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COALITION GRAPHS OF PATHS, CYCLES, AND TREES

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Abstract

A *coalition* in a graph $G = (V, E)$ consists of two disjoint sets of vertices V_1 and V_2, neither of which is a dominating set of G but whose union $V_1 \cup V_2$ is a dominating set of G. A *coalition partition* in a graph G of order $n = |V|$

is a vertex partition $\pi = \{V_1, V_2, \dots, V_k\}$ of V such that every set V_i either is a dominating set consisting of a single vertex of degree $n-1$, or is not a dominating set but forms a coalition with another set V_j which is not a dominating set. Associated with every coalition partition π of a graph G is a graph called the *coalition graph* of G with respect to π, denoted $CG(G, \pi)$, the vertices of which correspond one-to-one with the sets $V_1, V_2, \dots, V_k$ of π and two vertices are adjacent in $CG(G, \pi)$ if and only if their corresponding sets in π form a coalition. In this paper we study coalition graphs, focusing on the coalition graphs of paths, cycles, and trees. We show that there are only finitely many coalition graphs of paths and finitely many coalition graphs of cycles and we identify precisely what they are. On the other hand, we show that there are infinitely many coalition graphs of trees and characterize this family of graphs.

Keywords: vertex partition, dominating set, coalition.

2020 Mathematics Subject Classification: 05C69.

1. Introduction

In order to introduce the concepts of coalitions in graphs, coalition partitions, and coalition graphs, we will need the following definitions. Let $G = (V, E)$ be a graph with vertex set V and edge set E. The *open neighborhood* of a vertex $v \in V$ is the set $N(v) = \{u \mid uv \in E\}$, and its *closed neighborhood* is the set $N[v] = N(v) \cup \{v\}$. Each vertex $u \in N(v)$ is called a *neighbor* of v, and $|N(v)|$ is the *degree* of v, denoted $\deg(v)$. In a graph G of order $n = |V|$, a vertex of degree $n-1$ is called a *full* or *universal* vertex, while a vertex of degree 0 is an *isolate*. The minimum degree of G is denoted by $\delta(G)$ and the maximum degree by $\Delta(G)$. A subset $V_i \subseteq V$ is called a *singleton set* if $|V_i| = 1$ or a *non-singleton set* if $|V_i| \geq 2$. A set $S \subseteq V$ is a *dominating set* of a graph G if every vertex in $V - S$ is adjacent to at least one vertex in S. A set $S \subseteq V$ is an independent set if the vertices of S are pairwise nonadjacent, and the *vertex independence number* $\alpha(G)$ is the maximum cardinality of an independent set of G.

Coalitions and coalition partitions were introduced by the authors in 2020 [1], where they defined the concepts in terms of general graph properties but focused on the property of being a dominating set as follows.

Definition 1. A *coalition* π in a graph G consists of two disjoint sets of vertices V_1 and V_2, neither of which is a dominating set of G but whose union $V_1 \cup V_2$ is a dominating set of G. We say that the sets V_1 and V_2 *form a coalition* and that they are *coalition partners* in π.

Definition 2. A *coalition partition*, henceforth called a *c-partition*, in a graph G is a vertex partition $\pi = \{V_1, V_2, \dots, V_k\}$ such that every set V_i of π is either a singleton dominating set of G, or is not a dominating set of G but forms a coalition with another non-dominating set $V_j \in \pi$. The *coalition number* $C(G)$ equals the maximum order k of a c-partition of G.

Naturally associated with a graph G and a c-partition of G is a graph called a *coalition graph*. Coalition graphs were defined in [1] and first studied in [2].

Definition 3. Given a graph G with a c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G, the *coalition graph* $CG(G, \pi)$ is the graph with k vertices labeled $V_1, V_2, \ldots, V_k$, corresponding one-to-one with the elements of π, and two vertices V_i and V_j are adjacent in $CG(G, \pi)$ if and only if the sets V_i and V_j are coalition partners in π, that is, neither V_i nor V_j is a dominating set of G, but $V_i \cup V_j$ is a dominating set of G.

Consider a graph G with c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ and its associated coalition graph $CG(G, \pi)$. As defined, the vertices of $CG(G, \pi)$ are in one-to-one correspondence with the sets of π and are labeled the same, that is, $CG(G, \pi)$ has vertex set $\{V_1, V_2, \ldots, V_k\}$. Note that we use V_i to simultaneously denote a set in G and its corresponding vertex in $CG(G, \pi)$, depending on the context to make it clear. Note also that a vertex V_i is an isolate in the coalition graph $CG(G, \pi)$ if and only if V_i is a singleton set of π containing a full vertex of G.

For the remainder of the paper, we denote the family of paths, cycles, and complete graphs of order n by P_n, C_n, and K_n, respectively, and the complete bipartite graph having r vertices in one partite set and s vertices in the other by $K_{r,s}$. The double star $S(r, s)$, for $r, s \geq 1$, is a tree with exactly two (adjacent) vertices that are not leaves, one of which has r leaf neighbors and the other s leaf neighbors. The *union* $G \cup H$ of two disjoint graphs G and H is the disconnected graph with components G and H. The *join* $G + H$ of two disjoint graphs G and H is the graph obtained from the union of G and H by adding every possible edge between the vertices of $V(G)$ and the vertices of $V(H)$. The *corona* $G \circ K_1$ of a graph G is the graph obtained from G by adding for each vertex $v \in V$ a new vertex v' and the edge vv'. For an integer k, we use the standard notation $i \in [k]$ to mean that i is an integer and $1 \leq i \leq k$. To aid in discussion, we sometimes use a coloring of the vertices of V to represent a partition of V, that is, we assign to each vertex a color $i \in [k]$ and define the partition to be $\{V_1, V_2, \ldots, V_k\}$, where V_i is the set of vertices colored i.

