]$ the *subgraph* of G induced by U . Throughout the paper, subgraphs are understood to be induced subgraphs. A vertex set $U \subseteq V$ is a *clique* in G if the vertices in U are pairwise adjacent. Let $\overline{G} = (V, \overline{E})$ denote *the complement graph* of a graph G . A vertex set $U \subseteq V$ is a *stable set* in G if U is a clique in

\overline{G} . A vertex set $V' \subseteq V$ in a graph $G = (V, E)$ is a *dominating* set in G , if for all vertices $u \in V \setminus V'$, there is a vertex $v \in V'$ such that $uv \in E$. Let $C \subseteq V$, a vertex $x \in V \setminus C$ is called *k-vertex* of C if x is adjacent to exactly k vertices in C . Let C_k denote an induced cycle with k vertices and let C_k -free graph be a graph that does not have a C_k as an induced subgraph.

A $2K_2$ is a graph induced by two disjoint cliques of size 2.

A *split graph* is a graph whose vertices can be partitioned into a clique K and a stable set S .

A (k, l) *split graph* is a graph whose vertices can be partitioned into k cliques and l stable sets.

A graph G is *chordal* if it does not contains any induced cycle C_k for $k \geq 4$.

A graph G is *weakly chordal* if neither the graph nor its complement contains a chordless cycle with five or more vertices as an induced subgraph.

A *module* M in a graph $G = (V, E)$ is defined as a vertex set $M \subseteq V$ such that every vertex outside M is either adjacent to all vertices of M or to none of them. If a module M in G has at least two vertices and is not the entire vertex set of G , it is called a *homogeneous set* in G . G is *prime* if it contains no homogeneous sets. Note that a module in G is a module in \overline{G} as well, and the complement of a prime graph is also prime. A module is *maximal* if it can not be contained in any other module. It is well known that in a connected graph $G = (V, E)$ with a connected complement $\overline{G} = (V, \overline{E})$, the maximal homogeneous sets are pairwise disjoint. In this case, the graph G^* obtained from G by contracting every maximal module set to a single vertex is called *the characteristic graph* G^* of G . It is easy to see that G^* is connected and prime. Subsequently, we need the following useful lemma given in [25] to characterize the prime graphs of SP_5 -free graphs.

Lemma 1 [25]. *If a prime graph contains an induced C_4 , then it contains an induced house, A or domino (see Figure 1).*

In its complement version, lemma 2 means the following.

Lemma 2 [25]. *If a prime graph contains an induced $2K_2$, then it contains an induced P_5 , co-A or co-domino (see Figure 1).*

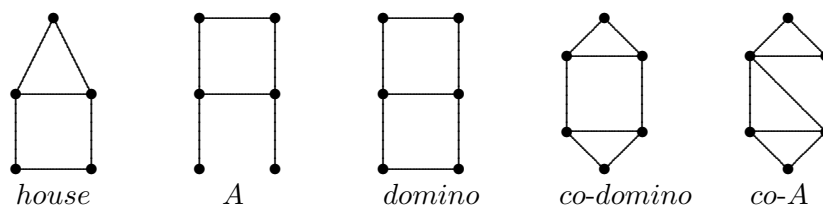


Figure 1. House, A, domino, co-domino and co-A.

G is a split graph if and only if G is $(2K_2, C_4, C_5)$ -free [16]. It is therefore simple to see that a graph G is locally split if and only if G is (J_1, J_2, J_3) -free (see Figure 2).

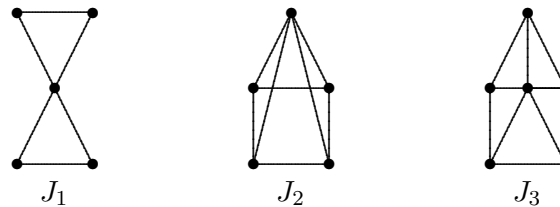


Figure 2. J_1 , J_2 and J_3 .

3. THE STRUCTURE OF SP_5 -FREE GRAPHS

In this section we first study two prime graphs, namely the prime P_5 -free graph and prime \overline{SC}_k -free graph. We then analyze the prime \overline{S} chordal graph.

3.1. Prime SP_5 -free graphs containing a $2K_2$

Our aim is to build the prime graphs of SP_5 -free graphs. We first start with the prime graphs containing a $2K_2$.

Theorem 3. *If G is a prime graph containing an induced $2K_2$, then G is SP_5 -free if and only if G is a Γ_1 graph (Figure 6) or is a chordal $(2, 1)$ split graph or is a weakly chordal $(2, 1)$ split graph.*

We construct the prime graphs G containing a $2K_2$ from a co-domino or a co-A. By Lemma 2, G contains a co-domino denoted D induced by the vertex set $\{1, 2, 3, 4, 5, 6\}$ or a co-A denoted B induced by the vertex set $\{a, b, c, d, e, f\}$ (see Figure 3).

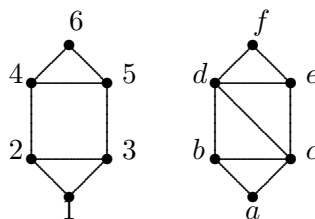


Figure 3. co-domino and co-A.

For any vertex $x \in V(G) \setminus D$, all possible adjacency cases of a vertex x in D are represented in Figure 4 with boldface edges indicating P_5 , J_1 , J_2 and J_3 . A vertex x is of type M_i if it is represented in Figure 4 in the graph M_i .

Before we give the proof of this theorem in page 14, the following 14 lemmas are presented and their proofs provided.

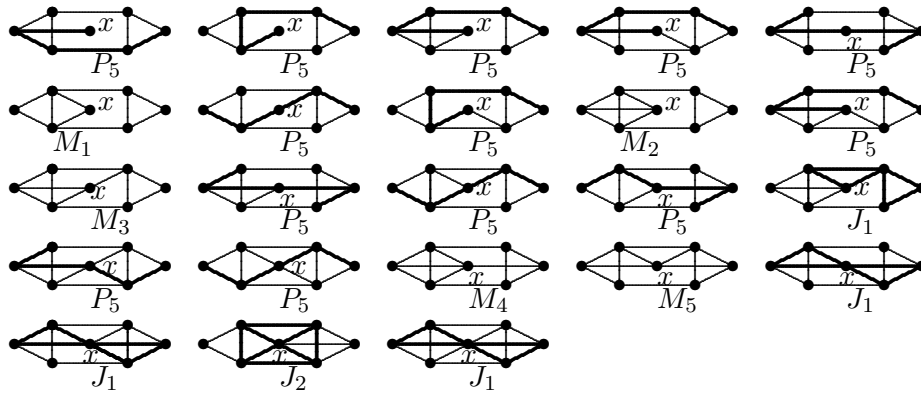


Figure 4. Adjacency of one vertex to a co-domino.

Lemma 4. Any co-domino D dominates the graph G .

Proof. Suppose the opposite (i.e., that the co-domino does not dominate the graph G) so, there exists a vertex $y \in V \setminus D$ such that for any vertex v in D ($v \in \{1, 2, \dots, 6\}$) y is at distance 2 to D . Let x be a vertex of type M_i , all possible cases are induced by the subgraph $M_i \cup \{y\}$. It is worth noting that fixing the neighbors of the vertex x does not influence the reasoning of our proof. This is because if the neighbors change, then it can be used instead symmetry. We have the following.

For the subgraph $M_1 \cup \{y\}$ (respectively, $M_2 \cup \{y\}$ and $M_3 \cup \{y\}$) the vertex set $\{y, x, 2, 4, 6\}$ induced a P_5 . This is a contradiction as G is a SP_5 -free.

For the subgraph $M_4 \cup \{y\}$ the vertex set $\{y, x, 2, 4, 5\}$ induced a P_5 . This is also a contradiction as G is a SP_5 -free.

For the subgraph $M_5 \cup \{y\}$ the vertex set $\{y, x, 1, 2, 4\}$ induced a P_5 . This is also a contradiction as G is a SP_5 -free.

All these cases are represented in Figure 5 where boldface edges indicating a P_5 . ■

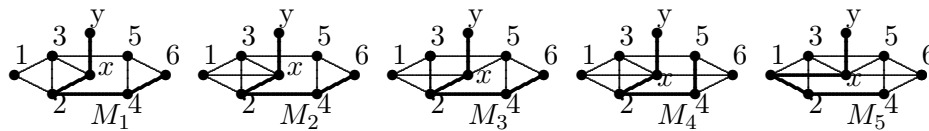


Figure 5. Co dominoes are dominating.

Lemma 5. *Let G be a prime SP_5 -free graph containing a co-domino. Then the vertex set $A_2 = N(1) \cap N(3) \cap N(4) \cap \overline{N}(5) \cap \overline{N}(6)$ is a module of size 1.*

Proof. Let G be a prime SP_5 -free graph containing a co-domino. Suppose that A_2 is not a module. So for any $x \in A_2$ such that x together with the vertex set $\{1, 3, 4, 5, 6\}$ induces a co-domino denoted by D_x in G , there exists a vertex z not belonging to $A_2 \cup \{1, 3, 4, 5, 6\}$ such that $zx \in E$ and $z2 \in \overline{E}$. By lemma 4 z is adjacent to D_x (respectively to D_2). Then z cannot be of type M_1, M_2, M_3 or M_4 otherwise z will be adjacent to 2. This is a contradiction with the initial hypothesis. If z is of type M_5 , then the vertex set $\{1, z, x, 5, 6\}$ induces a J_1 , which is also a contradiction as G is a SP_5 -free graph. ■

By symmetry, Lemma 5 holds as well for the sets: $A_3 = N(1) \cap N(2) \cap N(5) \cap \overline{N}(4) \cap \overline{N}(6)$, $A_4 = N(6) \cap N(5) \cap N(2) \cap \overline{N}(3) \cap \overline{N}(1)$, $A_5 = N(6) \cap N(4) \cap N(3) \cap \overline{N}(2) \cap \overline{N}(1)$ are modules of size 1. Which implies that the case of adjcence M_3 is impossible.

Let G be a graph represented in Figure 4 such that $V(G) = V(D) \cup L = K_1 \cup K_2 \cup L$ where $K_1 = \{1, 2, 3\}$, $K_2 = \{4, 5, 6\}$ and $L = \{v : v \in V \text{ is adjacent to } K_1 \cup K_2\}$. The following results will allow us to define the prime graphs induced by the vertex set $\{L \cup K_1 \cup K_2\}$.

- Lemma 6.**
1. *If L contains a vertex x of type M_5 , then L cannot contain vertices of type M_1, M_2 or M_4 ;*
 2. *If L contains a vertex x of type M_4 , then*
 - (a) *L cannot contain a vertex v of type M_1 with $xv \in \overline{E}$;*
 - (b) *L cannot contain a vertex v of type M_2 with $xv \in \overline{E}$ and $N(v) \subseteq N(x)$;*
 - (c) *L cannot contain a vertex v of type M_2 with $xv \in E$ and $N(v) \not\subseteq N(x)$;*
 3. *If L contains a vertex x of type M_2 , then L cannot contain a vertex v of type M_1 with $xv \in E$ and $N(v) \not\subseteq N(x)$.*

Proof. Let us consider Case 1. Suppose that L contains a vertex x of type M_5 , without loss of generality let $N(x) = \{1, 3, 5, 6\}$. We have the following.

