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BOUNDS ON THE TOTAL DOUBLE ROMAN DOMINATION NUMBER OF GRAPHS

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Abstract

Let G be a simple graph with no isolated vertex and let $\gamma_{tdR}(G)$ be the total double Roman domination number of G. In this paper, we present lower and upper bounds on $\gamma_{tdR}(G)$ of a graph G in terms of the order, open packing number and the numbers of support vertices and leaves, and we characterize all extremal graphs. We also prove that for any connected graph G of order n with minimum degree at least two, $\gamma_{tdR}(G) \leq \left\lfloor \frac{4n}{3} \right\rfloor$.

Keywords: double Roman domination, total double Roman domination, open packing.

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1. Introduction

Throughout this paper, G is a simple graph with vertex set $V(G)$ and edge set $E(G)$. The order $|V(G)|$ of G is denoted by $n = n(G)$. The *open neighborhood* of a vertex v in G is the set $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ and its *closed neighborhood* is the set $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex v in G is $d(v) = d_G(v) = |N_G(v)|$. The minimum degree and maximum degree among all vertices of G are denoted by $\delta(G)$ and $\Delta(G)$, respectively.

For a subset S of vertices of G, we denote by $G[S]$ the subgraph induced by S. The *distance* $d(u, v)$ between two vertices u and v of a connected graph G is the length of a shortest (u, v)-path in G. The *eccentricity* of a vertex v in a connected graph G is the maximum of the distances from v to the other vertices of G and is denoted by $\mathrm{ecc}_G(v)$. The *diameter* of G is the maximum eccentricity taken over all vertices of G and is denoted by $\mathrm{diam}(G)$. If the length of a path P of G is equal to $\mathrm{diam}(G)$, then we call P a *diametral path* of G. A subset S of vertices of G is an *independent set* if no two vertices of S are adjacent in G.

We write P_n for the path of order n, C_n for the cycle of length n, K_n for the complete graph of order n and $K_{p,q}$ for the complete bipartite graph with two partite sets having p and q vertices. A vertex of degree one is referred as a *leaf* and its unique neighbor is called a *support vertex*. A *strong support vertex* is a support vertex adjacent to at least two leaves, while a *weak support vertex* is a support vertex adjacent to precisely one leaf. A *star* S_n of order $n \geq 2$ is the complete bipartite graph $K_{1,n-1}$. We call the *center* of a star to be a vertex of maximum degree. A *double star* is the tree with exactly two vertices that are not leaves. The *corona graph* $cor(H)$ of a graph H is the graph obtained from H by attaching one pendent edge at each vertex of H.

A *subdivision* of an edge uv is obtained by removing the edge uv, adding a new vertex w, and adding edges uw and wv. For a set S of vertices in a graph G, the subgraph obtained from G by deleting all vertices in S and all edges incident with vertices in S is denoted by $G-S$. If $S = \{x\}$, then we simply denote $G-\{x\}$ by $G - x$. For two disjoint graphs G and H, the *union* of G and H, denoted by $G \cup H$, is the graph with vertex set $V(G) \cup V(H)$ and edge set $E(G) \cup E(H)$.

A subset S of vertices in a graph G is an *open packing* if the open neighborhoods of vertices in S are pairwise disjoint. The *open packing number* $\rho^o(G)$ is the maximum cardinality of an open packing. For a real-valued function $f : V(G) \to \mathbb{R}$ and $S \subseteq V(G)$, we define $f(S) = \sum_{x \in S} f(x)$.

A *double Roman dominating function* (DRDF) on a graph G is a function $f : V(G) \to \{0, 1, 2, 3\}$ having the property that if $f(v) = 0$, then the vertex v must be adjacent to at least two vertices assigned 2 or one vertex assigned 3 under f, whereas if $f(v) = 1$, then the vertex v must be adjacent to at least one vertex assigned 2 or 3. The *weight* of a DRDF f is the value $\omega(f) = f(V(G))$.

The concept of double Roman domination in graphs is now well studied in [1–3, 5–7, 9, 10, 13, 14] and elsewhere.

As a new variant of the double Roman domination, the total double Roman domination was introduced by Hao *et al.* [8]. The concept was studied further in, for example, [4, 11, 12]. The *total double Roman dominating function* (TDRDF) on a graph G with no isolated vertex is a DRDF f on G with the additional property that the subgraph of G induced by the set $\{v \in V(G) : f(v) \neq 0\}$ has no isolated vertices. The *total double Roman domination number* $\gamma_{tdR}(G)$ is the minimum weight of a TDRDF on G. A TDRDF on G with weight $\gamma_{tdR}(G)$ is called a $\gamma_{tdR}(G)$-*function*.

In this paper, we continue the study of the total double Roman domination number and we establish some new bounds for it. Some of our results improve the previous bounds.

Now, we are in a position to give the main results of this paper.

Let $\mathcal{G}$ be the family of all graphs that can be obtained from a graph G of order n with $\Delta(G) = n-1$ by adding a pendent edge to a vertex v of G, where $d_G(v) = n-1$.

Theorem 1. *For any connected graph G of order at least three,*

$$\gamma_{tdR}(G) \geq 3\rho^o(G)/2 + 1$$

with equality if and only if $G \in \mathcal{G}$.

We now give a lower bound for the total double Roman domination number of a tree in terms of its order and the number of leaves, and characterize all trees achieving equality for the proposed bound.

Theorem 2. *For any tree T of order $n(T) \geq 2$ with $l(T)$ leaves,*

$$\gamma_{tdR}(T) \geq \frac{6(n(T) - l(T) + 2)}{5}$$

with equality if and only if T is a path of order $n(T) \equiv 0 \pmod 5$.

We next turn our attention to investigate an upper bound on the total double Roman domination number of a tree. Let $\mathcal{T} = \{cor(T) : T$ is a tree of order at least two$\}$.

Theorem 3. *For any tree T of order $n(T) \geq 3$ with $s(T)$ support vertices,*

$$\gamma_{tdR}(T) \leq \frac{6n(T) + 3s(T)}{5}$$

with equality if and only if $T \in \mathcal{T}$.

Remark 4. We note that for an arbitrary tree T of order $n(T) \geq 3$, T has at most $n(T)/2$ support vertices and hence by Theorem 3,

$$\gamma_{tdR}(T) \leq \frac{6n(T) + 3s(T)}{5} \leq \frac{3n(T)}{2},$$

implying that Theorem 3 improves the known result due to Shao *et al.* [12], that is, $\gamma_{tdR}(T) \leq \frac{3n(T)}{2}$.

Remark 5. We remark that if G is a connected graph of order $n(G) \geq 3$ with $s(G)$ support vertices that is not a tree, then it is not necessarily true that $\gamma_{tdR}(G) \leq \frac{6n(G)+3s(G)}{5}$. For example, Hao *et al.* [8] determined the exact value of $\gamma_{tdR}(C_n)$, that is, $\gamma_{tdR}(C_n) = \lceil \frac{6n}{5} \rceil$ for any integer $n \geq 3$. Thus if $n \not\equiv 0 \pmod 5$, then

$$\gamma_{tdR}(C_n) = \left\lceil \frac{6n}{5} \right\rceil > \frac{6n}{5} = \frac{6n + 3s(C_n)}{5}.$$

Finally, we derive an upper bound on $\gamma_{tdR}(G)$ for connected graphs G with minimum degree at least two.

Theorem 6. *For any connected graph G of order n with $\delta(G) \geq 2$, $\gamma_{tdR}(G) \leq \lfloor \frac{4n}{3} \rfloor$ and this bound is sharp for C_n ($n \in \{3, 4, 5, 6, 7, 8, 11\}$).*

2. Proof of Theorem 1

In this section, we prove Theorem 1. For this purpose, we shall need the following result due to Hao *et al.* [8].

Proposition 7 [8]**.** *For any graph G of order $n \geq 3$ with no isolated vertex, $\gamma_{tdR}(G) = 4$ if and only if $\Delta(G) = n - 1$.*

We are now in a position to present a proof of Theorem 1.

Proof. Let f be a $\gamma_{tdR}(G)$-function and let X be an open packing of cardinality $\rho^o(G)$ in G. Since $N_G(u) \cap N_G(v) = \emptyset$ for any two distinct vertices $u, v \in X$, we have $G[X] = mK_2 \cup (\rho^o(G) - 2m)K_1$, where $0 \leq m \leq \rho^o(G)/2$. It can be easily verified that the following two facts are true.

Fact 1. *For any isolated vertex x in $G[X]$, $f(N_G[x]) \geq 3$ and for any two adjacent vertices x and y in $G[X]$, $f(N_G(x) \cup N_G(y)) \geq 3$.*

Fact 2. *For any vertex $x \in V(G)$, $|N_G(x) \cap X| \leq 1$.*

We now have the following claims as follows.

Claim 1. *If $\rho^o(G) \geq 2m + 1$, then $\gamma_{tdR}(G) \geq 3\rho^o(G)/2 + 3/2$.*

Proof. Let $X_1 = \{x : x \text{ is an isolated vertex in } G[X]\}$. In this case, we have $|X_1| = \rho^o(G) - 2m \geq 1$. Thus by Facts 1 and 2, we have

$$\begin{aligned}\gamma_{tdR}(G) &= f(V(G)) \geq \sum_{xy\in E(G[X])} f(N_G(x) \cup N_G(y)) + \sum_{x\in X_1} f(N_G[x]) \\ &\geq 3m + 3|X_1| = 3m + 3(\rho^o(G) - 2m) \geq 3\rho^o(G)/2 + 3/2,\end{aligned}$$

and so Claim 1 holds. □

In the following, by Claim 1, we may assume that $\rho^o(G) = 2m$. This implies that $G[X] = mK_2$.

Claim 2. *If there exists some edge, say uv, in $E(G[X])$ such that $f(N_G(u) \cup N_G(v)) = 3$, then $\gamma_{tdR}(G) \geq 3\rho^o(G)/2 + 2$.*

Proof. Since $f(N_G(u) \cup N_G(v)) = 3$, it is easy to verify that $f(u) + f(v) = 3$ and $f(x) = 0$ for each $x \in (N_G(u) \cup N_G(v))\backslash\{u, v\}$. Without loss of generality, assume that $f(u) = 1$ and $f(v) = 2$. Noticing that G is a connected graph of order at least three, we may assume that there exists some vertex, say w, in $N_G(v)\backslash\{u\}$ (the case when there exists some vertex in $N_G(u)\backslash\{v\}$ is similar). Clearly $f(w) = 0$. This forces that there exists some vertex, say w_1, in $N_G(w)\backslash\{v\}$ such that $f(w_1) \geq 2$.