In Section 2, we discuss properties of coalition graphs. In Sections 3 and 4, we show that there is a finite number of coalition graphs of paths and cycles, respectively, and determine all of them, while in Section 5 we characterize the infinite family of coalition graphs of trees.

2. Coalition Graphs

Throughout we will use a coloring of the vertices of a graph G to form a partition π and determine the corresponding coalition graph $CG(G, \pi)$. For an example of a coalition graph of a path, let the partition $\pi = \{V_1, V_2, V_3, V_4, V_5\}$ be defined according to the coloring of the path P_6 on the left in Figure 1. That is, $V_1 = \{v_1, v_6\}$ and $V_i = \{v_i\}$ for $2 \le i \le 5$. Then $CG(P_6, \pi) \simeq P_2 \cup P_3$ as shown on the right, where for example, the vertices colored 1 and 3 form a dominating set of P_6 and therefore V_1 and V_3 are adjacent in $CG(P_6, \pi)$. Similarly for the vertices colored 1 and 4, V_1 and V_4 are adjacent in $CG(P_6, \pi)$. But the vertices colored 2 and 4 do not form a dominating set, and hence V_2 and V_4 are not adjacent in $CG(P_6, \pi)$.

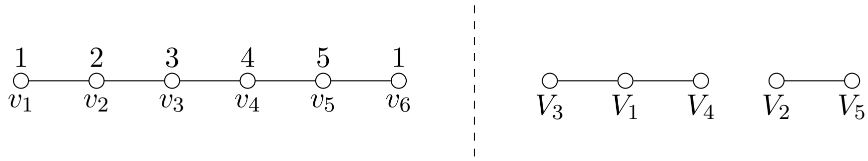

Figure 1. A c-partition π of P_6 and the coalition graph $CG(P_6, \pi)$.

We next prove an upper bound on the maximum degree of a coalition graph.

Proposition 1. *Let G be a graph and let π be a c-partition of G. Then*

$$\Delta(CG(G, \pi)) \le \Delta(G) + 1.$$

Proof. Let G be a graph and let π be a c-partition of G, and let V_i be any set in π. If V_i is a dominating set of G, then it is a singleton set containing a full vertex of G. Thus, the vertex V_i is an isolate in $CG(G, \pi)$. Hence, we may assume that V_i is not a dominating set of G, and so there is a vertex $v \in V(G)$ that is not dominated by V_i. Any set in π that forms a coalition with V_i must contain a vertex in $N_G[v]$ in G. Since there are $\deg(v) + 1 \le \Delta(G) + 1$ vertices in $N_G[v]$, the degree of V_i in the coalition graph is at most $\Delta(G) + 1$. ■

Corollary 2. *If G is a path or a cycle with c-partition π, then $\Delta(CG(G, \pi)) \le 3$.*

The next result indicates the limited structure of coalition graphs of graphs having a vertex of degree 1, and so coalition graphs of trees, in particular of paths, have this limitation.

Proposition 3. *If G is a graph with a vertex of degree 1 and π is a c-partition of order k in G, then $CG(G, \pi)$ is a spanning subgraph of the join $K_2 + \overline{K}_{k-2}$.*

Proof. Let G be a graph with a leaf vertex x and let y be the neighbor of x. Given a c-partition $\pi = \{V_1, V_2, \ldots, V_k\}$ of G, let V_1 be the set containing x. If $y \in V_1$, then since V_1 is not a singleton set, y is not a full vertex, and hence, G has no full vertex. Thus, all other sets in π must form a coalition with V_1 in order to dominate x. In this case, since no two sets V_i and V_j, for $i, j \ne 1$, form a coalition, the corresponding vertices in $CG(G, \pi)$ are not adjacent. Hence, $CG(G, \pi)$ is the star $K_{1,k-1}$, which is a spanning subgraph of $K_2 + \overline{K}_{k-2}$ as desired.

Otherwise, let V_2 be the set containing y. Again to dominate x, every coalition in π must include at least one of V_1 and V_2. It follows that the only edges in $CG(G, \pi)$ are the edges from V_i, for $3 \leq i \leq k-2$, to either V_1 or V_2 or both and possibly the edge V_1V_2. Thus, $CG(G, \pi)$ is a spanning subgraph of $K_2 + \overline{K}_{k-2}$. ■

Corollary 4. *If H is a coalition graph of a nontrivial tree and H has order k, then H is a spanning subgraph of $K_2 + \overline{K}_{k-2}$.*

Corollary 5. *If H is a coalition graph of a nontrivial tree and H has order k, then the vertex independence number of H is at least $k-2$, that is, $\alpha(H) \geq k-2$.*

We next prove a lower bound on the vertex independence number of coalition graphs.