If there exists a vertex v of type M_1 with $N(v) = \{2, 3\}$ (respectively of type M_2 with $N(v) = \{1, 2, 3\}$ or M_4 with $N(v) = \{1, 2, 3, 6\}$) in L , then we have two possibilities. Either $xv \in \overline{E}$ and then the vertex set $\{v, 3, 2, 5, x\}$ induces J_1 (respectively the vertex set $\{v, 1, x, 6, 4\}$ induces P_5 or the vertex set $\{2, v, 3, x, 5\}$ induces J_1), or $xv \in E$ and then the vertex set $\{v, x, 1, 3, 2\}$ induces J_2 (respectively the vertex set $\{v, 1, x, 5, 6\}$ induces J_1 or the vertex set $\{3, v, x, 6, 5\}$ induces J_2).

Case 2. Suppose that L contains a vertex x of type M_4 with $N(x) = \{1, 2, 3, 6\}$. Without loss of generality we have the following.

1. If there exists a vertex v of type M_1 with $xv \in \overline{E}$ in L , then we have two possibilities. Either $N(v) \subseteq N(x)$ and then the vertex set $\{v, 2, x, 6, 5\}$ induces P_5 , or $N(v) \not\subseteq N(x)$ and then the vertex set $\{v, 5, 6, x, 1\}$ induces P_5 .
2. If there exists a vertex v of type M_2 with $xv \in \overline{E}$ and $N(v) \subseteq N(x)$, then the vertex set $\{v, 2, x, 6, 5\}$ induces P_5 .
3. If there exists a vertex v of type M_2 with $xv \in E$ and $N(v) \not\subseteq N(x)$, then the vertex set $\{1, 2, x, v, 6\}$ induces J_1 .

Case 3. Suppose that L contains a vertex x of type M_2 and a vertex v of type M_1 with $xv \in E$ and $N(v) \not\subseteq N(x)$. Then the vertex set $\{1, x, v, 4, 6\}$ induces P_5 . ■

Lemma 7. L contains at most two non adjacent vertices of type M_5 .

Proof. Suppose the opposite, (i.e., that L contains at least three vertices of type M_5). Without loss of generality, let x, y and z be vertices of type M_5 with $N(x) = N(y) = \{1, 2, 4, 6\}$ and $N(z) = \{1, 3, 5, 6\}$. There exists at least a vertex v of type M_i for $i \in \{1, 2, 4\}$ such that $vx \in E$ and $vy \in \overline{E}$, otherwise $\{x, y\}$ induces a module. But by Lemma 6, we cannot have a vertex of type M_5 with a vertex of type M_i for $i \in \{1, 2, 4\}$ in L without inducing forbidden configurations, which is a contradiction with the hypothesis. On the other hand, $xz \in \overline{E}$ otherwise the vertex set $\{6, 5, 4, z, x\}$ induces J_2 . ■

Lemma 8. L contains at most two non adjacent vertices of type M_4 .

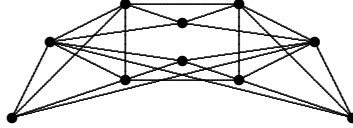
Proof. Suppose the opposite, (i.e., that L contains at least three vertices of type M_4). Without loss of generality, let x, y and z be vertices of type M_4 with $N(x) = N(y) = \{1, 2, 3, 6\}$ and $N(z) = \{1, 4, 5, 6\}$. There exists at least a vertex v of type M_i for $i \in \{1, 2\}$ such that $vx \in E$ and $vy \in \overline{E}$, otherwise $\{x, y\}$ induces a module.

If v is of type M_1 , then by Lemma 6(2a) $vx \in E$, $vy \in E$ and $vz \in E$ which is a contradiction with the hypothesis $vy \in \overline{E}$.

If v is of type M_2 , then by Lemma 6(2b) $vx \in E$, $vy \in E$ and $vz \in E$ which is a contradiction with the hypothesis $vy \in \overline{E}$. On the other hand, $xz \in \overline{E}$ otherwise the vertex set $\{z, x, 6, 4, 2\}$ induces J_2 . ■

By Lemmas 7 and 8, if L contains vertices of type M_4 and M_5 , then $|L| \leq 4$. The graph induced by the co-domino D and these vertices is called Γ_1 and is represented in Figure 6.

Lemma 9. If L contains at least two vertices of type M_2 , then L cannot contain a vertex of type M_4 .

Figure 6. Graph Γ_1 .

Proof. Let x_1, x_2 and x_3 be vertices of type M_2 . We have without loss of generality $N(x_1) = N(x_2) = \{1, 2, 3\}$ and $N(x_4) = \{4, 5, 6\}$.

There exists at least a vertex v of type M_4 such that $vx_1 \in E$ and $vx_2 \in \overline{E}$, otherwise $\{x_1, x_2\}$ is a module. But by Lemma 6(2b, 2c), either $N(v) \subseteq N(x_1)$, then $vx_1 \in E$ and $vx_2 \in E$ which is a contradiction with the hypothesis $vx_2 \in \overline{E}$, or $N(v) \subsetneq N(x_1)$ and then $vx_1 \in \overline{E}$ and $vx_2 \in \overline{E}$, which is a contradiction with the hypothesis $vx_1 \in E$. ■

Lemma 10. *If L contains at least two vertices of type M_2 , then the graph induced by $L \cup K_1 \cup K_2$ is a weakly chordal (2, 1) split graph.*

Proof. For any $\{x_1, \dots, x_k\}$ vertices of type M_2 with the same neighbors $\{1, 2, 3\}$, there are vertices $\{v_1, \dots, v_{k-1}\}$ with $N(v_i) \subseteq N(x_i)$ such that x_1 is adjacent to v_1 and not adjacent to any other vertices v_i for $i = \{2, \dots, k-1\}$, x_2 is adjacent to v_2 and not adjacent to any other vertices v_i for $i = \{1, \dots, k-1\} \setminus \{2\}$ and so on x_{k-1} is adjacent to v_{k-1} and not adjacent to any other vertices v_i for $i = \{1, \dots, k-2\}$. By Lemmas 6 and 9, each vertex v_i cannot be of type M_4 or M_5 , so v_i is of type M_1 and for any v_i and v_j we have $v_i v_j \in \overline{E}$, otherwise the vertex set $\{2, v_i, v_j, x_k, 1\}$ induces J_1 . Moreover for any x_i and x_j we have $x_i x_j \in E$, otherwise the vertex set $\{3, x_i, v_i, x_j, v_j\}$ induces J_1 . Finally, the vertex set $\{x_1, x_2, \dots, x_{k-1}, x_k, 1, 2, 3\}$ induces a clique and the vertex set $\{v_1, \dots, v_{k-1}\}$ induces a stable set.

For any $\{x'_1, \dots, x'_l\}$ vertices of type M_2 with the same neighbors $\{4, 5, 6\}$, there are vertices $\{v'_1, \dots, v'_{l-1}\}$ with $N(v'_i) \subseteq N(x'_i)$ such that x'_1 is adjacent to v'_1 and not adjacent to other vertices v'_i for $i = \{2, \dots, l-1\}$, x'_2 is adjacent to v'_2 and not adjacent to other vertices v'_i for $i = \{1, \dots, l-1\} \setminus \{2\}$ and so on x'_{l-1} is adjacent to v'_{l-1} and not adjacent to other vertices v'_i for $i = \{1, \dots, l-2\}$. By Lemma 6, $v'_i x_i \in \overline{E}$ and $v_i x'_i \in \overline{E}$.

The same reasoning can be done to find that $\{x'_1, x'_2, \dots, x'_{l-1}, x'_l, 4, 5, 6\}$ induces a clique. This is because we have $x_i x'_j \in \overline{E}$ (otherwise the vertex set $\{6, x'_j, x_i, 3, v_i\}$ induces P_5) and $v_i v'_j \in \overline{E}$ (otherwise the vertex set $\{1, x_i, v_i, v'_j, x'_j\}$ induces P_5). Then the vertex set $\{v_1, \dots, v_{k-1}, v'_1, \dots, v'_{l-1}\}$ induces a stable

set.

Now suppose that the subgraph $H = G[L \cup K_1 \cup K_2]$ is not weakly chordal (i.e., H or \overline{H} contains a chordless cycle of length ≥ 5).

- If there exists at least a cycle C of length ≥ 5 in H , then the only possible case is $C = \{x_i, x_j, x'_i, x'_j, v'_j\}$, but we have shown previously that $x_i x'_j \in \overline{E}$ which is a contradiction.

- Consider now the subgraph $\overline{H} = \overline{G[L \cup K_1 \cup K_2]} = G[K' \cup S_1 \cup S_2]$ where $K' = \overline{L} = \{v_1, \dots, v_{k-1}, v'_1, \dots, v'_{l-1}\}$ induces a clique, $S_1 = \overline{K_1} = \{x_1, x_2, \dots, x_{k-1}, x_k, 1, 2, 3\}$ and $S_2 = \overline{K_2} = \{x'_1, x'_2, \dots, x'_{l-1}, x'_l, 4, 5, 6\}$ induces two stable sets in \overline{H} . By definition of \overline{H} , the vertices $\{x_i, 1\}$ belonging to S_1 (respectively, $\{x'_i, 6\}$ belonging to S_2) are adjacent to all vertices of S_2 (respectively, are adjacent to all vertices of S_1) and the vertices $(3, 2)$ (respectively, the vertices $(4, 5)$) are adjacent respectively to $S_2 \setminus \{5\}$ and $S_2 \setminus \{4\}$ (respectively to $S_2 \setminus \{2\}$ and $S_2 \setminus \{3\}$). Let $C = \{y_1, \dots, y_p\}$ be a chordless cycle in \overline{H} , by hypothesis $p \geq 5$. We have two scenarios.

If for any $i \in \{1, \dots, p\}$, $y_i \in S_1 \cup S_2$, then the longest chain in $S_1 \cup S_2$ is of type $\{y_i, 3, 4, 2, 5\}$, but as $y \in \{x_i, x'_i, 1, 6\}$, the vertex set $\{y_i, 3, 4, 2, 5\}$ contains a chord.

If there exists at least a vertex $y_i \in K'$ ($y_i \in \{v_i, v'_i\}$), v_i (respectively, v'_i) is adjacent to all vertices of the vertex set $S_1 \cup S_2 \setminus \{x_i, 2, 3\}$ (respectively of the vertex set $S_1 \cup S_2 \setminus \{x'_i, 4, 5\}$), then without loss of generality the longest chain in \overline{H} is either of type $\{v_i, 5, 2, 4\}$ but $v_i 4 \in E$, or of type $\{v'_i, v_i, 5, 2\}$ but $v'_i 2 \in E$. Then the length of the chordless cycle in \overline{H} is at most 4, which is a contradiction with the hypothesis.

Hence, $H = G[L \cup K_1 \cup K_2]$ is weakly chordal. ■

For any vertex $x \in V(G) \setminus B$, all possible adjacency cases of the vertex x in B are represented in Figure 7 with boldface edges indicating P_5, J_1, J_2 and J_3 .