First, suppose that w_1 is not adjacent to a vertex of X in G. Then by Facts 1 and 2, we have

$$\begin{aligned}\gamma_{tdR}(G) &= f(V(G)) \geq f(w_1) + \sum_{xy\in E(G[X])} f(N_G(x) \cup N_G(y)) \\ &\geq 3m + 2 = 3\rho^o(G)/2 + 2.\end{aligned}$$

Second, suppose that w_1 is adjacent to a vertex, say u_1, of X in G. It is easy to verify that $u_1 \neq u$ (for otherwise, it is a contradiction to the fact that $f(x) = 0$ for each $x \in (N_G(u) \cup N_G(v))\backslash\{u, v\}$). From our earlier assumption, we note that $G[X] = mK_2$. Thus there exists some vertex, say v_1, of X such that $u_1v_1 \in E(G[X])$. Moreover, since $f(w_1) \geq 2$ and $w_1 \in N_G(u_1)$, it can be easily checked that $f(N_G(u_1) \cup N_G(v_1)) \geq 5$. Consequently, by Facts 1 and 2, we have

$$\begin{aligned}\gamma_{tdR}(G) &= f(V(G)) \geq f(N_G(u_1) \cup N_G(v_1)) + \sum_{xy\in E(G[X])\backslash\{u_1v_1\}} f(N_G(x) \cup N_G(y)) \\ &\geq 5 + 3(m - 1) = 3\rho^o(G)/2 + 2.\end{aligned}$$

Thus Claim 2 holds. □

In the following, by Fact 1 and Claim 2, we may assume that for any edge $xy \in E(G[X])$, $f(N_G(x) \cup N_G(y)) \geq 4$.

Claim 3. *If $m \geq 2$, then $\gamma_{tdR}(G) \geq 3\rho^o(G)/2 + 2$.*

Proof. By Fact 2, we have

$$\gamma_{tdR}(G) = f(V(G)) \geq \sum_{xy \in E(G[X])} f(N_G(x) \cup N_G(y)) \geq 4m \geq 3\rho^o(G)/2 + 2$$

and hence Claim 3 holds. □

Claim 4. *If $m = 1$, then $\gamma_{tdR}(G) \geq 3\rho^o(G)/2 + 1$ with equality if and only if $G \in \mathcal{G}$.*

Proof. In this case, we have $G[X] = K_2$. Now let uv be the unique edge in $E(G[X])$. As previously mentioned, $f(N_G(u) \cup N_G(v)) \geq 4$. Then by Fact 2, we have

$$\gamma_{tdR}(G) = f(V(G)) \geq f(N_G(u) \cup N_G(v)) \geq 4 = 3\rho^o(G)/2 + 1, \tag{1}$$

establishing the desired lower bound.

Suppose next that $\gamma_{tdR}(G) = 3\rho^o(G)/2 + 1$. Then we have equality throughout the inequality chain (1). In particular, $\gamma_{tdR}(G) = 4$ and hence by Proposition 7, we have $\Delta(G) = n(G) - 1$. If $d(x) \geq 2$ for each $x \in V(G)$, then for any two distinct vertices $u, v \in V(G)$, we have $N_G(u) \cap N_G(v) \neq \emptyset$, this implies that $\rho^o(G) = 1$, a contradiction to the fact that $\rho^o(G) = 2m = 2$. Therefore, there must be a vertex of degree one. This forces that $G \in \mathcal{G}$. Conversely, suppose that $G \in \mathcal{G}$. It is easy to verify that $\gamma_{tdR}(G) = 4$ and $\rho^o(G) = 2$, implying that $\gamma_{tdR}(G) = 3\rho^o(G)/2 + 1$. Thus Claim 4 holds. □

The proof is completed. ■

3. Proof of Theorem 2

In this section, we prove Theorem 2. Before presenting our proof, we list below two preliminary observations whose proofs are easy to see and a known result that will be useful in proving our results later.

Observation 8. *Let T be a tree with diameter at least three and let v be a strong support vertex of T. Then there exists a $\gamma_{tdR}(T)$-function f such that $f(v) = 3$ and $f(x) = 0$ for every leaf adjacent to v.*

Observation 9. *Let T be a tree and let v be a weak support vertex adjacent to the unique leaf u. Then there exists a $\gamma_{tdR}(T)$-function f such that $f(u) + f(v) = 3$. In particular, if v has degree 2, then there exists a $\gamma_{tdR}(T)$-function f such that $f(u) = 1$ and $f(v) = 2$.*

Hao *et al.* [8] determined the total double Roman domination number of paths as follows.

Proposition 10 [8]**.** *For* $n \geq 2$*,*

$$\gamma_{tdR}(P_n) = \begin{cases} 6, & \text{if } n = 4, \\ \left\lceil \frac{6n}{5} \right\rceil, & \text{otherwise.} \end{cases}$$

We now give a proof of Theorem 2.

Proof. We proceed by induction on the number $n(T)$. If $n(T) \in \{2,3,4,5\}$ and $T \neq P_5$, then it is easy to verify that $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$. If $T = P_5$, then it follows from Proposition 10 that $\gamma_{tdR}(T) = 6 = \frac{6(n(T)-l(T)+2)}{5}$. Assume, then, that $n(T) \geq 6$ and that for any tree T' of order $n(T')$ with $l(T')$ leaves and $2 \leq n(T') < n(T)$, $\gamma_{tdR}(T') \geq \frac{6(n(T')-l(T')+2)}{5}$ with equality if and only if T' is a path of order $n(T') \equiv 0 \pmod 5$. Let T be a tree of order $n(T)$ with $l(T)$ leaves. If $\operatorname{diam}(T) = 2$, that is, if T is a star, then $\gamma_{tdR}(T) = 4 > \frac{6(n(T)-l(T)+2)}{5}$. If $\operatorname{diam}(T) = 3$, that is, if T is a double star, then $\gamma_{tdR}(T) = 6 > \frac{6(n(T)-l(T)+2)}{5}$. So in the following we may assume that $\operatorname{diam}(T) \geq 4$.

Now let $P = v_1 v_2 \cdots v_{d+1}$ be a diametral path of T ($d = \operatorname{diam}(T)$). If $T \neq P$, then let v_k be the first vertex of P such that $d_T(v_k) \geq 3$. Without loss of generality, we choose the diametral path P such that k is as small as possible. By symmetry, we may assume that $2 \leq k \leq \lceil (d+1)/2 \rceil$. Let f be a $\gamma_{tdR}(T)$-function.

Claim 1. *If* $d_T(v_2) \geq 3$*, then* $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$*.*

Proof. By Observation 8, we may assume that $f(v_2) = 3$ and $f(x) = 0$ for each leaf adjacent to v_2. Clearly, $f(v_3) \geq 1$.

First, suppose that $f(v_3) \geq 2$. Let $T' = T - (N_T(v_2) \backslash \{v_3\})$. Then the function f' defined by $f'(v_2) = 1$ and $f'(x) = f(x)$ for each $x \in V(T') \backslash \{v_2\}$, is a TDRDF on T'. Note that $n(T) = n(T') + d_T(v_2) - 1$ and $l(T) = l(T') + d_T(v_2) - 2$. Then by the induction hypothesis, we have

$$\begin{aligned} \gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(N_T(v_2) \backslash \{v_3\}) + f(v_2) - f'(v_2) \\ &\geq \frac{6(n(T') - l(T') + 2)}{5} + 2 \\ &= \frac{6((n(T) - d_T(v_2) + 1) - (l(T) - d_T(v_2) + 2) + 2)}{5} + 2 \\ &> \frac{6(n(T) - l(T) + 2)}{5}, \end{aligned}$$

as desired.

Second, suppose that $f(v_3) = 1$ and $d_T(v_3) \geq 3$. Let $T' = T - (N_T[v_2]\backslash\{v_3\})$. Note that $f(v_3) = 1$. One can verify that if v_3 is a support vertex of T' adjacent to a leaf, say u, in $V(T')\backslash V(P)$, then $f(u) = 2$; and if v_3 is adjacent to a support vertex, say w, in $V(T')\backslash V(P)$, then we may assume that $f(w) = 3$ and $f(x) = 0$ for every leaf x adjacent to w. In either case, the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + d_T(v_2)$ and $l(T) = l(T') + d_T(v_2) - 1$. Then by the induction hypothesis, we have

$$\begin{aligned} \gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(N_T[v_2]\backslash\{v_3\}) \\ &\geq \frac{6(n(T') - l(T') + 2)}{5} + 3 \\ &= \frac{6((n(T) - d_T(v_2)) - (l(T) - d_T(v_2) + 1) + 2)}{5} + 3 \\ &> \frac{6(n(T) - l(T) + 2)}{5}, \end{aligned}$$

as desired.

Finally, suppose that $f(v_3) = 1$ and $d_T(v_3) = 2$. Assume now that $f(v_4) \leq 1$. Let $T' = T - N_T[v_2]$. Then the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + d_T(v_2) + 1$ and $l(T') + d_T(v_2) - 2 \leq l(T) \leq l(T') + d_T(v_2) - 1$. Then by the induction hypothesis, we have

$$\begin{aligned} \gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(N_T[v_2]) \\ &\geq \frac{6(n(T') - l(T') + 2)}{5} + 4 \\ &\geq \frac{6((n(T) - d_T(v_2) - 1) - (l(T) - d_T(v_2) + 2) + 2)}{5} + 4 \\ &> \frac{6(n(T) - l(T) + 2)}{5}, \end{aligned}$$

as desired.

Assume next that $f(v_4) \geq 2$. Let $T' = T - (N_T[v_2]\backslash\{v_3\})$. Then the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + d_T(v_2)$ and $l(T) = l(T') + d_T(v_2) - 2$. Then by the induction hypothesis, we have

$$\begin{aligned} \gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(N_T[v_2]\backslash\{v_3\}) \\ &\geq \frac{6(n(T') - l(T') + 2)}{5} + 3 \\ &= \frac{6((n(T) - d_T(v_2)) - (l(T) - d_T(v_2) + 2) + 2)}{5} + 3 \\ &> \frac{6(n(T) - l(T) + 2)}{5}, \end{aligned}$$

as desired.

By the above discussions, Claim 1 holds. □

In the following, by Claim 1, we may assume that $d_T(v_2) = 2$ and so by Observation 9, we may assume that $f(v_1) = 1$ and $f(v_2) = 2$.