Proposition 6. *Let G be a graph with a c-partition π having order k. If $k \geq \delta(G) + 2$, then $\alpha(CG(G, \pi)) \geq k - \delta(G) - 1$.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$, for $k \geq \delta(G) + 2$, be a c-partition of a graph G. Let v be a vertex of minimum degree $\delta(G)$ in G. Note that at most $\delta(G) + 1$ sets of π contain a vertex from $N_G[v]$. Thus, at least $k - \delta(G) - 1 \geq 1$ sets of π do not dominate v in G, and so no two of these sets form a coalition of π. Thus, their corresponding vertices form an independent set in $CG(G, \pi)$, implying that $\alpha(CG(G, \pi)) \geq k - \delta(G) - 1$. ■

Corollary 7. *If H is a coalition graph of a path and H has order $k \geq 3$, then $\alpha(H) \geq k-2$.*

Corollary 8. *If H is a coalition graph of a cycle and H has order $k \geq 4$, then $\alpha(H) \geq k-3$.*

Proposition 9. *Let G be a graph with a c-partition π having order k. If $k \geq \Delta(G) + 3$, then every vertex of $CG(G, \pi)$ is included in an independent set of cardinality at least $k - \Delta(G) - 1$.*

Proof. Let $\pi = \{V_1, V_2, \ldots, V_k\}$ for $k \geq \Delta(G) + 3$ be a c-partition of a graph G. Consider any vertex, say V_i, of $CG(G, \pi)$. By Proposition 1, $\Delta(CG(G, \pi)) \leq \Delta(G) + 1$. Thus, V_i has at most $\Delta(G) + 1$ neighbors in $CG(G, \pi)$. Since $k \geq \Delta(G) + 3 \geq \Delta(CG(G, \pi)) + 2$, there is at least one vertex of $CG(G, \pi)$, say V_j, that is not adjacent to V_i in $CG(G, \pi)$. This means that $V_i \cup V_j$ is not a dominating set of G. Hence, there is some vertex u in G that is not dominated by $V_i \cup V_j$. At most $\Delta(G) + 1$ sets of π dominate u, so at least $k - \Delta(G) - 1 \geq 2$ sets, including V_i and V_j, do not dominate u. Since no two of these $k - \Delta(G) - 1$ sets are coalition partners in π, the vertices corresponding to these sets must therefore be independent in G. Thus, the vertex V_i is in an independent set of G of cardinality at least $k - \Delta(G) - 1$. Since V_i is an arbitrary vertex of $CG(G, \pi)$, the result holds. ∎

3. Coalition Graphs of Paths

In this section we characterize the coalition graphs of paths. In fact, we show that there are only finitely many and we identify all of them. We begin with a result from [1] that gives the coalition numbers of paths.

Theorem 10 (Haynes *et al.* [1])**.** *For the path P_n,*

$$C(P_n) = \begin{cases} n & \textit{if} \quad n \leq 4, \\ 4 & \textit{if} \quad n = 5, \\ 5 & \textit{if} \quad 6 \leq n \leq 9, \\ 6 & \textit{if} \quad n \geq 10. \end{cases}$$

Corollary 11. *The coalition graph of a path has at most six vertices.*

Corollary 12. *There are only finitely many coalition graphs of paths.*

By Corollary 11, coalition graphs of paths have order at most 6. The number of graphs of order at most 6 is $1 + 2 + 4 + 11 + 34 + 156 = 208$, where the ith summand of this sum is the number of graphs of order i. It remains, therefore, to determine which of these graphs are coalition graphs of paths, henceforth called *CP-graphs*. We show that exactly 18 of the eligible graphs are CP-graphs.

Let the diamond be the graph $K_4 - e$, that is, the graph formed from the complete graph K_4 by removing an arbitrary edge e. Let the bull graph B be the graph formed by adding a leaf adjacent to two vertices in a triangle as shown in Figure 2, and let the graphs F_1 and F_2 be the graphs depicted in Figure 3. Let $\mathcal{F}$ be the family consisting of the following graphs: K_1, K_2, $\overline{K}_2$, $K_1 \cup K_2$, P_3, K_3, $K_{1,3}$, $2K_2$, P_4, C_4, F_1, $K_4 - e$, $P_2 \cup P_3$, F_2, B, P_5, $S(1, 2)$, and $S(2, 2)$. We show that the CP-graphs are precisely the graphs in $\mathcal{F}$.

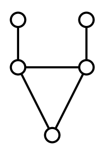

Figure 2. The bull graph B.

Theorem 13. *A graph G is a CP-graph if and only if $G \in \mathcal{F}$.*

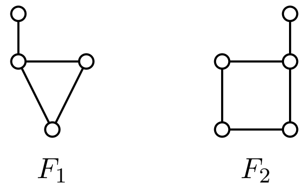

Figure 3. Graphs F_1 and F_2.

Proof. Let G be a CP-graph of order n. By Corollaries 2 and 11, G has order $n \leq 6$ and maximum degree $\Delta(G) \leq 3$. Further, by Corollary 7, if $n \geq 3$, then $\alpha(G) \geq n-2$.

Based on these properties of CP-graphs, we can eliminate many of the 208 graphs of order $n \leq 6$ immediately. To complete the proof, by order n, we determine all CP-graphs by eliminating the ones that are not CP-graphs and showing how to obtain the ones that are CP-graphs from a c-partition of a path. We denote the coloring of the vertices of a path $P_j = (u_1, u_2, \ldots, u_j)$ by the sequence $(\ell_1, \ell_2, \ldots, \ell_j)$, where it is understood that vertex u_i is assigned color ℓ_i.