The vertex x is of type M'_i if it is represented in Figure 7 in the graph M'_i .

Lemma 11. B dominates G for M'_1, M'_2, M'_3 and M'_4 adjacency types.

Proof. Suppose the opposite (i.e., that the co-A does not dominate the graph G for M'_1, M'_2, M'_3 and M'_4 adjacency types). So, there exists a vertex $y \in V \setminus B$ such that for any vertex v in B ($v \in \{a, b, c, d, e, f\}$), y is at a distance 2 to B . Let x be a vertex of type M'_i and all possible cases are induced by the subgraph $M'_i \cup \{y\}$. We have then the following.

For the subgraph $M'_1 \cup \{y\}$ (respectively for the subgraph $M'_2 \cup \{y\}$), the vertex set $\{y, x, c, d, f\}$ induced P_5 . This is a contradiction as G is a SP_5 -free.

For the subgraph $M'_3 \cup \{y\}$, the vertex set $\{y, x, c, e, f\}$ induced a P_5 , this is also a contradiction as G is a SP_5 -free.

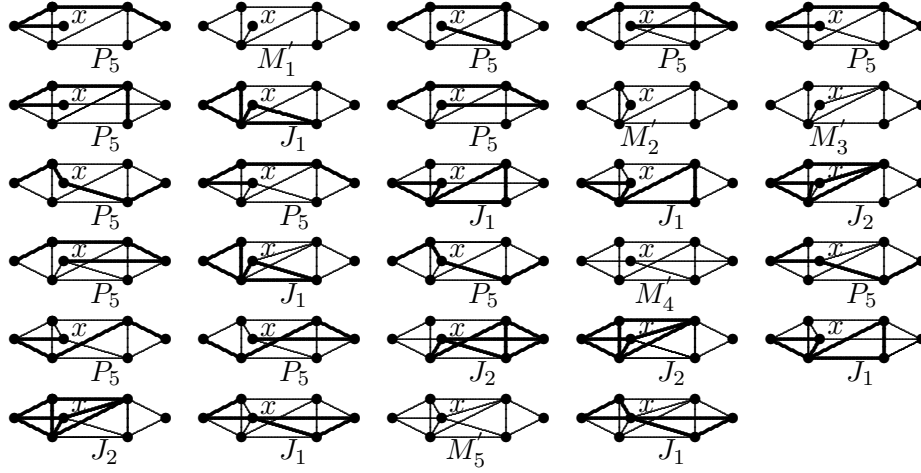
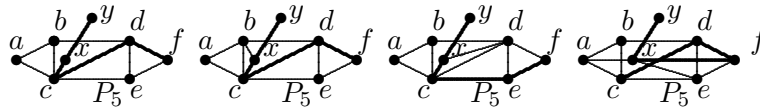


Figure 7. Adjacency of one vertex to a co-A.

For the subgraph $M'_4 \cup \{y\}$, the vertex set $\{y, x, f, d, c\}$ induced P_5 , this is also a contradiction as G is a SP_5 -free.

All these cases are represented in Figure 8 with boldface edges indicating a P_5 . ■

Figure 8. Co-A are dominating for M'_1, M'_2 and M'_3 .

Lemma 12. B does not dominate G for M'_5 adjacency type, but each vertex not adjacent to B is of distance 2 from B .

Proof. Suppose there exists a vertex $y \in V \setminus B$ such that y is of distance 3 to B and x of type M'_5 . Then the vertex set $\{y, z, x, e, f\}$ induces P_5 . ■

Lemma 13. Let G be a prime SP_5 -free graph containing a co-A. Then the vertex set $A_a = N(b) \cap N(c) \cap \overline{N}(d) \cap \overline{N}(e) \cap \overline{N}(f)$ is a module of size 1.

Proof. Let G be a prime SP_5 -free graph containing a co-A. Suppose that A_a is not a module. So for any $x \in A_a$ such that x together with the vertex set $\{b, c, d, e, f\}$ induces a co-A denoted by B_x in G , there exists a vertex z not belonging to $A_a \cup \{b, c, d, e, f\}$ such that $zx \in E$ and $za \in \overline{E}$. By Lemma 11, for M'_1, M'_2, M'_3 and M'_4 we have z adjacent to B_x (respectively to B_a). Also z cannot be of type M'_4 or M'_5 , otherwise z will be adjacent to a , which is a contradiction with the hypothesis. If z is of type M'_1 or M'_2 , then the vertex set $\{c, e, d, x, z\}$

induces J_1 . If z is of type M'_3 , the vertex set $\{c, x, z, b, d\}$ induces J_2 , then A_a is a module of size 1. ■

By symmetry, Lemma 13 holds as well for the set $A_f = N(d) \cap N(e) \cap \overline{N}(c) \cap \overline{N}(b) \cap \overline{N}(a)$ which is a module of size 1.

Then by this claim, the adjacency type M'_2 is impossible.

Let G be a graph represented in Figure 7 such that $V(G) = V(B) \cup L = K_1 \cup K_2 \cup L$ where $K_1 = \{a, b, c\}$, $K_2 = \{d, e, f\}$ and $L = \{v : v \in V \text{ is adjacent to } K_1 \cup K_2\}$. The following results will allow us to define the prime graph induced by the set $\{L \cup K_1 \cup K_2\}$.

In the following lemma, we study the types of adjacencies M'_i, M'_j with $i \neq j$ which induce forbidden configurations in prime SP_5 -free graphs starting from a co-A. We fix the neighborhood of a vertex of type M'_i without loss of generality, because if the neighborhood changes the reasoning remains the same by symmetry.

Lemma 14. 1. If L contains a vertex x of type M'_5 such that $N(x) = \{a, b, c, d, e\}$, then

- (a) L cannot contain a vertex v of type M'_5 with $xv \in \overline{E}$;
 - (b) L cannot contain a vertex v of type M'_4 with $N(x) = \{a, f, e\}$;
 - (c) L cannot contain a vertex of type M'_4 with $N(x) = \{a, b, f\}$ and $xv \in E$.
2. If L contains a vertex x of type M'_4 such as $N(x) = \{a, b, f\}$, then
- (a) L cannot contain a vertex v of type M'_3 ;
 - (b) L cannot contain a vertex v of type M'_1 with $N(v) = \{d\}$;
 - (c) L cannot contain a vertex v of type M'_1 with $N(v) = \{c\}$.
3. If L contains a vertex x of type M'_3 , then L cannot contain a vertex v of type M'_1 with $xv \in E$.

Proof. Case 1. Suppose that L contains a vertex x of type M'_5 such that $N(x) = \{a, b, c, d, e\}$. Then we have three scenarios.

- (a) If there exists a vertex v of type M'_5 in L , then either $N(v) = \{b, c, d, e, f\}$ and then $xv \in E$, otherwise the vertex set $\{b, x, e, v, d\}$ induces J_2 , or $N(v) = \{a, b, c, d, e\}$, then $xv \in E$, otherwise the vertex set $\{a, x, e, v, c\}$ induces J_2 .
- (b) If there exists a vertex v of type M'_4 in L with $N(v) = \{a, f, e\}$, then either $xv \in \overline{E}$ so the vertex set $\{x, c, e, f, v\}$ induces J_1 , or $xv \in E$ so the vertex set $\{a, b, d, e, v, x\}$ induces J_3 .
- (c) If there exists a vertex v of type M'_4 in L with $N(v) = \{a, b, f\}$, then $xv \in \overline{E}$, otherwise the vertex set $\{a, v, x, d, e\}$ induces J_1 .

Case 2. Suppose that L contains a vertex x of type M'_4 such that $N(x) = \{a, b, f\}$. Then we have also three scenarios.

(a) If there exists a vertex v of type M'_3 in L with $N(v) = \{c, d\}$, then either $xv \in \overline{E}$ so the vertex set $\{a, x, f, d, v\}$ induces P_5 , or $xv \in E$ so the vertex set $\{a, x, v, d, e\}$ induces P_5 .

(b) If there exists a vertex v of type M'_1 in L with $N(v) = \{d\}$, then either $xv \in \overline{E}$ so the vertex set $\{v, d, f, x, a\}$ induces P_5 , or $xv \in E$ so the vertex set $\{v, x, a, c, e\}$ induces P_5 ;

(c) If there exists a vertex v of type M'_1 in L with $N(v) = \{c\}$, then either $xv \in \overline{E}$ so the vertex set $\{x, f, d, c, v\}$ induces a P_5 , or $xv \in E$ so the vertex set $\{v, x, b, d, e\}$ induces P_5 .

Case 3. Suppose that L contains a vertex x of type M'_3 with $N(x) = \{c, d\}$. Then if there exists a vertex v of type M'_1 in L , we have $xv \in \overline{E}$, otherwise the vertex set $\{v, x, d, b, a\}$ induces P_5 . ■

Lemma 15. *If L contains at least two vertices of type M'_5 , then the graph induced by $L \cup K_1 \cup K_2$ is a chordal $(2, 1)$ split graph.*

Proof. For any vertices $\{x_1, \dots, x_k\}$ of type M'_5 with the same neighbors $\{a, b, c, d, e\}$, there are vertices $\{v_1, \dots, v_{k-1}\}$ such that x_1 is adjacent to v_1 and not adjacent to other vertices v_i for $i = \{2, \dots, k-1\}$, x_2 is adjacent to v_2 and not adjacent to other vertices v_i for $i = \{1, \dots, k-1\} \setminus \{2\}$ and so on x_{k-1} is adjacent to v_{k-1} and not adjacent to other vertices v_i for $i = \{1, \dots, k-2\}$. By Lemma 14, we have $x_i x_j \in E$ and each vertex v_i are of type M_4 , M_3 or M_1 . We therefore cannot have two types of M_i for $i \in \{1, 3, 4\}$ adjacencies at the same time leading to the following cases.

1. Suppose that v_i is of type M_4 . By Lemma 14(2b, 2c), $N(v_i) = \{a, b, f\}$ with $x_i v_i \in \overline{E}$ for $i = \{2, \dots, k\}$. But by hypothesis x_1 is adjacent to v_1 which is a contradiction, therefore v_i cannot be of type M_4 .

2. Suppose that v_i is of type M_3 , for any v_i and v_j we have $N(v_i) = N(v_j) = \{c, d\}$ and $v_i v_j \in \overline{E}$, otherwise the vertex set $\{v_i, v_j, d, e, f\}$ induces J_1 . Then $\{x_1, x_2, \dots, x_{k-1}, a, b, c\}$ induces a clique and the vertex set $\{v_1, \dots, v_{k-1}, x_k\}$ induces a stable set.

3. Suppose that v_i for $i = \{2, \dots, k-1\}$ is of type M_1 . Then $v_i v_j \in \overline{E}$ for $i \neq j$, otherwise either $N(v_i) = \{c\}$ and $N(v_j) = \{d\}$ with $v_i v_j \in E$ so the vertex set $\{v_i, v_j, d, b, a\}$ induces P_5 , or without loss of generality $N(v_i) = N(v_j) = \{c\}$ so the vertex set $\{c, v_i, v_j, a, b\}$ induces J_1 . Finally $\{x_1, x_2, \dots, x_{k-1}, x_k, a, b, c\}$ induces a clique and the vertex set $\{v_1, \dots, v_{k-1}\}$ induces a stable set.