Claim 2. *If $d_T(v_3) \geq 3$, then $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$.*

Proof. First, suppose that v_3 is a strong support vertex of T. By Observation 8, we may assume that $f(v_3) = 3$ and $f(x) = 0$ for every leaf x adjacent to v_3. Let $T' = T - v_1$. Then the function f' defined by $f'(v_2) = 1$ and $f'(x) = f(x)$ for each $x \in V(T')\backslash\{v_2\}$, is a TDRDF on T'. Note that $n(T) = n(T') + 1$ and $l(T) = l(T')$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(v_1) + 1 \geq \frac{6(n(T') - l(T') + 2)}{5} + 2\\ &= \frac{6((n(T) - 1) - l(T) + 2)}{5} + 2 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

Second, suppose that v_3 is a weak support vertex of T. Let u be the unique leaf adjacent to v_3. By Observation 9, we may assume that $f(u) + f(v_3) = 3$. Let $T' = T - \{v_1, v_2\}$. Then the function f' defined by $f'(u) = 1$, $f'(v_3) = 3$ and $f'(x) = f(x)$ for each $x \in V(T')\backslash\{u, v_3\}$, is a TDRDF on T'. Note that $n(T) = n(T') + 2$ and $l(T) = l(T') + 1$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(v_1) + f(v_2) - 1 \geq \frac{6(n(T') - l(T') + 2)}{5} + 2\\ &= \frac{6((n(T) - 2) - (l(T) - 1) + 2)}{5} + 2 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

Finally, suppose that v_3 is not a support vertex of T. By the choice of the diametral path $P = v_1 v_2 \cdots v_{d+1}$, we may assume that $u_i \in N_T(v_3)\backslash\{v_2, v_4\}$ for each $i \in \{1, 2, \ldots, t\}$, where $t = d_T(v_3) - 2$, and that w_i is the unique leaf adjacent to u_i. Then by Observation 9, we may assume that $f(u_i) = 2$ and $f(w_i) = 1$ for each $i \in \{1, 2, \ldots, t\}$.

Assume now that $f(v_3) \geq 1$ or $d_T(v_3) \geq 4$. Let $T' = T - \{v_1, v_2\}$. Then the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + 2$ and $l(T) = l(T') + 1$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(v_1) + f(v_2) \geq \frac{6(n(T') - l(T') + 2)}{5} + 3\\ &= \frac{6((n(T) - 2) - (l(T) - 1) + 2)}{5} + 3 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

Assume next that $f(v_3) = 0$ and $d_T(v_3) = 3$. This implies that $t = 1$. Let $T' = T - \{v_1, v_2, w_1\}$. Then the function f' defined by $f'(v_3) = 1$ and $f'(x) = f(x)$ for each $x \in V(T')\backslash\{v_3\}$, is a TDRDF on T'. Note that $n(T) = n(T') + 3$ and $l(T) = l(T') + 1$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(v_1) + f(v_2) + f(w_1) - 1 \\ &\geq \frac{6(n(T') - l(T') + 2)}{5} + 3 = \frac{6((n(T) - 3) - (l(T) - 1) + 2)}{5} + 3 \\ &> \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

By the above discussions, Claim 2 holds. $\square$

In the following, by Claim 2, we may assume that $d_T(v_3) = 2$.

Claim 3. *If $f(v_3) \geq 1$, then $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$.*

Proof. We distinguish three cases as follows.

Case 1. $f(v_3) \geq 2$. Let $T' = T - v_1$. Since $f(v_1) = 1$ and $f(v_2) = 2$ by our earlier assumption, we have that the function f' defined by $f'(v_2) = 1$ and $f'(x) = f(x)$ for each $x \in V(T')\backslash\{v_2\}$, is a TDRDF on T'. Note that $n(T) = n(T') + 1$ and $l(T) = l(T')$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(v_1) + 1 \geq \frac{6(n(T') - l(T') + 2)}{5} + 2 \\ &= \frac{6((n(T) - 1) - l(T) + 2)}{5} + 2 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

Case 2. $f(v_3) = 1$ and $f(v_4) \leq 1$. Let $T' = T - \{v_1, v_2, v_3\}$. It is easy to verify that the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + 3$ and $l(T') \leq l(T)$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(v_1) + f(v_2) + f(v_3) \\ &\geq \frac{6(n(T') - l(T') + 2)}{5} + 4 \geq \frac{6((n(T) - 3) - l(T) + 2)}{5} + 4 \\ &> \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

Case 3. $f(v_3) = 1$ and $f(v_4) \geq 2$. Let $T' = T - \{v_1, v_2\}$. Then the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T')+2$ and $l(T) = l(T')$.

By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + f(v_1) + f(v_2) \geq \frac{6(n(T') - l(T') + 2)}{5} + 3 \\ &= \frac{6((n(T) - 2) - l(T) + 2)}{5} + 3 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

By the above discussions, Claim 3 holds. □

In the following, by Claim 3, we may assume that $f(v_3) = 0$.

Claim 4. *If $f(v_4) = 3$, or $d_T(v_4) \geq 3$ and $f(v_4) = 2$, then $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$.*

Proof. If $f(v_4) = 3$, then by the method similar to Case 3 of Claim 3, it is easy to verify that $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$. Assume next that $d_T(v_4) \geq 3$ and $f(v_4) = 2$. Let $T' = T - \{v_1, v_2, v_3\}$. Clearly the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + 3$ and $l(T) = l(T') + 1$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + \sum_{i=1}^{3} f(v_i) \geq \frac{6(n(T') - l(T') + 2)}{5} + 3 \\ &= \frac{6((n(T) - 3) - (l(T) - 1) + 2)}{5} + 3 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired. Thus Claim 4 holds. □

By our earlier assumption, $f(v_2) = d_T(v_3) = 2$ and $f(v_3) = 0$. This forces $f(v_4) \geq 2$. Moreover, by Claim 4, we may assume that $d_T(v_4) = f(v_4) = 2$ in the following.

Claim 5. *If $f(v_5) \geq 2$, then $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$.*

Proof. Let $T' = T - \{v_1, v_2, v_3\}$. Then the function f' defined by $f'(v_4) = 1$ and $f'(x) = f(x)$ for each $x \in V(T')\backslash\{v_4\}$, is a TDRDF on T'. Note that $n(T) = n(T') + 3$ and $l(T) = l(T')$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + \sum_{i=1}^{3} f(v_i) + 1 \geq \frac{6(n(T') - l(T') + 2)}{5} + 4 \\ &= \frac{6((n(T) - 3) - l(T) + 2)}{5} + 4 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired. Thus Claim 5 holds. □

By our earlier assumption, $f(v_3) = 0$ and $d_T(v_4) = f(v_4) = 2$. This forces $f(v_5) \geq 1$. Moreover, by Claim 5, we may assume that $f(v_5) = 1$ in the following. Since $d_T(v_1) = 1$, $d_T(v_2) = d_T(v_3) = d_T(v_4) = 2$ and $n(T) \geq 6$ by our assumptions, we have $d_T(v_5) \geq 2$.

Claim 6. *If $d_T(v_5) = 2$, then $\gamma_{tdR}(T) \geq \frac{6(n(T)-l(T)+2)}{5}$ with equality if and only if T is a path of order $n(T) \equiv 0 \pmod 5$.*

Proof. First, suppose that $f(v_6) \geq 2$. Let $T' = T - \{v_1, v_2, v_3, v_4\}$. It is easy to verify that the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + 4$ and $l(T) = l(T')$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + \sum_{i=1}^{4} f(v_i) \geq \frac{6(n(T') - l(T') + 2)}{5} + 5 \\ &= \frac{6((n(T) - 4) - l(T) + 2)}{5} + 5 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

Second, suppose that $f(v_6) \leq 1$. Let $T' = T - \{v_1, v_2, v_3, v_4, v_5\}$. It is easy to verify that the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + 5$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + \sum_{i=1}^{5} f(v_i) \geq \frac{6(n(T') - l(T') + 2)}{5} + 6 \\ &\geq \frac{6((n(T) - 5) - l(T) + 2)}{5} + 6 = \frac{6(n(T) - l(T) + 2)}{5}, \qquad (2)\end{aligned}$$

establishing the desired lower bound.

Assume next that $\gamma_{tdR}(T) = \frac{6(n(T)-l(T)+2)}{5}$. Then we have equality throughout the inequality chain (2). In particular, $l(T') = l(T)$ and $f'(V(T')) = \frac{6(n(T')-l(T')+2)}{5}$. This implies that $\gamma_{tdR}(T') \leq f'(V(T')) = \frac{6(n(T')-l(T')+2)}{5}$. Moreover, since $\gamma_{tdR}(T') \geq \frac{6(n(T')-l(T')+2)}{5}$ by the induction hypothesis, this forces $\gamma_{tdR}(T') = \frac{6(n(T')-l(T')+2)}{5}$. Again by the induction hypothesis, we have $T' = P_{n(T')} = P_{n(T)-5}$, where $n(T') \equiv 0 \pmod 5$. As shown earlier, $l(T') = l(T)$. Thus $d_T(v_6) = 2$, implying that $T = P_{n(T)}$ and $n(T) \equiv 0 \pmod 5$. On the other hand, if $T = P_{n(T)}$ and $n(T) \equiv 0 \pmod 5$, then by Proposition 10, we have $\gamma_{tdR}(T) = \frac{6n(T)}{5} = \frac{6(n(T)-l(T)+2)}{5}$.

By the above discussions, Claim 6 holds. □

Claim 7. *If $d_T(v_5) \geq 3$, then $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$.*

Proof. Let H be the connected component of $T - \{v_4, v_6\}$ containing the vertex v_5. Since $P = v_1 v_2 \cdots v_{d+1}$ is a diametral path of T, we have $1 \leq \mathrm{ecc}_H(v_5) \leq 4$. Let $u_1 u_2 \cdots u_k$ be a path of H such that $k = \mathrm{ecc}_H(v_5)$ and u_k is adjacent to v_5 in H.

Case 1. $\mathrm{ecc}_H(v_5) = 1$. In this case, v_5 is adjacent to the leaf u_1. Since $f(v_5) = 1$ by our earlier assumption, we have $f(u_1) = 2$. Let $T' = T - \{v_1, v_2, v_3, v_4\}$. Then the restriction f' of f on $V(T')$ is a TDRDF on T'. Note that $n(T) = n(T') + 4$ and $l(T) = l(T') + 1$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &= f(V(T)) = f'(V(T')) + \sum_{i=1}^{4} f(v_i) \geq \frac{6(n(T') - l(T') + 2)}{5} + 5 \\ &= \frac{6((n(T) - 4) - (l(T) - 1) + 2)}{5} + 5 > \frac{6(n(T) - l(T) + 2)}{5},\end{aligned}$$

as desired.

Case 2. $2 \leq \mathrm{ecc}_H(v_5) \leq 4$. If $d_T(u_2) \geq 3$, then by the similar method to Claim 1, we have $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$. So in the following we may assume that $d_T(u_2) = 2$. Then by Observation 9, we may assume that $f(u_1) = 1$ and $f(u_2) = 2$. If $\mathrm{ecc}_H(v_5) = 2$, then by the similar method to Claim 2, we have $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$.