We note that the only CP-graphs with isolated vertices are the coalition graphs of the paths P_1, P_2, and P_3, since these paths are the only paths with a full vertex. It follows that any CP-graph of order at least 4 has no isolated vertices. Now we consider $n \in [6]$.

For $n = 1$, the trivial graph K_1 is clearly the CP-graph of P_1 and $K_1 \in \mathcal{F}$. Both graphs of order $n = 2$, namely $\overline{K}_2$ and K_2, are CP-graphs as can be seen by the coloring $(1,2)$ for the path P_2 and the coloring $(1,1,2,2)$ of the path P_4, respectively. Hence, the result holds for $n = 2$.

Of the four graphs of order $n = 3$, only $\overline{K}_3$ is not a CP-graph, since it has three isolated vertices. The other three graphs, namely, $K_1 \cup K_2$, P_3, and K_3, are CP-graphs as can be seen by the coloring $(1,2,3)$ of the path P_3, coloring $(1,2,3,3)$ of the path P_4, and coloring $(1,2,3,1,3,2)$ of the path P_6, respectively.

Of the 11 graphs of order $n = 4$, four have an isolated vertex and are eliminated, as is K_4 since $\alpha(K_4) = 1 < n-2 = 2$. All six remaining graphs G are CP-graphs as can be seen below, where we list a path P_j, a coloring defining the c-partition π of P_j, and the CP-graph $CG(P_j, \pi) \simeq G$:

P_6, $(1,1,2,3,4,1)$, $K_{1,3}$,

P_8, $(1,2,4,4,3,3,1,2)$, $2K_2$,

P_7, $(1,2,4,2,1,3,1)$, P_4,

P_4, $(1,2,3,4)$, C_4,

P_5, $(1,2,3,4,1)$, the graph F_1 as shown in Figure 3,

P_6, $(1,2,3,4,1,2)$, the diamond $K_4 - e$.

There are 34 graphs of order $n = 5$. Of these, 11 have an isolated vertex. Of the remaining 23 graphs, 11 have a vertex of degree 4 and six do not have vertex independence number of at least $n-2=3$, leaving six graphs, five of which are CP-graphs as shown below:

P_6, $(1,2,3,4,5,1)$, $P_2 \cup P_3$,

P_9 (1,2,3,4,5,3,4,2,1), the graph F_2 as shown in Figure 3,

P_9, $(1,2,5,1,3,2,4,1,2)$, the graph bull B as shown in Figure 2,

P_7, $(1,2,3,4,5,1,2)$, P_5,

P_9, $(2,1,3,4,1,2,5,2,1)$, $S(1,2)$.

This leaves one unresolved graph, namely $K_{2,3}$, having order 5. We show that $K_{2,3}$ is not a CP-graph.

Claim 14. *The graph $K_{2,3}$ is not a CP-graph.*

Proof. Suppose to the contrary that $K_{2,3}$ is a CP-graph of a path $P_j = (u_1, u_2, \ldots, u_j)$ with c-partition $\pi = (V_1, V_2, V_3, V_4, V_5)$. Without loss of generality, let V_1 and V_2 be the vertices of $K_{2,3}$ that form the smaller partite set of $K_{2,3}$. Since the vertices V_1 and V_2 are not adjacent in $K_{2,3}$, V_1 and V_2 do not form a coalition of π, that is, $V_1 \cup V_2$ does not dominate P_j. Let u_i be a vertex in the path P_j such that u_i is not dominated by $V_1 \cup V_2$. Since each of V_3, V_4, and V_5 is in a coalition with V_1 and with V_2, it follows, without loss of generality, that $u_{i-1} \in V_3$, $u_i \in V_4$, and $u_{i+1} \in V_5$. But then it is impossible for both $V_1 \cup V_5$ and $V_2 \cup V_5$ to dominate vertex u_{i-1} in P_j. □

Finally, let us consider the 156 graphs G of order $n=6$, by their size $m=|E|$. First note that Corollary 4 and Corollary 2 imply that $m \leq 6$. The only such graph G having $m \leq 3$ and no isolated vertices is the graph $3K_2$, for which $\alpha(G) = 3 < 4 = n-2$, a contradiction. Hence, we may assume that $4 \leq m \leq 6$. By Proposition 9, every vertex of G is in an independent set with cardinality at least $n - \Delta(G) - 1 = n - 3 = 3$.

For $m=4$ there are nine graphs; six have an isolated vertex and one has an independence number of only 3, leaving two graphs, namely, $2P_3$ and $K_2 \cup K_{1,3}$, unresolved. Note the center of the $K_{1,3}$ component of $K_2 \cup K_{1,3}$ is not in an independent set of cardinality 3, so $K_2 \cup K_{1,3}$ is not a CP-graph.

To see that the graph $G \simeq 2P_3$ is not a CP-graph, assume to the contrary that G is the coalition graph $CG(P_j, \pi)$ of some c-partition $\pi = \{V_1, V_2, V_3, V_4, V_5, V_6\}$ of a path P_j, for some $j \geq 6$. Without loss of generality, let V_1 and V_2 be the vertices of degree 2 in G. Since V_1 and V_2 are not adjacent in G, the sets V_1 and V_2 do not form a coalition of π in P_j, that is, $V_1 \cup V_2$ is not a dominating set of P_j. Let u be a vertex of P_j that is not dominated by $V_1 \cup V_2$. Now each of the remaining four vertices of G is adjacent to one of V_1 and V_2, that is, their corresponding sets form coalitions with either V_1 or V_2 in π. Thus, each of these four sets must contain a vertex that dominates vertex u in the path P_j, a contradiction since at most three sets have a vertex from the closed neighborhood of u in P_j. Thus, $G \simeq 2P_3$ is not a CP-graph.