For any vertices $\{x'_1, \dots, x'_l\}$ of type M'_5 with the same neighbors $\{b, c, d, e, f\}$, there are vertices $\{v'_1, \dots, v'_{l-1}\}$ such that x'_1 is adjacent to v'_1 and not adjacent to any other vertices v'_i for $i = \{2, \dots, l-1\}$, x'_2 is adjacent to v'_2 and not adjacent to any other vertices v'_i for $i = \{1, \dots, l-1\} \setminus \{2\}$ and so on x'_{l-1} is adjacent to

v'_{l-1} and not adjacent to any other vertices v'_i for $i = \{1, \dots, l-2\}$. The same reasoning can be done to find that $\{x'_1, x'_2, \dots, x'_{l-1}, x_l, d, e, f\}$ induces a clique, as we have shown previously $v_i v'_j \in \overline{E}$, then the vertex set $\{v_1, \dots, v_{k-1}, v'_1, \dots, v'_{l-1}\}$ induces a stable set.

On the other hand by Lemma 12, B does not dominate G for M'_5 . So, for any $\{y_1, \dots, y_k, y'_1, \dots, y'_l\}$ of distance 2 to B such that $x_i y_i \in E$ and $x'_i y'_i \in E$, either $x_i y'_i \in E$ and then $y_i y'_i \in \overline{E}$, otherwise the vertex set $\{x_i, y_i, y'_i, a, b\}$ induces J_1 or $x_i y_i \in \overline{E}$ and then $y_i y'_i \in \overline{E}$, otherwise the vertex set $\{y'_i, y_i, x_i, c, v'_i\}$ induces P_5 . In summary a $(2, 1)$ split graph induces $K_1 = \{x_1, x_2, \dots, x_{k-1}, x_k, a, b, c\}$, $K_2 = \{x'_1, x'_2, \dots, x'_{l-1}, x'_l, d, e, f\}$ and a stable set $\{v_1, \dots, v_{k-1}, v'_1, \dots, v'_{l-1}, y_1, \dots, y_k, y'_1, \dots, y'_l\}$.

Now suppose that $L \cup K_1 \cup K_2$ is not chordal. So there is at least a cycle of length at least 4, then $L \cup K_1 \cup K_2$ contains a C_4 or a C_5 . If $L \cup K_1 \cup K_2$ contains a C_4 , then $C_4 = \{x_i, x_j, x'_i, x'_j\}$, $C_4 = \{x_i, x_j, x'_i, v'_i\}$ or $C_5 = \{x_i, x_j, x'_i, x'_j, v'_j\}$, but previously we have shown that $x_i x'_j \in E$ for any $(i, j) \in \{1, \dots, k\}^2$ which is a contradiction with the hypothesis. ■

Lemma 16. L contains at most two vertices of type M'_4 .

Proof. Suppose the opposite (i.e., that L contains at least three vertices of type M'_4). Without loss of generality, let x, y and z be vertices of type M'_4 with $N(x) = N(y) = \{a, b, f\}$ and $N(z) = \{a, e, f\}$. There exists at least a vertex v of type M_i such that $vx \in E$ and $vy \in \overline{E}$, otherwise $\{x, y\}$ induces a module, by Lemma 14 the vertex v is of type M'_5 . Without loss of generality let $N(v) = \{a, b, c, d, e\}$ (otherwise reason by symmetry) by Lemma 14 as $N(x) = N(y) = \{a, b, f\}$, then $xv \in \overline{E}$ and $yv \in \overline{E}$ which is a contradiction with the hypothesis.

In summary, $L \cup K_1 \cup K_2$ has a bounded number of vertices. ■

Lemma 17. If L contains at least two vertices of type M'_3 , then the graph induced by $L \cup K_1 \cup K_2$ is a chordal $(2, 1)$ split graph.

Proof. Suppose that L contains two vertices x, y of type M'_3 with $N(x) = N(y) = \{c, d\}$, There exists at least a vertex v of type M_i such that $vx \in E$ and $vy \in \overline{E}$, otherwise $\{x, y\}$ induces a module. By Lemma 14, $i \in \{1, 5\}$, and we have two cases.

If v is of type M_1 , then by Lemma 14(3), $xv \in \overline{E}$ and $yv \in \overline{E}$, which is a contradiction with the hypothesis.

If v is of type M_5 , then by Lemma 14, $L \cup K_1 \cup K_2$ is a chordal $(2, 1)$ split graph. ■

Proof of Theorem 3. Let G be a prime graph containing an induced $2K_2$.

⇐ It is clear that if G is a Γ_1 graph represented in Figure 6, then G is a prime SP_5 -free graph containing a $2K_2$. Otherwise G is chordal $(2, 1)$ split graph

(respectively weakly chordal $(2, 1)$ split graph) of which any vertex x in G , as constructed in Lemma 15 (respectively, by Lemma 10), is split. Moreover G is chordal (respectively weakly chordal), then G is P_5 -free.

\Rightarrow Let G be a prime SP_5 -free graph containing an induced $2K_2$. We build the prime graphs of G from a co-domino or a co-A. By Lemmas 4 to 11, we find a Γ_1 graph or chordal $(2, 1)$ split graph or weakly chordal $(2, 1)$ split graph if we start from a co-domino. By Lemmas 12 to 17, we find a chordal $(2, 1)$ split graph if we start from a co-A. ■

In what follows, we consider the prime SP_5 -free graph without a $2K_2$ as an induced subgraph which is an $S2K_2$ -free graph. We then study the complement \overline{SC}_4 -free graph, which is a C_4 -free graph where the non neighbors of each vertex induce a split graph.

Note that if a graph is the \overline{S} , then it is $(\overline{J}_1, \overline{J}_2, \overline{J}_3)$ -free (see Figure 10). Thus \overline{S} graph is C_k -free for $k \geq 8$.

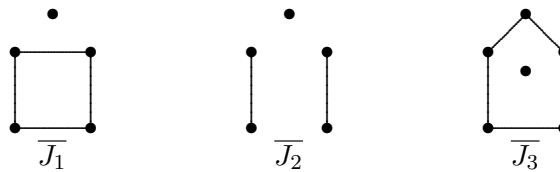


Figure 9. A \overline{J}_1 , \overline{J}_2 and \overline{J}_3 .

3.2. Prime \overline{SC}_4 -free graph containing a C_7 , a C_6 or a C_5

Let a cycle C_p be induced by the vertex set $\{1, 2, \dots, p\}$ for $p \in \{5, 6, 7\}$ and edges $\{i, i+1\}$ for $i \in \{1, \dots, p-1\}$, and let V_k be a vertex set of k -vertices of C_p . We will say that a vertex x has consecutive neighbors if $N(x) = \{i, i+1, i+2, \dots\}$ for $i \in \{1, \dots, p-1\}$. Note that G is \overline{S} , then it is \overline{J}_2 -free, so C_p dominates G .

Theorem 18. *Let G be a prime \overline{SC}_4 -free graph containing C_7 , C_6 or C_5 . Then G has a bounded number of vertices or is $[(2, 2)$ split, (C_4, C_5, C_7) -free] graph, or is $[(2, 1)$ split, (C_4, C_6, C_7, P_6) -free] graph.*

To prove this result we need to prove the following lemmas.

Lemma 19. *Let G be a prime \overline{SC}_4 -free graph containing C_7 . Then G contains at most 18 vertices.*

Proof. Let G be a prime \overline{SC}_4 -free graph containing a C_7 . The straightforward case analysis shows that there are only three admissible cases for a vertex being adjacent to C_7 : k -vertices with consecutive neighbors for $k \in \{4, 5, 7\}$. Otherwise for $k \in \{1, 2, 3, 6\}$, let $u \in V_1$ with $N(u) = \{2\}$ (respectively, $u \in V_2$ with $N(u) =$

$\{2, 3\}$ or $u \in V_3$ with $N(u) = \{1, 2, 3\}$ or $u \in V_6$ with $N(u) = \{1, 2, 3, 4, 5, 6\}$. Then the vertex set $\{u, 1, 7, 3, 4\}$ (respectively, the vertex set $\{u, 1, 7, 4, 5\}$ or the vertex set $\{u, 2, 4, 6, 7\}$ or the vertex set $\{u, 1, 6, 7\}$) induces $\overline{J_2}$ (respectively, induces $\overline{J_2}$ or C_4). So V_k is empty for $k \in \{1, 2, 3, 6\}$. We start by calculating the number of vertices having different neighbors in C_7 in each vertex set V_k , then we are interested in the vertices having the same neighbors in C_7 which risk inducing module.

1. Let us calculate $|V_k|$ for $k \in \{4, 5, 7\}$ for vertices having different neighbors in C_7 .

- Let us show that $|V_7| \leq 1$. Suppose the opposite (i.e., $|V_7| \geq 2$). Let $(u, v) \in V_7$ such that $N(u) = N(v) = \{1, 2, 3, 4, 5, 6, 7\}$ with $uv \in E$, otherwise $\{v, 3, u, 7\}$ induces C_4 . But $\{u, v\}$ induce a module which is a contradiction with the hypothesis.

- Let us show that $|V_5| \leq 3$. Suppose the opposite (i.e., $|V_5| \geq 4$). For any $(u, v) \in V_5$, (u, v) have three or four neighbors in common in C_7 . We have different cases.

If (u, v) have three consecutive neighbors in common (i.e., without loss of generality let $N(u) = \{7, 1, 2, 3, 4\}$ and $N(v) = \{2, 3, 4, 5, 6\}$), then the vertex set $\{7, 6, v, u\}$ induces C_4 for $uv \in E$ and the vertex set $\{v, 2, u, 4\}$ induces C_4 for $uv \in \overline{E}$, then this case is impossible.

If (u, v) have three non consecutive neighbors in common (i.e., without loss of generality let $N(u) = \{1, 2, 3, 4, 5\}$ and $N(v) = \{1, 3, 4, 5, 6\}$), then $uv \in E$, otherwise the vertex set $\{1, u, 4, v\}$ induces C_4 .

If (u, v) have four consecutive neighbors in common (i.e., without loss of generality let $N(u) = \{1, 2, 3, 4, 5\}$ and $N(v) = \{7, 1, 2, 3, 4\}$), then $uv \in E$, otherwise the vertex set $\{u, 1, v, 3\}$ induces C_4 .

Let the vertices $(u, v, w) \in V_5$. The straightforward cases shows that either (u, v) have four consecutive neighbors in common in C_7 and then (u, w) and (v, w) have three non consecutive neighbors in common, or there exists a vertex $y \in V_5$ such that y has three consecutive neighbors in common with u, v or w . By what precedes this is impossible so, $|V_5| \leq 3$.

- Let show that $|V_4| \leq 7$. Suppose the opposite (i.e., $|V_4| \geq 8$). For any $(u, v) \in V_4$, (u, v) have one, two or three neighbors in common in C_7 . If $|V_4| \geq 8$, then two vertices (u, v) in V_4 have the same neighbors in C_7 , which is a contradiction with the hypothesis.