Assume now that $\mathrm{ecc}_H(v_5) = 3$. If $d_T(u_3) \geq 3$, then again by the similar method to Claim 2, we have $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$. Now let $d_T(u_3) = 2$. This implies that u_3 has exactly two neighbors u_2 and v_5. Moreover, since $f(u_2) = 2$ and $f(v_5) = 1$, this forces $f(u_3) \geq 1$. Thus by the similar method to Claim 3, we have $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$.

Assume next that $\mathrm{ecc}_H(v_5) = 4$. Note that $P' = u_1u_2u_3u_4v_5v_6 \cdots v_{d+1}$ is also a diametral path of T. By the choice of the diametral path $P = v_1v_2 \cdots v_{d+1}$, we have $d_T(u_2) = d_T(u_3) = d_T(u_4) = 2$. If $f(u_3) \geq 1$, then by the similar method to Claim 3, we have $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$. Hence we may assume that $f(u_3) = 0$. This forces $f(u_4) \geq 2$. If $f(u_4) = 3$, then by the similar method to Claim 4, we have $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$. Thus it suffices for us to consider the last case when $f(u_4) = 2$. In this case, by the similar method to Case 1 of this claim, we have $\gamma_{tdR}(T) > \frac{6(n(T)-l(T)+2)}{5}$.

By the above discussions, Claim 7 holds. □

The proof is completed. ■

4. Proof of Theorem 3

We now give a proof of Theorem 3.

Proof. We proceed by induction on the number $n(T)$. If $n(T) = 3$, that is, if $T = P_3 \notin \mathcal{T}$, then $\gamma_{tdR}(T) = 4 < \frac{6n(T)+3s(T)}{5}$. If $T = P_4$, then $T \in \mathcal{T}$ and $\gamma_{tdR}(T) = 6 = \frac{6n(T)+3s(T)}{5}$. If $T = S_4$, then $T \notin \mathcal{T}$ and $\gamma_{tdR}(T) = 4 < \frac{6n(T)+3s(T)}{5}$. Assume, then, that $n(T) \geq 5$ and that for any tree T' of order $n(T')$ with $s(T')$ support vertices and $3 \leq n(T') < n(T)$, we have $\gamma_{tdR}(T') \leq \frac{6n(T')+3s(T')}{5}$ with equality if and only if $T' \in \mathcal{T}$. Now let T be a tree of order $n(T) \geq 5$ with $s(T)$ support vertices. If $\mathrm{diam}(T) = 2$, that is, if $T = S_n$, then $T \notin \mathcal{T}$ and $\gamma_{tdR}(T) = 4 < \frac{6n(T)+3s(T)}{5}$. If $\mathrm{diam}(T) = 3$, that is, if T is a double star, then $T \notin \mathcal{T}$ since $n(T) \geq 5$ and $\gamma_{tdR}(T) = 6 < \frac{6n(T)+3s(T)}{5}$. So in the following we may assume that $\mathrm{diam}(T) \geq 4$.

Claim 1. *If T has a strong support vertex, then $\gamma_{tdR}(T) < \frac{6n(T)+3s(T)}{5}$.*

Proof. Let v be a strong support vertex of T, u be a leaf adjacent to v, $T' = T - u$ and let f' be a $\gamma_{tdR}(T')$-function. Note that v is a support vertex of T'. If v is a weak support vertex of T' adjacent to a unique leaf w, then by Observation 9, we may assume that $f'(v) + f'(w) = 3$ and we may assume, without loss of generality, that $f'(w) < f'(v)$; and if v is a strong support vertex of T', then by Observation 8, we may assume that $f'(v) = 3$ and $f'(x) = 0$ for each leaf x adjacent to v in T'. In either case, it is easy to see that the function f defined by $f(u) = 1$ and $f(x) = f'(x)$ for each $x \in V(T')$, is a TDRDF on T. Note that $n(T) = n(T') + 1$ and $s(T) = s(T')$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &\leq f(V(T)) = f'(V(T')) + f(u) \leq \frac{6n(T') + 3s(T')}{5} + 1 \\ &= \frac{6(n(T)-1) + 3s(T)}{5} + 1 < \frac{6n(T) + 3s(T)}{5},\end{aligned}$$

and so Claim 1 holds. □

By Claim 1, we may assume that every support vertex of T is a weak support vertex. If T is a path, then clearly $T \notin \mathcal{T}$ since $n(T) \geq 5$ and so by Proposition 10, we have $\gamma_{tdR}(T) = \left\lceil \frac{6n(T)}{5} \right\rceil < \frac{6n(T)+3s(T)}{5}$, as desired. Hence we may assume that T is not a path, that is, there exists some vertex of degree at least 3 in T. Let $P = v_1 v_2 \cdots v_{d+1}$ be a diametral path of T, where $d = \mathrm{diam}(T) \geq 4$, such that v_k is the first vertex of P with $d_T(v_k) \geq 3$. By our earlier assumption, we note that $k \geq 3$.

Claim 2. *If $k = 3$, then $\gamma_{tdR}(T) \leq \frac{6n(T)+3s(T)}{5}$ with equality if and only if $T \in \mathcal{T}$.*

Proof. Let $T' = T - \{v_1, v_2\}$ and let f' be a $\gamma_{tdR}(T')$-function. Then the function f defined by $f(v_1) = 1$, $f(v_2) = 2$ and $f(x) = f'(x)$ for each $x \in V(T')$, is a TDRDF on T. Note that $n(T) = n(T') + 2$ and $s(T) = s(T') + 1$. If $T' \notin \mathcal{T}$, then clearly $T \notin \mathcal{T}$ and by the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &\leq f(V(T)) = f'(V(T')) + f(v_1) + f(v_2) < \frac{6n(T') + 3s(T')}{5} + 3 \\ &= \frac{6(n(T)-2) + 3(s(T)-1)}{5} + 3 = \frac{6n(T) + 3s(T)}{5}.\end{aligned}$$

Suppose now that $T' \in \mathcal{T}$. Noticing that $d_T(v_3) \geq 3$, we have $T \in \mathcal{T}$. Let f be a $\gamma_{tdR}(T)$-function. By Observation 9, we may assume that $f(u) + f(v) = 3$ for any support vertex v of T and a unique leaf u adjacent to v. Moreover, since $n(T) = 2s(T)$, we have $\gamma_{tdR}(T) = 3s(T) = \frac{6n(T)+3s(T)}{5}$. Thus Claim 2 holds. $\square$

Claim 3. *If $k = 4$, then $\gamma_{tdR}(T) < \frac{6n(T)+3s(T)}{5}$.*

Proof. Let $T' = T - \{v_1, v_2, v_3\}$ and let f' be a $\gamma_{tdR}(T')$-function. Then the function f defined by $f(v_1) = f(v_3) = 1$, $f(v_2) = 2$ and $f(x) = f'(x)$ for each $x \in V(T')$, is a TDRDF on T. Note that $n(T) = n(T') + 3$ and $s(T) = s(T') + 1$. By the induction hypothesis, we have

$$\begin{aligned} \gamma_{tdR}(T) &\leq f(V(T)) = f'(V(T')) + \sum_{i=1}^{3} f(v_i) \leq \frac{6n(T') + 3s(T')}{5} + 4 \\ &= \frac{6(n(T) - 3) + 3(s(T) - 1)}{5} + 4 < \frac{6n(T) + 3s(T)}{5} \end{aligned}$$

and so Claim 3 holds. $\square$

Claim 4. *If $k = 5$, then $\gamma_{tdR}(T) < \frac{6n(T)+3s(T)}{5}$.*

Proof. First, suppose that v_5 is a weak support vertex in T. Let $T' = T - \{v_1, v_2, v_3, v_4\}$ and let f' be a $\gamma_{tdR}(T')$-function. Since v_5 is a weak support vertex in T, we have that v_5 is also a weak support vertex in T'. Let $u \in V(T) \backslash V(P)$ be the unique leaf adjacent to v_5. By Observation 9, we may assume that $f'(v_5) + f'(u) = 3$. Further, we may assume that $f'(v_5) = 2$ and $f'(u) = 1$, or $f'(v_5) = 3$ and $f'(u) = 0$. In either case, the function f defined by $f(v_1) = 1$, $f(v_2) = f(v_3) = 2$, $f(v_4) = 0$ and $f(x) = f'(x)$ for each $x \in V(T')$, is a TDRDF on T. Note that $n(T) = n(T') + 4$ and $s(T) = s(T') + 1$. By the induction hypothesis, we have

$$\begin{aligned} \gamma_{tdR}(T) &\leq f(V(T)) = f'(V(T')) + \sum_{i=1}^{4} f(v_i) \leq \frac{6n(T') + 3s(T')}{5} + 5 \\ &= \frac{6(n(T) - 4) + 3(s(T) - 1)}{5} + 5 < \frac{6n(T) + 3s(T)}{5}. \end{aligned}$$

Second, suppose that v_5 is not a support vertex in T. Let T_1 be the connected component of $T - v_6$ containing the vertex v_5. If there exists some vertex in $V(T_1) \backslash \{v_5\}$ of degree at least 3 in T_1, then by the similar method to Claim 2 or 3, we have $\gamma_{tdR}(T) < \frac{6n(T)+3s(T)}{5}$. Hence we may assume that every vertex of $V(T_1) \backslash \{v_5\}$ has degree 1 or 2. If there exists some leaf at distance 2 or 3 from v_5 in T_1, then again by the similar method to Claim 2 or 3, we have $\gamma_{tdR}(T) < \frac{6n(T)+3s(T)}{5}$. So in the following we may assume that T_1 is the tree that can be obtained from a star s_t with center v_5 of order $t \geq 3$ by subdividing every edge three times.