For $m = 5$, there are 15 graphs; six have isolated vertices and two others have a vertex of degree 4 or more, and five others have independence numbers less than 4, leaving only two graphs to consider: the double star $S(2,2)$ and the *broom* B_4, where B_4 is obtained from a path P_4 by attaching two leaves to one endvertex. The vertex of degree 3 in B_4 is not in any independent set having cardinality at least 3, so B_4 is not a CP-graph. Using notation previously adopted, we show that $S(2,2)$, which is in $\mathcal{F}$, is a CP-graph: the path P_{10} with partition π defined by the coloring $(2,1,5,6,1,2,4,3,2,1)$ yields $CG(P_{10},\pi) \simeq S(2,2)$.

For $m = 6$, there are 21 graphs; six have isolates, four of the remaining 15 graphs have a vertex of degree at least 4, and ten of the remaining 11 graphs have independence number less than 4. This leaves only one unresolved graph, namely, the graph G with one leaf attached to each of two nonadjacent vertices of a cycle C_4. Since neither vertex of degree 3 of G is in an independent set of cardinality 3, G is not a CP-graph.

Thus, in the final analysis, only one graph of order $n = 6$ is a CP-graph, namely the double star $S(2,2)$, and the result follows. ■

4. COALITION GRAPHS OF CYCLES

In this section we characterize the coalition graphs of cycles, henceforth called *CC-graphs*. Similar to the result with paths, we show that the number of CC-graphs is finite and and we identify all of them. We begin with a result from [1] that gives the coalition numbers of cycles.

Theorem 15 (Haynes *et al.* [1]). *For the cycle C_n,*

$$C(C_n) = \begin{cases} n & \text{if} \quad n \leq 6, \\ 5 & \text{if} \quad n = 7, \\ 6 & \text{if} \quad n \geq 8. \end{cases}$$

Corollary 16. *The coalition graph of a cycle has at most six vertices.*

Corollary 17. *There are only finitely many coalition graphs of cycles.*

By Corollary 16, CC-graphs have order at most 6. In this section, we show that exactly 27 of the 208 graphs of order at most 6 are CC-graphs.

Let $\mathcal{H}$ be the family consisting of the following 27 graphs: K_2, $\overline{K}_3$, P_3, K_3, $2K_2$, P_4, C_4, $K_{1,3}$, the graphs F_1 and F_2 shown in Figure 3, the diamond $K_4 - e$, K_4, $P_2 \cup P_3$, $K_2 \cup K_3$, P_5, the house graph H shown in Figure 4, the double star $S(1,2)$, the bull graph B shown in Figure 2, the graphs H_1, H_2, and H_3 shown in Figure 5, C_5, $3K_2$, $K_2 \cup P_4$, the corona $P_3 \circ K_1$, the double star $S(2,2)$, and the corona $K_3 \circ K_1$. We show that the CC-graphs are precisely the graphs in $\mathcal{H}$.

It is worth mentioning that with the exception of the three graphs $K_1, \overline{K}_2$, and $K_1 \cup K_2$, CP-graphs are also CC-graphs. In other words, 15 of the 18 graphs in family $\mathcal{F}$ are in $\mathcal{H}$.

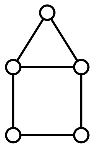

Figure 4. The house graph H.

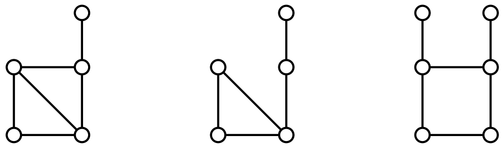

Figure 5. Graphs H_1, H_2, and H_3.

Theorem 18. *A graph G is a CC-graph if and only if $G \in \mathcal{H}$.*

Proof. Let G be a CC-graph of order n. By Corollaries 2 and 16, G has order $n \leq 6$ and maximum degree $\Delta(G) \leq 3$. We note that the trivial graph is not the CC-graph of any cycle, so $2 \leq n \leq 6$. Moreover, Corollary 8 implies that if $n \geq 4$, then $\alpha(G) \geq n - 3$. Also note that the only cycle whose CC-graph G could have an isolate is the cycle C_3. Thus, if G is not the CC-graph of the singleton partition of the cycle C_3, G has no isolated vertices, that is, $\delta(G) \geq 1$.

Hence, the trivial graph is not a CC-graph, so $2 \leq n \leq 6$. To complete the proof, by order n, we determine all CC-graphs by eliminating the ones that are not CC-graphs and showing how to obtain the ones that are CC-graphs from a c-partition of a cycle. We use the notation used in the proof for paths and show that a graph is a CC-graph by listing a cycle C_j along with a coloring defining a c-partition π of C_j and the CC-graph $CG(C_j, \pi) \simeq G$. We denote the coloring of the vertices of a cycle $C_j = (u_1, u_2, \ldots, u_j, u_1) = (u_1, u_2, \ldots, u_j)$ by the sequence $(\ell_1, \ell_2, \ldots, \ell_j)$, where it is understood that vertex u_i is assigned color ℓ_i. Now we consider the graphs G having order n for $2 \leq n \leq 6$.