2. Let $(u, v) \in V_k$ for $k \in \{4, 5, 7\}$ with $N(u) = N(v)$. There exists a vertex $z \in V_j$, for $j \neq k$ and $j \in \{4, 5, 7\}$ such that $uz \in E$ and $vz \in \overline{E}$, otherwise $\{u, v\}$ induces a module. We have different scenarios.

- Let $(u, v) \in V_7$ with $N(u) = N(v) = \{1, 2, 3, 4, 5, 6, 7\}$ and $uv \in E$, otherwise $\{u, 1, v, 3\}$ induces C_4 . Let $z \in V_k$ for $k \in \{4, 5\}$, without loss of generality

let $N(z) = \{1, 2, 3, 4\}$ or $N(z) = \{1, 2, 3, 4, 5\}$. Then the vertex set $\{1, z, 3, v\}$ induces C_4 , which is a contradiction as G is \overline{SC}_4 -free graph.

- Let $(u, v) \in V_5$ with $N(u) = N(v) = \{1, 2, 3, 4, 5\}$ and $uv \in E$, otherwise $\{u, 1, v, 3\}$ induces C_4 . Let $z \in V_k$ for $k \in \{4\}$, without loss of generality let $N(z) = \{1, 2, 3, 4\}$. Then the vertex set $\{1, z, 3, v\}$ induces C_4 , which is a contradiction as G is \overline{SC}_4 -free graph.

- Let $(u, v) \in V_4$ with $N(u) = N(v) = \{1, 2, 3, 4\}$. It is clear from the above that if there exists a vertex $z \in V_7$ (respectively, $z \in V_5$) such that $vz \in \overline{E}$, then we induce C_4 .

So, the vertex set adjacent to the cycle C_7 is of order at most 11 vertices. ■

Lemma 20. *Let G be a prime $\overline{S}(C_4, C_7)$ -free graph containing C_6 . Then*

1. *either $|V_6| = \emptyset$ (respectively, $|V_2| = \emptyset$) and then G contains at most 21 vertices (respectively, 16 vertices);*
2. *or G is $[(2, 2)$ split, (C_4, C_5, C_7) -free] graph.*

Proof. The straightforward case analysis shows that there are only four admissible cases for a vertex being adjacent to C_6 : k -vertices with consecutive neighbors for $k \in \{2, 3, 4, 6\}$. Otherwise for $k \in \{1, 5\}$, let $u \in V_1$ ($u \in V_5$) with $N(u) = \{2\}$ (respectively, without loss of generality with $N(u) = \{1, 2, 3, 4, 5\}$), the vertex set $\{u, 1, 6, 3, 4\}$ induces \overline{J}_2 (respectively the vertex set $\{u, 1, 5, 6\}$ induces a C_4). So V_k is empty for $k \in \{1, 5\}$. We start by calculating the number of vertices having different neighbors in C_6 in each vertex set V_k , then we are interested in the vertices having the same neighbors in C_6 which risk inducing module.

1. Let us calculate the size $|V_k|$ of V_k whose vertices have different neighbors in C_6 for $k \in \{2, 3, 4, 6\}$.

- Let us show that $|V_6| \leq 1$. Suppose the opposite (i.e., $|V_6| \geq 2$). Let $(u, v) \in V_6$ such that $N(u) = N(v) = \{1, 2, 3, 4, 5, 6\}$ with $uv \in E$, otherwise $\{u, 1, 3, v\}$ induces a C_4 . But $\{u, v\}$ induce a module which is a contradiction with the hypothesis.

- Let us build the set V_4 to show that $|V_4| \leq 3$. First, for any $u \in V_4$ the neighbors of u are consecutive, otherwise let $N(u) = \{2, 3, 4, 6\}$ (respectively, $N(u) = \{1, 3, 4, 5\}$). Then the vertex set $\{1, 2, u, 6\}$ (respectively, $\{3, 4, 5, u\}$) induces C_4 .

Let $(u, v) \in V_4$ with $N(u) = \{1, 2, 3, 4\}$. Then we have two possibilities. If (u, v) have two consecutive (respectively non consecutive) neighbors in common in C_6 , without loss of generality, let $N(v) = \{3, 4, 5, 6\}$ (respectively, $N(v) = \{4, 5, 6, 1\}$), then $uv \in \overline{E}$ (respectively, $uv \in E$) otherwise the vertex set $\{u, v, 1, 6\}$ (respectively, $\{1, u, 4, v\}$) induces C_4 . On the other hand, if (u, v) have three consecutive neighbors in common in C_6 (otherwise without loss of generality, let $N(v) = \{1, 3, 4, 5\}$, then the vertex set $\{1, v, 5, 6\}$ induces a C_4), without

loss of generality, let $N(v) = \{2, 3, 4, 5\}$, then $uv \in E$, otherwise the vertex set $\{u, 4, v, 2\}$ induces a C_4 .

We have 6 vertices in V_4 adjacent to C_6 with three neighbor in common (the maximum possible so as not to induce a module). But from the above, any pair of vertices (v_i, v_j) , $1 \leq (i, j) \leq 6$, $v_i v_j \in E$ for $j = i + 1$ or $j = i + 3$ and $v_i v_j \in \overline{E}$ for $j = i + 2$. Then the vertex set $\{v_i, v_{i+1}, v_{i+2}, v_{i+3}\}$ induces a C_4 . So, we have at most three vertices in V_4 .

- Let us build the set V_3 to show that $|V_3| \leq 6$. First, for any $u \in V_3$ the neighbors of u are consecutive or without loss of generality $N(u) = \{1, 2, 5\}$, otherwise let $N(u) = \{1, 2, 4\}$. Then the vertex set $\{u, 2, 3, 4\}$ induce C_4 .

Let $(u, v) \in V_3$, (u, v) have no neighbors in common in C_6 . Then $uv \in \overline{E}$, otherwise let $N(u) = \{1, 2, 3\}$ and $N(v) = \{4, 5, 6\}$, and then the vertex set $\{u, 3, v, 4\}$ induces C_4 . We have 6 vertices in V_3 adjacent to C_6 with two neighbor in common (the maximum possible so as not to induce a module), then we have at most six vertices in V_3 .

- Let us build the set V_2 to show that $|V_2| \leq 6$. First, for any $u \in V_2$ the neighbors of u are consecutive, otherwise without loss of generality either $N(u) = \{1, 3\}$, and then the vertex set $\{1, u, 3, 2\}$ induces C_4 or $N(u) = \{1, 4\}$, and then the vertex set $\{2, 3, 5, 6, u\}$ induces \overline{J}_1 , or $N(u) = \{2, 5\}$. Then the vertex set $\{1, 6, 4, 3, u\}$ induces \overline{J}_1 .

Let $(u, v) \in V_2$, (u, v) have one neighbor (respectively no neighbor) in common in C_6 . Then $uv \in E$. Otherwise, without loss of generality, let $N(u) = \{1, 2\}$ and $N(v) = \{2, 3\}$ (respectively, $N(v) = \{3, 4\}$). Then the vertex set $\{u, 1, v, 3, 5\}$ induces \overline{J}_2 . We have 6 vertices in V_2 adjacent to C_6 with one neighbor in common (the maximum possible so as not to induce a module), then we have at most six vertices in V_2 .

2. Let $(u, v) \in V_k$ for $k \in \{2, 3, 4, 6\}$ with $N(u) = N(v)$. There exists a vertex $z \in V_j$, for $j \neq k$ and $j \in \{2, 3, 4, 6\}$ such that $uz \in E$ and $vz \in \overline{E}$, otherwise $\{u, v\}$ induces a module. We have different scenarios.

- Let $(u, v) \in V_6$ with $N(u) = N(v) = \{1, 2, 3, 4, 5, 6\}$ and $uv \in E$. Let $z \in V_k$ for $k \in \{2, 3, 4\}$. If $k = 4$ (respectively, $k = 3$), without loss of generality let $N(z) = \{1, 2, 3, 4\}$ (respectively, $N(z) = \{1, 2, 3\}$), then the vertex set $\{1, z, 3, v\}$ induces C_4 . If $k = 2$, let $(x_1, \dots, x_k) \in V_6$ with the same neighbors $\{1, 2, 3, 4, 5, 6\}$ and let $\{v_1, \dots, v_{k-1}\}$ such that x_1 is adjacent to v_1 and not adjacent to any other vertices v_i for $i = \{2, \dots, k-1\}$, x_2 is adjacent to v_2 and not adjacent to any other vertices v_i for $i = \{1, \dots, k-1\} \setminus \{2\}$ and so on x_{k-1} is adjacent to v_{k-1} and not adjacent to any other vertices v_i for $i = \{1, \dots, k-2\}$ with $x_i x_j \in E$ and $v_i v_j \in E$ for any (i, j) . Then the sets of vertices $\{x_1, x_2, \dots, x_k, 3, 4\}$, $\{v_1, 1, 2\}$ induce cliques K_1, K_2 and the sets of vertices $\{v_2, \dots, v_{k-1}, 3\}$, $\{6\}$ induce stable sets S_1, S_2 . Suppose now that the graph G induce by $K_1 \cup K_2 \cup S_1 \cup S_2$ contains C_5 . Then it is induced either by the vertex set $\{x_i, x_j, v_1, 1, v_i\}$ but $x_i 1 \in E$

and $x_j1 \in E$, or $\{x_i, v_1, 1, 6, v_j\}$ but $x_i1 \in E$ and $16 \in E$, or $\{1, 2, x_i, v_j, 6\}$ but $x_i1 \in E$ and $v_j1 \in E$. So, G is $[(2, 2)$ split, (C_4, C_5, C_7) -free] graph.

- Let $(u, v) \in V_4$ with $N(u) = N(v) = \{1, 2, 3, 4\}$ and $uv \in E$. Let $z \in V_k$ for $k \in \{2, 3\}$. If $k = 3$, without loss of generality let $N(z) = \{1, 2, 3\}$ (respectively, $N(z) = \{1, 2, 6\}$ or $N(z) = \{1, 5, 6\}$), then $zv \in E$, otherwise the vertex set $\{1, z, 3, v\}$ induces C_4 (respectively, then $zv \in E$, otherwise the vertex set $\{1, z, v, 3, 5\}$ induces $\overline{J_2}$ or $zv \in \overline{E}$, otherwise the vertex set $\{z, v, 6, 5\}$ induces C_4), which is a contradiction with the hypothesis. If $k = 2$, without loss of generality let $N(z) = \{1, 2\}$ (respectively, $N(z) = \{2, 5\}$), then $zv \in E$, otherwise the vertex set $\{v, 3, z, 1, 5\}$ induces $\overline{J_2}$ (respectively, then $zv \in E$, otherwise the vertex set $\{z, v, 6, 5\}$ induces C_4). So this case is impossible, then $|V_4| \leq 3$.