Now let $T_2 = T - T_1$ and let f' be a $\gamma_{tdR}(T_2)$-function. Since $P = v_1 v_2 \cdots v_{d+1}$ is a diametral path, this forces that $n(T_2) \geq 4$. Observe that the function f defined by $f(x) = 2$ for every vertex x at distance 1 or 3 from v_5 in T_1, $f(x) = 0$ for every vertex x at distance 2 from v_5 in T_1, $f(x) = 1$ for other vertices x of T_1 and $f(x) = f'(x)$ for $x \in V(T_2)$, is a TDRDF on T. Note that $n(T) = n(T_2) + 4t - 3$ and $s(T_2) + t - 2 \leq s(T)$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &\leq f(V(T)) = f(V(T_1)) + f'(V(T_2)) \leq 5(t-1) + 1 + \frac{6n(T_2) + 3s(T_2)}{5} \\ &\leq 5t - 4 + \frac{6(n(T) - 4t + 3) + 3(s(T) - t + 2)}{5} < \frac{6n(T) + 3s(T)}{5}\end{aligned}$$

and so Claim 4 holds. □

Claim 5. *If $k = 6$, then $\gamma_{tdR}(T) < \frac{6n(T)+3s(T)}{5}$.*

Proof. Let $T' = T - \{v_1, v_2, v_3, v_4, v_5\}$ and let f' be a $\gamma_{tdR}(T')$-function. Then the function f defined by $f(v_1) = f(v_5) = 1$, $f(v_2) = f(v_4) = 2$, $f(v_3) = 0$ and $f(x) = f'(x)$ for any $x \in V(T')$, is a TDRDF on T. Note that $n(T) = n(T') + 5$ and $s(T) = s(T') + 1$. By the induction hypothesis, we have

$$\begin{aligned}\gamma_{tdR}(T) &\leq f(V(T)) = f'(V(T')) + \sum_{i=1}^{5} f(v_i) \leq \frac{6n(T') + 3s(T')}{5} + 6 \\ &= \frac{6(n(T) - 5) + 3(s(T) - 1)}{5} + 6 < \frac{6n(T) + 3s(T)}{5}\end{aligned}$$

and so Claim 5 holds. □

Claim 6. *If $k \in \{7, 8\}$, then $\gamma_{tdR}(T) < \frac{6n(T)+3s(T)}{5}$.*

Proof. Let $T' = T - \{v_1, v_2, v_3, v_4, v_5\}$. First, suppose that $T' \notin \mathcal{T}$. Let f' be a $\gamma_{tdR}(T')$-function. By the induction hypothesis, we have $f'(V(T')) = \gamma_{tdR}(T') < \frac{6n(T')+3s(T')}{5}$. Then the function f defined by $f(v_1) = f(v_5) = 1$, $f(v_2) = f(v_4) = 2$, $f(v_3) = 0$ and $f(x) = f'(x)$ for each $x \in V(T')$, is a TDRDF on T. Note that $n(T) = n(T') + 5$ and $s(T') \leq s(T)$. Thus we have

$$\begin{aligned}\gamma_{tdR}(T) &\leq f(V(T)) = f'(V(T')) + \sum_{i=1}^{5} f(v_i) < \frac{6n(T') + 3s(T')}{5} + 6 \\ &\leq \frac{6(n(T) - 5) + 3s(T)}{5} + 6 = \frac{6n(T) + 3s(T)}{5},\end{aligned}$$

as desired.

Second, suppose that $T' \in \mathcal{T}$. By the induction hypothesis, we have $\gamma_{tdR}(T') = \frac{6n(T')+3s(T')}{5}$. One can verify that the function f' defined by $f'(x) = 2$ for every support vertex of T' and $f'(x) = 1$ for every leaf x of T', is a $\gamma_{tdR}(T')$-function with $f'(V(T')) = \frac{6n(T')+3s(T')}{5}$. In particular, if $k = 7$, then $f'(v_6) = 1$ and $f'(x) = 2$ for each $x \in N_T[v_7]\backslash\{v_6\}$ and if $k = 8$, then $f'(v_6) = 1$ and $f'(v_7) = f'(v_8) = 2$. If $k = 7$, then define a function f on T by $f(v_1) = 1$, $f(v_2) = f(v_3) = f(v_5) = 2$, $f(v_4) = f(v_7) = 0$ and $f(x) = f'(x)$ for each $x \in V(T')\backslash\{v_7\}$. If $k = 8$, then define a function f on T by $f(v_1) = f(v_5) = 1$, $f(v_2) = f(v_4) = f(v_6) = 2$, $f(v_3) = f(v_7) = 0$ and $f(x) = f'(x)$ for each $x \in V(T')\backslash\{v_6, v_7\}$. In either case, one can check that the resulting function f is a TDRDF on T. Note that $n(T) = n(T') + 5$ and $s(T) = s(T')$. Thus we have

$$\begin{aligned}\gamma_{tdR}(T) &\leq f(V(T)) = f'(V(T')) - f'(v_6) - f'(v_7) + \sum_{i=1}^{7} f(v_i)\\ &= \frac{6n(T') + 3s(T')}{5} + 5 = \frac{6(n(T) - 5) + 3s(T)}{5} + 5 < \frac{6n(T) + 3s(T)}{5},\end{aligned}$$

as desired.

By the above discussions, Claim 6 holds. □

Claim 7. *If $k \geq 9$, then $\gamma_{tdR}(T) < \frac{6n(T)+3s(T)}{5}$.*

Proof. Let $T' = T - \{v_1, v_2, v_3, v_4, v_5\}$ and let f' be a $\gamma_{tdR}(T')$-function. Then the function f defined by $f(v_1) = f(v_5) = 1$, $f(v_2) = f(v_4) = 2$, $f(v_3) = 0$ and $f(x) = f'(x)$ for any $x \in V(T')$, is a TDRDF on T. Note that $n(T) = n(T') + 5$ and $s(T) = s(T')$. Noticing that $T' \notin \mathcal{T}$, it follows from the induction hypothesis that

$$\begin{aligned}\gamma_{tdR}(T) &\leq f(V(T)) = f'(V(T')) + \sum_{i=1}^{5} f(v_i) < \frac{6n(T') + 3s(T')}{5} + 6\\ &= \frac{6(n(T) - 5) + 3s(T)}{5} + 6 = \frac{6n(T) + 3s(T)}{5}\end{aligned}$$

and so Claim 7 holds. □

The proof is completed. ■

5. Proof of Theorem 6

First we recall a result proved in [8].

Proposition 11 [8]**.** *For $n \geq 3$, $\gamma_{tdR}(C_n) = \lceil \frac{6n}{5} \rceil$.*

The proof of Theorem 6 is based on the following two lemmas. For positive integers $k \geq 3$ and $l \geq 1$, let $C_{k,l}$ be the graph obtained from a cycle $C_k = x_1x_2\cdots x_kx_1$ by adding a pendant path $x_1y_1y_2\cdots y_l$.

Lemma 12. For positive integers $k \geq 3$ and $l \geq 1$, $\gamma_{tdR}(C_{k,l}) \leq \frac{4(k+l)}{3}$.

Proof. If $k+l=4$, then obviously $\gamma_{tdR}(C_{k,l}) = 4 < \frac{4(k+l)}{3}$. Let $k+l \geq 5$. Since adding an edge cannot increase the total double Roman domination number of the path $x_2\cdots x_kx_1y_1\cdots y_l$, Proposition 10 implies that

$$\gamma_{tdR}(C_{k,l}) \leq \gamma_{tdR}(P_{k+l}) \leq \left\lceil \frac{6(k+l)}{5} \right\rceil \leq \frac{4(k+l)}{3},$$

as desired. ■

Now define $\mathcal{R}_1$ to be the family of all connected loopless multigraphs with minimum degree at least three and define $\mathcal{R}$ to be the family of all graphs that can be obtained from a graph in $\mathcal{R}_1$ by subdividing every edge t times, where $t \in \{1,2,4\}$. Observe that any graph in $\mathcal{R}$ has order at least five.

We shall adopt the following definitions and notations. Let G be a simple graph with minimum degree at least two and let $M = \{x \in V(G) : d(x) \geq 3\}$. An M*-ear path* in G is a path $P = v_1v_2\cdots v_k$ such that $\{sv_1, v_kr\} \subseteq E(G)$ and $d(v_i) = 2$ for $1 \leq i \leq k$, where $s, r \in M$. And we also say that "s is connected to the M-ear path P". Let $\mathcal{P}_i = \{P : P$ is an M-ear path of order i in $G\}$ for each $i \geq 1$ and let $\mathcal{P} = \bigcup_{i\geq 1} \mathcal{P}_i$. If $G \in \mathcal{R}$, then we have $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_4$.

Lemma 13. *Every graph $G \in \mathcal{R}$ of order n has a* TDRDF *h that assigns two to every vertex of degree at least three and $\omega(h) \leq \frac{4n}{3}$.*

Proof. We proceed by induction on n. The result is immediate for $n = 5$. Assume, then, that $n \geq 6$ and that the result holds for all graphs $G' \in \mathcal{R}$ of order less than n. Let $G \in \mathcal{R}$ be a graph of order n and let $M = \{x \in V(G) : d(x) \geq 3\}$.

Assume now that there exists an M-ear path $x_1x_2x_3x_4 \in \mathcal{P}_4$ such that $sx_1, x_4r \in E(G)$, where $s, r \in M$. Let G' be the graph obtained from $G - \{x_2, x_3, x_4\}$ by adding a new edge x_1r. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(s) = g(r) = 2$. It is easy to check that the function h defined by $h(x_3) = h(x_4) = 1$, $h(x_2) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) = \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

So in the following we may assume that $\mathcal{P}_4 = \emptyset$, implying that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$. Suppose that $|M| = 2$. It is easy to check that the function h defined by $h(x) = 2$ for each $x \in M$ and $h(x) = 1$ for each $x \in V(G)\backslash M$, is a TDRDF on G and $\omega(h) = n + 2 \leq \frac{4n}{3}$, as desired.

Henceforth, we may assume that $|M| \geq 3$. Suppose next that there exist two vertices $s, r \in M$ with $\min\{d(s), d(r)\} \geq 4$ such that $P = x_1 \in \mathcal{P}_1$ (or $P = x_1x_2 \in \mathcal{P}_2$) is an M-ear path in G, where $sx_1, x_1r \in E(G)$ (or $sx_1, x_2r \in E(G)$). Let $G' = G - V(P)$. Obviously, every connected component of G' belongs to $\mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4|V(G')|}{3} = \frac{4(n-|V(P)|)}{3}$. In particular, $g(s) = g(r) = 2$. It is easy to check that the function h defined by $h(x) = 1$ for each $x \in V(P)$ and $h(x) = g(x)$ for each $x \in V(G)\backslash V(P)$, is a TDRDF on G and $\omega(h) = \omega(g) + |V(P)| \leq \frac{4(n-|V(P)|)}{3} + |V(P)| < \frac{4n}{3}$, as desired.

Hence we may assume that for any two vertices $s, r \in M$, if there exists an M-ear path $x \in \mathcal{P}_1$ (or $xy \in \mathcal{P}_2$) such that $sx, xr \in E(G)$ (or $sx, yr \in E(G)$), then one of $d(s)$ and $d(r)$ is equal to 3 and the other is equal to at least 3. We now have the following claims.

Claim 1. *If $\mathcal{P}_2 \neq \emptyset$, then G has a* TDRD *h that assigns two to every vertex of degree at least three and $\omega(h) \leq \frac{4n}{3}$.*

Proof. Let x_1x_2 be an M-ear path in $\mathcal{P}_2$ such that $sx_1, x_2r \in E(G)$, where $s, r \in M$. From our earlier assumption, we may assume that $d(s) = 3$ and $d(r) \geq 3$.

Case 1. s is connected to three M-ear paths in $\mathcal{P}_2$. Let y_1y_2 and z_1z_2 be two M-ear paths in $\mathcal{P}_2$ different from the M-ear path x_1x_2, such that sy_1, sz_1, y_2t, z_2t' $\in E(G)$, where $t, t' \in M\backslash\{s\}$.