Of the two graphs of order $n = 2$, clearly, $\overline{K}_2$ is not a CC-graph since it has an isolated vertex and is not the CC-graph of C_3. However, K_2 is a CC-graph as can be seen with the coloring $(1, 1, 1, 2, 2, 2)$ of the cycle C_6.

Of the four graphs of order $n = 3$, $K_1 \cup K_2$ is not a CC-graph because it has an isolated vertex. All three remaining graphs of order $n = 3$ are CC-graphs as can be seen by:

C_3, $(1, 2, 3)$, $\overline{K}_3$,

C_6, $(1, 1, 1, 2, 2, 3)$, P_3,

C_6, $(1, 2, 3, 2, 3, 1)$, K_3.

Of the 11 graphs of order $n = 4$, four have an isolated vertex and are eliminated. All seven remaining graphs are CC-graphs as can be seen by the following:

C_8, $(1, 2, 4, 4, 3, 3, 1, 2)$, $2K_2$,

C_7, $(1, 2, 4, 2, 1, 3, 1)$, P_4,

C_6, $(1, 1, 2, 3, 4, 1)$, $K_{1,3}$,

C_5, $(1, 2, 3, 4, 1)$, the graph F_1 shown in Figure 3,

C_6, $(3, 1, 3, 2, 4, 2)$, C_4,

C_6, $(1, 2, 3, 4, 1, 2)$, $K_4 - e$,

C_4, $(1, 2, 3, 4)$, K_4.

Of the 34 graphs of order $n = 5$, 11 have an isolated vertex and another 11 have a vertex of degree at least 4. Ten of the remaining 12 graphs are CC-graphs as can be seen by the following:

C_6, $(1, 2, 3, 4, 5, 5)$, $P_2 \cup P_3$,

C_{12}, $(1, 2, 5, 3, 1, 4, 3, 1, 5, 2, 3, 4)$, $K_2 \cup K_3$,

C_7, $(1, 2, 3, 4, 5, 1, 2)$, P_5,

C_{12}, $(1, 2, 5, 3, 2, 4, 3, 1, 4, 3, 2, 5)$, the house graph H as shown in Figure 4,

C_9, $(1, 2, 1, 3, 4, 1, 2, 5, 2)$, $S(1, 2)$,

C_7, $(1, 2, 5, 1, 3, 2, 4)$, the bull graph B as shown in Figure 2,

C_{12}, $(1, 2, 5, 4, 2, 1, 5, 3, 1, 5, 2, 4)$, the graph H_1 as shown in Figure 5,

C_9, $(1, 2, 4, 3, 1, 5, 3, 2, 4)$, the graph H_2 as shown in Figure 5,

C_5, $(1, 2, 3, 4, 5)$, C_5,

C_9, $(1, 2, 3, 4, 5, 3, 4, 2, 1)$, the graph F_2 as shown in Figure 3.

The remaining two, namely the $K_{2,3}$ and the graph H' shown in Figure 6, are not CC-graphs. The argument to Claim 14 in the proof to Theorem 13, which shows that $K_{2,3}$ is not a CP-graph, also shows that $K_{2,3}$ is not a CC-graph. Moreover, the same argument extends to the graph H' shown in Figure 6 since H' has three vertices adjacent to two nonadjacent vertices. Thus, the result holds for graphs of order $n \leq 5$.

We will use the following property of CC-graphs having order $n = 6$.

Claim 19. *A CC-graph of order $n = 6$ has at most one perfect matching.*

Proof. Let G be a CC-graph of order $n = 6$, defined by a c-partition $\pi = \{V_1, V_2, \ldots, V_6\}$ of a cycle C_j for some positive integer $j \geq 6$. Assume that G has a perfect matching $M = \{V_1V_2, V_3V_4, V_5V_6\}$, that is, each of the sets $S_1 = V_1 \cup V_2$, $S_2 = V_3 \cup V_4$, and $S_3 = V_5 \cup V_6$ is a dominating set of G. The only way a cycle C_j can have three disjoint dominating sets is if j is a multiple of 3. Moreover, each of the three disjoint dominating sets S_1, S_2, and S_3 of C_j must include exactly one vertex of $N[u_i]$ for every vertex u_i of C_j. Without loss of generality, the only possible coloring of C_j to produce π is $(1,3,5,2,4,6,1,3,5,2,4,6,\ldots)$. It follows that the perfect matching M is unique. □

Let G be a CC-graph of order $n=6$. By Claim 19, G has at most one unique perfect matching, and by Proposition 9, every vertex of G is in an independent set of cardinality at least $n-3=3$. Let us consider the 156 graphs of order $n=6$ in terms of their size $m=|E|$.

If $m \leq 3$, then there is only one graph without an isolated vertex, which is $3K_2$ and this is a CC-graph as can be seen by the cycle C_6 with coloring $(1,2,3,4,5,6)$.

Of the nine graphs having size $m=4$, six have an isolated vertex, leaving only the three graphs, $2P_3$, $K_2 \cup K_{1,3}$, and $K_2 \cup P_4$, for consideration.

The argument in the proof of Theorem 13 that shows $2P_3$ is not a CP-graph also shows that $2P_3$ is not a CC-graph.