- Let $(u, v) \in V_3$ with $N(u) = N(v) = \{1, 2, 3\}$. Let $z \in V_2$, without loss of generality let $N(z) = \{1, 2\}$ (respectively, $N(z) = \{2, 5\}$). Then $zv \in E$, otherwise the vertex set $\{v, 3, 5, 6, z\}$ induces $\overline{J_2}$ (respectively, $zv \in E$, otherwise the vertex set $\{v, 1, 3, 4, z\}$ induces $\overline{J_2}$). So, this case is impossible, then $|V_3| \leq 6$. ■

Lemma 21. *Let G be a prime $\overline{S}(C_4, C_6, C_7)$ -free graph containing C_5 . Then G is $[(2, 1)$ split, (C_4, C_6, C_7, P_6) -free] graph.*

Proof. The straightforward case analysis shows that there are only four admissible cases for a vertex being adjacent to C_5 : k -vertices with consecutive neighbors for $k \in \{1, 2, 3, 5\}$. Otherwise for $k \in \{4\}$, let $u \in V_4$, without loss of generality with $N(u) = \{1, 2, 3, 4\}$, the vertex set $\{u, 1, 4, 5\}$ induces C_4 . So V_k is empty for $k \in \{4\}$. Let $(u, v) \in V_k$ for $k \in \{2, 3, 5\}$ with $N(u) = N(v)$. There exists a vertex $z \in V_j$, for $j \neq k$ and $j \in \{2, 3, 5\}$ such that $uz \in E$ and $vz \in \overline{E}$, otherwise $\{u, v\}$ induces a module. We start by calculating the number of vertices having different neighbors in C_6 in each vertex set V_k , then we study the pairs of vertices belonging to $V_i \times V_j$ for $i \neq j$. Finally, we are interested in the vertices having the same neighbors in C_5 which risk inducing module.

1. Let us calculate the size $|V_k|$ of V_k whose vertices have different neighbors in C_5 for $k \in \{1, 2, 3, 5\}$.

(a) Let us show that $|V_5| \leq 1$. Suppose the opposite (i.e., $|V_5| \geq 2$). Let $(u, v) \in V_5$ such that $N(u) = N(v) = \{1, 2, 3, 4, 5\}$ with $uv \in E$, otherwise $\{u, 2, v, 5\}$ induces C_4 . But $\{u, v\}$ induces a module, which is a contradiction with the hypothesis.

(b) Let us build the set V_3 to show that $|V_3| \leq 5$. First, for each $u \in V_3$ the neighbors of u are consecutive, otherwise let $N(u) = \{1, 2, 3, 5\}$, then the vertex set $\{u, 2, 3, 4\}$ induces C_4 .

Let $(u, v) \in V_3$. If (u, v) have one consecutive neighbors in common in C_5 , then $uv \in \overline{E}$. Otherwise, without loss of generality, let $N(u) = \{1, 2, 3\}$ and $N(v) = \{3, 4, 5\}$, then the vertex set $\{u, v, 5, 1\}$ induces C_4 . We have 5 vertices

in V_3 adjacent to C_5 with two neighbors in common (the maximum possible so as not to induce a module), so we have at most five vertices in V_4 .

Note that for any vertex $(u, v) \in V_3$ having two neighbors in common we can have $uv \in E$ or $uv \in \overline{E}$.

(c) Let us build the set V_2 to show that $|V_2| \leq 2$. First, for each $u \in V_2$ the neighbors of u are consecutive, otherwise without loss of generality let $N(u) = \{2, 4\}$, then the vertex set $\{u, 2, 3, 4\}$ induce C_4 .

Let $(u, v) \in V_2$, (u, v) have one neighbor (respectively no neighbor) in common in C_5 , then $uv \in \overline{E}$. Otherwise, without loss of generality, let $N(u) = \{1, 2\}$ and $N(v) = \{1, 5\}$ (respectively, $N(v) = \{3, 4\}$), then the vertex set $\{u, v, 5, 4, 3, 2\}$ (respectively the vertex set $\{u, 2, 3, v\}$) induces C_6 (respectively, C_4).

Suppose now that $|V_2| \geq 3$. Let the vertices $(u, v, w) \in V_2$, at least two vertices among the vertices (u, v, w) have a neighbor in common. If two vertices among the vertices (u, v, w) have a neighbor in common, without loss of generality let $N(u) = \{2, 4\}$, $N(u) = \{1, 5\}$ and $N(u) = \{3, 4\}$, then the vertex set $\{2, u, v, 5, w\}$ induces \overline{J}_2 (because according to what precedes the vertices (u, v, w) are not adjacent two by two). If three vertices (u, v, w) have a neighbor in common, without loss of generality let $N(u) = \{1, 2\}$, $N(u) = \{1, 5\}$ and $N(u) = \{2, 3\}$, then the vertex set $\{v, 5, w, 3, u\}$ induces \overline{J}_2 , which is a contradiction as G is $\overline{S}(C_4, C_7, C_6)$ -free graph.

(d) Let us build the set V_1 to show that $|V_1| \leq 2$. Let $(u, v) \in V_1$. If (u, v) have consecutive neighbors in C_5 , then $uv \in \overline{E}$. Otherwise, without loss of generality, let $N(u) = \{1\}$ and $N(v) = \{2\}$, then the vertex set $\{u, v, 2, 1\}$ induces C_4 . If there is a vertex $w \in V_1$ such that (u, w) have non consecutive neighbors in C_5 , let $N(w) = \{3\}$, then $uw \in E$, otherwise the vertex set $\{u, 1, w, 3, v\}$ induces \overline{J}_2 .

Suppose now that $|V_1| \geq 3$, let the vertices $(u, v, w) \in V_1$ such that $N(u) \neq N(v) \neq N(w)$, without loss of generality, let $N(u) = \{1\}$, $N(v) = \{2\}$ and $N(w) = \{3\}$. Then $uv \in \overline{E}$, otherwise the vertex set $\{u, v, 1, 2\}$ induces C_4 and $uw \in \overline{E}$, otherwise the vertex set $\{1, u, w, 3, 4, 5\}$ induces C_6 , but in this case the vertex set $\{u, 1, 3, w, v\}$ induces \overline{J}_2 , which is a contradiction as G is $\overline{S}(C_4, C_7, C_6)$ -free graph.

2. Let $(u, v) \in V_i \times V_j$ for $i \neq j$. We have different scenarios.

(a) For $(u, v) \in V_5 \times V_j$ and $j \neq 5$. If $j = 3$, without loss of generality, let $N(u) = \{1, 2, 3, 4, 5\}$ and $N(v) = \{1, 2, 3\}$, then $uv \in E$. If $j = 2$ (respectively, $j = 1$) non-adjacencies or adjacencies are possible between vertices.

(b) For $(u, v) \in V_3 \times V_j$ and $j \neq 3$. If $j = 2$ (respectively, $j = 1$), then non-adjacencies or adjacencies are possible between vertices for $|N(u) \cap N(v)| \geq 1$ (respectively, $|N(u) \cap N(v)| = 1$). Otherwise $|N(u) \cap N(v)| = 0$, without loss of generality, let $N(u) = \{1, 2, 3\}$ and $N(v) = \{4, 5\}$ (respectively, $N(v) = \{5\}$)

then $uv \in \overline{E}$, otherwise the vertex set $\{u, 1, 5, v\}$ induces C_4 .

(c) For $(u, v) \in V_2 \times V_j$ and $j \neq 3$. All non-adjacencies or adjacencies are possible between vertices whose neighborhoods $(N(u), N(v))$ are non-consecutive. Otherwise, either $N(u) \cap N(v) = 1$, without loss of generality, let $N(u) = \{1, 2\}$ and $N(v) = \{1\}$, then $uv \in E$ (otherwise the vertex set $\{u, 2, 4, 5, v\}$ induces $\overline{J_2}$), or $N(u) \cap N(v) = 1$ where $(N(u), N(v))$ are consecutive, without loss of generality, let $N(u) = \{1, 2\}$ and $N(v) = \{5\}$, then $uv \in \overline{E}$ (otherwise the vertex set $\{1, u, v, 5\}$ induces C_4 and in this case the vertex set $\{u, 1, 3, 4, v\}$ induces $\overline{J_2}$).

3. Let $(x_1, \dots, x_k) \in V_j$ for $j \in \{1, 2, 3, 4\}$ with $N(x_1) = \dots = N(x_k)$. There exists $(v_1, \dots, v_{k-1}) \in V_i$ for $i \neq j$ such that x_1 is adjacent to v_1 and not adjacent to any other vertices v_i for $i = \{2, \dots, k-1\}$, x_2 is adjacent to v_2 and not adjacent to any other vertices v_i for $i = \{1, \dots, k-1\} \setminus \{2\}$ and so on x_{k-1} is adjacent to v_{k-1} and not adjacent to any other vertices v_i for $i = \{1, \dots, k-2\}$. We have different scenarios.

(a) Let $(x_1, \dots, x_k) \in V_5$. Then $x_i x_j \in E$, otherwise the vertex set $\{x_i, 1, x_j, 3\}$ induces C_4 . From the previous point (2.a) we have $(v_1, \dots, v_{k-1}) \in V_i$ for $i \in \{1, 2\}$ with $N(v_i) = N(v_j)$ or $N(v_i) \neq N(v_j)$. Moreover, $v_i v_j \in \overline{E}$ for $(i, j) \in \{1, 2\}$, otherwise the vertex set $\{x_i, x_j, v_i, v_j\}$ induces C_4 . From the previous points (1.c) and (1.d) we have at most two vertices (that we note) v_1, v_2 with $N(v_1) \neq N(v_2)$ and for any $i \in \{3, \dots, k-1\}$ we have $N(v_i) = N(v_j)$.

For $(v_1, \dots, v_{k-1}) \in V_1$ (respectively, $(v_1, \dots, v_{k-1}) \in V_2$), without loss of generality, let $N(v_1) = \{1\}$, $N(v_2) = \{4\}$ (respectively, $N(v_1) = \{1, 2\}$ and $N(v_2) = \{2, 3\}$ or $N(v_1) = \{1, 2\}$ and $N(v_2) = \{3, 4\}$) and $N(v_i) = N(v_1)$ or $N(v_i) = N(v_2)$ for $i \in \{3, \dots, k-1\}$. Then the graph G is induced by the cliques $K_1 = \{x_1, \dots, x_k, 1, 2\}$ and $K_1 = \{3, 4\}$ and the stable set $S = \{v_1, \dots, v_{k-1}, 5\}$.

(b) Let $(x_1, \dots, x_k) \in V_3$. Then $x_i x_j \in E$, otherwise the vertex set $\{x_i, 1, x_j, 3\}$ induces C_4 . From the previous point (2.b) we have $(v_1, \dots, v_{k-1}) \in V_i$ for $i \in \{1, 2\}$ with $N(v_i) = N(v_j)$ or $N(v_i) \neq N(v_j)$. Moreover, $v_i v_j \in \overline{E}$ for $(i, j) \in \{1, 2\}$, otherwise the vertex set $\{x_i, x_j, v_i, v_j\}$ induces C_4 . From the previous points (1.c) and (1.d) we have at most two vertices (that we note) v_1, v_2 with $N(v_1) \neq N(v_2)$ and for any $i \in \{3, \dots, k-1\}$ we have $N(v_i) = N(v_j)$.