First, suppose that $|\{r, t, t'\}| \in \{2, 3\}$. Without loss of generality, assume that $t' \notin \{r, t\}$. This implies that r, t and t' are distinct or $r = t \neq t'$. Let G' be the graph obtained from $G - \{s, y_1, z_1\}$ by adding two new edges x_1t' and y_2z_2. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(r) = g(t) = g(t') = 2$. Moreover, since y_2 and z_2 have degree two in G', $g(y_2) + g(z_2) \geq 2$. Then the function h defined by $h(y_1) = h(y_2) = h(z_1) = h(z_2) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

Second, suppose that $r = t = t'$. Since $|M| \geq 3$, there exists an M-ear path w_1w_2 in $\mathcal{P}_2$ such that $rw_1, w_2t'' \in E(G)$ or there exists an M-ear path w in $\mathcal{P}_1$ such that $rw, wt'' \in E(G)$, where $t'' \in M\backslash\{s, r\}$. Assume now that the former holds. Let G' be the graph obtained from $G - \{s, x_1, x_2\}$ by adding two new edges y_1t'' and z_1t''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(r) = g(t'') = 2$. Moreover, since w_1 and w_2 have degree two in G', $g(w_1) + g(w_2) \geq 2$. Then the function h defined by $h(x_1) = h(x_2) = h(w_1) = h(w_2) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

So in the following we may assume that there exists an M-ear path w in $\mathcal{P}_1$ such that $rw, wt'' \in E(G)$, where $t'' \in M\backslash\{s, r\}$. Suppose now that $d_G(r) \geq 5$. Let G' be the graph obtained from $G - \{s, x_1, x_2, y_1, y_2, z_1\}$ by adding a new edge z_2t''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-6)}{3}$. In particular, $g(r) = g(t'') = 2$. Then the function h defined by $h(x_1) = h(x_2) = h(y_1) = h(y_2) = h(z_1) = h(z_2) = 1$, $h(w) = \max\{g(z_2), g(w)\}$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 8 \leq \frac{4(n-6)}{3} + 8 = \frac{4n}{3}$, as desired.

Suppose next that $d_G(r) = 4$. Since $d_G(t'') \geq 3$, there exists an M-ear path $P = w_1'w_2' \in \mathcal{P}_2$ such that $t''w_1', w_2't''' \in E(G)$ or there exists an M-ear path $P = w' \in \mathcal{P}_1$ such that $t''w', w't''' \in E(G)$, where $t''' \in M\backslash\{s, r, t''\}$. Let G' be the graph obtained from $G - \{s, x_1, x_2, y_1, y_2, z_1, z_2\}$ by adding a new edge rt'''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-7)}{3}$. In particular, $g(t'') = g(t''') = 2$. Moreover, $g(r) + g(w) \geq 2$ since r and w have degree two in G' and if $P \in \mathcal{P}_2$, then $g(w_1') + g(w_2') \geq 2$ since w_1' and w_2' have degree two in G'. Then the function h defined by $h(w) = 0$, $h(x_1) = h(x_2) = h(y_1) = h(y_2) = h(z_1) = h(z_2) = h(x) = 1$ for each $x \in V(P)$, $h(s) = h(r) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 9 \leq \frac{4(n-7)}{3} + 9 < \frac{4n}{3}$, as desired.

Case 2. s is connected to two M-ear paths in $\mathcal{P}_2$ and an M-ear path in $\mathcal{P}_1$. Let y_1y_2 be an M-ear path in $\mathcal{P}_2$ different from the M-ear path x_1x_2 and let z be an M-ear path in $\mathcal{P}_1$ such that $sy_1, sz, y_2t, zt' \in E(G)$, where $t, t' \in M\backslash\{s\}$.

Subcase 2.1. r, t and t' are distinct. Let G' be the graph obtained from $G - \{s, x_1, x_2,\}$ by adding two new edges y_1r and zt. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(r) = g(t) = g(t') = 2$. Moreover, since y_1 and y_2 have degree two in G', $g(y_1) + g(y_2) \geq 2$. Then the function h defined by $h(x_1) = h(x_2) = h(y_1) = h(y_2) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g)+4 \leq \frac{4(n-3)}{3}+4 = \frac{4n}{3}$, as desired.

Subcase 2.2. $r = t \neq t'$. Suppose now that $d_G(r) \geq 4$. Let G' be the graph obtained from $G-\{s, x_1, x_2, y_1\}$ by adding a new edge y_2z. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-4)}{3}$. In particular, $g(r) = g(t') = 2$. Moreover, since y_2 and z have degree two in G', $g(y_2)+g(z) \geq 2$. Then the function h defined by $h(x_1) = h(x_2) = h(y_1) = h(y_2) = h(z) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 5 \leq \frac{4(n-4)}{3} + 5 < \frac{4n}{3}$, as desired.

Suppose next that $d_G(r) = 3$. Since $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2$, there exists an M-ear path $w_1w_2 \in \mathcal{P}_2$ such that $rw_1, w_2t'' \in E(G)$ or there exists an M-ear path $w \in \mathcal{P}_1$ such that $rw, wt'' \in E(G)$, where $t'' \in M \backslash \{s, r\}$. If the former holds, then by the method similar to Case 1, the assertion is trivial. Hence we may assume that the latter holds.

Now assume that $t'' \neq t'$. Let G' be the graph obtained from $G - \{s, x_1, x_2, y_1, y_2, z\}$ by adding a new edge rt'. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-6)}{3}$. In particular, $g(t') = g(t'') = 2$. Moreover, since r and w have degree two in G', $g(r) + g(w) \geq 2$. Then the function h defined by $h(x_1) = h(x_2) = h(y_1) = h(y_2) = h(z) = h(w) = 1$, $h(s) = h(r) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 8 \leq \frac{4(n-6)}{3} + 8 = \frac{4n}{3}$, as desired.

So in the following we may assume that $t'' = t'$. Since $t' \in M$, $d_G(t') \geq 3$ and hence there exists an M-ear path $P = w_1'w_2' \in \mathcal{P}_2$ such that $t'w_1', w_2't''' \in E(G)$ or there exists an M-ear path $P = w' \in \mathcal{P}_1$ such that $t'w', w't''' \in E(G)$, where $t''' \in M \backslash \{s, r, t'\}$. Let G' be the graph obtained from $G - \{x_1, x_2, y_1, y_2\}$ by adding two new edges st''' and rt'''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-4)}{3}$. In particular, $g(t') = g(t''') = 2$. Moreover, since s, z, r, w and every vertex of $V(P)$ have degree two in G', we have $\min\{g(s) + g(z), g(r) + g(w)\} \geq 2$ and if $P \in \mathcal{P}_2$, then $g(w_1') + g(w_2') \geq 2$. It is easy to check that the function h defined by $h(z) = h(w) = 0$, $h(x_1) = h(x_2) = h(y_1) = h(y_2) = h(x) = 1$ for each $x \in V(P)$, $h(s) = h(r) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 5 \leq \frac{4(n-4)}{3} + 5 < \frac{4n}{3}$, as desired.

Subcase 2.3. $r = t' \neq t$ (the case $t = t' \neq r$ is similar). Let G' be the graph obtained from $G - \{s, x_1, x_2\}$ by adding two new edges zt and y_1r. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(r) = g(t) = 2$. Moreover, since y_1 and y_2 have degree two in G', $g(y_1) + g(y_2) \geq 2$. Then the function h defined by $h(z) = 0$, $h(x_1) = h(x_2) = h(y_1) = h(y_2) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

Subcase 2.4. $r = t = t'$. Since $|M| \geq 3$, there exists an M-ear path $P = w_1w_2 \in \mathcal{P}_2$ such that $rw_1, w_2t'' \in E(G)$ or there exists an M-ear path $P = w \in \mathcal{P}_1$ such that $rw, wt'' \in E(G)$, where $t'' \in M \backslash \{s, r\}$.

Suppose now $d_G(r) \geq 5$. Let G' be the graph obtained from $G - \{s, x_1, x_2, y_1, y_2\}$ by adding a new edge zt''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-5)}{3}$. In particular, $g(r) = g(t'') = 2$. Define the function h

by $h(z) = 0$, $h(x_1) = h(x_2) = h(y_1) = h(y_2) = 1$, $h(w_2) = \max\{g(z), g(w_2)\}$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x when $P \in \mathcal{P}_2$; and by $h(z) = 0$, $h(x_1) = h(x_2) = h(y_1) = h(y_2) = 1$, $h(w) = \max\{g(z), g(w)\}$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x when $P \in \mathcal{P}_1$. In either case, it is easy to check that h is a TDRDF on G and $\omega(h) \leq \omega(g) + 6 \leq \frac{4(n-5)}{3} + 6 < \frac{4n}{3}$, as desired.

So in the following we may assume that $d_G(r) = 4$. First, assume that $P = w_1w_2 \in \mathcal{P}_2$. Let G' be the graph obtained from $G - \{s, x_1, x_2\}$ by adding two new edges y_1t'' and zt''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(r) = g(t'') = 2$. Moreover, since w_1 and w_2 have degree two in G', we have $g(w_1) + g(w_2) \geq 2$. Then the function h defined by $h(z) = 0$, $h(x_1) = h(x_2) = h(w_1) = h(w_2) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

Second, assume that $P = w \in \mathcal{P}_1$. Since $t'' \in M\backslash\{s, r\}$, $d_G(t'') \geq 3$ and hence there exists an M-ear path $P' = w'_1w'_2 \in \mathcal{P}_2$ such that $t''w'_1, w'_2t''' \in E(G)$ or there exists an M-ear path $P' = w' \in \mathcal{P}_1$ such that $t''w', w't''' \in E(G)$, where $t''' \in M\backslash\{s, r, t''\}$. Let G' be the graph obtained from $G - \{s, x_1, x_2, y_1, y_2, z\}$ by adding a new edge rt'''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-6)}{3}$. In particular, $g(t'') = g(t''') = 2$. Moreover, since r, w and every vertex of $V(P')$ have degree two in G', we have $g(r) + g(w) \geq 2$ and $g(w'_1) + g(w'_2) \geq 2$ when $P' \in \mathcal{P}_2$. Then the function h defined by $h(z) = h(w) = 0$, $h(x_1) = h(x_2) = h(y_1) = h(y_2) = 1$, $h(x) = 1$ for each $x \in V(P')$, $h(s) = h(r) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 7 \leq \frac{4(n-6)}{3} + 7 < \frac{4n}{3}$, as desired.