Further, the graph $K_2 \cup K_{1,3}$ is not a CC-graph since the center of the star $K_{1,3}$ component is not in an independent set of cardinality at least 3.

This only leaves to be resolved the graph $K_2 \cup P_4$, which is a CC-graph as can be seen by:

C_9, $(1,2,3,4,2,6,1,5,3)$, $K_2 \cup P_4$.

For $m=5$, there are 15 graphs; six have isolated vertices and two others have $\Delta \geq 4$, leaving only seven graphs to consider. Three of these have a vertex that is not in an independent set of cardinality at least 3, leaving only four graphs. One of these, namely $K_2 \cup C_4$, has two distinct perfect matchings. An argument similar to the one that shows $2P_3$ is not a CC-graph also shows that the path P_6 is not a CC-graph. This leaves only the following two graphs, both of which are CC-graphs:

C_9, $(1,2,3,4,5,3,1,2,6)$, $P_3 \circ K_1$,

C_{10}, $(1,2,1,3,4,1,2,5,6,2)$, $S(2,2)$.

For $m=6$, there are 21 graphs; six have isolates. Four of those remaining have a vertex of degree greater than 3. Of the remaining 11 graphs, three have two distinct perfect matchings, leaving eight. Of these, six have a vertex not in an independent set of cardinality three or more. The remaining two are CC-graphs as can be seen by the following:

C_{12}, $(1,2,3,4,2,3,1,5,3,1,2,6)$, $K_3 \circ K_1$,

C_{12}, $(1,2,3,1,5,3,4,5,3,1,2,6)$, the graph H_3 as shown in Figure 5.

For $m=7$, there are 24 graphs; four have an isolate and ten others have a vertex of degree greater than 3. Of the remaining ten graphs, seven have more than one perfect matching, and the other three have a vertex not in an independent set of cardinality at least three. Thus, there are no CC-graphs of order $n=6$ and size $m=7$.

For $m = 8$, there are 24 graphs; two have isolates and 17 others have a vertex of degree greater than 3. All of the remaining five graphs have more than one perfect matching. Thus, there are no CC-graphs of order $n = 6$ and size $m = 8$.

For $m = 9$, there are 21 graphs; one has an isolated vertex and eighteen others have a vertex of degree greater than 3. Each of the remaining two graphs have more than one perfect matching. Thus, there are no CC-graphs of order $n = 6$ and size $m = 9$.

For $m \geq 10$, all graphs have a vertex of degree greater than 3. Thus, there are no additional CC-graphs, and we have completed the characterization. ∎

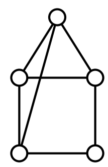

Figure 6. The graph H'.

5. Coalition Graphs of Trees

In this section we characterize the coalition graphs of trees, henceforth called *CT-graphs*. Unlike the finite families of CP-graphs and CC-graphs, we show that the family of CT-graphs is infinite.

Let $\mathcal{T}$ be the family of graphs G of order n defined as follows. For $n \leq 3$, let $G \in \{K_1, K_2, \overline{K}_2, K_1 \cup K_2, P_3, K_3\}$, and for $n \geq 4$, let G be any spanning subgraph of $K_2 + \overline{K}_{n-2}$ that has no isolated vertices.

Theorem 20. *A graph G of order n is a CT-graph if and only if $G \in \mathcal{T}$.*

Proof. Let G be a CT-graph of order n, and let T be a tree with a c-partition π such that $G \simeq CG(T, \pi)$. The only graph G having $n \leq 3$ that is not in $\mathcal{T}$ is the graph $\overline{K}_3$. But no CT-graph G can have three isolated vertices since no tree has three full vertices. If $n \leq 3$ and $G \neq \overline{K}_3$, then the result holds by Theorem 13.

Henceforth, we may assume that $n \geq 4$, and so the tree T must also have order at least 4. Since stars $K_{1,k}$ are the only trees with a full vertex, the only way for G to have an isolated vertex is if T is a star. If T is a star of order at least 3, then $G = CG(T, \pi) \simeq K_1 \cup K_2$ for every c-partition π of T, and so $G \in \mathcal{T}$ and the result holds. Therefore, we may assume that T is not a star, and so G has no isolated vertex. By Proposition 3, G is a spanning subgraph of the join $K_2 + \overline{K}_{n-2}$. To complete the proof for $n \geq 4$, it suffices to show that for every non-star, spanning subgraph G of $K_2 + \overline{K}_{n-2}$ with no isolated vertices, there exists a tree T with a c-partition $\pi = \{V_1, V_2, \ldots, V_n\}$ such that $CG(T, \pi) \simeq G$.

Let V_1 and V_2 be the two vertices of G in the subgraph K_2 of the join $K_2 + \overline{K}_{n-2}$, and let the remaining $n-2$ vertices be $V_3, V_4, \ldots, V_n$. Let $S = \{V_3, \ldots, V_{n-2}\}$, and without loss of generality, let S_1 be the vertices of S that are adjacent to V_1 but not adjacent to V_2, let S_2 be the vertices of S that are adjacent to V_2 but not adjacent to V_1, and let $S_{1,2}$ be the vertices of S that are adjacent to both V_1 and V_2 in G. Note that any of S_1, S_2, and $S_{1,2}$ could be empty, but since G has no isolated vertices, we have $|S_1| + |S_2| + |S_{1,2}| = |S| = n-2 \geq 2$.