For $(v_1, \dots, v_{k-1}) \in V_1$ (respectively, $(v_1, \dots, v_{k-1}) \in V_2$), without loss of generality, from the previous points (1.c) and (2.b), let $N(v_1) = \{1\}$, $N(v_2) = \{4\}$ (respectively, $N(v_1) = \{1, 2\}$ and $N(v_2) = \{1, 5\}$ or $N(v_1) = \{1, 2\}$ and $N(v_2) = \{1, 5\}$) and $N(v_i) = N(v_1)$ or $N(v_i) = N(v_2)$ for $i \in \{3, \dots, k-1\}$. Then the graph G is induced by the cliques $K_1 = \{x_1, \dots, x_k, 1, 2\}$ and $K_1 = \{3, 4\}$ and the stable set $S = \{v_1, \dots, v_{k-1}, 5\}$.

(c) Let $(x_1, \dots, x_k) \in V_2$. From the previous point (2.c) we have $(v_1, \dots, v_{k-1}) \in V_i$ for $i \in \{1\}$ with $N(v_i) = N(v_j)$ or $N(v_i) \neq N(v_j)$ and vertices (x_i, v_i) have non-consecutive neighbors in C_5 . So, without loss of generality, let $N(x_i) = \{1, 2\}$

and $N(v_i) = \{4\}$ for any $i \in \{1, \dots, k-1\}$, moreover $z_i z_j \in \overline{E}$, otherwise the vertex set $\{v_i, v_j, x_i, 1, 3\}$ induces $\overline{J_2}$. On the other hand, $x_i x_j \in E$ for $i \neq j$ and $i \in \{1, \dots, k-1\}$, otherwise the vertex set $\{x_i, v_i, x_j, v_j, x_k\}$ induces $\overline{J_2}$. Finally, the graph G is induced by the cliques $K_1 = \{x_1, \dots, x_{k-1}, 1, 2\}$ and $K_1 = \{3, 4\}$ and the stable set $S = \{v_1, \dots, v_{k-1}, x_k, 5\}$.

Now we show that G is P_6 -free. Suppose the opposite i.e., that G is $(2, 1)$ split and contains a P_6 . Then we distinguish three types of chains that we will note $P^1 = \{v_i, x_i, x_{i+1}, 3, 4, 5\}$, $P^2 = \{v_i, x_i, x_{i+1}, 3, 4, v_i\}$ and $P^3 = \{v_i, x_i, v_j, 3, 4, 5\}$. If $x_i \in V_5$ (respectively, $x_i \in V_3$), then x_i is adjacent to the vertices 3, 4 and 5. If $x_i \in V_2$ (respectively, $x_i \in V_1$), then the only possible chain is of type P^3 and in this case we have x_i is adjacent to v_i and v_j , which is the contradiction with the hypothesis. ■

Proof of Theorem 18. Let G be a prime \overline{SC}_4 -free graph.

So, based on our lemmas, we have constructed the prime graphs starting from cycles of lengths k , for $k = \{7, 6, 5\}$ and in each case we obtain either a graph with a bounded number of vertices or $[(2, 2)$ split, (C_4, C_5, C_7) -free] graph or $[(2, 1)$ split, P_6 -free] graph. ■

3.3. Prime \overline{S} chordal graph

Theorem 22. *If G is a prime \overline{S} chordal, then G is a split graph or a chordal $(2, 1)$ split graph.*

Proof. A vertex v is simplicial in G if its neighborhood in G is a clique (see [20] for properties of chordal graphs). If G is chordal, then G contains a simplicial vertex denoted v . Let N_i be the set of connected components of $G \setminus N(v) \cup \{v\}$.

Case 1. $|N_i| = 1$ for all $i \in \{1, \dots, k\}$. It is clear that $\bigcup_{i=1}^k N_i \cup \{v\}$ is a stable and as $N(v)$ is a clique, then the graph G is split.

Case 2. There is at least one N_i for all $i \in \{1, \dots, k\}$ such that $|N_i| \neq 1$. Let N_i and N_j , $i \neq j$, $(i, j) \in \{1, \dots, k\}$ be such that $|N_i| \neq 1$ and $|N_j| \neq 1$. Then it is clear that if $u_i v_i$ is an edge in N_i and $u_j v_j$ is an edge in N_j , then $\{u_i, v_i, u_j, v_j, v\}$ induces $\overline{J_1}$. Thus there is one connected component N_1 which contains at least two vertices. Note that N_1 is $2K_2$ -free, otherwise the $2K_2$ with the vertex v induces $\overline{J_1}$.

So N_1 is a $(2K_2, C_4)$ -free graph (i.e., G is a pseudo split graph, such a graph which can be partitioned into a clique A , a stable B and C which induce C_5 or an empty set with edges between A and C and no edges between A and B [22, 30]). As G is chordal, then C is an empty set and the set $V = \{v\} \cup N(v) \cup A \cup B$ is a partition of the vertex set of the graph G . We set $K_1 = N(v)$, $K_2 = A$ and $B \cup \{v\} \cup N_i = S$ for $i = \{2, \dots, k\}$, where G is a $(2, 1)$ split chordal graph. ■

Using the previous results, we summarize the structure of prime SP_5 -free graph in the following theorem.

Theorem 23 (Structure of prime SP_5 -free). *Every prime P_5 -free locally split graph G satisfies one of the following conditions.*

1. G has a bounded number of vertices;
2. G is a split graph;
3. G is a chordal $(2, 1)$ split graph;
4. G is a weakly chordal $(2, 1)$ split graph;
5. G is a $[(2, 2)$ split, (C_4, C_5, C_7) -free] graph;
6. G or is a $[(2, 1)$ split, (C_4, C_6, C_7, P_6) -free] graph.

4. POLYNOMIALS ALGORITHMS OF SP_5 -FREE GRAPHS

Let $G = (V, E)$ be an arbitrary graph. The optimization problems discussed in this section include the following.

- An independent set S (also called stable set) in a graph G is a set of pairwise non-adjacent vertices. The maximum independent set (MIS) problem is to find an independent set in G of maximum cardinality. This number is denoted by $\alpha(G)$. In the weighted case, each node $v \in V$ has an associated non-negative weight $w(v)$ and the goal is to find a maximum weight independent set $\alpha_w(G) = \sum_{v \in S} w(v)$.
- Let $c = (S_1, \dots, S_k)$ be a vertex k -coloring of G where S_i corresponds to the stable set of vertices colored i . The minimum coloring problem (MC) is to find a vertex coloring minimizing k . This number is denoted by $\chi(G)$. In the weighted case, each node $v \in V$ has an associated non-negative weight $w(v)$ and the goal is to find a weighted minimum coloring i.e., the weight of a color i is defined as $w(S_i) = \max_{v \in S_i} w(v)$ and the minimum weight of a coloring of G is $\chi_w(G) = \min \sum w(S_i)$.
- A clique cover is a collection $\zeta = (K_1, \dots, K_k)$ of clique covering all the vertices of G . A minimum clique cover of G is a clique cover of minimum cardinality. This number is denoted by $\theta(G)$. A weighted clique cover of G is a collection ζ of cliques K_i , with a positive weight x_k assigned to each clique K_i in the collection, such that, for each vertex v of G , $\sum_{K \in \zeta, v \in K} x_k \geq w(v)$. A minimum weighted clique cover of G is a weighted clique cover minimizing $\sum_{K \in \zeta} x_K$.

Recall a theorem given in [17] on the modular decomposition.

Theorem 24 [17]. *Let $G = (V, E)$ be a graph with at least two vertices. Then exactly one of the following conditions holds.*

1. G is not connected, and it can be decomposed into its connected components;

2. \overline{G} is not connected, and G can be decomposed into the connected components of \overline{G} ;
3. G is connected and co-connected. There is some $U \subseteq V$ and a unique partition P of V such that
 - (a) $|U| > 3$,
 - (b) $G[U]$ is a maximal prime induced subgraph of G , and
 - (c) For every class S of the partition P , S is a module of G and $|S \cap U| = 1$.

The decomposition theorem applied to a graph G gives a unique modular decomposition tree T . The leaves of the modular decomposition tree are the vertices of G . The interior nodes of T called the parallel node, series node and prime node. In other words.

1. If G is disconnected, then decompose it into its connected components G_i . The node of the tree is called parallel node and the characteristic graph G^* is a stable set.
2. If \overline{G} is disconnected, then decompose G into G_i , where $\overline{G_i}$ are the connected components of \overline{G} . The node of the tree is called serie node and the characteristic graph G^* is a clique.
3. If G is connected and co-connected, then its maximal homogeneous sets are pairwise disjoint and they form the partition P of V , the node of the tree is called prime node. The characteristic graph G^* is obtained from G by contracting every maximal homogeneous set of G to a single vertex.

An important class of graph in modular decomposition is the cograph (also called P_4 -free graph)[11]. This class of graphs is completely decomposable by a modular decomposition (i.e., the only nodes that appear in the modular decomposition tree is the parallel and series nodes), for more details see [35]. The problems, namely, the Maximum (Weight) Stable Set, the Maximum (Weight) Clique, the Minimum Coloring and the Minimum Clique Cover are linear of cograph [12].

Let G a SP_5 -free graph. We start with a linear time algorithm to compute the modular decomposition tree. If the tree contains no prime node, then the graph is a cograph, otherwise, we check each connected component of the graph to see if it is either a cograph or has a modular decomposition tree where only the root is a prime node. Finally, we check which class is the prime node.

4.1. Recognition

First, note that either a graph is disconnected and then it is SP_5 -free if and only if each of its components is SP_5 -free, or the complement of a graph is disconnected, then it is SP_5 -free if and only if it is a split graph (G is SP_5 -free graph if and

only if \overline{G} is \overline{S} house-free graph if and only if \overline{G} is $(house, \overline{J}_1, \overline{J}_2, \overline{J}_3)$ -free graph if and only if \overline{G} is $(C_4, C_5, 2K_2)$ -free graph).

If the input graph is disconnected (respectively, the input graph has a disconnected complementary graph), then we run the algorithm on each component (respectively, we accept if it is a split and otherwise we reject). Now, it is just enough to check the recognition of prime graphs appearing in the structure of SP_5 -free graphs. The recognition of a graph with a constant number of vertices is done in $O(1)$. The recognition of split graph and chordal $(2, 1)$ split graphs can be done in linear time. A time bound $O(n^3)$ (respectively, $O(n^4)$) is given for the recognition of $(2, 1)$ split graph (respectively, $(2, 2)$ split graph) [4]. So we can easily deduce the next result.

Theorem 25. *The recognition of SP_5 -free graphs is done in polynomial time.*

4.2. Maximum Weight Independent Set

The operations to solve the maximum independent set for each node of the tree T are given in [10, 33]. This algorithm has been modified in order to solve the problem of the maximum stable. For details see the following references [34]. So it is clear that solving the Maximum Weight Independent Set (MWIS) on prime graphs in polynomial time can solve this problem on the original graph.

We look at the different classes of the prime graph listed in Theorem 23. For graphs with a bounded number of vertices, the (MWIS) is done in $O(1)$ since it suffices to check all stable sets of these graphs. The result of Alekseev and Lozin [3] allows to solve the MWIS of (p, q) split graph for $q \leq 2$ in polynomial time. Finally, as split and chordal split $(2, 1)$ graph are chordal, then the problem becomes linear in this case [15]. So we deduce the following result.