Case 3. s is connected to an M-ear path in $\mathcal{P}_2$ and two M-ear paths in $\mathcal{P}_1$. Let y and z be two M-ear paths in $\mathcal{P}_1$ such that $sy, sz, yt, zt' \in E(G)$, where $t, t' \in M\backslash\{s\}$. First, suppose that r, t and t' are distinct. Let G' be the graph obtained from $G - \{s, x_1, x_2\}$ by adding two new edges yr and zr. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(r) = g(t) = g(t') = 2$. Observe that the function h defined by $h(x_1) = h(x_2) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is an TDRDF on G and $\omega(h) = \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

Second, suppose that $r = t \neq t'$ (the case $r = t' \neq t$ is similar). Let G' be the graph obtained from $G - s$ by adding two new edges x_1t' and yz. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-1)}{3}$. In particular, $g(r) = g(t') = 2$. Moreover, since x_1, x_2, y and z have degree two in G', $g(x_1) + g(x_2) \geq 2$

and $g(y) + g(z) \geq 2$. Observe that the function h defined by $h(y) = 0$, $h(x_1) = h(x_2) = h(z) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is an TDRDF on G and $\omega(h) \leq \omega(g) + 1 \leq \frac{4(n-1)}{3} + 1 < \frac{4n}{3}$, as desired.

Third, suppose that $t = t' \neq r$. Let G' be the graph obtained from $G - \{s, x_1, x_2\}$ by adding two new edges yr and zr. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(r) = g(t) = 2$. Observe that the function h defined by $h(x_1) = h(x_2) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is an TDRDF on G and $\omega(h) = \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

Finally, assume that $r = t = t'$. Since $|M| \geq 3$, there exists an M-ear path $P = w_1w_2 \in \mathcal{P}_2$ such that $rw_1, w_2t'' \in E(G)$ or there exists an M-ear path $P = w \in \mathcal{P}_1$ such that $rw, wt'' \in E(G)$, where $t'' \in M \backslash \{s, r\}$. Let G' be the graph obtained from $G - \{s, x_1, x_2\}$ by adding two new edges yt'' and zt''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(r) = g(t'') = 2$. Define the function h by $h(y) = h(z) = 0$, $h(x_1) = h(x_2) = 1$, $h(w_2) = \max\{g(y), g(z), g(w_2)\}$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x when $P \in \mathcal{P}_2$; and by $h(y) = h(z) = 0$, $h(x_1) = h(x_2) = 1$, $h(w) = \max\{g(y), g(z), g(w)\}$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x when $P \in \mathcal{P}_1$. In either case, it is easy to check that h is a TDRDF on G and $\omega(h) \leq \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

By the above arguments, Claim 1 is true. □

Claim 2. *If $\mathcal{P} = \mathcal{P}_1$, then G has a* TDRDF *h that assigns two to every vertex of degree at least three and $\omega(h) \leq \frac{4n}{3}$.*

Proof. Let x_1, y_1 and z_1 be three M-ear paths in $\mathcal{P}_1$ such that sx_1, sy_1, sz_1, x_1r, y_1t and $z_1t' \in E(G)$, where $s, r, t, t' \in M$. From our earlier assumption, we may assume, without loss of generality, that $d(s) = 3$.

Case 1. $|\{r, t, t'\}| = 2$. Without loss of generality, assume that $r = t \neq t'$. First, suppose that $d_G(r) = 3$ and there exists some vertex, say w_1, in G that is adjacent to r and t'. Since $t' \in M$, we have $d_G(t') \geq 3$ and hence there exists some vertex, say w_2, in G that is adjacent to t' and t'', where $t'' \in M \backslash \{s, r, t'\}$. Let G' be the graph obtained from $G - \{s, x_1, y_1\}$ by adding two new edges rt'' and z_1t''. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-3)}{3}$. In particular, $g(t') = g(t'') = 2$. Moreover, since r and w_1 have degree two in G', $g(r) + g(w_1) \geq 2$. Then the function h defined by $h(y_1) = h(z_1) = h(w_1) = 0$, $h(x_1) = h(w_2) = 1$, $h(s) = h(r) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

Second, suppose that $d_G(r) = 3$ and there exists some vertex, say w, in G that is adjacent to r and t'', where $t'' \in M\backslash\{s, r, t'\}$. Let G' be the graph obtained from $G - \{s, x_1, y_1, r\}$ by adding a new edge wz_1. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \le \frac{4(n-4)}{3}$. In particular, $g(t') = g(t'') = 2$. Then the function h defined by $h(x_1) = 0$, $h(y_1) = 1$, $h(s) = h(r) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) = \omega(g) + 5 \le \frac{4(n-4)}{3} + 5 < \frac{4n}{3}$, as desired.

Finally, suppose that $d_G(r) \ge 4$. Let G' be the graph obtained from $G - \{s, x_1, y_1\}$ by adding a new edge rz_1. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \le \frac{4(n-3)}{3}$. In particular, $g(r) = g(t') = 2$. Then the function h defined by $h(x_1) = 0$, $h(y_1) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) = \omega(g) + 3 \le \frac{4(n-3)}{3} + 3 < \frac{4n}{3}$, as desired.

Case 2. r, t and t' are distinct. Assume now that $d_G(r) \ge 4$. Let G' be the graph obtained from $G - \{s, x_1, y_1\}$ by adding a new edge z_1t. Obviously, every connected component of G' belongs to $\mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \le \frac{4(n-3)}{3}$. In particular, $g(r) = g(t) = g(t') = 2$. Then the function h defined by $h(x_1) = 0$, $h(y_1) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) = \omega(g) + 3 \le \frac{4(n-3)}{3} + 3 < \frac{4n}{3}$, as desired.

So in the following we may assume that $d_G(r) = 3$. Let w_1 and w_2 be two M-ear paths in $\mathcal{P}_1$ such that rw_1, rw_2, w_1t'' and $w_2t''' \in E(G)$, where $t'', t''' \in M\backslash\{s, r\}$. If $t'' = t'''$, then by the method similar to Case 1 of Claim 2, the assertion is trivial. Hence we may assume that $t'' \neq t'''$.

First, suppose that $\{t'', t'''\} \subseteq M\backslash\{s, r, t, t'\}$. Let G' be the graph obtained from $G - \{s, x_1, r\}$ by adding two new edges y_1z_1 and w_1w_2. Obviously, every connected component of G' belongs to $\mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \le \frac{4(n-3)}{3}$. In particular, $g(t) = g(t') = g(t'') = g(t''') = 2$. Moreover, since y_1, z_1, w_1 and w_2 have degree two in G', $g(y_1) + g(z_1) \ge 2$ and $g(w_1) + g(w_2) \ge 2$. Then the function h defined by $h(x_1) = 0$, $h(y_1) = h(z_1) = h(w_1) = h(w_2) = 1$, $h(s) = h(r) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \le \omega(g) + 4 \le \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

Second, suppose that $t'' \in \{t, t'\}$ and $t''' \in M\backslash\{s, r, t, t'\}$ (the case $t''' \in \{t, t'\}$ and $t'' \in M\backslash\{s, r, t, t'\}$ is similar). Without loss of generality, assume that $t'' = t$ and $t''' \in M\backslash\{s, r, t, t'\}$. Let G' be the graph obtained from $G - \{s, x_1, r\}$ by adding two new edges y_1z_1 and w_1w_2. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \le \frac{4(n-3)}{3}$. In particular, $g(t) = g(t') = g(t''') = 2$. Moreover, since

y_1, z_1, w_1 and w_2 have degree two in G', $g(y_1)+g(z_1) \geq 2$ and $g(w_1)+g(w_2) \geq 2$. Then the function h defined by $h(x_1) = 0$, $h(y_1) = h(z_1) = h(w_1) = h(w_2) = 1$, $h(s) = h(r) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$, as desired.

Finally, assume that $t'' = t$ and $t''' = t'$ (the case $t'' = t'$ and $t''' = t$ is similar). Let G' be the graph obtained from $G - s$ by adding two new edges x_1t and y_1z_1. Obviously, $G' \in \mathcal{R}$ and by the induction hypothesis, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-1)}{3}$. In particular, $g(r) = g(t) = g(t') = 2$. Moreover, $g(y_1) + g(z_1) \geq 2$ since y_1 and z_1 have degree two in G' and $g(x_1)+g(w_1)+g(w_2) \geq 1$ since $N_{G'}(r) = \{x_1, w_1, w_2\}$. Then the function h defined by $h(x_1) = h(y_1) = h(w_2) = 0$, $h(z_1) = h(w_1) = 1$, $h(s) = 2$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and $\omega(h) \leq \omega(g) + 1 \leq \frac{4(n-1)}{3} + 1 < \frac{4n}{3}$, as desired.

Case 3. $r = t = t'$. From our earlier assumptions, we note that $\mathcal{P} = \mathcal{P}_1$, $|M| \geq 3$ and for any two vertices $s, r \in M$, if there exists an M-ear path $x \in \mathcal{P}_1$ (or $xy \in \mathcal{P}_2$) such that $sx, xr \in E(G)$ (or $sx, yr \in E(G)$), then one of $d(s)$ and $d(r)$ is equal to 3 and the other is equal to at least 3. Noticing Cases 1 and 2, it suffices to consider the graph G, where $V(G) = \{u\} \cup \{v_i : 1 \leq i \leq k\} \cup \{w_i^j : 1 \leq i \leq k$ and $1 \leq j \leq 3\}$ and $E(G) = \{uw_i^j, v_iw_i^j : 1 \leq i \leq k$ and $1 \leq j \leq 3\}$. Observe that $n = 4k + 1$. It is easy to check that the function h defined by $h(u) = h(v_i) = 2$ for $1 \leq i \leq k$, $h(w_i^1) = 1$ for $1 \leq i \leq k$ and $h(x) = 0$ for other vertices x, is a TDRDF on G and $\omega(h) = 3k + 2 < \frac{4n}{3}$, as desired.

By the above arguments, Claim 2 is true. □

The proof is completed. ■

We are now in a position to present a proof of Theorem 6.

Proof. Since $\gamma_{tdR}(G)$ is integer, it suffices to show that $\gamma_{tdR}(G) \leq \frac{4n}{3}$. We proceed by induction on $n + |E(G)|$. If $n + |E(G)| = 6$, then $G = C_3$ and the result is trivial. Assume, then, that $n + |E(G)| \geq 7$ and that the result holds for all connected graphs G' with $|V(G')| + |E(G')| < n + |E(G)|$ and $\delta(G') \geq 2$. Let G be a connected graph of order n with $\delta(G) \geq 2$ and let $M = \{x \in V(G) : d(x) \geq 3\}$. If $M = \emptyset$, then G is a cycle and hence by Proposition 11, we have $\gamma_{tdR}(G) = \left\lceil \frac{6n}{5} \right\rceil \leq \frac{4n}{3}$. Suppose next that $M \neq \emptyset$.