In order to construct a tree T and a c-partition π of T such that $G \simeq CG(T, \pi)$, we first construct three colored trees T_1, T_2, and T_3. For $i \in [2]$, let T_i be the tree formed by subdividing the edges of a $K_{1,|S_i|+|S_{1,2}|}$ with center x_i exactly twice. Color the vertices of T_i for $i \in [2]$ as follows: assign color $3-i$ to the vertex x_i and also to the leaves of T_i, assign color i to the support vertices of T_i, and assign each of the remaining vertices in T_1, that is, each of the neighbors of x_1 a different color from the colors $3, 4, \ldots, |S_1| + |S_{1,2}| + 2$, while coloring the remaining vertices of T_2, that is, each of the neighbors of x_2, a different color from the colors $|S_1| + 3, \ldots, n$.

Let T_3 be obtained from the union of $n-3$ copies of the path P_5, where n is the order of G, by adding a new vertex x_3 and attaching x_3 to the center of each P_5. We call these original $n-3$ paths *base paths*. We assign color n to x_3 and then we color each base path with $(1, 2, i, 1, 2)$ such that i is a color from $3, \ldots, n-1$ and no two base paths have the same color center. In other words, each neighbor of x_3 is assigned a unique color i for $3 \leq i \leq n-1$.

Finally, we build the desired tree T with partition π from the colored trees T_1, T_2, and T_3 as follows. We note that it is possible for x_i to a be a leaf in T_i for $i \in [3]$. Let u_i be a leaf of T_i such that $u_i \neq x_i$ for $i \in [3]$.

If V_1 is adjacent to V_2 in G, then let T be the tree obtained from $T_1 \cup T_2$ by adding the edge u_1u_2. Figure 7 is an example of a coalition graph G and the tree T built by this construction.

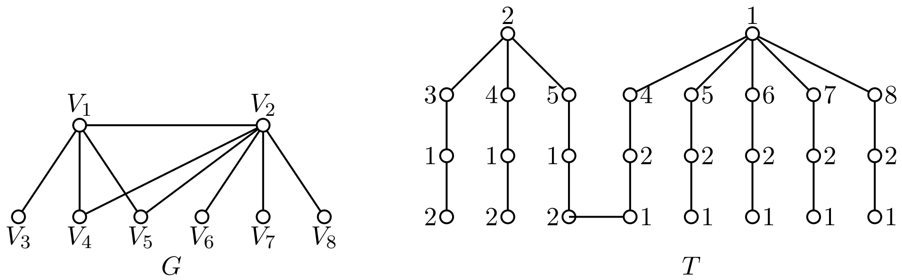

Figure 7. Graph G and tree T.

If V_1 is not adjacent to V_2 in G, then let T be the tree obtained by $T_1 \cup T_2 \cup T_3$ by adding the edges u_1u_2 and u_2u_3. Figure 8 is an example of a coalition graph

G and the tree T built by this construction.

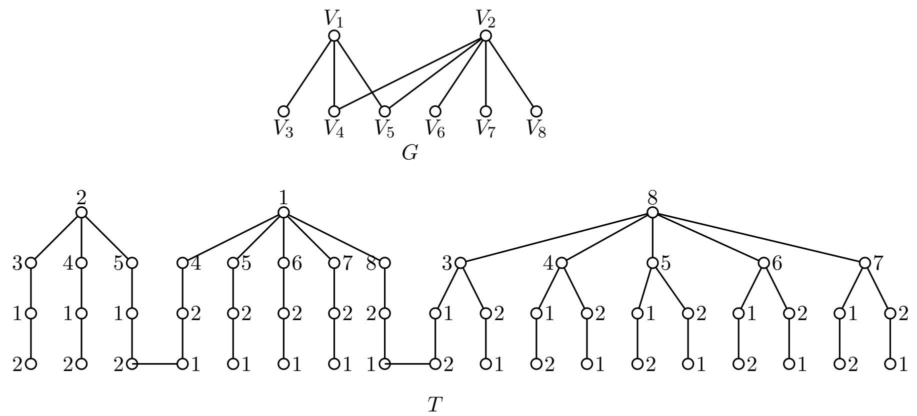

Figure 8. Graph G and tree T.

Let π be the partition of the vertices defined by the coloring of T preassigned to the subtrees T_1, T_2, and T_3. It is straightforward to verify that π is a c-partition of T and that $G \simeq CG(T, \pi)$, as desired. ■

6. CONCLUDING REMARKS

We have shown that there are only finitely many coalition graphs of paths and finitely many coalition graphs of cycles and we have characterized them. We have also characterized the infinite family of coalition graphs of trees. This raises several questions concerning the coalition graphs.

1. Can you characterize the coalition graphs of cubic graphs?
2. What can you say about the coalition graphs of r-regular graphs?
3. What can you say about the coalition graphs of grid graphs $P_m \Box P_n$? cylinders $P_m \Box C_n$? and tori $C_m \Box C_n$?
4. What can you say about the coalition graphs of n-cubes Q_n?
5. Given a positive integer k, how many coalition graphs can be defined by c-partitions of path P_k?
6. Does there exist a positive integer k such that all 18 CP-graphs can be defined by c-partitions of P_k? If so, what is the smallest universal coalition path?
7. Does there exist a universal coalition cycle, that is, a cycle C_k on which all 27 CC-graphs can be defined? If so, what is the smallest such integer k?

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