Theorem 26. *The maximum weight independent set of SP_5 -free graphs is done in polynomial time.*

Recall that the MWIS of P_5 -free graphs is done in $O(n^{12}m)$ [28]. Noting that subcubic P_5 -free graphs are a subclass of SP_5 -free graphs. Then we can deduce that the maximum independent set of subcubic P_5 -free graphs is done in polynomial time. Recently, in [29] this problem has been solved differently for the subcubic H -free graphs for the case when every connected component of H has a certain form. Also, different researches exist on MIS of superclass [5] and subclass of *triangle*-free graph [7, 14, 24].

4.3. Weighted minimum coloring

The operations to solve for each node of the tree T are given in [10, 33]. We look at the different classes of the prime graph listed in Theorem 23.

For graphs of constant size the problem can be solved in linear time by formulating them as an integer linear programming problem which is solved by the branch and bound method. This improves the work of Mc Diarmid and Reed [32] who showed that the weighted minimum coloring (WMC) for the graphs of constant size can be solved in polynomial time.

Let us recall the result of Hoáng [26] in the following theorem.

Theorem 27. *The weighted minimum coloring of chordal graph is done in $O(n^2)$.*

As chordal $(2, 1)$ split (respectively, split) graphs are subclass of chordal class, so we deduce that their (WMC) is done in polynomial time. We also have a weakly chordal $(2, 1)$ split graphs are subclass of weakly chordal class so, we deduce that their (WMC) is done in polynomial time [23].

For a $[(2, 2)$ split, (C_4, C_5, C_7) -free] graphs, it is easy to see that it is a subclass of (banner, odd hole)-free graphs (a banner is a graph that consists of a hole on four vertices and a single vertex with precisely one neighbor on the hole) [27], for which the a minimum coloring is done in polyomial time.

The chromatic number of any (C_4, P_6) -free graph can be found in polynomial time [18], as $[(2, 1)$ split, (C_4, C_6, C_7, P_6) -free] graphs form a subclass of (C_4, P_6) -free graphs. Then we have the same result for this subclass. Thus we deduce the following result.

Theorem 28. *The minimum coloring of SP_5 -free graphs is done in polynomial time.*

The (MC) of *triangle*-free graph is NP-complete. Many researchers work on this problem for the subclass of *triangle*-free graph [9, 13, 31].

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REFERENCES

- [1] H. Ait Haddadene and H. Issaadi, Coloring perfect split neighbourhood graphs (CIRO 05', 2005).
- [2] H. Ait Haddadene and H. Issaadi, *Perfect graphs and vertex coloring problem*, IAENG Int. J. Appl. Math. **39** (2009) 128–133.
- [3] V.E. Alekseev and V. Lozin, *Independent sets of maximum weight in $(p; q)$ -colorable graphs*, Discrete Math. **265** (2003) 351–356.
[https://doi.org/10.1016/S0012-365X\(02\)00877-4](https://doi.org/10.1016/S0012-365X(02)00877-4)
- [4] A. Brandstädt, *Partitions of graphs into one or two independent sets and cliques*, Discrete Appl. Math. **152** (1996) 47–54.
[https://doi.org/10.1016/0012-365X\(94\)00296-U](https://doi.org/10.1016/0012-365X(94)00296-U)

- [5] A. Brandstädt, *(P_5 , diamond)-free graphs revisited: structure and linear time optimization*, Discrete Appl. Math. **138** (2004) 13–27.
[https://doi.org/10.1016/S0166-218X\(03\)00266-X](https://doi.org/10.1016/S0166-218X(03)00266-X)
- [6] A. Brandstädt and D. Kratsch, *On the structure of (P_5, gem) -free graphs*, Discrete Appl. Math. **145** (2005) 155–166.
<https://doi.org/10.1016/j.dam.2004.01.009>
- [7] S. Brandt, *Triangle-free graphs whose independence number equals the degree*, Discrete Math. **310** (2010) 662–669.
<https://doi.org/10.1016/j.disc.2009.05.021>
- [8] H.L. Bodlaender, A. Brandstädt, D. Kratsch, M. Rao and J. Spinrad, *On algorithms for (P_5, gem) -free graphs*, Theoret. Comput. Sci. **349** (2005) 2–21.
<https://doi.org/10.1016/j.tcs.2005.09.026>
- [9] F. Bonomo, M. Chudnovsky, P. Maceli, O. Schaudt, M. Stein and M. Zhong, *3-colouring graphs without triangles or induced paths on seven vertices* (2014), submitted.
- [10] V. Chvátal, *On certain polytopes associated with graphs*, J. Combin. Theory Ser. B **18** (1975) 138–154.
[https://doi.org/10.1016/0095-8956\(75\)90041-6](https://doi.org/10.1016/0095-8956(75)90041-6)
- [11] D.G. Corneil, H. Lerchs and L. Stewart-Burlingham, *Complement reducible graphs*, Discrete Appl. Math. **3** (1981) 163–174.
[https://doi.org/10.1016/0166-218X\(81\)90013-5](https://doi.org/10.1016/0166-218X(81)90013-5)
- [12] D.G. Corneil, Y. Perl and L.K. Stewart, *Cographs: recognition, applications and algorithms*, Congr. Numer. **43** (1984) 249–258.
- [13] K. Dabrowski, V. Lozin, R. Raman and B. Ries, *Colouring vertices of triangle-free graphs without forests*, Discrete Math. **312** (2012) 1372–1385.
<https://doi.org/10.1016/j.disc.2011.12.012>
- [14] Z. Dvořák and M. Mnich, *Large independent sets in triangle-free planar graphs*, Lecture Notes in Comput. Sci. **8737** (2014) 346–357.
https://doi.org/10.1007/978-3-662-44777-2_29
- [15] A. Frank, *Some polynomial algorithms for certain graphs and hypergraphs*, in: Proc. Fifth British Combinatorial Conf. Aberdeen 1975, C.Nash-Williams and J. Sheehan (Ed(s)), Congr. Numer. **XV** (1976) 211–226.
- [16] S. Földes and P.L. Hammer, *Split graphs*, in: 8th S-E Conf.on Combinatorics, Graph Theory and Computing, F. Hoffman *et al.* (Ed(s)), Congr. Numer. **XIX** (1977) 311–315.
- [17] T. Gallai, *Transitiv orientatierbare graphen*, Acta Math. Acad. Sci. Hungar. **18** (1967) 25–66.
<https://doi.org/10.1007/BF02020961>
- [18] S. Gaspers, S. Haung and D. Paulusma, *Colouring square-free graphs without long induced paths*, J. Comput. System Sci. **106** (2019) 60–79.
<https://doi.org/10.1016/j.jcss.2019.06.002>

- [19] V. Giakoumakis and I. Rusu, *Weighted parameters in $(P_5, \overline{P_5})$ graphs*, Discrete Appl. Math. **80** (1997) 255–261.
[https://doi.org/10.1016/S0166-218X\(97\)00093-0](https://doi.org/10.1016/S0166-218X(97)00093-0)
- [20] M.C. Golumbic, *Algorithmic Graph Theory and Perfect Graphs* (Academic Press, 1980).
<https://doi.org/10.1016/C2013-0-10739-8>
- [21] M. Habib and C. Paul, *A survey of the algorithmic aspects of modular decomposition*, Comput. Sci. Rev. **4** (2010) 41–59.
<https://doi.org/10.1016/j.cosrev.2010.01.001>
- [22] P.L. Hammer and B. Simeone, *The splittance of a graph*, Combinatorica **1** (1981) 275–284.
<https://doi.org/10.1007/BF02579333>
- [23] R. Hayward, C. Hoàng and F. Maffray, *Optimizing weakly triangulated graphs*, Graphs Combin. **5** (1989) 339–349.
<https://doi.org/10.1007/BF01788689>
- [24] C.C. Heckman and R. Thomas, *Independent sets in triangle-free cubic planar graphs*, J. Combin. Theory Ser. B **96** (2006) 253–275.
<https://doi.org/10.1016/j.jctb.2005.07.009>
- [25] C.T. Hoàng and B.A. Reed, *Some classes of perfectly orderable graphs*, J. Graph Theory **13** (1989) 445–463.
<https://doi.org/10.1002/jgt.3190130407>
- [26] C.T. Hoàng, *Efficient algorithms for minimum weighted colouring of some classes of perfect graphs*, Discrete Appl. Math. **55** (1994) 133–143.
[https://doi.org/10.1016/0166-218X\(94\)90004-3](https://doi.org/10.1016/0166-218X(94)90004-3)
- [27] C.T. Hoàng, *On the structure of (banner, odd hole)-free graphs*, J. Graph Theory **89** (2018) 395–412.
<https://doi.org/10.1002/jgt.22258>
- [28] D. Lokshtanov, M. Vatshelle and Y. Villanger, *Independent set in P_5 -free graphs in polynomial time*, in: Proc. of the 25th Annual ACM-SIAM Symposium on Discrete Algorithms, C. Chekuri (Ed(s)), (SIAM, 2014) 570–58.
<https://doi.org/10.1137/1.9781611973402.43>
- [29] V. Lozin, J. Monnot and B. Ries, *On the maximum independent set problem in subclasses of subcubic graphs*, J. Discrete Algorithms **31** (2015) 104–112.
<https://doi.org/10.1016/j.jda.2014.08.005>
- [30] F. Maffray and M. Preissmann, *Linear recognition of pseudo-split graphs*, Discrete Appl. Math. **52** (1994) 307–312.
[https://doi.org/10.1016/0166-218X\(94\)00022-0](https://doi.org/10.1016/0166-218X(94)00022-0)
- [31] N. Matsumoto and M. Tanaka, *The chromatic number of triangle-free and broom-free graphs in terms of the number of vertices*, Aequationes Math. **95** (2021) 319–328.
<https://doi.org/10.1007/s00010-020-00760-z>

- [32] C. McDiarmid and B.A. Reed, *Channel assignment and weighted coloring*, Networks **36** (2000) 114–117.
[https://doi.org/10.1002/1097-0037\(200009\)36:2;114::AID-NET6;3.0.CO;2-G](https://doi.org/10.1002/1097-0037(200009)36:2;114::AID-NET6;3.0.CO;2-G)
- [33] R. H. Möhring and F. J. Radermacher, *Substitution decomposition for discrete structures and connections with combinatorial optimization*, North-Holland Math. Studies **95** (1984) 257–355.
[https://doi.org/10.1016/S0304-0208\(08\)72966-9](https://doi.org/10.1016/S0304-0208(08)72966-9)
- [34] M. Rao, *Décompositions de Graphes et Algorithmes Efficaces*, Ph.D. Thesis (Université Paul Verlaine, Metz, 2006).
- [35] P.D. Sumner, *Graphs indecomposable with respect to the X-join*, Discrete Math. **6** (1973) 281–298.
[https://doi.org/10.1016/0012-365X\(73\)90100-3](https://doi.org/10.1016/0012-365X(73)90100-3)

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