If there exists an edge $e \in E(G)$ joining two vertices in M such that $G - e$ is connected, then by the induction hypothesis, $\gamma_{tdR}(G) \leq \gamma_{tdR}(G - e) \leq \frac{4n}{3}$. If there exists an edge $e \in E(G)$ joining two vertices in M such that $G - e$ is disconnected with connected components G_1 and G_2, then by the induction hypothesis, $\gamma_{tdR}(G) \leq \gamma_{tdR}(G_1)+\gamma_{tdR}(G_2) \leq \frac{4|V(G_1)|}{3}+\frac{4|V(G_2)|}{3} = \frac{4n}{3}$. Henceforth, we may assume that M is an independent set. We now have the following claims.

Claim 1. *If there exists an M-ear path P in $\mathcal{P}$ whose leaves are adjacent to the same vertex, say s, in M such that $d_G(s) = 3$, then $\gamma_{tdR}(G) \leq \frac{4n}{3}$.*

Proof. Since G is simple, P has order at least two. Let $N_G(s)\backslash V(P) = \{t\}$. Then there exists a unique M-ear path P' such that t is a leaf of P'. Let $G' = G - (V(P) \cup V(P') \cup \{s\})$. Then $\delta(G') \geq 2$ and hence by the induction hypothesis, $\gamma_{tdR}(G') \leq \frac{4|V(G')|}{3}$. On the other hand, since $G[V(P) \cup V(P') \cup \{s\}] \cong C_{|V(P)|+1,|V(P')|}$, we have $\gamma_{tdR}(G[V(P) \cup V(P') \cup \{s\}]) \leq \frac{4|V(P)\cup V(P')\cup\{s\}|}{3}$ by Lemma 12. As a result, we have

$$\begin{aligned}\gamma_{tdR}(G) &\leq \gamma_{tdR}(G') + \gamma_{tdR}(G[V(P) \cup V(P') \cup \{s\}])\\ &\leq \frac{4|V(G')|}{3} + \frac{4|V(P) \cup V(P') \cup \{s\}|}{3} = \frac{4n}{3}\end{aligned}$$

and hence Claim 1 is true. □

Therefore for any M-ear path $P \in \mathcal{P}$, if two leaves of P are adjacent to the same vertex, say s, in M, then we may assume that $d_G(s) \geq 4$.

Claim 2. *If there exists an M-ear path P in $\mathcal{P}_k$ ($k = 3$ or $k \geq 5$), then $\gamma_{tdR}(G) \leq \frac{4n}{3}$.*

Proof. Let $G' = G - V(P)$. Then every connected component of G' has minimum degree at least two and hence by Proposition 10 and the induction hypothesis, we have

$$\gamma_{tdR}(G) \leq \gamma_{tdR}(P) + \gamma_{tdR}(G') \leq \left\lceil \frac{6|V(P)|}{5} \right\rceil + \frac{4|V(G')|}{3} \leq \frac{4n}{3}$$

and hence Claim 2 holds. □

By Claim 2, we may assume that $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_4$ in the following.

Claim 3. *If two leaves of any M-ear path are adjacent to distinct vertices in M, then $\gamma_{tdR}(G) \leq \frac{4n}{3}$.*

Proof. Observe that $G \in \mathcal{R}$ and hence by Lemma 13, we have $\gamma_{tdR}(G) \leq \frac{4n}{3}$ and so Claim 3 holds. □

For $k \in \{2, 4\}$, let $\mathcal{P}'_k = \{P \in \mathcal{P}_k$: two leaves of P are adjacent to the same vertex in $M\}$. Note that G is simple. By Claim 3, we may assume that $\mathcal{P}'_2 \cup \mathcal{P}'_4 \neq \emptyset$.

Claim 4. *If there are two M-ear paths P and P' in $\mathcal{P}'_2 \cup \mathcal{P}'_4$ such that every leaf of P and P' is adjacent to the same vertex, say s, in M, then $\gamma_{tdR}(G) \leq \frac{4n}{3}$.*

Proof. Let $G' = G - V(P')$. Note that $\delta(G') \geq 2$. By the induction hypothesis, we have $\gamma_{tdR}(G') \leq \frac{4(n-|V(P')|)}{3}$. Let g be a $\gamma_{tdR}(G')$-function. First, suppose that $P = u_1u_2 \in \mathcal{P}'_2$. Observe that $g(u_1) + g(u_2) + g(s) \geq 3$. Define the function h by $h(u_1) = h(u_2) = 0$, $h(x) = 1$ for each $x \in V(P')$, $h(s) = 3$ and $h(x) = g(x)$ for other vertices x when $P' \in \mathcal{P}'_2$; and by $h(u_1) = h(u_2) = h(v_1) = 0$, $h(v_2) = h(v_4) = 1$, $h(v_3) = 2$, $h(s) = 3$ and $h(x) = g(x)$ for other vertices x when $P' = v_1v_2v_3v_4 \in \mathcal{P}'_4$. In either case, it is easy to check that h is a TDRDF on G and hence $\gamma_{tdR}(G) \leq \omega(h) \leq \omega(g) + |V(P')| \leq \frac{4(n-|V(P')|)}{3} + |V(P')| < \frac{4n}{3}$, as desired.

Second, suppose that $P = u_1u_2u_3u_4 \in \mathcal{P}'_4$. If $g(s) \geq 1$ and $P' \in \mathcal{P}'_2$, then the function h defined by $h(x) = 0$ for each $x \in V(P')$, $h(s) = 3$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and so $\gamma_{tdR}(G) \leq \omega(h) \leq \omega(g) + 2 \leq \frac{4(n-2)}{3} + 2 < \frac{4n}{3}$, as desired. If $g(s) \geq 1$ and $P' = v_1v_2v_3v_4 \in \mathcal{P}'_4$, then the function h defined by $h(v_1) = h(v_4) = 0$, $h(v_2) = 1$, $h(v_3) = 2$, $h(s) = 3$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and so $\gamma_{tdR}(G) \leq \omega(h) \leq \omega(g) + 5 \leq \frac{4(n-4)}{3} + 5 < \frac{4n}{3}$, as desired. Hence we may assume that $g(s) = 0$. One can verify that $g(u_1)+g(u_2)+g(u_3)+g(u_4) \geq 6$. Define the function h by $h(u_1) = h(u_4) = 0$, $h(u_2) = 1$, $h(u_3) = 2$, $h(x) = 1$ for each $x \in V(P')$, $h(s) = 3$ and $h(x) = g(x)$ for other vertices x when $P' \in \mathcal{P}'_2$; and by $h(u_1) = h(u_4) = h(v_1) = 0$, $h(u_2) = h(v_2) = h(v_4) = 1$, $h(u_3) = h(v_3) = 2$, $h(s) = 3$ and $h(x) = g(x)$ for other vertices x when $P' = v_1v_2v_3v_4 \in \mathcal{P}'_4$. In either case, it is easy to check that h is a TDRDF on G and hence $\gamma_{tdR}(G) \leq \omega(h) \leq \omega(g) + |V(P')| \leq \frac{4(n-|V(P')|)}{3} + |V(P')| < \frac{4n}{3}$, as desired.

Thus Claim 4 is true. □

Claim 5. *If there are no two M-ear paths P and P' in $\mathcal{P}'_2 \cup \mathcal{P}'_4$ such that every leaf of P and P' is adjacent to the same vertex in M, then $\gamma_{tdR}(G) \leq \frac{4n}{3}$.*

Proof. First, suppose that there exists an M-ear path $P \in \mathcal{P}'_2 \cup \mathcal{P}'_4$ whose leaves are adjacent to the same vertex, say s, in M such that $d_G(s) = 4$. Let $N_G(s) \backslash V(P) = \{w_1, w_2\}$ and let G' be the graph obtained from $G - (V(P) \cup \{s\})$ by adding a new edge w_1w_2. Obviously, G' has minimum degree at least two and hence by the induction hypothesis, $\gamma_{tdR}(G') \leq \frac{4|V(G')|}{3}$. Let g be a $\gamma_{tdR}(G')$-function. If $P = u_1u_2 \in \mathcal{P}'_2$, then the function h defined by $h(u_1) = 0$, $h(u_2) = 1$, $h(s) = 3$ and $h(x) = g(x)$ for other vertices x, is a TDRDF on G and hence $\gamma_{tdR}(G) \leq \omega(h) = \omega(g) + 4 = \gamma_{tdR}(G') + 4 \leq \frac{4(n-3)}{3} + 4 = \frac{4n}{3}$. If $P = u_1u_2u_3u_4 \in \mathcal{P}'_4$, then the function h defined by $h(u_1) = 0$, $h(u_3) = h(u_4) = 1$, $h(u_2) = h(s) = 2$ and $h(x) = g(x)$ for other vertices x when $g(w_1) + g(w_2) = 0$, and by $h(u_1) = h(u_4) = 0$, $h(u_2) = 1$, $h(u_3) = 2$, $h(s) = 3$ and $h(x) = g(x)$ for other vertices x when $g(w_1) + g(w_2) \geq 1$, is a TDRDF on G and hence $\gamma_{tdR}(G) \leq \omega(h) = \omega(g) + 6 = \gamma_{tdR}(G') + 6 \leq \frac{4(n-5)}{3} + 6 < \frac{4n}{3}$, as desired.

Second, suppose that for any M-ear path $P \in \mathcal{P}'_2 \cup \mathcal{P}'_4$, two leaves of P are adjacent to the same vertex in M of degree at least 5. For $k \in \{2, 4\}$, let $m_k = |\mathcal{P}'_k|$ and let $\mathcal{P}'_4 = \left\{v_1^i v_2^i v_3^i v_4^i : 1 \leq i \leq m_4\right\}$. From our earlier assumption, we note that $\mathcal{P}_2' \cup \mathcal{P}_4' \neq \emptyset$. Let $X = \bigcup_{P \in \mathcal{P}_2' \cup \mathcal{P}_4'} V(P)$ and let $G' = G - X$. Since $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_4$, we have $G' \in \mathcal{R}$ and hence by Lemma 13, G' has a TDRDF g that assigns two to every vertex of degree at least three and $\omega(g) \leq \frac{4(n-|X|)}{3}$. It is easy to check that the function h defined by $h(x) = 1$ for each $x \in \bigcup_{P \in \mathcal{P}_2'} V(P)$, $h(v_1^i) = h(v_4^i) = 0$ for $1 \leq i \leq m_4$, $h(v_2^i) = h(v_3^i) = 2$ for $1 \leq i \leq m_4$ and $h(x) = g(x)$ for each $x \in V(G')$, is a TDRDF on G and hence $\gamma_{tdR}(G) \leq \omega(h) = \omega(g) + |X| \leq \frac{4(n-|X|)}{3} + |X| < \frac{4n}{3}$, implying that Claim 5 is true. □

The proof is completed. ■

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